The Enumeration of Nowhere-Zero Integral Flows on Graphs

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Flows on Graphs

A : abelian group

A-flow on a (bridgeless) graph $\Gamma = (V, E)$: mapping $x : E \to A$ such that for every node $v \in V$

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Theorem

(Tutte 1954) $\overline{\varphi}_\Gamma(|A|)$ is a polynomial in $|A|$.

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Remarks: $\overline{\varphi}_\Gamma(k) > 0$ if and only if $\varphi_\Gamma(k) > 0$.
For a plane graph, $\overline{\varphi}_\Gamma(k) = \chi_{\Gamma^*}(k) := \#\,(\text{proper } k\text{-colorings of } \Gamma^*)$.

Enter Geometry

Let $C \subseteq \mathbb{R}^E$ denote the real cycle space of Γ defined by the equations

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and let $\Box = (-1,1)^E$. Then a k-flow on Γ is a point in $\Box \cap C \cap \frac{1}{k}\mathbb{Z}^E$, that is, a k-fractional lattice point in the polytope $\Box \cap C$.

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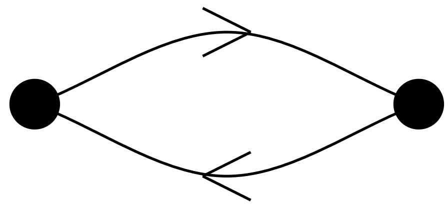
Moreover, if we let $\mathcal{H}$ denote the set of coordinate hyperplanes in $\mathbb{R}^E$, then a nowhere-zero k-flow on Γ is a k-fractional lattice point in

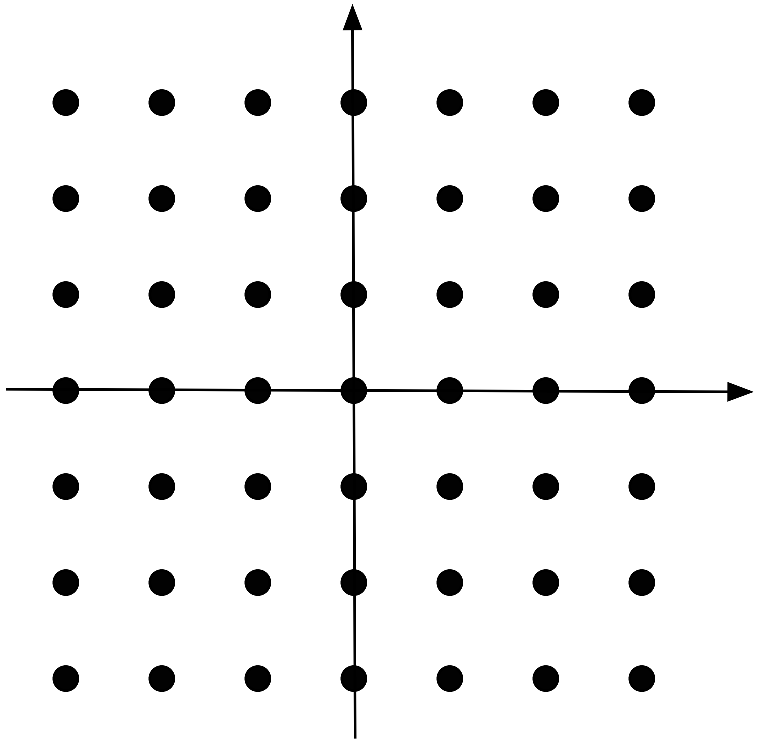
$$(\Box \cap C) \setminus \bigcup \mathcal{H} ,$$

an instance of an inside-out polytope.

A Simple Example



$2K_2$



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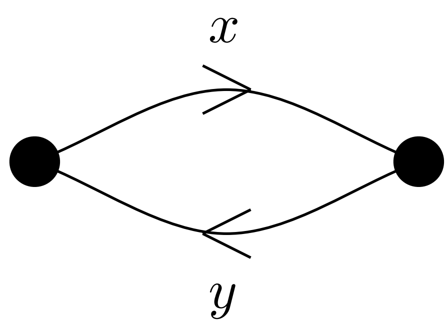

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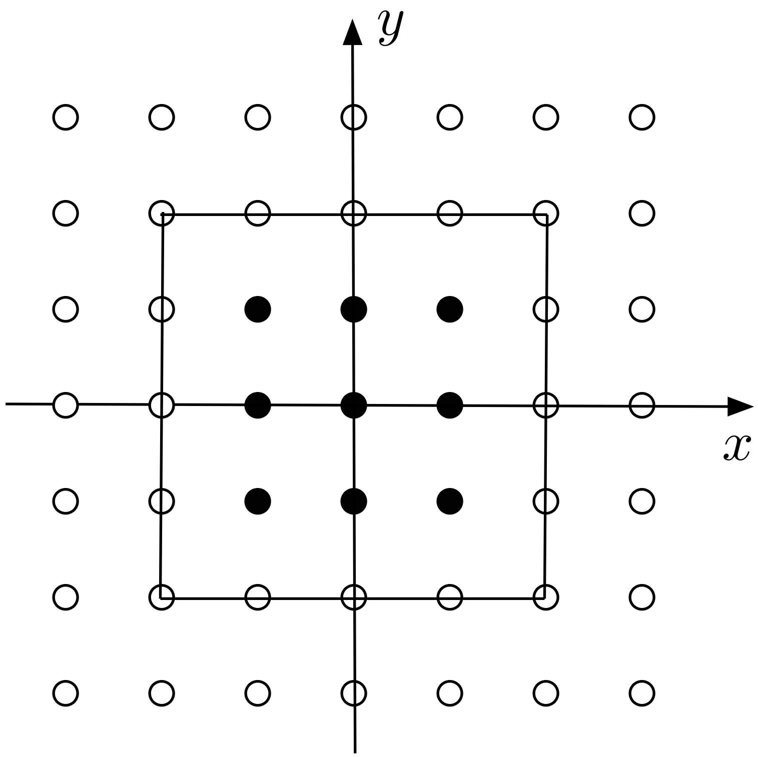

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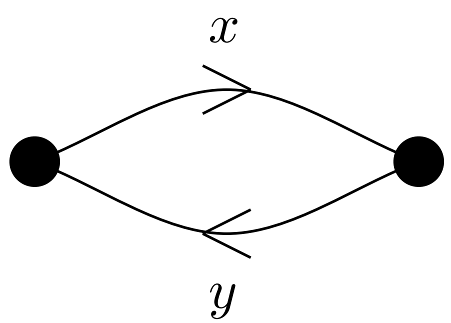

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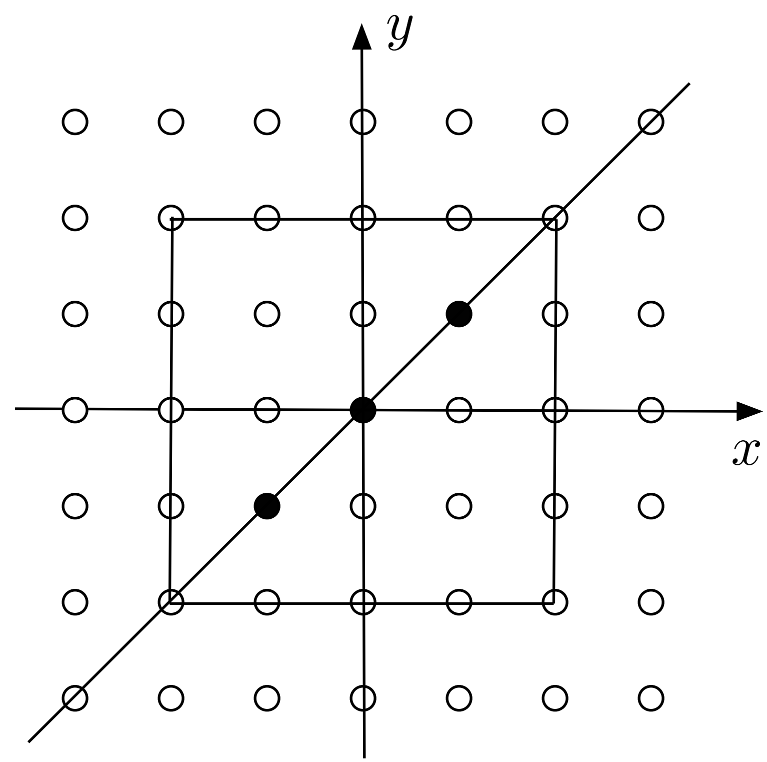

$$\phi^0_{2K_2}(k) = 2k - 1$$

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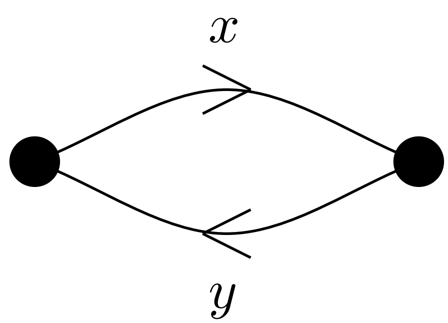

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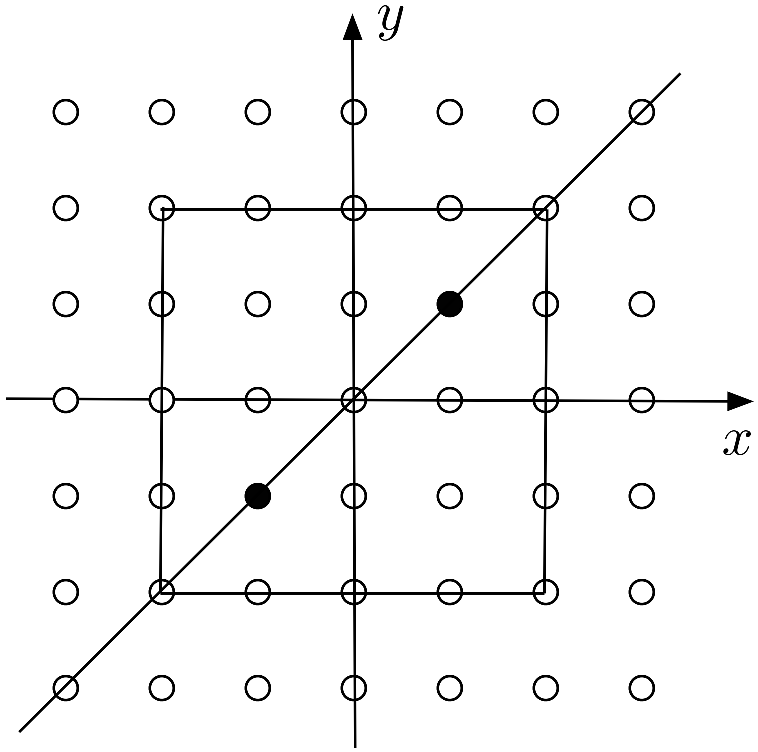

$$\phi_{2K_2}(k) = 2k - 2$$

Ehrhart Polynomials

$\mathcal{P} \subset \mathbb{R}^d$ – convex integral polytope

For $t \in \mathbb{Z}_{>0}$ let $\mathrm{Ehr}_{\mathcal{P}}(t) := \# \left(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d \right)$

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(Ehrhart 1962) $\mathrm{Ehr}_{\mathcal{P}}(t)$ is a polynomial in t of degree $\dim \mathcal{P}$ with leading term $\mathrm{vol}\, \mathcal{P}$ (normalized to $\mathrm{aff}\, \mathcal{P} \cap \mathbb{Z}^d$).
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Ehrhart Theory has recently seen a flurry of applications in various areas of mathematics. One class of applications comes from the enumeration of lattice points in a polytope $\mathcal{P}$ but off a hyperplane arrangement $\mathcal{H}$—an inside-out polytope $(\mathcal{P}, \mathcal{H})$.

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Proof: Recall that $C \subseteq \mathbb{R}^E$ is defined by $\sum_{h(e)=v} x(e) = \sum_{t(e)=v} x(e)$, $\square = (-1,1)^E$, and $\mathcal{H}$ contains the coordinate hyperplanes in $\mathbb{R}^E$; then

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The matrix defining the cycle space C is totally unimodular, and hence $\Box \cap C$ is an integral polytope. For the same reason, any of the connected components of $\Box \cap C \setminus \bigcup \mathcal{H}$ is an integral polytope.

Totally Cyclic Orientations

An orientation of Γ is totally cyclic if every edge lies in a coherent circle, that is, where the edges are oriented in a consistent direction around the circle. A totally cyclic orientation τ and a flow x are compatible if $x \geq 0$ when it is expressed in terms of τ.

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Recall $\varphi_\Gamma(k) := \#\,(\text{nowhere-zero } k\text{-flows})$

Theorem (B–Z) $|\varphi_\Gamma(-k)|$ equals the number of pairs (τ, x) consisting of a totally cyclic orientation τ and a compatible $(k+1)$-flow x. In particular, $|\varphi_\Gamma(0)|$ counts the totally cyclic orientations of Γ.

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A (Classic) Dual Theorem

$\Gamma = (V, E)$ – (loopless) graph

k-coloring of Γ : mapping $x : V \to \{1, 2, \ldots, k\}$
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An orientation α of Γ and a k-coloring x are compatible if $x_j \geq x_i$ whenever there is an edge oriented from i to j. An orientation is acyclic if it has no directed cycles.

Theorem (Stanley 1973) $|\chi_\Gamma(-k)|$ equals the number of pairs (α, x) consisting of an acyclic orientation α of Γ and a compatible k-coloring. In particular, $|\chi_\Gamma(-1)|$ counts the acyclic orientations of Γ.

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Idea of proof: Recall that

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(1) Use Ehrhart–Macdonald Reciprocity for the connected components of $\Box \cap C \setminus \bigcup \mathcal{H}$

(2) Realize that each orthant of $\mathbb{R}^E$ corresponds to a totally cyclic orientation of Γ (Greene–Zaslavsky)

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Prove that for any bridgeless graph $\varphi_\Gamma(k) > 0$ for all $k \geq 5$.

Graph coloring a la Ehrhart

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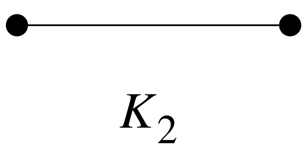

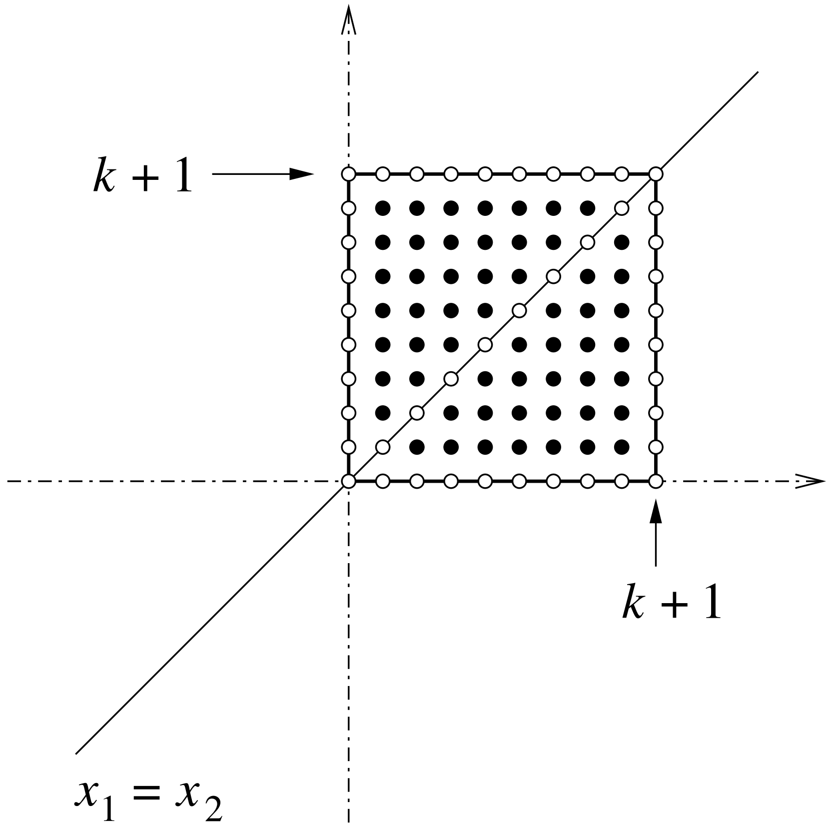

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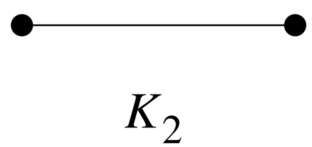

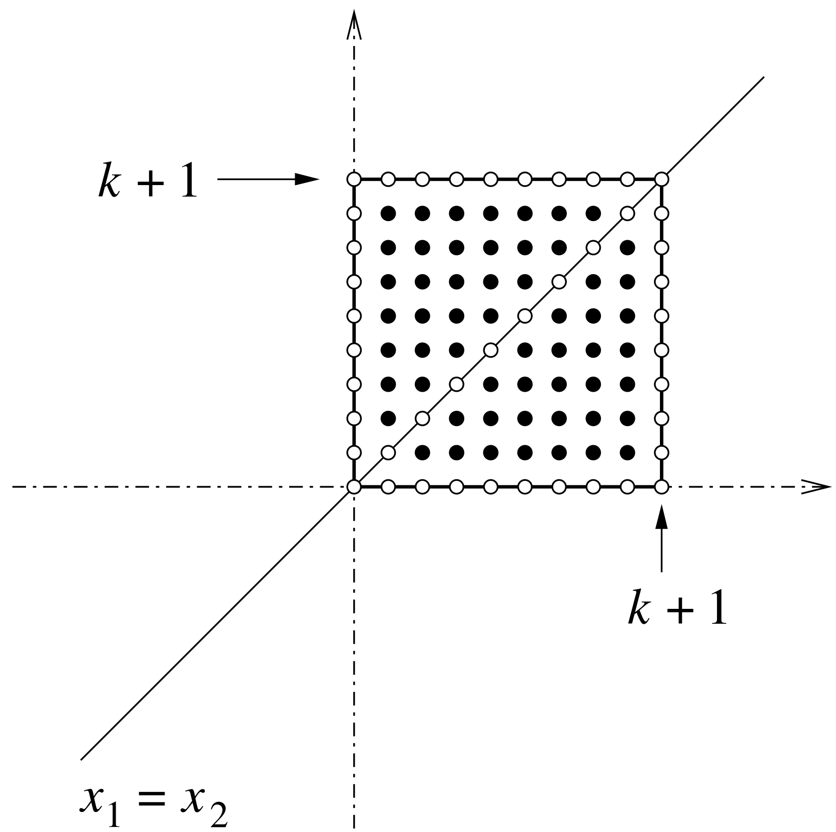

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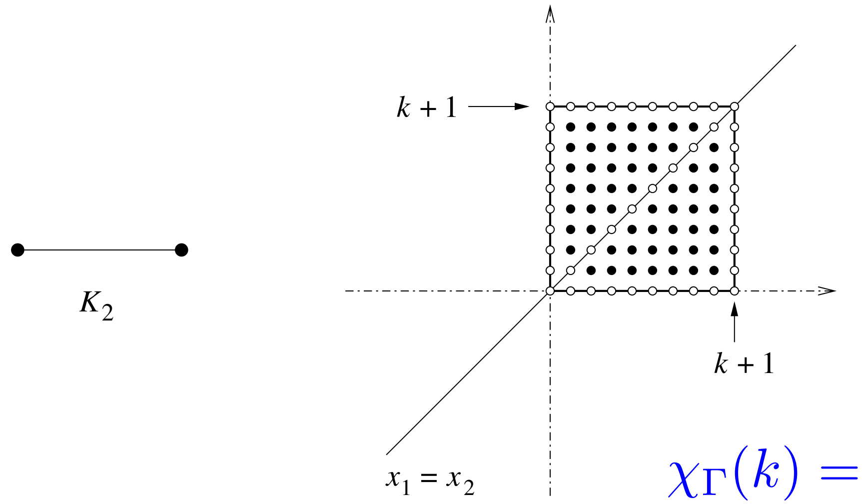

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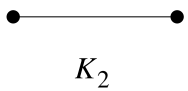

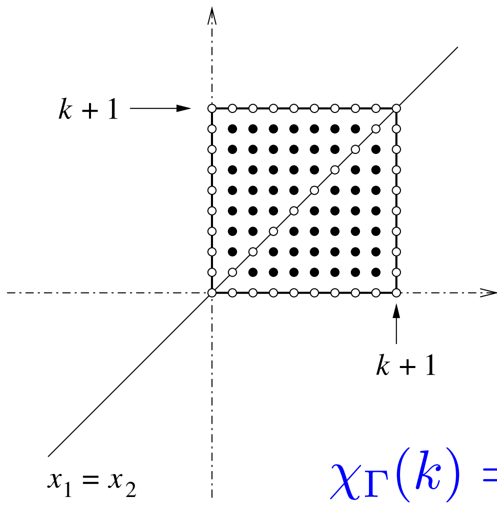

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Greene's observation

$$\text{region of } \mathcal{H}(\Gamma) \quad \Longleftrightarrow \quad \text{acyclic orientation of } \Gamma$$

$$x_i < x_j \quad \Longleftrightarrow \quad i \longrightarrow j$$