

Combinatorial Reciprocity Theorems

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based on joint work with



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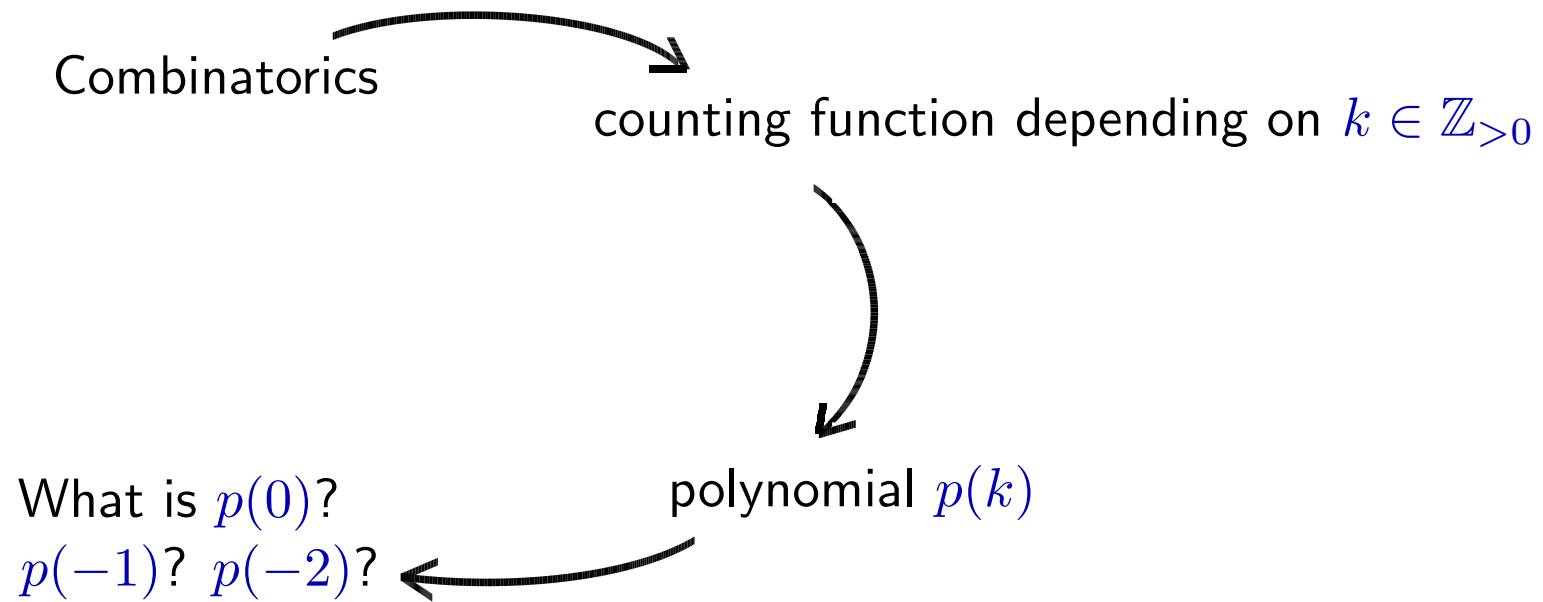
Universität Frankfurt

Thomas Zaslavsky

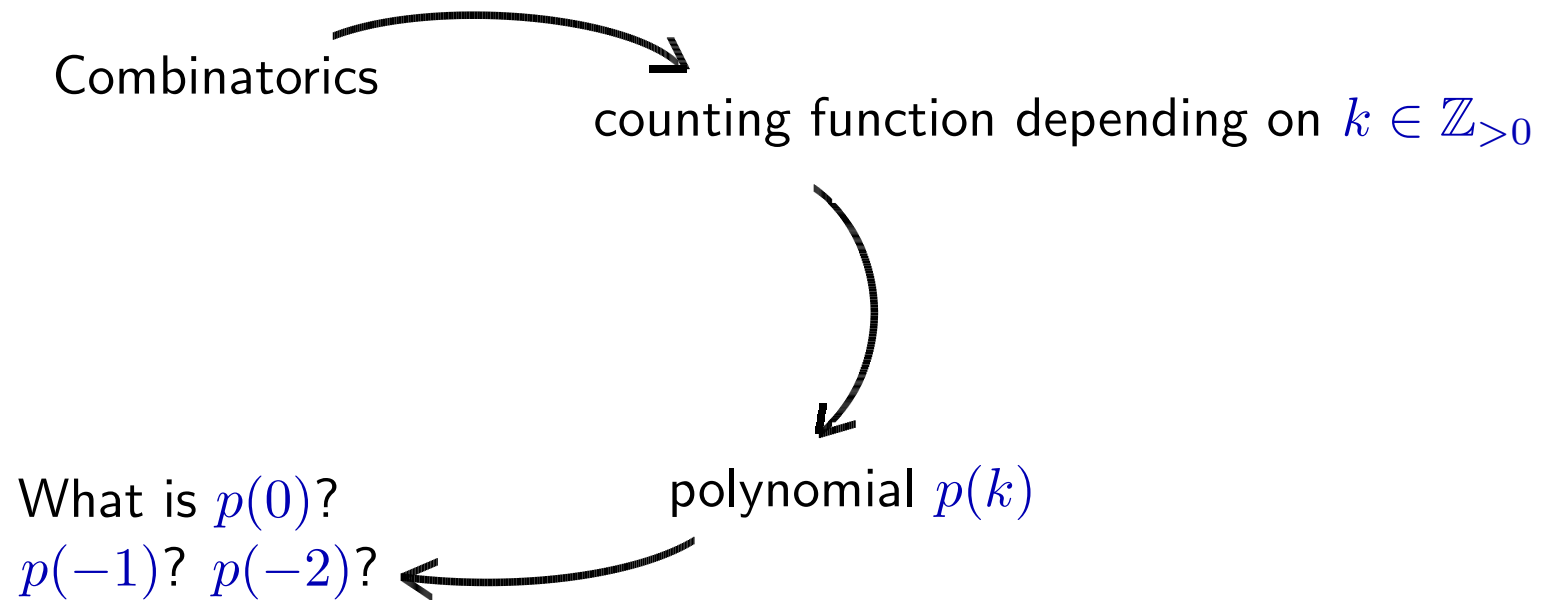
Binghamton University



The Theme



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- ▶ Two-for-one charm of combinatorial reciprocity theorems
- ▶ “Big picture” motivation: understand/classify these polynomials

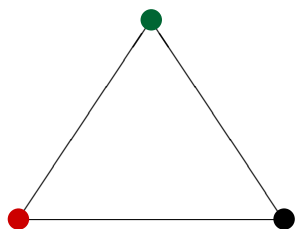
Chromatic Polynomials

$G = (V, E)$ — graph (without loops)

Proper k -coloring of G — $x \in [k]^V$ such that $x_i \neq x_j$ if $ij \in E$

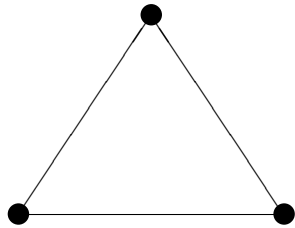
$\chi_G(k) := \#$ (proper k -colorings of G)

Example:



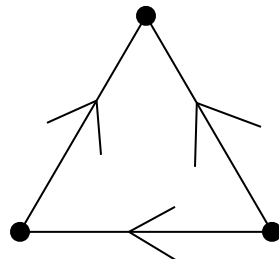
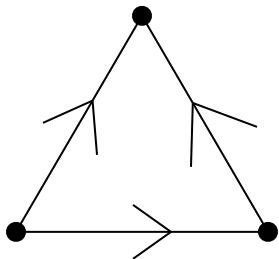
$$\chi_{K_3}(k) = k(k-1)(k-2)$$

Chromatic Polynomials



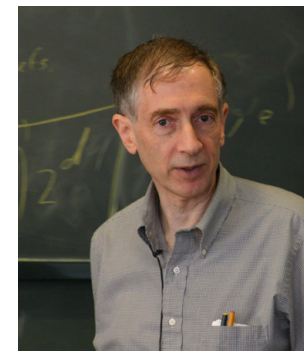
$$\chi_{K_3}(k) = k(k-1)(k-2)$$

Theorem (Birkhoff 1912, Whitney 1932)
 $\chi_G(k)$ is a polynomial in k .



$|\chi_{K_3}(-1)| = 6$ counts the number of **acyclic orientations** of K_3 .

Theorem (Stanley 1973) $(-1)^{|V|} \chi_G(-k)$ equals the number of pairs (α, \mathbf{x}) consisting of an acyclic orientation α of G and a compatible k -coloring \mathbf{x} . In particular, $(-1)^{|V|} \chi_G(-1)$ equals the number of acyclic orientations of G .



Order Polynomials

(Π, \preceq) — finite poset

$$\Omega_{\Pi}(k) := \# \{ \phi \in [k]^{\Pi} : a \preceq b \implies \phi(a) \leq \phi(b) \}$$

$$\Omega_{\Pi}^{\circ}(k) := \# \{ \phi \in [k]^{\Pi} : a \prec b \implies \phi(a) < \phi(b) \}$$

Example: $\Pi = [d] \longrightarrow \Omega_{\Pi}^{\circ}(k) = \binom{k}{d}$ and $\binom{-k}{d} = (-1)^d \binom{k+d-1}{d}$

Order Polynomials

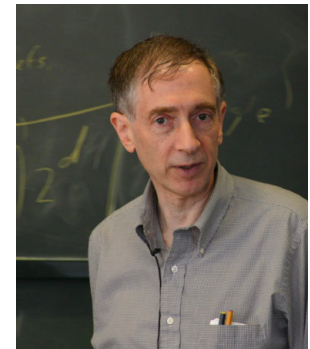
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Theorem (Stanley 1970) $\Omega_{\Pi}(k)$ and $\Omega_{\Pi}^{\circ}(k)$ are polynomials related via $\Omega_{\Pi}^{\circ}(-k) = (-1)^{|\Pi|} \Omega_{\Pi}(k)$.



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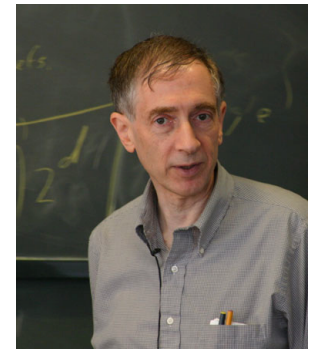
$$\sum_{k \geq 0} \Omega_{\Pi}^{(\circ)}(k) z^k = \frac{h_{\Pi}^{(\circ)}(z)}{(1-z)^{|\Pi|+1}}$$

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Equivalently, $z^{|\Pi|+1} h_{\Pi}^{\circ}(\frac{1}{z}) = h_{\Pi}(z)$.

Eulerian Simplicial Complexes

Γ — simplicial complex (collection of subsets of a finite set, closed under taking subsets)

Γ is **Eulerian** if it is pure and every interval has as many elements of even rank as of odd rank

$f_j := \# (j+1)\text{-subsets} = \# \text{ faces of dimension } j$

$$h(z) := \sum_{j=0}^{d+1} f_{j-1} z^j (1-z)^{d+1-j}$$

Theorem (Everyone 19xy) If Γ is Eulerian then $z^{d+1} h(\frac{1}{z}) = h(z)$.

Key example (Dehn–Sommerville): $\Gamma =$ boundary complex of a simplicial polytope

Ehrhart Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in \mathbb{Z}^d

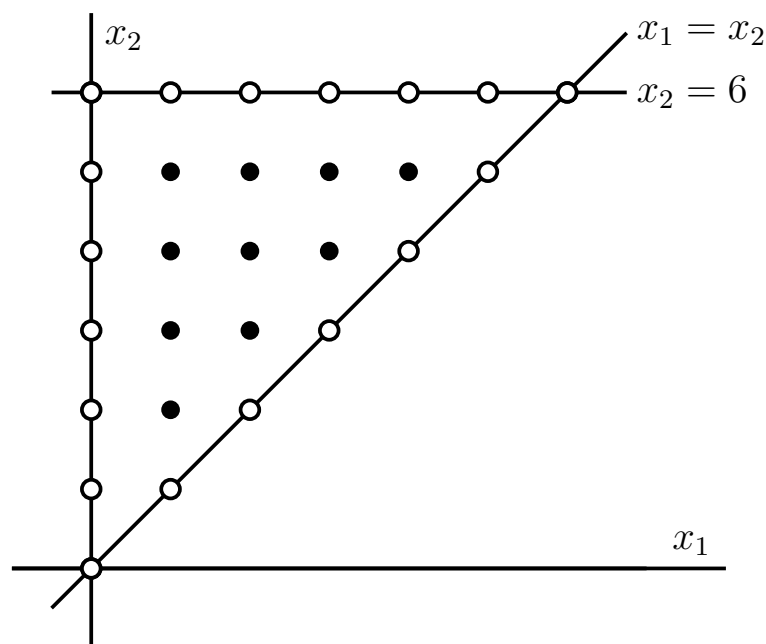
For $k \in \mathbb{Z}_{>0}$ let $\text{ehr}_{\mathcal{P}}(k) := \#(k\mathcal{P} \cap \mathbb{Z}^d)$

Example:

$$\begin{aligned} \Delta &= \text{conv} \{(0, 0), (0, 1), (1, 1)\} \\ &= \{(x, y) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2 \leq 1\} \end{aligned}$$

$$\text{ehr}_{\Delta}(k) = \binom{k+2}{2} = \frac{1}{2}(k+1)(k+2)$$

$$\text{ehr}_{\Delta}(-k) = \binom{k-1}{2} = \text{ehr}_{\Delta^{\circ}}(k)$$



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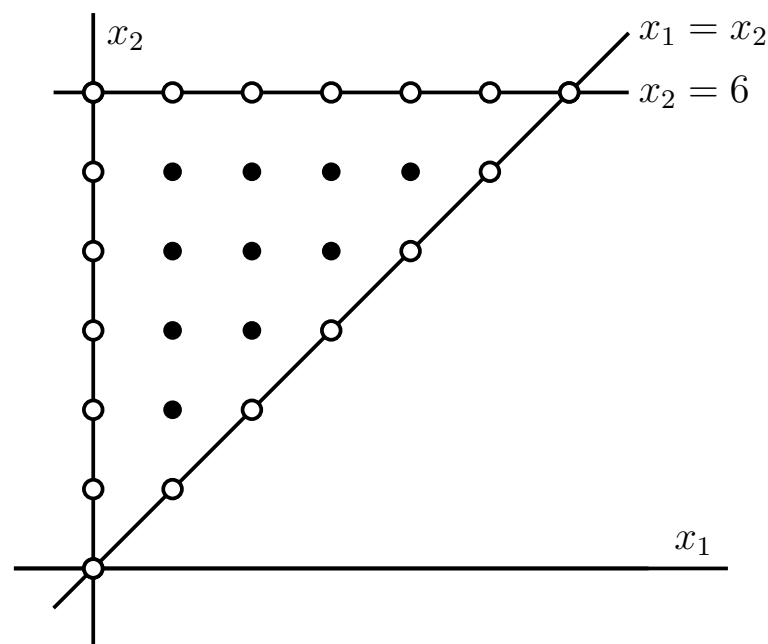
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For example, the evaluations $\text{ehr}_{\Delta}(-1) = \text{ehr}_{\Delta}(-2) = 0$ point to the fact that neither Δ nor 2Δ contain any interior lattice points.

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EH
1959

Theorem (Ehrhart 1962) $\text{ehr}_{\mathcal{P}}(k)$ is a polynomial in k .

Theorem (Macdonald 1971) $(-1)^{\dim \mathcal{P}} \text{ehr}_{\mathcal{P}}(-k)$ enumerates the **interior** lattice points in $k\mathcal{P}$.



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Theorem (Ehrhart 1962) $\text{ehr}_{\mathcal{P}}(k)$ is a polynomial in k .

$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{k>0} \text{ehr}_{\mathcal{P}}(k) z^k = \frac{h_{\mathcal{P}}^*(z)}{(1-z)^{\dim(\mathcal{P})+1}}$$

Theorem (Macdonald 1971) $(-1)^{\dim \mathcal{P}} \text{ehr}_{\mathcal{P}}(-k)$ enumerates the **interior** lattice points in $k\mathcal{P}$.

$$z^{\dim(\mathcal{P})} h_{\mathcal{P}}^*\left(\frac{1}{z}\right) = h_{\mathcal{P}^\circ}^*(z)$$



Combinatorial Reciprocity

Common theme: a combinatorial function, which is a priori defined on the positive integers,

- (1) can be algebraically extended beyond the positive integers (e.g., because it is a polynomial), and
- (2) has (possibly quite different) meaning when evaluated at negative integers.

Generating-function version: evaluate at reciprocals.

Ehrhart \longrightarrow Order Polynomials

(Π, \preceq) — finite poset

Order polytope $\mathcal{O}_\Pi := \{\phi \in [0, 1]^\Pi : a \preceq b \implies \phi(a) \leq \phi(b)\}$

$$\Omega_\Pi(k) = \#\{\phi \in [k]^\Pi : a \preceq b \implies \phi(a) \leq \phi(b)\} = \text{ehr}_{\mathcal{O}_\Pi}(k - 1)$$

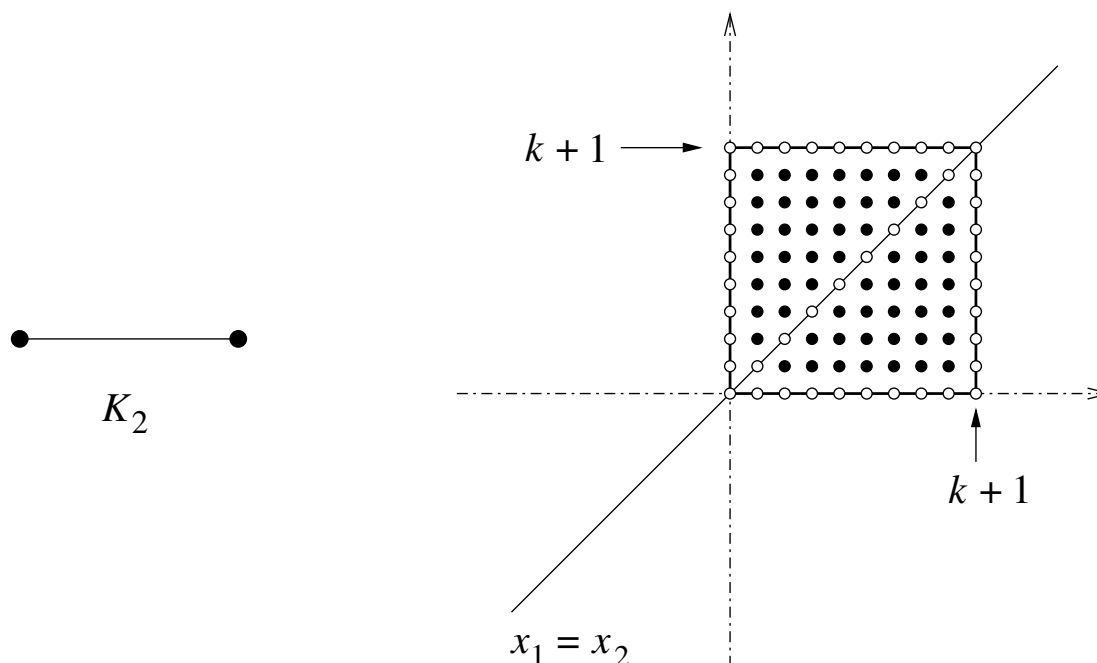
$$\Omega_\Pi^\circ(k) = \#\{\phi \in [k]^\Pi : a \prec b \implies \phi(a) < \phi(b)\} = \text{ehr}_{\mathcal{O}_\Pi^\circ}(k + 1)$$

and so $\text{ehr}_{\mathcal{P}}^\circ(-k) = (-1)^{\dim \mathcal{P}} \text{ehr}_{\mathcal{P}}(k)$ implies $\Omega_\Pi^\circ(-k) = (-1)^{|\Pi|} \Omega_\Pi(k)$

Order \longrightarrow Chromatic Polynomials

$$\chi_G(k) = \#(\text{proper } k\text{-colorings of } G) = \sum_{\Pi \text{ acyclic}} \Omega_{\Pi}^{\circ}(k)$$

$$(-1)^{|V|} \chi_G(-k) = \sum_{\Pi \text{ acyclic}} (-1)^{|\Pi|} \Omega_{\Pi}^{\circ}(-k) = \sum_{\Pi \text{ acyclic}} \Omega_{\Pi}(k)$$



Chain Partitions

(Π, \preceq) — finite graded poset with $\hat{0}$ and $\hat{1}$

$\phi : \Pi \setminus \{\hat{0}, \hat{1}\} \rightarrow \mathbb{Z}_{>0}$ order preserving

(Π, ϕ) -**chain partition** of $n \in \mathbb{Z}_{>0}$: $n = \phi(c_m) + \phi(c_{m-1}) + \cdots + \phi(c_1)$
for some multichain $\hat{1} \succ c_m \succeq c_{m-1} \succeq \cdots \succeq c_1 \succ \hat{0}$

$$\text{cp}_{\Pi, \phi}(k) := \#(\text{chain partitions of } k) \qquad \text{CP}_{\Pi, \phi}(z) := 1 + \sum_{k>0} \text{cp}_{\Pi, \phi}(k) z^k$$

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Example: $A = \{a_1 < a_2 < \cdots < a_d\} \subset \mathbb{Z}_{>0}$ $\Pi = [d]$ $\phi(j) := a_j$

\longrightarrow $\text{cp}_{\Pi, \phi}(k)$ is the **restricted partition function** with parts in A

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ϕ is **ranked** if $\text{rank}(a) = \text{rank}(b) \implies \phi(a) = \phi(b)$

Theorem If Π is Eulerian then

$$(-1)^{\text{rank}(\Pi)} \text{CP}_{\Pi, \phi} \left(\frac{1}{z} \right) = z^{\sum \text{distinct ranks}} \text{CP}_{\Pi, \phi}(z)$$

Chain Partitions for Simplicial Complexes

$\Pi = \Gamma \cup \{\hat{1}\}$ for a d -simplicial complex Γ with ground set V

$$\phi(\sigma) = \text{rank}(\sigma) = |\sigma|$$

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$$\text{cp}_{\Pi, \phi}(k) := \#(\text{chain partitions of } k) = \sum_{j=0}^{d+1} f_{j-1} \binom{k}{j}$$

Canonical geometric realization of Γ in \mathbb{R}^V :

$$\mathcal{R}[\Gamma] := \{\text{conv}\{\mathbf{e}_v : v \in \sigma\} : \sigma \in \Gamma\}$$

An Ehrhartian Interlude for Polytopal Complexes

\mathcal{C} — d -dimensional complex of lattice polytopes
with Euler characteristic $1 - (-1)^{d+1}$

Ehrhart polynomial $\text{ehr}_{\mathcal{C}}(k) := \#(k|\mathcal{C}| \cap \mathbb{Z}^d)$

We call \mathcal{C} **self reciprocal** if $\text{ehr}_{\mathcal{C}}(-k) = (-1)^d \text{ehr}_{\mathcal{C}}(k)$. Equivalently,

$$\text{Ehr}_{\mathcal{C}}(z) := 1 + \sum_{k>0} \text{ehr}_{\mathcal{C}}(k) z^k = \frac{h_{\mathcal{C}}^*(z)}{(1-z)^{d+1}} \text{ satisfies } z^{d+1} h_{\mathcal{C}}^*\left(\frac{1}{z}\right) = h_{\mathcal{C}}^*(z)$$

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Key examples (via Ehrhart–Macdonald reciprocity):

\mathcal{C} = boundary complex of a lattice polytope

\mathcal{C} = Eulerian complex of lattice polytopes

Back to Chain Partitions for Simplicial Complexes

$\Pi = \Gamma \cup \{\hat{1}\}$ for a $(d-1)$ -simplicial complex Γ with ground set V

$$\phi(\sigma) = \text{rank}(\sigma) = |\sigma| \qquad \mathcal{R}[\Gamma] := \{\text{conv}\{\mathbf{e}_v : v \in \sigma\} : \sigma \in \Gamma\}$$

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Observation 1 $\text{cp}_{\Pi, \phi}(k) := \#(\text{chain partitions of } k) = \text{ehr}_{\mathcal{R}[\Gamma]}(k)$

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Observation 1 $\text{cp}_{\Pi, \phi}(k) := \#(\text{chain partitions of } k) = \text{ehr}_{\mathcal{R}[\Gamma]}(k)$

Observation 2 $\text{CP}_{\Pi, \phi}(k) = \text{Ehr}_{\mathcal{R}[\Gamma]}(z) = \frac{h_{\mathcal{R}[\Gamma]}^*(z)}{(1-z)^{d+1}} = \frac{h_{\Gamma}(z)}{(1-z)^{d+1}}$

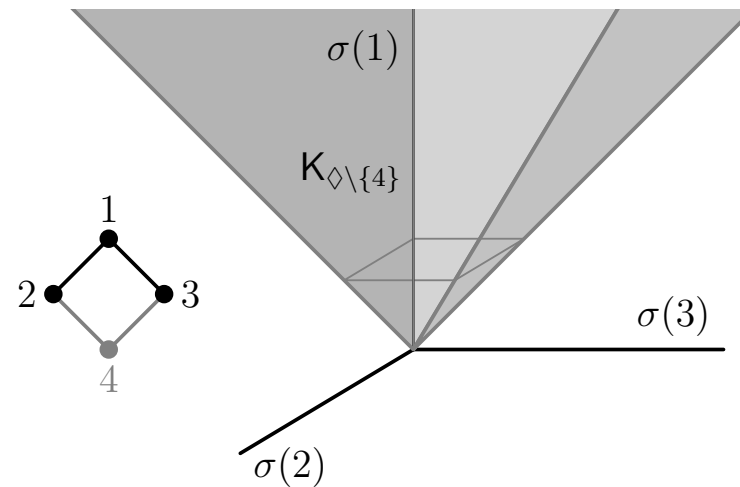
Corollary If Γ is Eulerian then $z^{d+1} h_{\Gamma}(\frac{1}{z}) = h_{\Gamma}(z)$.

The Answer To Most Of Life's Questions

(Π, \preceq) — finite poset

Order cone $\mathcal{K}_\Pi := \{\phi \in \mathbb{R}_{\geq 0}^\Pi : a \preceq b \implies \phi(a) \leq \phi(b)\}$

- ▶ Interesting geometry
- ▶ Linear extensions \longrightarrow triangulations
- ▶ Order polynomials
- ▶ P -partitions
- ▶ Euler–Mahonian statistics



For much more. . .

math.sfsu.edu/beck/crt.html



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