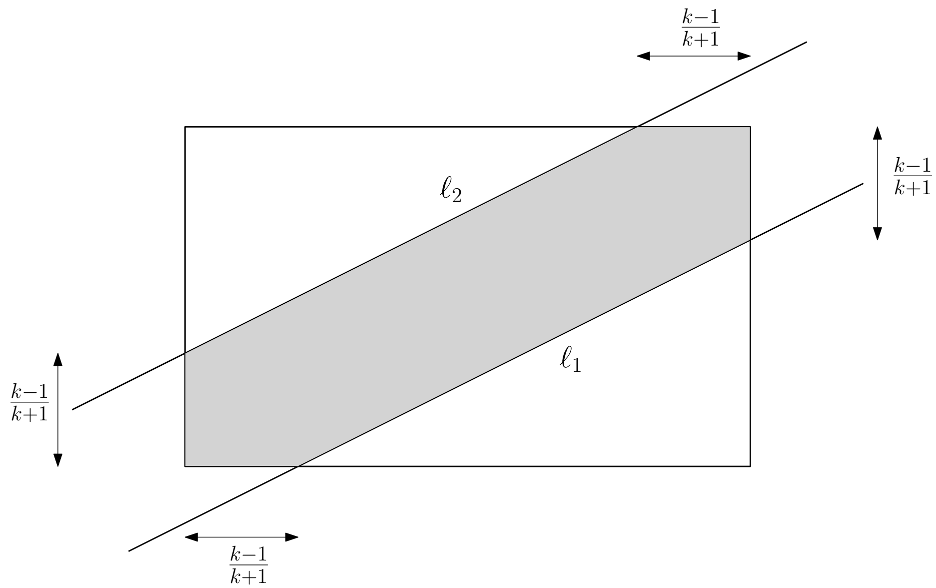
Lonely Runner Polyhedra



Matthias Beck

San Francisco State University

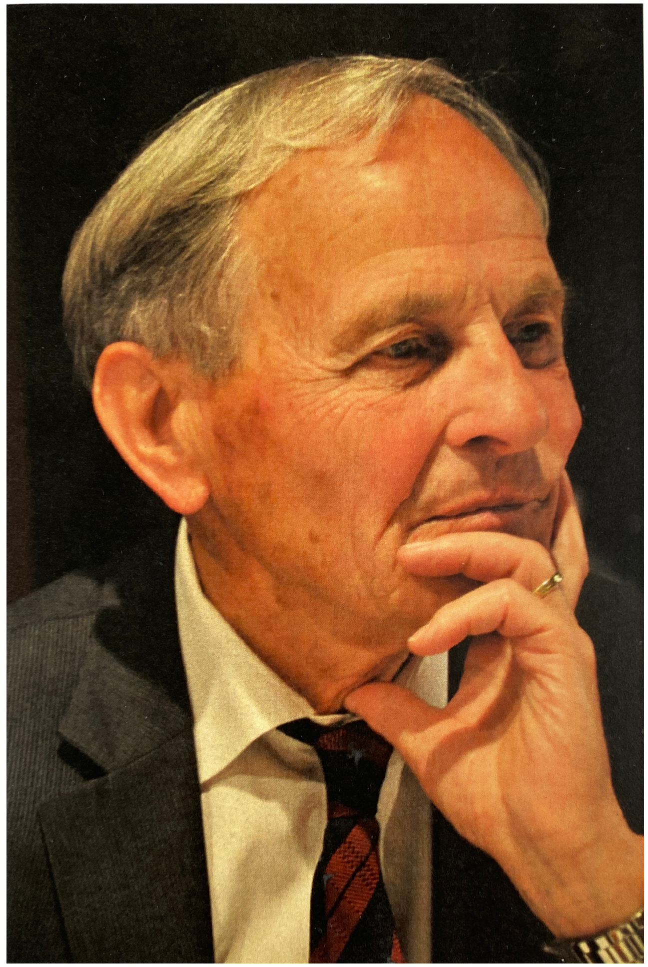
Freie Universität Berlin

Serkan Hoşten

San Francisco State University

Matthias Schymura

Brandenburgische Technische Universität Cottbus



In memoriam Günter Köhler (1940–2020)

The Lonely Runner Conjecture

$|| \cdot ||$ — distance to the nearest integer

Dirichlet's Approximation Theorem ($\sim$1840). For every $t \in \mathbb{R}$ and $k \in \mathbb{Z}_{>0}$ there exists $q \in \{1, 2, \ldots, k\}$ such that $||t\, q|| \leq \frac{1}{k+1}$.

Can this be improved by replacing $\{1, 2, \ldots, k\}$ with a different set?

Wills' Conjecture (1967). For every $\{n_1, n_2, \ldots, n_k\} \subset \mathbb{R}$ there exists $t \in \mathbb{R}$ such that $||t\, n_j|| \geq \frac{1}{k+1}$ for $1 \leq j \leq k$.

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Lonely Runner Model (Goddyn 1994).
If $k+1$ runners with different speeds move around a track of length 1 then each will at some point have distance $\frac{1}{k+1}$ to the other runners.

(Brian Weinstein @ fouriestseries ⟶)

Some History

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- Proved for $k \le 6$ (Betke–Wills 1972, Cusick–Pomerance 1984, Bohman–Holzman–Kleitman 2001, Baraja–Serra 2008)
- Known gaps of loneliness: $\frac{1}{2k}$ (exercise), $\frac{1}{2k} + \frac{c}{k^2}$ (Chen–Cusick 1999), $\frac{1}{2k} + \frac{c \log k}{k^2 (\log\log k)^2}$ (Tao 2018)

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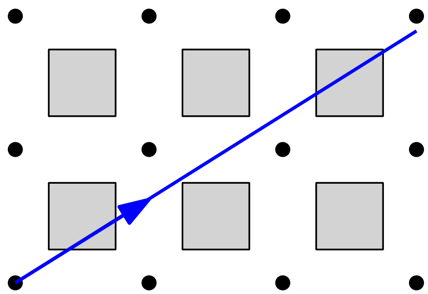
- We may assume $n_j \in \mathbb{Z}_{>0}$ (Henze–Malikiosis 2017) and thus also $\gcd(n_1, n_2, \dots, n_k) = 1$
- It suffices to consider $n_j \leq k^{ck^2}$ (Tao 2018)

Lonely Runner Geometry

Lonely Runner Conjecture (Wills 1967). For every $\mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k_{>0}$ there exists $t \in \mathbb{R}$ such that $||t\, n_j|| \geq \frac{1}{k+1}$ for $1 \leq j \leq k$.

View Obstruction Model (Cusick 1973). Every view direction $\mathbf{n} \in \mathbb{Z}^k_{>0}$ is obstructed by an integer translate of $[\frac{1}{k+1}, \frac{k}{k+1}]^k$.



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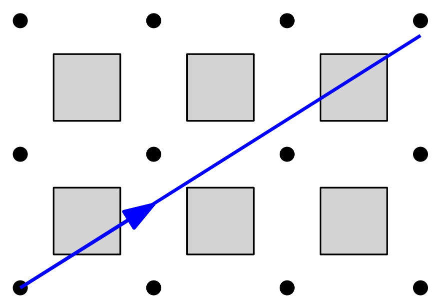

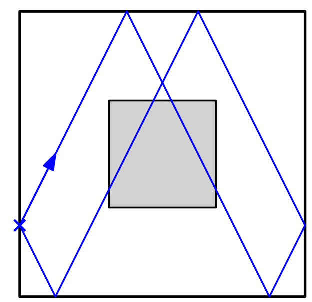

Billiard Model (Schoenberg 1976). Every billiard ball trajectory in direction $\mathbf{n} \in \mathbb{Z}_{>0}^k$ inside $[0,1]^k$ will meet $[\frac{1}{k+1}, \frac{k}{k+1}]^k$.

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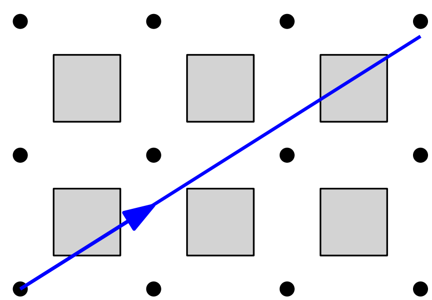

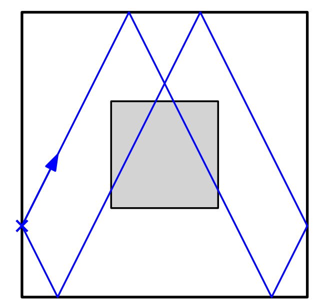

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Zonotope Model (Henze–Malikiosis 2017). For every $\mathbf{n} \in \mathbb{Z}^k_{>0}$, the zonotope $[\frac{1}{k+1}, \frac{k}{k+1}]^k \,\big|\, \mathbf{n}^\perp$ meets the lattice $\mathbb{Z}^k \,\big|\, \mathbf{n}^\perp$.

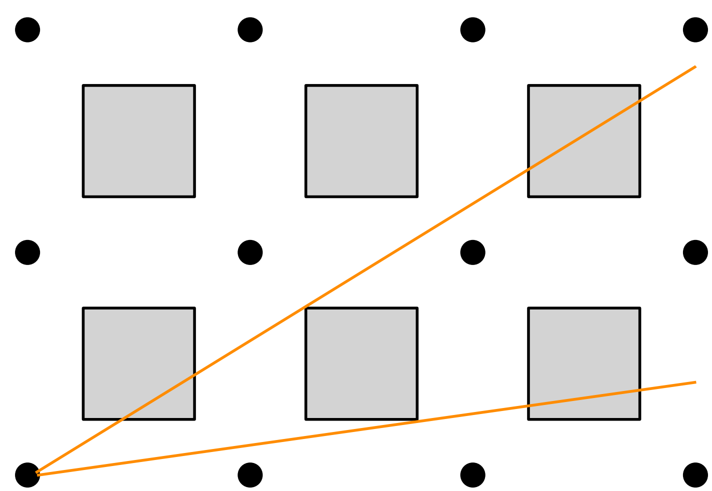
Lonely Runner Polyhedra

Lonely Runner Conjecture. For every $\mathbf{n} \in \mathbb{Z}_{>0}^k$ there exists $\mathbf{m} \in \mathbb{Z}_{\geq 0}^k$ such that

$$\mathbf{n} \;\in\; \mathcal{K}(\mathbf{m}) \;:=\; \operatorname{cone}\left(\mathbf{m} + \left[\frac{1}{k+1}, \frac{k}{k+1}\right]^k\right)$$

which is, in turn, equivalent to

$$\mathbf{m} \;\in\; \mathcal{P}(\mathbf{n}) \;:=\; \mathbb{R}\,\mathbf{n} - \left[\frac{1}{k+1}, \frac{k}{k+1}\right]^k .$$



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Polyhedral Model. $\mathcal{P}(\mathbf{n}) \cap \mathbb{Z}^k \neq \varnothing$ for every $\mathbf{n} \in \mathbb{Z}^k_{>0}$.

Goal. Given $\mathbf{n} \in \mathbb{Z}^k_{>0}$ with $\gcd(\mathbf{n}) = 1$ and $n_1 > n_2 > \cdots > n_k$, study

$$\mathcal{P}(\mathbf{n}) = \left\{\mathbf{x} \in \mathbb{R}^k : \frac{n_i - k\,n_j}{k+1} \leq n_j\,x_i - n_i\,x_j \leq \frac{k\,n_i - n_j}{k+1}\,,\; 1 \leq i < j \leq k\right\}$$

Lonely Runner Polygons

Lonely Runner Conjecture. For every $\mathbf{n} \in \mathbb{Z}_{>0}^{k}$, the polyhedron

$$\mathcal{P}(\mathbf{n}) = \left\{ \mathbf{x} \in \mathbb{R}^k : \frac{n_i - k\, n_j}{k+1} \le n_j\, x_i - n_i\, x_j \le \frac{k\, n_i - n_j}{k+1}, \ 1 \le i < j \le k \right\}$$

contains an integer lattice point.

Example (3 Runners). Here the lonely runner polyhedron is

$$\mathcal{P}(\mathbf{n}) \ = \ \left\{ \mathbf{x} \in \mathbb{R}^2 : \, n_1 - 2\, n_2 \ \le \ 3\, n_2\, x_1 - 3\, n_1\, x_2 \ \le \ 2\, n_1 - n_2 \right\}.$$

With $\gcd(n_1, n_2) = 1$ we can invoke Bézout's Lemma; note that

$$2\, n_1 - n_2 - (n_1 - 2\, n_2) + 1 \ = \ n_1 + n_2 + 1 \ \ge \ 3\,.$$

Easy Lonely Runners

Lonely Runner Conjecture. For every $\mathbf{n} \in \mathbb{Z}^k_{>0}$, the polyhedron

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Example 0. $n_1 \le k\, n_k$ if and only if $\mathbf{0} \in \mathcal{P}(\mathbf{n})$.

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Example 0. $n_1 \le k\, n_k$ if and only if $\mathbf{0} \in \mathcal{P}(\mathbf{n})$.

This includes the "extreme" case $\mathbf{n} = (k, k-1, \ldots, 1)$ which, in turn, lies on the boundary of

$$\mathcal{K}(\mathbf{0}) = \left\{ \mathbf{x} \in \mathbb{R}^k : \frac{1}{k} \le \frac{x_j}{x_i} \le k \ \text{ for } 1 \le i < j \le k \right\}.$$

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contains an integer lattice point.

Example (small divisors). If there exists an integer $\le k+1$ that does not divide any of $n_1, n_2, \ldots, n_k$ then $\mathcal{P}(\mathbf{n}) \cap \mathbb{Z}^k \neq \varnothing$.

If a is such an integer, set $m_j := \left\lfloor \frac{n_j}{a} \right\rfloor = \frac{n_j}{a} - \left\{ \frac{n_j}{a} \right\} \implies \mathbf{m} \in \mathcal{P}(\mathbf{n})$

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Example^2. If all n_j are odd then $\mathcal{P}(\mathbf{n}) \cap \mathbb{Z}^k \ne \varnothing$.

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contains an integer lattice point. Note that $\mathbf{m} \in \mathcal{P}(\mathbf{n})$ is equivalent to

$$\frac{(k+1)\, m_j + 1}{(k+1)\, m_i + k} \;\le\; \frac{n_j}{n_i} \;\le\; \frac{(k+1)\, m_j + k}{(k+1)\, m_i + 1}, \; 1 \le i < j \le k\,.$$

Example[2]. If $\mathbf{n} = 2\,\mathbf{m} + 1$ then $\mathbf{m} \in \mathcal{P}(\mathbf{n}) \cap \mathbb{Z}^k$.

Easy Lonely Runners

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contains an integer lattice point.

Theorem (MB–Hoșten–Schymura). Let $E := \{ j \in [k] : n_j \text{ is even} \}$ and $O := [k] \setminus E$. If

$$\begin{aligned} \max \{ n_j : j \in O \} &\le \tfrac{k-1}{2} \min \{ n_j : j \in E \} \\ \max \{ n_j : j \in E \} &\le k \min \{ n_j : j \in E \} \end{aligned}$$

then $\mathcal{P}(\mathbf{n}) \cap \mathbb{Z}^k \neq \varnothing$.

Corollary. If $n_2, n_3, \ldots, n_k$ are odd then $\mathcal{P}(\mathbf{n}) \cap \mathbb{Z}^k \neq \varnothing$.

Try Polyhedral Geometry

Lonely Runner Conjecture. For every $\mathbf{n} \in \mathbb{Z}^k_{>0}$, the polyhedron

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contains an integer lattice point.

- Lineality space $\mathbb{R}\,\mathbf{n}$ $\longrightarrow$ for existence questions of integer lattice points, we may bound $\mathcal{P}(\mathbf{n})$
- Iterative construction through projection?
- Iterative construction through cross section?

A Projection Ansatz

Lonely Runner Conjecture. For every $\mathbf{n} \in \mathbb{Z}^k_{>0}$, the polyhedron

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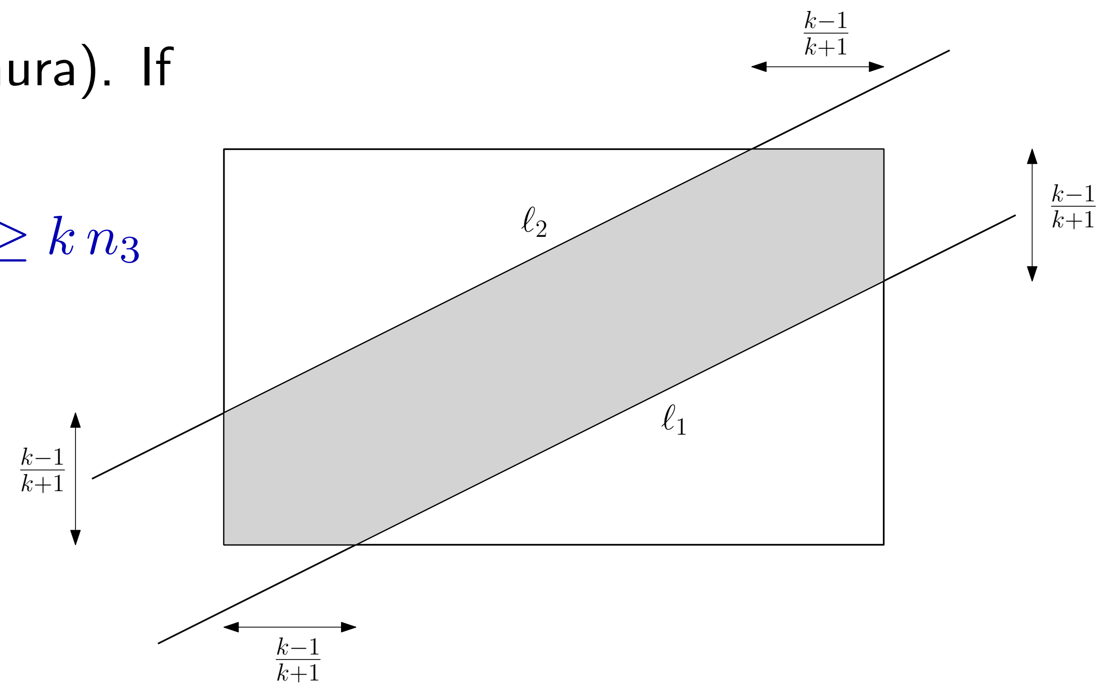
contains an integer lattice point.

Theorem (MB–Hoșten–Schymura). If

(a) $n_2 \le (k-2)\, n_k$ or

(b) $n_3 \le (k-2)\, n_k$ and $n_2 \ge k\, n_3$

then $\mathcal{P}(\mathbf{n}) \cap \mathbb{Z}^k \ne \varnothing$.



A Cross Section Ansatz

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contains an integer lattice point.

Theorem (MB–Hoșten–Schymura). If $\frac{n_j}{n_{j+1}} \ge \frac{2k}{k-1}$ for $2 \le j \le k-2$, and $\gcd(n_{k-1}, n_k) \le \frac{k-1}{k+1}(n_{k-1} - n_k)$, then $\mathcal{P}(\mathbf{n}) \cap \mathbb{Z}^k \ne \varnothing$.

The lacunary conditions allow us to iteratively construct integers $m_k, m_{k-1}, \ldots, m_1$ with $\mathbf{m} \in \mathcal{P}(\mathbf{n})$ (and the first and last step are easy).

A Few More Directions

$$\mathcal{P}(\mathbf{n}) = \left\{ \mathbf{x} \in \mathbb{R}^k : \frac{n_i - k\, n_j}{k+1} \le n_j\, x_i - n_i\, x_j \le \frac{k\, n_i - n_j}{k+1},\ 1 \le i < j \le k \right\}$$

- Is it true that each translate of $\mathcal{P}(\mathbf{n})$ meets $\mathbb{Z}^k$? (This would mean the runners can start at different places.)

- For this more general conjecture, it is crucial that the m_j are distinct: relaxing both conditions yields a theorem of Schoenberg (1976) with optimal gap of loneliness $\frac{1}{2k}$.

- For $k = 2$ the cones $\mathcal{K}(\mathbf{m})$ with $m_1\, m_2 = 0$ already cover $\mathbb{R}^2_{\ge 0}$. Is there anything remotely like this true for general k?

- Study the cones $\mathcal{K}(\mathbf{m})$

$$\left\{ \mathbf{x} \in \mathbb{R}^k : \frac{(k+1)\, m_j + 1}{(k+1)\, m_i + k} \le \frac{x_j}{x_i} \le \frac{(k+1)\, m_j + k}{(k+1)\, m_i + 1},\ 1 \le i < j \le k \right\}$$

Another Zonotope, Another Lattice

$$\widetilde{\mathbf{n}} := \frac{1}{n_k} (n_1, n_2, \ldots, n_{k-1})$$

$\mathcal{Z}$ — zonotope generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{k-1}, \widetilde{\mathbf{n}}$

Lonely Runner Conjecture. For every $\mathbf{n} \in \mathbb{Z}_{>0}^k$

$$\frac{1}{k+1} (\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{k-1} + \widetilde{\mathbf{n}}) \; + \; \frac{k-1}{k+1} \mathcal{Z} (\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_{k-1}, \widetilde{\mathbf{n}})$$

meets the lattice spanned by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{k-1}, \widetilde{\mathbf{n}}$.

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- The translate matters...
- Counterexamples, anyone?