The partial-fractions method for counting solutions to integral linear systems

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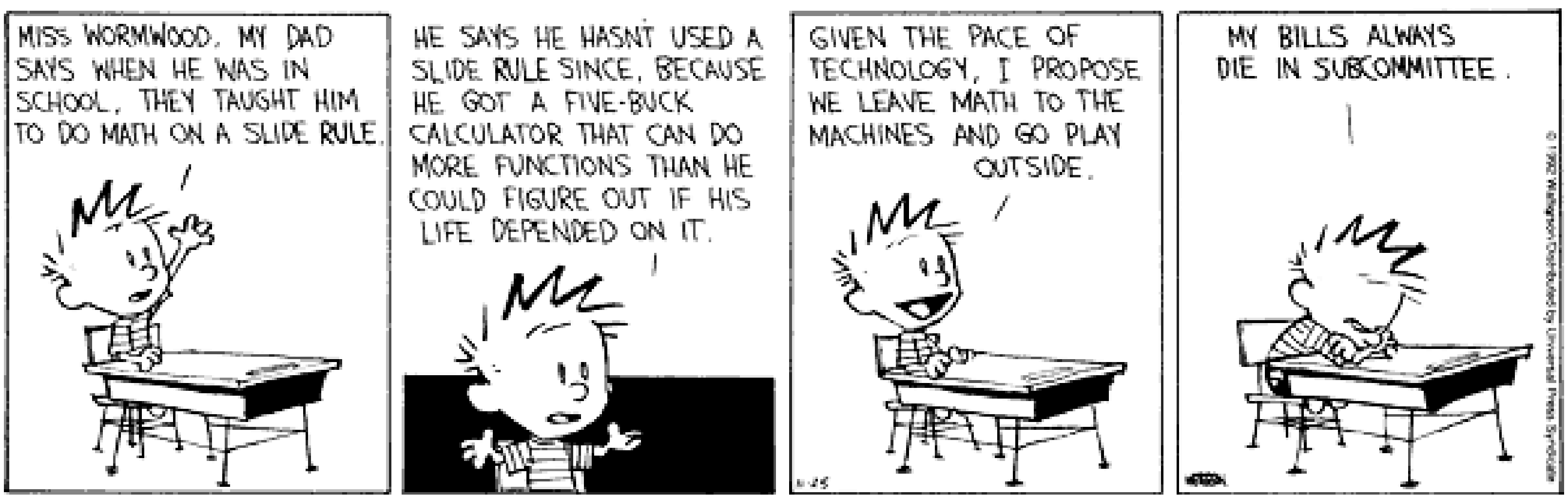

Vector partition functions

$\mathbf{A}$ – an $(m \times d)$-integral matrix
$\mathbf{b} \in \mathbb{Z}^m$

Goal: Compute vector partition function $\phi_{\mathbf{A}}(\mathbf{b}) := \# \left\{ \mathbf{x} \in \mathbb{Z}_{\geq 0}^d : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

(defined for $\mathbf{b}$ in the nonnegative linear span of the columns of $\mathbf{A}$)

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Applications in...

- Number Theory (partitions)
- Discrete Geometry (polyhedra)
- Commutative Algebra (Hilbert series)
- Algebraic Geometry (toric varieties)
- Representation Theory (tensor product multiplicities)
- Optimization (integer programming)
- Chemistry, Biology, Physics, Computer Science, Economics...

Ehrhart quasi-polynomials

Rational (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Q}^d$

Alternative description: $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}$

For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

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Quasi-polynomial – $c_d(t)\, t^d + c_{d-1}(t)\, t^{d-1} + \cdots + c_0(t)$ where $c_k(t)$ are periodic

Theorem (Ehrhart 1967) If $\mathcal{P}$ is a rational polytope, then the functions $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are quasi-polynomials in t of degree $\dim \mathcal{P}$. If $\mathcal{P}$ has integer vertices, then $L_{\mathcal{P}}$ and $L_{\mathcal{P}^\circ}$ are polynomials.

Theorem (Ehrhart, Macdonald 1970) $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

Vector partition theorems

$\phi_{\mathbf{A}}(\mathbf{b}) := \# \left\{ \mathbf{x} \in \mathbb{Z}_{\geq 0}^{d} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

Quasi-polynomial – a finite sum $\sum_{\mathbf{n}} c_{\mathbf{n}}(\mathbf{b})\, \mathbf{b}^{\mathbf{n}}$ with coefficients $c_{\mathbf{n}}$ that are functions of $\mathbf{b}$ which are periodic in every component of $\mathbf{b}$.

A matrix is unimodular if every square submatrix has determinant ± 1.

Theorem (Sturmfels 1995) $\phi_{\mathbf{A}}(\mathbf{b})$ is a piecewise-defined quasi-polynomial in $\mathbf{b}$ of degree $d - \operatorname{rank}(\mathbf{A})$. The regions of $\mathbb{R}^m$ in which $\phi_{\mathbf{A}}(\mathbf{b})$ is a single quasi-polynomial are polyhedral. If $\mathbf{A}$ is unimodular then $\phi_{\mathbf{A}}$ is a piecewise-defined polynomial.

Theorem (MB 2002) Let r_k denote the sum of the entries in the k^{th} row of $\mathbf{A}$, and let $\mathbf{r} = (r_1, \ldots, r_m)$. Then $\phi_{\mathbf{A}}(\mathbf{b}) = (-1)^{d - \operatorname{rank} \mathbf{A}} \phi_{\mathbf{A}}(-\mathbf{b} - \mathbf{r})$

Issues...

- Compute the regions of (quasi-)polynomiality of $\phi_{\mathbf{A}}(\mathbf{b})$
- Given one such region, compute the (quasi-)polynomial $\phi_{\mathbf{A}}(\mathbf{b})$
- Barvinok: $\displaystyle\sum_{t \geq 0} \phi_{\mathbf{A}}(t\mathbf{b})\, z^t$ can be computed in polynomial time

Euler's generating function

$$\phi_{\mathbf{A}}(\mathbf{b}) := \#\left\{ \mathbf{x} \in \mathbb{Z}_{\geq 0}^{d} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\} \qquad \mathbf{A} = \begin{pmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_d \\ | & | & & | \end{pmatrix}$$

$\phi_{\mathbf{A}}(\mathbf{b})$ equals the coefficient of $\mathbf{z}^{\mathbf{b}} := z_1^{b_1} \cdots z_m^{b_m}$ of the function

$$\frac{1}{\left(1 - \mathbf{z}^{\mathbf{c}_1}\right) \cdots \left(1 - \mathbf{z}^{\mathbf{c}_d}\right)}$$

expanded as a power series centered at $\mathbf{z} = 0$.

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Proof Expand each factor into a geometric series. ⬠

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Equivalently,

$$\phi_{\mathbf{A}}(\mathbf{b}) = \mathrm{const} \frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{\mathbf{b}}}$$

Partial fractions

$$\phi_{\mathbf{A}}(\mathbf{b}) = \text{const} \frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{\mathbf{b}}}$$

Expand into partial fractions in z_1:

$$\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{\mathbf{b}}} = \frac{1}{z_2^{b_2} \cdots z_m^{b_m}} \left(\sum_{k=1}^{d} \frac{A_k(\mathbf{z}, b_1)}{1-\mathbf{z}^{\mathbf{c}_k}} + \sum_{j=1}^{b_1} \frac{B_j(\mathbf{z})}{z_1^j} \right)$$

Here A_k and B_j are polynomials in z_1, rational functions in $z_2, \ldots, z_m$, and exponential in b_1.

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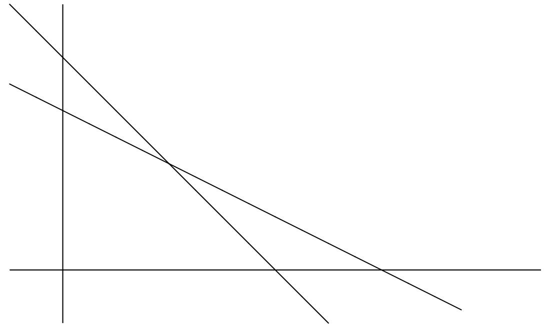
Advantages

- easy to implement
- allows symbolic computation
- constraints which define the regions of (quasi-)polynomiality are obtained "automatically"

An example

$$
\begin{aligned}
x_1, x_2, x_3, x_4 &\geq 0 \\
x_1 + 2x_2 + x_3 &= a \\
x_1 + x_2 + x_4 &= b
\end{aligned}
$$



$$\phi_{\mathbf{A}}(a, b) = \mathrm{const} \, \frac{1}{(1 - zw)(1 - z^2 w)(1 - z)(1 - w) z^a w^b}$$

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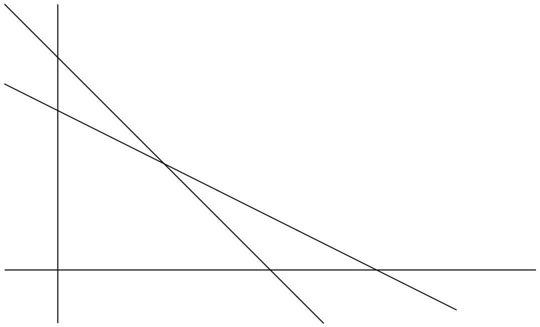
$$\begin{array}{rcl} x_1, x_2, x_3, x_4 & \geq & 0 \\ x_1 + 2x_2 + x_3 & = & a \\ x_1 + x_2 + x_4 & = & b \end{array}$$



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$$\frac{1}{(1-zw)(1-z^2w)(1-w)w^b} = -\frac{\frac{z^{b+1}}{(1-z)^2}}{1-zw} + \frac{\frac{z^{2b+3}}{(1-z)(1-z^2)}}{1-z^2w} + \frac{\frac{1}{(1-z)(1-z^2)}}{1-w} + \sum_{k=1}^{b} \frac{\cdots}{w^k}$$

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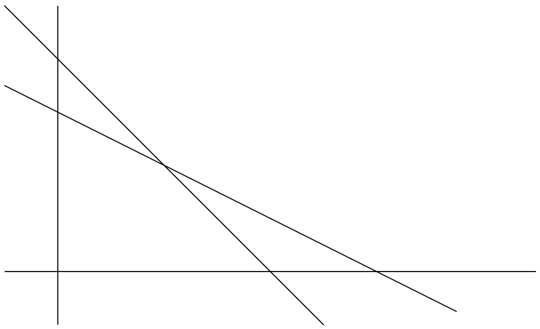

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$$\begin{aligned} \phi_{\mathbf{A}}(a,b) &= \operatorname{const} \frac{1}{(1-z)z^a} \left(-\frac{z^{b+1}}{(1-z)^2} + \frac{z^{2b+3}}{(1-z)(1-z^2)} + \frac{1}{(1-z)(1-z^2)} \right) \\ &= \operatorname{const} \left(-\frac{z^{b-a+1}}{(1-z)^3} + \frac{z^{2b-a+3}}{(1-z)^2(1-z^2)} + \frac{1}{(1-z)^2(1-z^2)z^a} \right) \end{aligned}$$

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If $2b-a+3 \le 0$ we expand into partial fractions again:

$$\begin{aligned}
&\operatorname{const} \frac{z^{2b-a+3}}{(1-z)^2(1-z^2)} \\
&= \operatorname{const}\left(\frac{1/2}{(1-z)^3} + \frac{\frac{a-2b-3}{2}+\frac{1}{4}}{(1-z)^2} + \frac{\frac{(a-2b-3)^2}{4}+\frac{a-2b-3}{2}+\frac{1}{8}}{1-z} + \frac{(-1)^{a+1}/8}{1+z} \right) \\
&= \frac{(a-2b)^2}{4} + \frac{2b-a}{2} + \frac{1+(-1)^{a+1}}{8}
\end{aligned}$$

An example

$$\phi_{\mathbf{A}}(a,b) = \begin{cases} \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} & \text{if } a \le b \\ ab - \frac{a^2}{4} - \frac{b^2}{2} + \frac{a+b}{2} + \frac{7+(-1)^a}{8} & \text{if } a > b > \frac{a-3}{2} \\ \frac{b^2}{2} + \frac{3b}{2} + 1 & \text{if } b \le \frac{a-3}{2} \end{cases}$$

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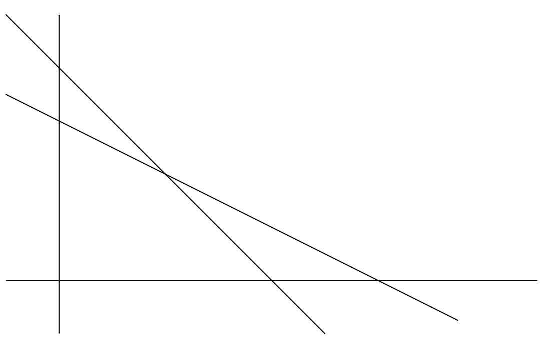

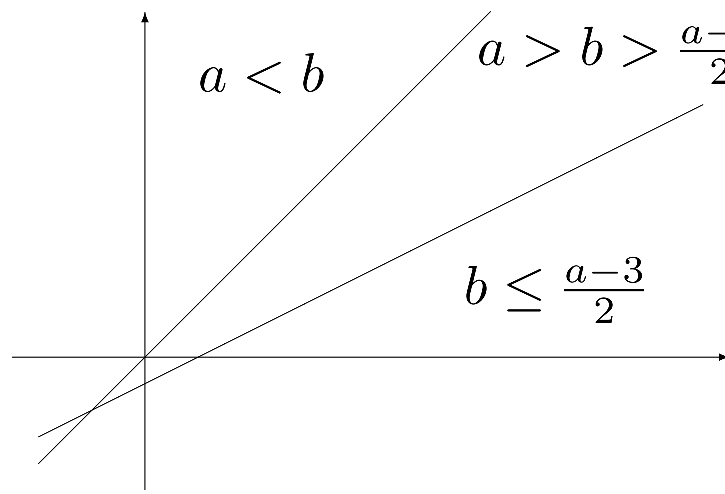

Open problems

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- Re-interpret each term as coming from a linear system and simplify

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Example: "second term" above

$$\mathrm{const}\,\frac{z^{2b-a+3}}{(1-z)^2(1-z^2)} = \#\left\{(x,y,z)\in\mathbb{Z}^3_{\geq 0}:\; x+y+2z=a-2b-3\right\}$$
$$= \#\left\{(x,y)\in\mathbb{Z}^2_{\geq 0}:\; x+2y\leq a-2b-3\right\}$$