The Lonely Runner
An Unsolved Mystery of Mathematics

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Math Encounters

MOMATH

NATIONAL MUSEUM OF MATHEMATICS

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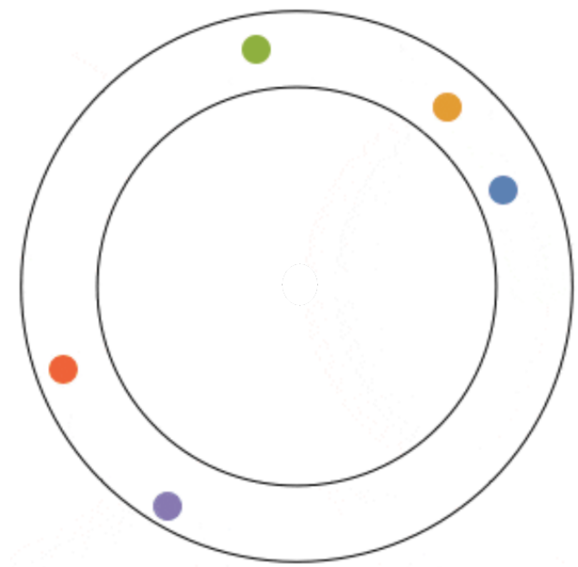

[Brian Weinstein @ fouriestseries]

The Lonely Runner Problem

Version 1. If $k+1$ runners start at the same point of a track of length 1 running with different speeds, then each runner will at some point have distance at least g to the other runners. Find a maximal g (depending only on k).

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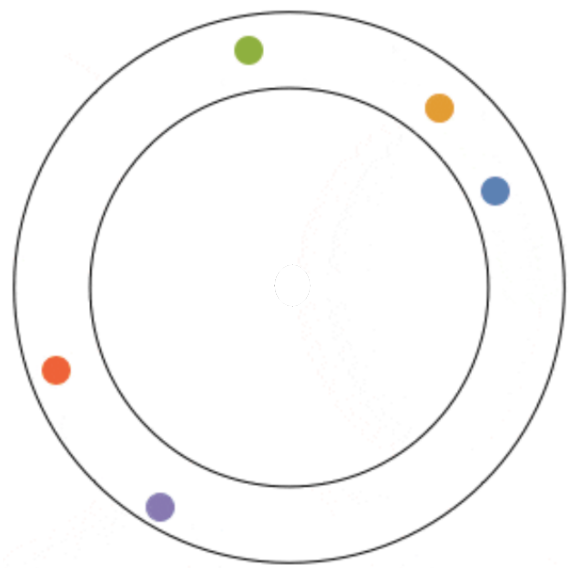
Version 2. If $k+1$ runners (not necessarily starting at the same point) on a track of length 1 run with different speeds, then each runner will at some point have distance at least g to the other runners. Find a maximal g (depending only on k).

Version 3. If $k+1$ runners (not necessarily starting at the same point) on a track of length 1 run (not necessarily with different speeds), then each runner will at some point have distance at least g to the other runners. Find a maximal g (depending only on k).

The Lazy Lonely Runner Problem

Version 1. If k runners start at the same point of a track of length 1 running with different speeds, then each runner will at some point have distance at least g to the starting point. Find a maximal g (depending only on k).



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The Lazy Lonely Runner Problem

Version 1. If k runners start at the same point of a track of length 1 running with different speeds, then each runner will at some point have distance at least g to the starting point. Find a maximal g (depending only on k).

Version 2. If k runners (not necessarily starting at the same point) on a track of length 1 run with different speeds, then each runner will at some point have distance at least g to a fixed point on the track. Find a maximal g (depending only on k).

Version 3. If k runners (not necessarily starting at the same point) on a track of length 1 run (not necessarily with different speeds), then each runner will at some point have distance at least g to a fixed point on the track. Find a maximal g (depending only on k).

The Lonely Runner Conjecture

Version 1. If k runners start at the same point of a track of length 1 running with different speeds, then each runner will at some point have distance at least g to the starting point. Find a maximal g (depending only on k).
Conjecture (Wills 1967) $g = \frac{1}{k+1}$. True for $k \le 6$ (Barajas & Serra 2008).

Version 2. If k runners (not necessarily starting at the same point) on a track of length 1 run with different speeds, then each runner will at some point have distance at least g to a fixed point on the track. Find a maximal g (depending only on k).
Conjecture (Wills) $g = \frac{1}{k+1}$. True for $k \le 3$ (Cslovijecsek, Malikiosis, Naszódi & Schymura 2022).

Version 3. If k runners (not necessarily starting at the same point) on a track of length 1 run (not necessarily with different speeds), then each runner will at some point have distance at least g to a fixed point on the track. Find a maximal g (depending only on k).
Theorem (Schoenberg 1976) $g = \frac{1}{2k}$.

How The Lonely Runner Got Started

$||\cdot||$ — distance to the nearest integer

Dirichlet's Approximation Theorem ($\sim$1840). For every $t \in \mathbb{R}$ and $k \in \mathbb{Z}_{>0}$ there exists $q \in \{1, 2, \ldots, k\}$ such that $||t\, q|| \le \frac{1}{k+1}$.

In 1967, Wills asked: Can this be improved by replacing $\{1, 2, \ldots, k\}$ with a different set?

The Lonely Runner Conjecture says it cannot: For every set $\{n_1, n_2, \ldots, n_k\}$ there exists $t \in \mathbb{R}$ such that $||t\, n_j|| \ge \frac{1}{k+1}$ for $1 \le j \le k$.

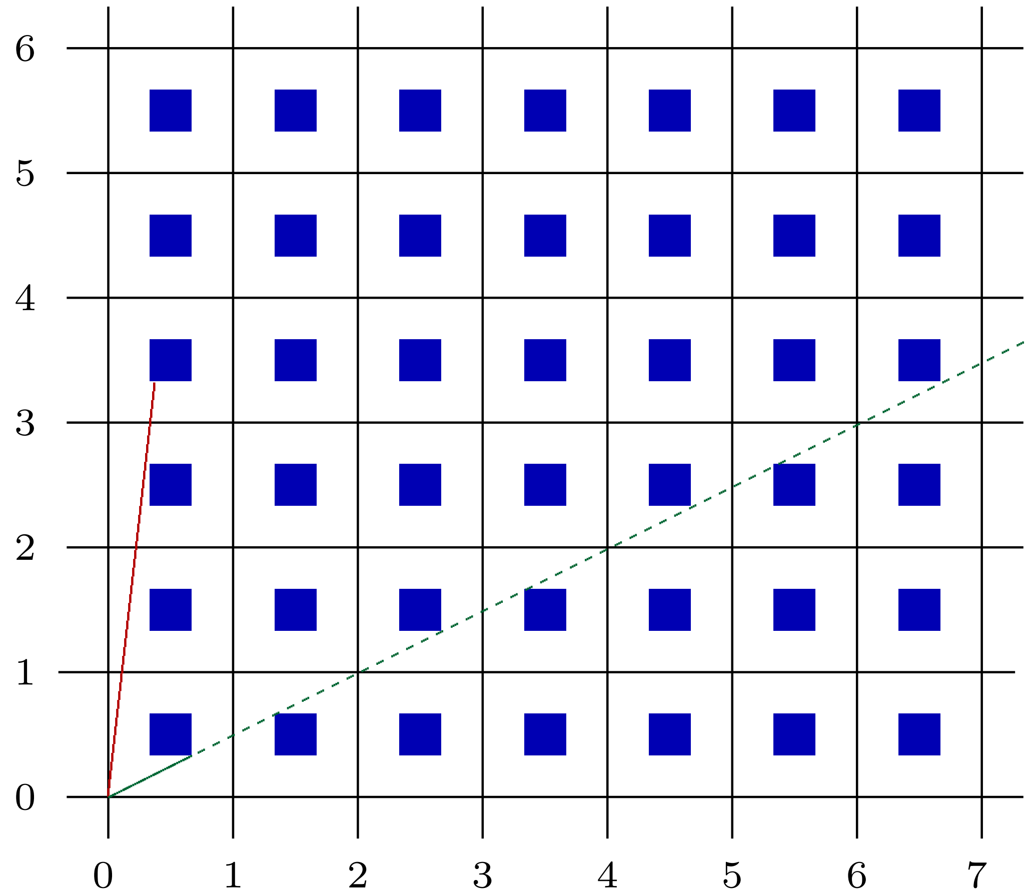
Some More History

Version 1. If k runners start at the same point of a track of length 1 running with different speeds, then each runner will at some point have distance at least g to the starting point. Find a maximal g (depending only on k).
Conjecture (Wills 1967) $g = \frac{1}{k+1}$.

- Proved for $k \le 6$ (Betke & Wills 1972, Cusick & Pomerance 1984, Bohman, Holzman & Kleitman 2001, Baraja & Serra 2008)
- Known gaps of loneliness: $\frac{1}{2k}$ (Wills 1967), $\frac{1}{2k} + \frac{c}{k^2}$ (Chen & Cusick 1999), $\frac{1}{2k} + \frac{c \log k}{k^2 (\log \log k)^2}$ (Tao 2018)
- We may assume $n_j \in \mathbb{Z}_{>0}$ (Henze & Malikiosis 2017) and thus also $\gcd(n_1, n_2, \ldots, n_k) = 1$
- It suffices to consider $n_j \le k^{ck^2}$ (Tao 2018)

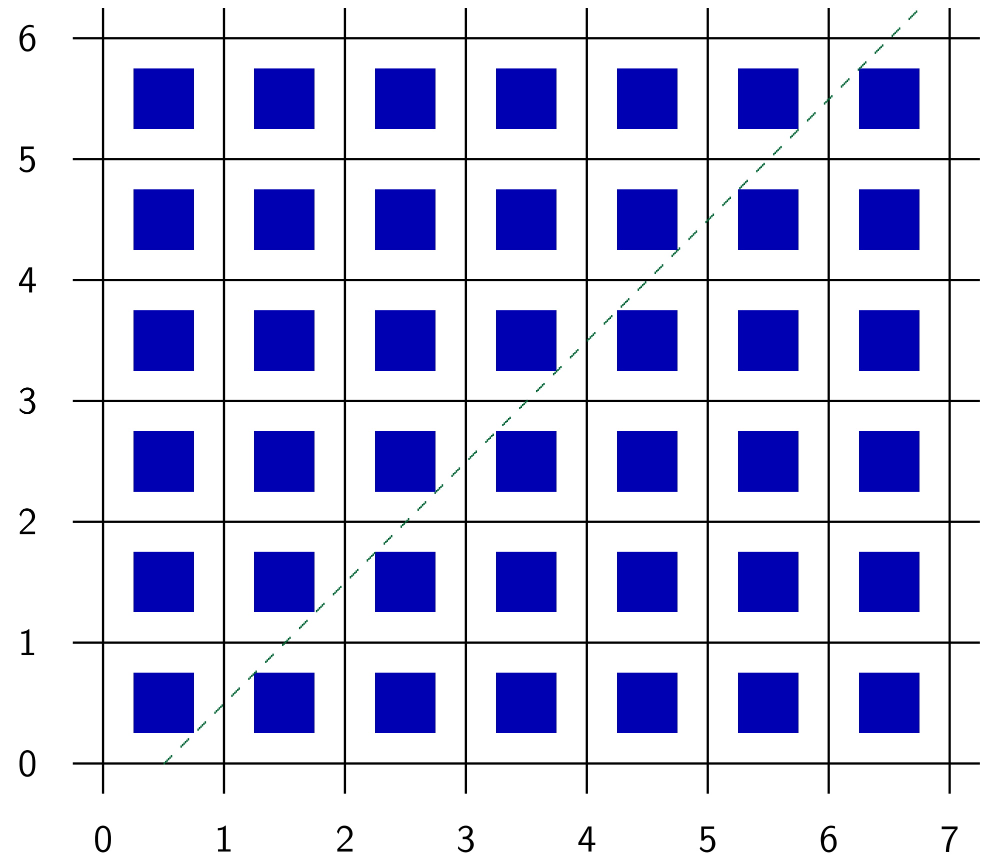
Lonely Runner Geometry: View Obstruction

Version 1. If 2 runners start at the same point of a track of length 1 running with different speeds, then each runner will at some point have distance at least $\frac{1}{3}$ to the starting point.



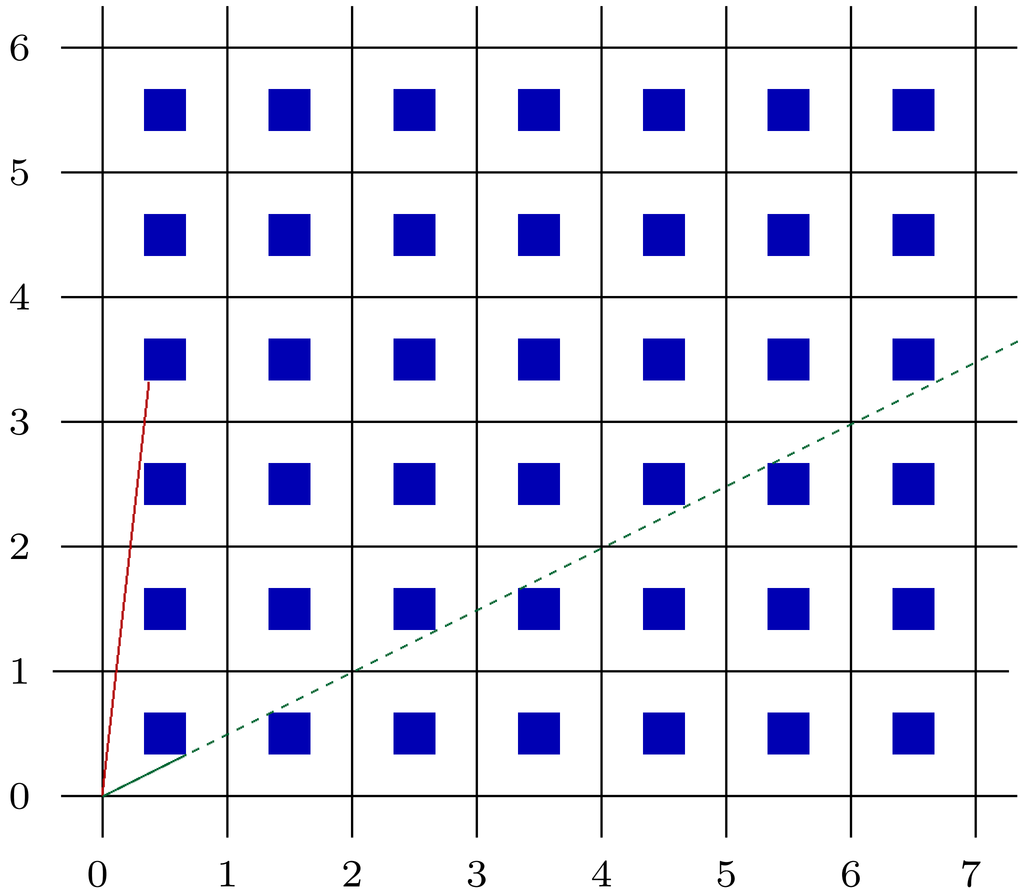
Lonely Runner Geometry: View Obstruction

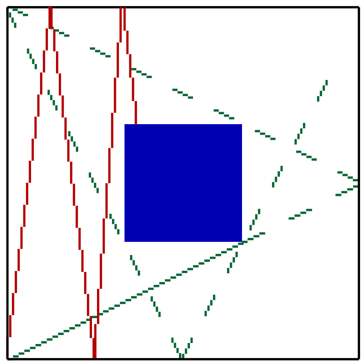
Version 3. If 2 runners (not necessarily starting at the same point) on a track of length 1 run (not necessarily with different speeds), then each runner has at some point distance at least $\frac{1}{4}$ to a fixed point on the track.



Lonely Runner Billiards

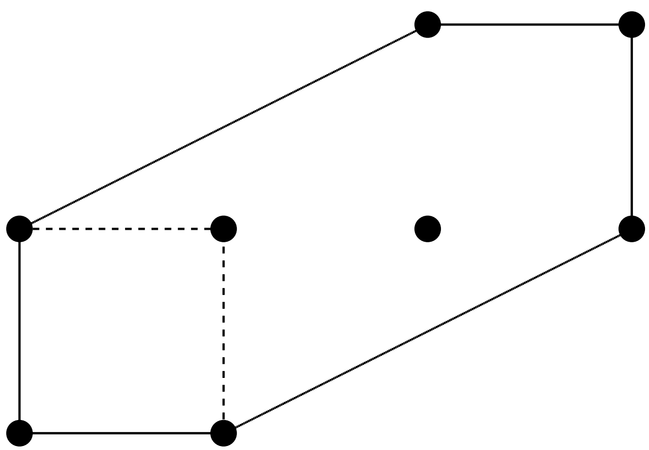
Version 1. If 2 runners start at the same point of a track of length 1 running with different speeds, then each runner will at some point have distance at least $\frac{1}{3}$ to the starting point.

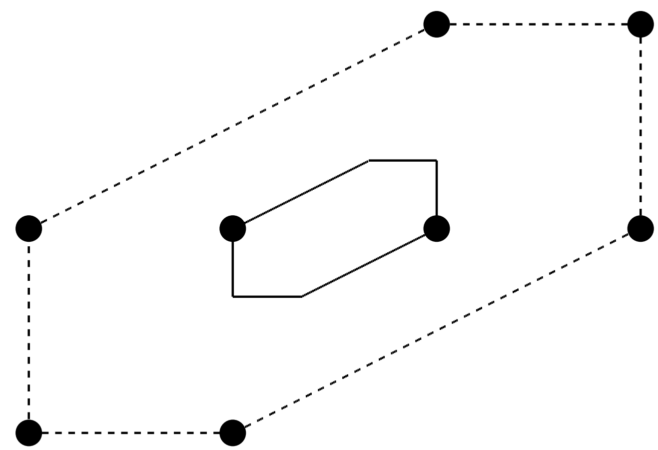


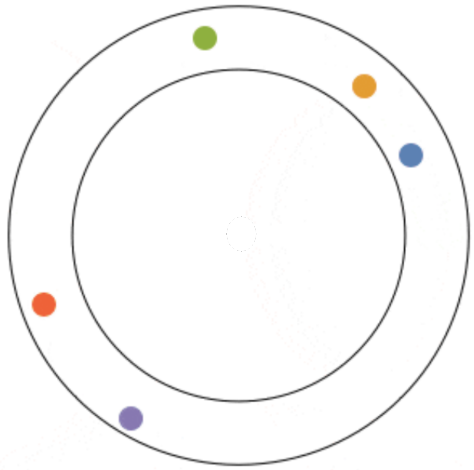


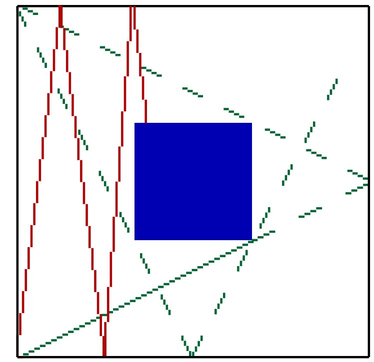
More Lonely Runner Geometry: Integer Programming

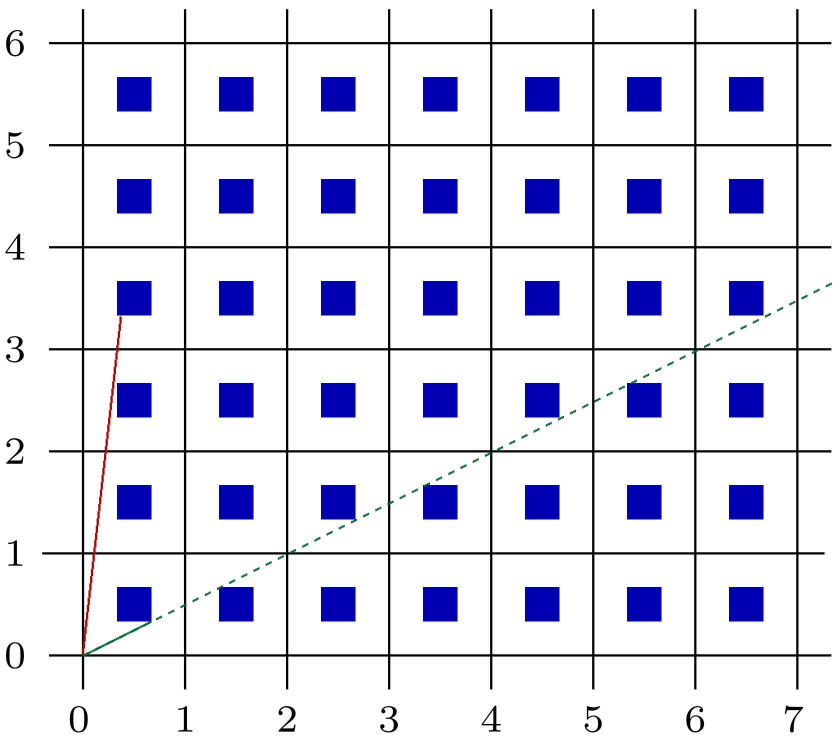
Version 7. Given distinct positive integers $n_1, n_2, \ldots, n_k$ we point-wise add the vector $(n_1, n_2, \ldots, n_k)$ to the k-dimensional unit cube. Shrink the resulting zonotope by a factor $\frac{k-1}{k+1}$, keeping its center of mass. Then the shrunken zonotope contains a point with integer coordinates.







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