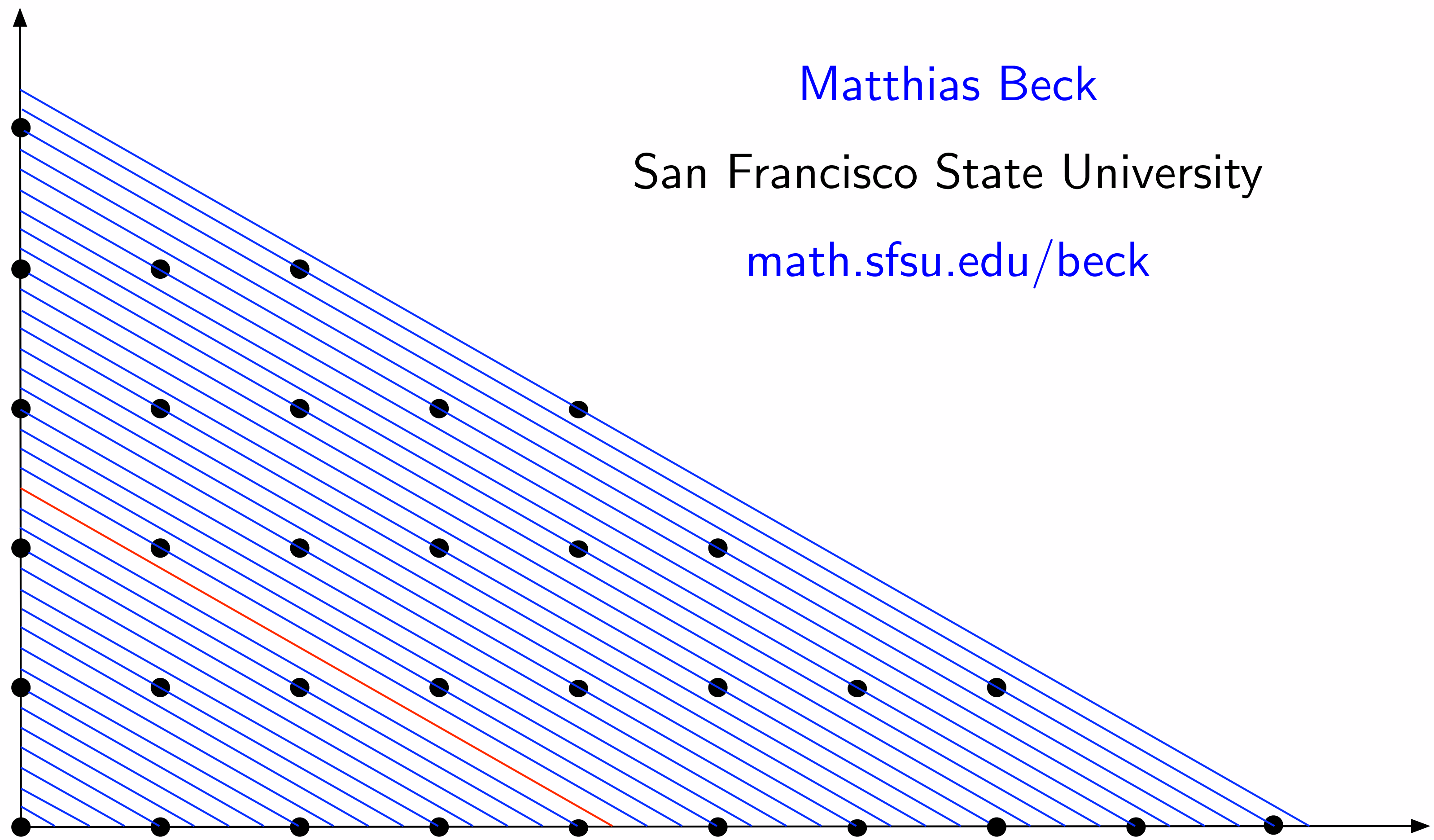
Enumerating integer points in polytopes: applications to number theory



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"It takes a village to count integer points."

Alexander Barvinok

Outline

- Ehrhart theory
- Dedekind sums
- “Coin-exchange problem” of Frobenius
- Roots of Ehrhart polynomials
- All kinds of magic

Joint work with...

- Sinai Robins (Ehrhart formulas, Dedekind sums)
- Ricardo Diaz and Sinai Robins (Frobenius problem)
- Jesus De Loera, Mike Develin, Julian Pfeifle, and Richard Stanley (roots)
- Dennis Pixton (Birkhoff polytope)
- Moshe Cohen, Jessica Cuomo, Paul Gribelyuk (weak magic)
- Thomas Zaslavsky (strong magic)

(Weak) semimagic squares

$$H_n(t) := \# \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = t \\ \sum_k x_{jk} = t \end{array} \right\}$$

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Theorem (Ehrhart, Stanley 1973, conjectured by Anand-Dumir-Gupta 1966) $H_n(t)$ is a polynomial in t of degree $(n-1)^2$. This polynomial satisfies

$$H_n(-t) = (-1)^{n-1} H_n(t-n) \quad \text{and} \quad H_n(-1) = \cdots = H_n(-n+1) = 0 \; .$$

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For example...

- $H_1(t) = 1$
- $H_2(t) = t + 1$
- (MacMahon 1905) $H_3(t) = 3\binom{t+3}{4} + \binom{t+2}{2} = \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$

Ehrhart theory

Integral (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right) = \# \left(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d \right)$

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Theorem (Ehrhart 1962) If $\mathcal{P}$ is an integral polytope, then...

- $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are polynomials in t of degree $\dim \mathcal{P}$
- Leading term: $\mathrm{vol}(P)$ (suitably normalized)
- (Macdonald 1970) $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

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Alternative description of a polytope:

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\} \rightleftarrows \left\{ \mathbf{x} \in \mathbb{R}^d_{\ge 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$$

A magic example: the Birkhoff polytope

$$\mathcal{B}_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2}_{\geq 0} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

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- $H_n(-t) = (-1)^{n-1} H_n(t-n)$ and $H_n(-1) = \cdots = H_n(-n+1) = 0$ follow with $L_{\mathcal{B}_n^\circ}(t) = H_n(t-n)$ and Ehrhart-Macdonald Reciprocity.

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- A close relative to $\mathcal{B}_n$ appeared recently in connections with pseudomoments of the ζ-function (Conrey–Gamburd 2005)

Ehrhart theory

Integral (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Z}^d$

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Theorem (Ehrhart 1962) If $\mathcal{P}$ is an integral polytope, then...

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Ehrhart theory

Rational (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Q}^d$

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Theorem (Ehrhart 1962) If $\mathcal{P}$ is an rational polytope, then...

- $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are quasi-polynomials in t of degree $\dim \mathcal{P}$
- Leading term: $\mathrm{vol}(P)$ (suitably normalized)
- (Macdonald 1970) $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

Alternative description of a polytope:

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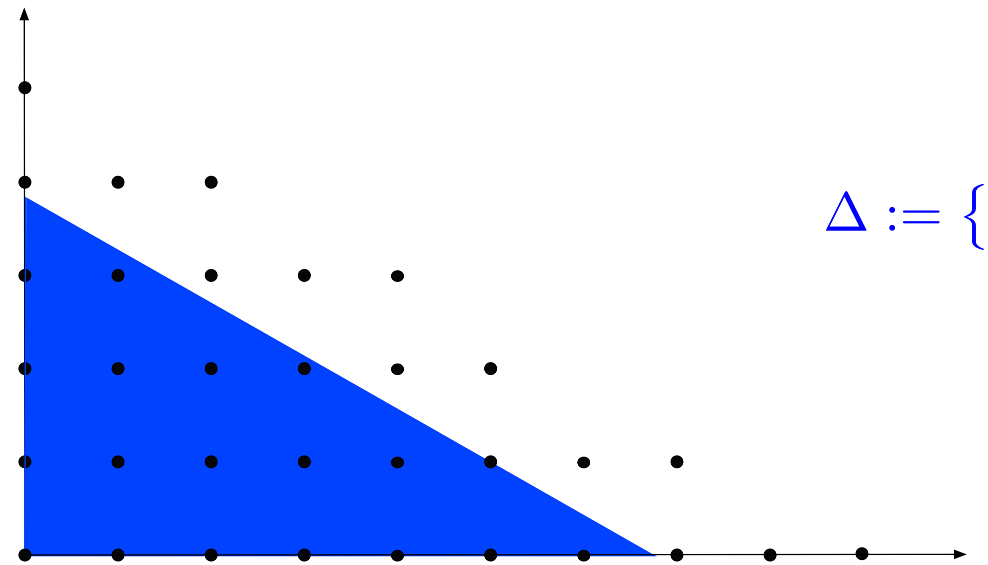
Quasi-polynomial – $c_d(t)\,t^d + c_{d-1}(t)\,t^{d-1} + \cdots + c_0(t)$ where $c_k(t)$ are periodic

An example in dimension 2



$$\Delta := \left\{ (x,y) \in \mathbb{R}^2_{\geq 0} : \; ax + by \leq 1 \right\}$$

$(a = 7, b = 4, t = 23)$

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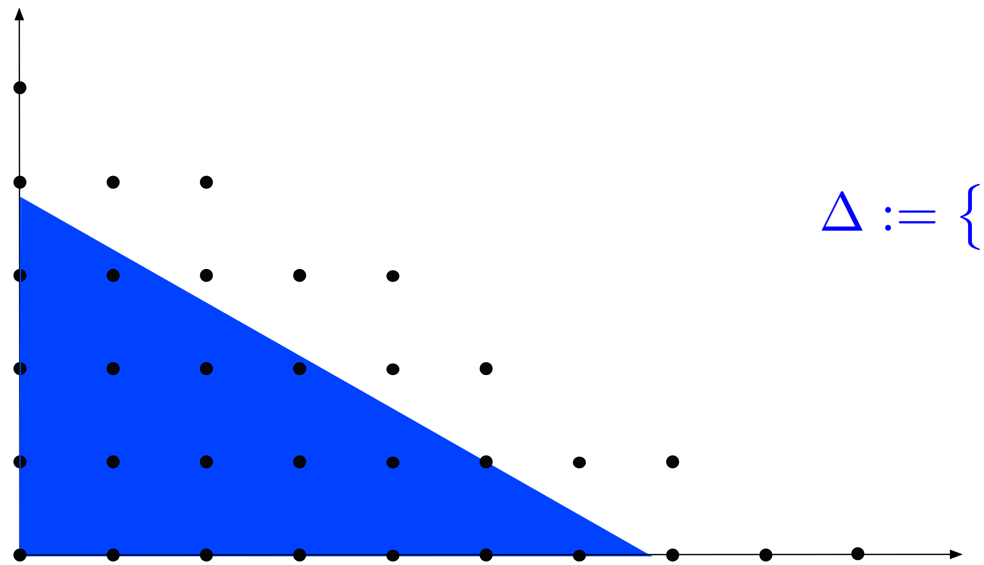

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$$L_\Delta(t) \quad = \quad \# \left\{ (m, n) \in \mathbb{Z}^2_{\geq 0} : \ am + bn \leq t \right\}$$

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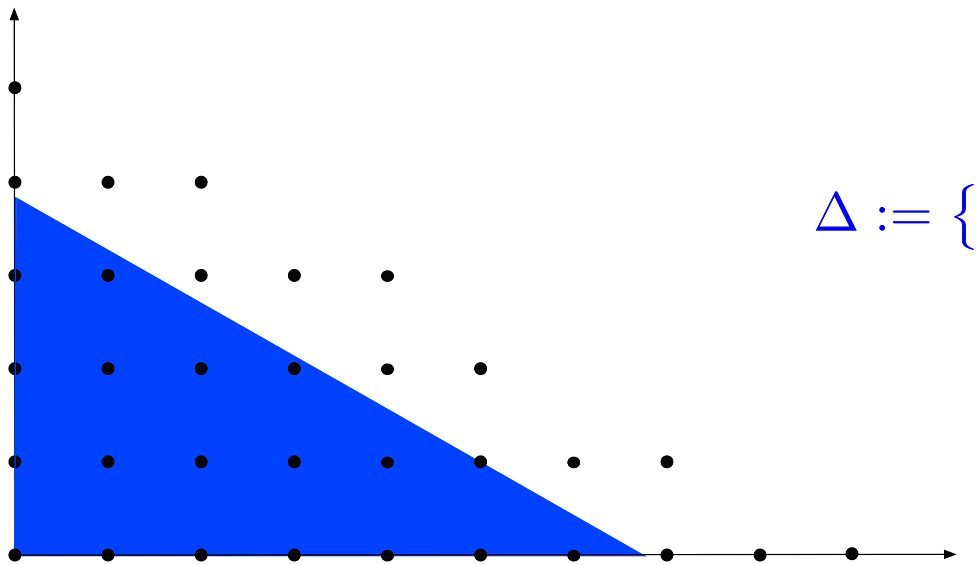

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$$\begin{aligned} L_\Delta(t) &= \# \left\{ (m,n) \in \mathbb{Z}^2_{\geq 0} : \ am + bn \leq t \right\} \\ &= \# \left\{ (m,n,s) \in \mathbb{Z}^3_{\geq 0} : \ am + bn + s = t \right\} \end{aligned}$$

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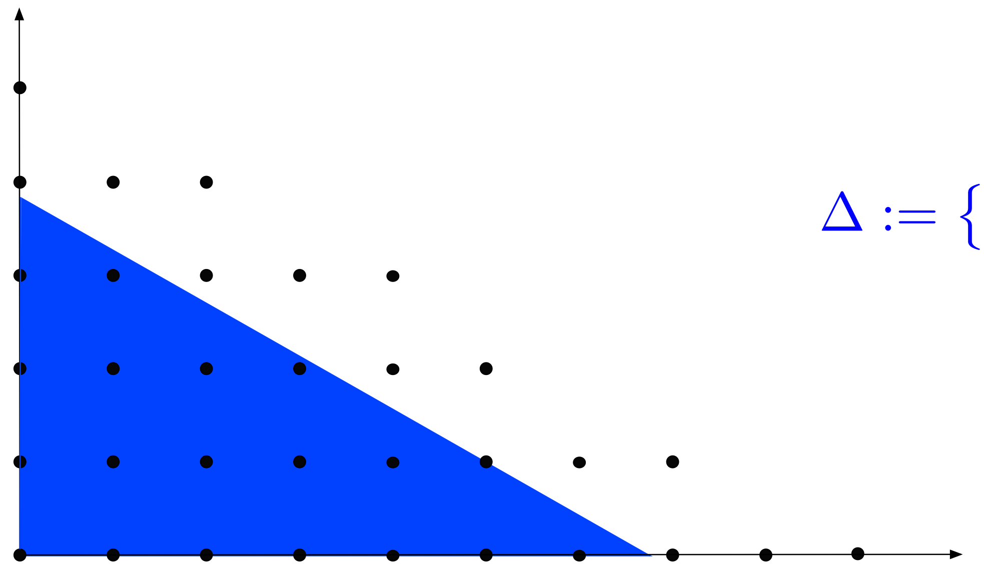

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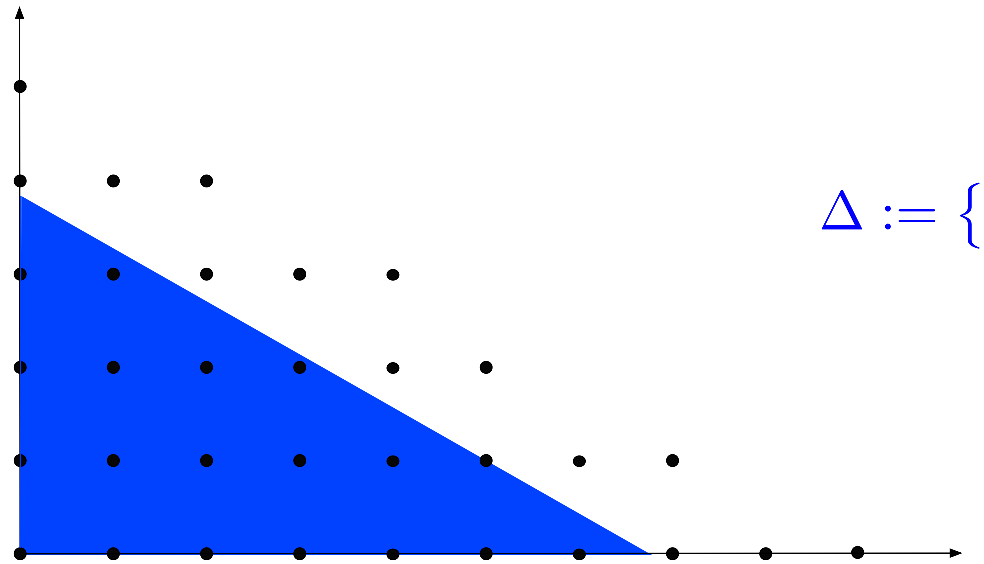

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$$\Delta := \left\{ (x,y) \in \mathbb{R}^2_{\geq 0} : \; ax + by \leq 1 \right\}$$

$$f(x) := \frac{1}{(1 - x^a)\,(1 - x^b)\,(1 - x)\,x^{t+1}}$$

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An example in dimension 2

$$\Delta := \left\{ (x,y) \in \mathbb{R}^2_{\geq 0} : \ ax + by \leq 1 \right\} \qquad \gcd(a,b) = 1$$

$$f(x) := \frac{1}{(1-x^a)\,(1-x^b)\,(1-x)\,x^{t+1}} \qquad \xi_a := e^{2\pi i/a}$$

$$\begin{aligned} L_\Delta(t) &= \frac{1}{2\pi i} \int_{|x|=\epsilon} f \, dx \\ &= \operatorname{Res}_{x=1}(f) + \sum_{k=1}^{a-1} \operatorname{Res}_{x=\xi_a^k}(f) + \sum_{j=1}^{b-1} \operatorname{Res}_{x=\xi_b^j}(f) \end{aligned}$$

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&= \frac{t^2}{2ab} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{a} + \frac{1}{b} \right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + 3 + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} \right) \\
&\quad + \frac{1}{a} \sum_{k=1}^{a-1} \frac{1}{(1-\xi_a^{kb})\,(1-\xi_a^k)\,\xi_a^{kt}} + \frac{1}{b} \sum_{j=1}^{b-1} \frac{1}{\left(1-\xi_b^{ja}\right)\left(1-\xi_b^j\right)\xi_b^{jt}}
\end{aligned}$$

An example in dimension 2

$$\Delta := \left\{ (x,y) \in \mathbb{R}^2_{\geq 0} : \; ax + by \leq 1 \right\}$$

(Pick's or) Ehrhart's Theorem implies that L_Δ has constant term $L_\Delta(0) = 1$

$$\frac{1}{a} \sum_{k=1}^{a-1} \frac{1}{\left(1 - \xi_a^{kb}\right)\left(1 - \xi_a^{k}\right)} + \frac{1}{b} \sum_{j=1}^{b-1} \frac{1}{\left(1 - \xi_b^{ja}\right)\left(1 - \xi_b^{j}\right)}$$

$$= 1 - \frac{1}{12} \left(\frac{3}{a} + \frac{3}{b} + 3 + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} \right)$$

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$$= 1 - \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + 3 + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right)$$

However...

$$\frac{1}{a} \sum_{k=1}^{a-1} \frac{1}{\left(1-\xi_a^{kb}\right)\left(1-\xi_a^{k}\right)} = -\frac{1}{4a} \sum_{k=1}^{a-1} \cot\left(\frac{\pi k b}{a}\right) \cot\left(\frac{\pi k}{a}\right) + \frac{a-1}{4a}$$

Dedekind sums

$$s(a,b) := \frac{1}{4b} \sum_{j=1}^{b-1} \cot\left(\frac{\pi j a}{b}\right) \cot\left(\frac{\pi j}{b}\right)$$

Since their introduction by Dedekind in the 1880's, these sums and their generalizations have appeared in various areas such as analytic (transformation law of η-function) and algebraic number theory (class numbers), topology (group action on manifolds), combinatorial geometry (lattice point problems), and algorithmic complexity (random number generators).

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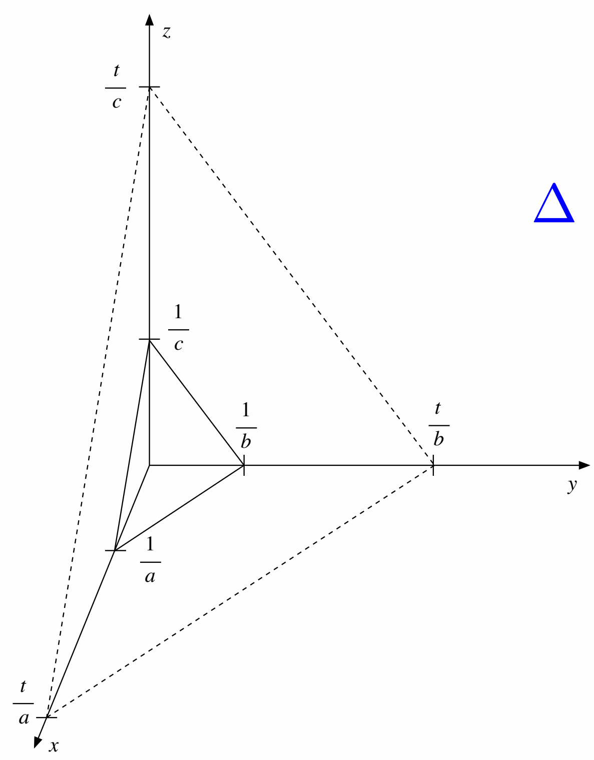
The identity $L_\Delta(0) = 1$ implies...

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right)$$

the Reciprocity Law for Dedekind sums.

A 2-dimensional example in dimension 3



$$\Delta := \left\{ (x,y,z) \in \mathbb{R}^3_{\geq 0} : \; ax + by + cz = 1 \right\}$$

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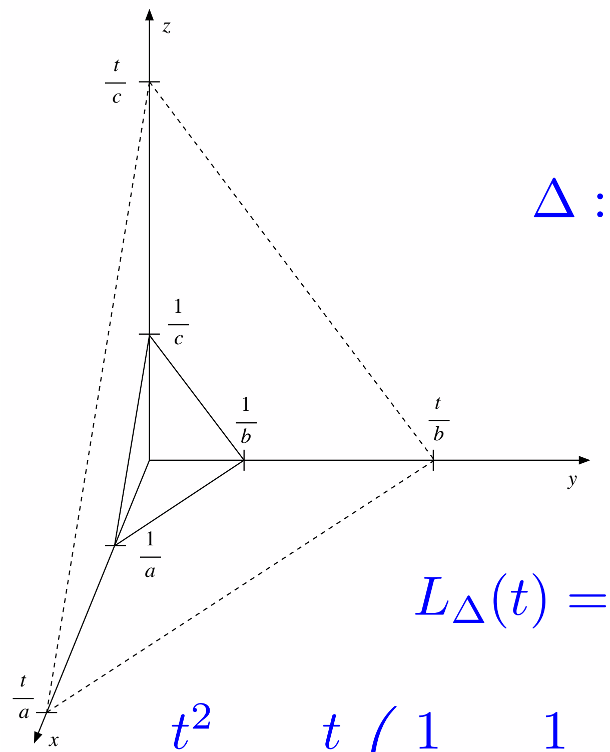

$$\Delta := \left\{ (x,y,z) \in \mathbb{R}^3_{\geq 0} : \; ax + by + cz = 1 \right\}$$

$$\gcd(a,b) = \gcd(b,c) = \gcd(c,a) = 1$$

$$L_\Delta(t) = \frac{1}{2\pi i} \int_{|x|=\epsilon} \frac{dx}{(1-x^a)(1-x^b)(1-x^c)\, x^{t+1}}$$

$$= \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right)$$

$$+ \frac{1}{a}\sum_{k=1}^{a-1} \frac{1}{(1-\xi_a^{kb})(1-\xi_a^{kc})\,\xi_a^{kt}} + \frac{1}{b}\sum_{k=1}^{b-1} \frac{1}{(1-\xi_b^{kc})(1-\xi_b^{ka})\,\xi_b^{kt}}$$

$$+ \frac{1}{c}\sum_{k=1}^{c-1} \frac{1}{(1-\xi_c^{ka})(1-\xi_c^{kb})\,\xi_c^{kt}}$$

More Dedekind sums

$$s(a,b;c) := \frac{1}{4c} \sum_{j=1}^{c-1} \cot\left(\frac{\pi j a}{c}\right) \cot\left(\frac{\pi j b}{c}\right)$$

The identity $L_\Delta(0) = 1$ implies Rademacher's Reciprocity Law

$$s(a,b;c) + s(b,c;a) + s(c,a;b) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right)$$

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Moreover,

$$t\Delta = \left\{(x,y,z) \in \mathbb{R}^3_{\geq 0} : \; ax + by + cz = t\right\}$$

has no interior lattice points for $0 < t < a+b+c$, so that Ehrhart-Macdonald Reciprocity implies that $L_\Delta(t) = 0$ for $-(a+b+c) < t < 0$, which is equivalent to the Reciprocity Law for Dedekind-Rademacher sums.

Even more Dedekind sums

The Ehrhart quasi-polynomial for $\Delta := \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \ a_1 x_1 + \cdots + a_d x_d = 1 \right\}$ gives rise to the Fourier-Dedekind sum (MB–Robins 2003)

$$\sigma_n \left(a_1, \ldots, a_d; a_0 \right) := \frac{1}{a_0} \sum_{\lambda^{a_0}=1} \frac{\lambda^n}{\left(1 - \lambda^{a_1}\right) \cdots \left(1 - \lambda^{a_d}\right)} .$$

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The identity $L_\Delta(0) = 1$ implies the Reciprocity Law for Zagier's "higher-dimensional Dedekind sums", whereas

$$L_\Delta(t) = 0 \qquad \text{for} \qquad -\left(a_1 + \cdots + a_d\right) < t < 0$$

gives a new reciprocity relation which is a "higher-dimensional" analog of that for the the Dedekind-Rademacher sum.

Partition functions and the Frobenius problem

The Ehrhart quasi-polynomial

$$L_\Delta(t) = \#\left\{(m_1, \ldots, m_d) \in \mathbb{Z}^d_{\geq 0} : \; m_1 a_1 + \cdots + m_d a_d = t\right\}$$

is the restricted partition function $p_A(t)$ for $A = \{a_1, \ldots, a_d\}$

Partition functions and the Frobenius problem

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Shameless plug

M. Beck & S. Robins

Computing the continuous discretely
Integer-point enumeration in polyhedra

To appear in Springer Undergraduate Texts in Mathematics

Preprint available at math.sfsu.edu/beck

MSRI Summer Graduate Program at Banff (August 6–20)

Coefficients and roots of Ehrhart polynomials

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Then $L_{\mathcal{P}}(t) = c_d\, t^d + \cdots + c_0$ is a polynomial in t.

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▶ The inequalities $f(x) \ge 0$ and $c_{d-1} > 0$ are currently the sharpest constraints on Ehrhart coefficients. Are there others?

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Easy fact: $L_{\mathcal{P}}$ has no integer roots besides $-d, -d+1, \ldots, -1$.

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(1) The roots of Ehrhart polynomials of lattice d-polytopes are bounded in norm by $1 + (d+1)!$.

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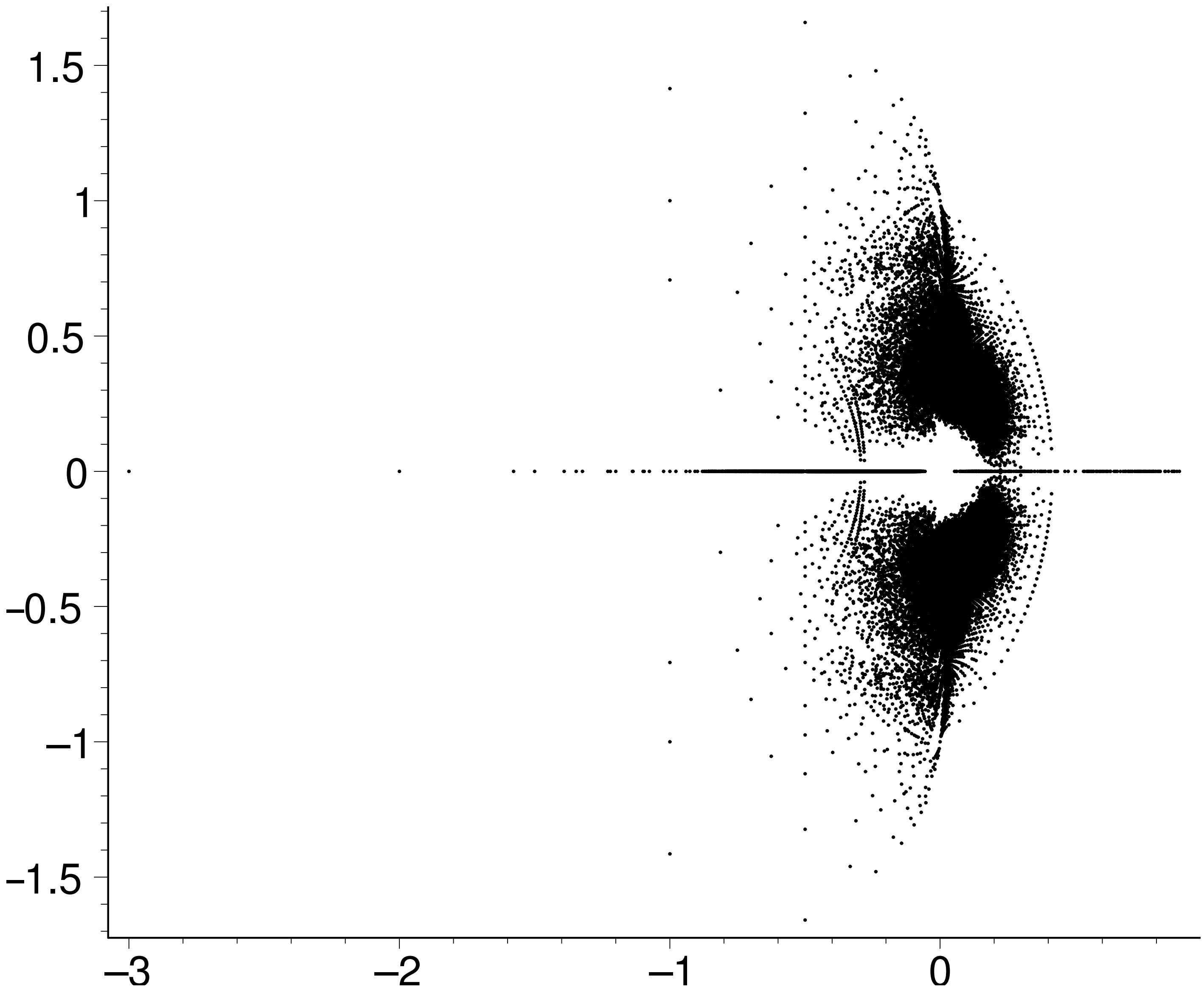
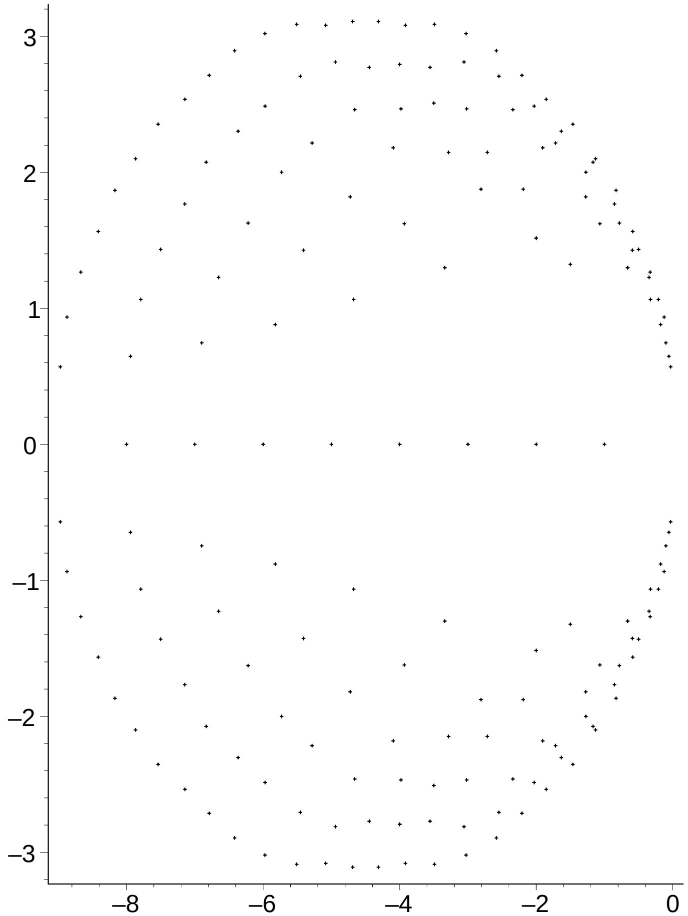
Conjecture: All roots α satisfy $-d \le \mathrm{Re}\ \alpha \le d - 1$.

Roots of some tetrahedra



Roots of the Birkhoff polytopes



(Weak) semimagic squares revisited

$$H_n(t) := \# \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = t \\ \sum_k x_{jk} = t \end{array} \right\}$$

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$$H_n(t) = \frac{1}{(2\pi i)^n} \int (z_1 \cdots z_n)^{-t-1} \times$$

$$\sum_{m_1+\cdots+m_n=n}^{*} \binom{n}{m_1, \ldots, m_n} \prod_{k=1}^{n} \left(\frac{z_k^{t+n-1}}{\prod_{j \neq k} (z_k - z_j)} \right)^{m_k} d\mathbf{z}$$

where $\sum^*$ denotes that we only sum over those n-tuples of non-negative integers satisfying $m_1 + \cdots + m_n = n$ and $m_1 + \cdots + m_r > r$ for $1 \leq r < n$

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- Computation of H_n for $n \leq 9$ and $\operatorname{vol} \mathcal{B}_n$ for $n \leq 10$

Birkhoff volumes

n	$\operatorname{vol} \mathcal{B}_n$
1	1
2	2
3	$9/8$
4	$176/2835$
5	$23590375/167382319104$
6	$9700106723/1319281996032 \cdot 10^6$
7	$\dfrac{77436678274508929033}{137302963682235238399868928 \cdot 10^8}$
8	$\dfrac{5562533838576105333259507434329}{12589036260095477950081480942693339803308928 \cdot 10^{10}}$
9	$\dfrac{559498129702796022246895686372766052475496691}{92692623409952636498965146712806984296051951329202419606108477153345536 \cdot 10^{14}}$

$$\frac{727291284016786420977508457990121862548823260052557333386607889}{828160860106766855125676318796872729344622463533089422677980721388055739956270293750883504892820848 64 \cdot 10^{7}}$$

Weak magic squares

$$M_n(t) := \# \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = t \\ \sum_k x_{jk} = t \\ \sum_j x_{jj} = t \\ \sum_j x_{j,n-j} = t \end{array} \right\}$$

is an (Ehrhart) quasi-polynomial in t

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- (MB–Cohen–Cuomo–Gribelyuk 2003) For $n \geq 3$, $\deg M_n = n^2 - 2n - 1$
- Open problem: What is the period of M_n? Is it always > 1 for $n \geq 2$?

Strong magic squares

$M_n^*(t)$ – # magic $n \times n$-squares with distinct entries and "magic sum" t

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Theorem (MB–Zaslavsky ~2006) $M_n^*(t)$ is the Ehrhart quasi-polynomial of an inside-out polytope, satisfying

$$M_n^*(t) = \sum_{u \in \mathcal{L}} \mu\left(\hat{0}, u\right) L_{u \cap \mathbb{R}^{n^2}_{>0}}(t) \ ,$$

where $\mathcal{L}$ is the intersection lattice of the hyperplane arrangement $\left\{x_i = x_j : \ 1 \le i < j \le n^2\right\}$ and μ is its Möbius function.

For example...

$$M^\circ(t) = \begin{cases} \frac{2t^2-32t+144}{9} = \frac{2}{9}(t^2-16t+72) & \text{if } t \equiv 0 \mod 18, \\ \frac{2t^2-32t+78}{9} = \frac{2}{9}(t-3)(t-13) & \text{if } t \equiv 3 \mod 18, \\ \frac{2t^2-32t+120}{9} = \frac{2}{9}(t-6)(t-10) & \text{if } t \equiv 6 \mod 18, \\ \frac{2t^2-32t+126}{9} = \frac{2}{9}(t-7)(t-9) & \text{if } t \equiv 9 \mod 18, \\ \frac{2t^2-32t+96}{9} = \frac{2}{9}(t-4)(t-12) & \text{if } t \equiv 12 \mod 18, \\ \frac{2t^2-32t+102}{9} = \frac{2}{9}(t^2-16t+51) & \text{if } t \equiv 15 \mod 18, \\ 0 & \text{if } t \not\equiv 0 \mod 3. \end{cases}$$

Magic dice (a Monthly problem)

Given a 3×3-square, we form three 3-sided dice, as follows: the sides of die i are labelled with the numbers in row i. We say die i beats die j if we expect die i to show a bigger number than die j more than half the time.

(a) Suppose the square is a (strong) magic square whose entries are $1, 2, \ldots, 9$. Prove that no die beats the other two and no die loses to the other two. Every die beats one die and loses to the other die.

(b) Show the same is true for any strong magic square.

(c) Suppose the square is a semimagic square whose entries are $1, 2, \ldots, 9$. Show the same conclusion holds as in (a) and (b).

(d) But, there are semimagic squares for which one die beats both other dice.