Integer Partitions From A Geometric Viewpoint

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"If things are nice there is probably a good reason why they are nice: and if you do not know at least one reason for this good fortune, then you still have work to do."

Richard Askey (Ramanujan and Important Formulas, *Srinivasa Ramanujan (1887–1920), a Tribute*, K. R. Nagarajan and T. Soundarajan, eds., Madurai Kamaraj University, 1987.)

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Polyhedral Geometry

Partition Analysis

Arithmetic

Integer Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ of an integer $k \geq 0$ satisfies

$$k = \lambda_1 + \lambda_2 + \cdots + \lambda_n \qquad \text{and} \qquad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

Example

$$\begin{aligned} 5 &= 1+1+1+1+1 \\ &= 1+1+1+2 \\ &= 1+2+2 \\ &= 1+1+3 \\ &= 2+3 \\ &= 1+4 \\ &= 5 \end{aligned}$$

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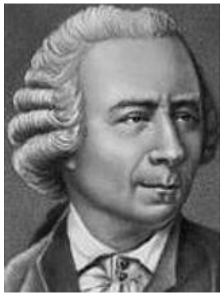
- Number Theory
- Combinatorics
- Symmetric functions
- Representation Theory
- Physics





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Goal Compute $\displaystyle\sum_{\lambda} q^{\lambda_1 + \cdots + \lambda_n}$

where the sum runs through your favorite partitions.

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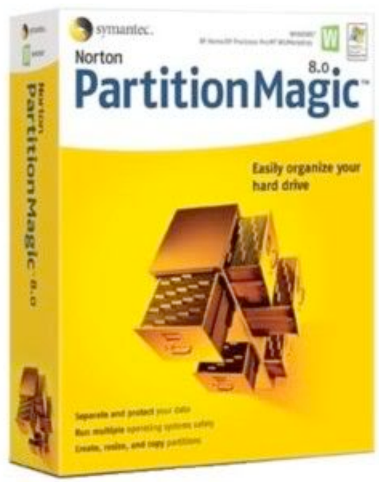

Example (Euler's mother-of-all-partition-identities)
$\#$ partitions of k into odd parts $=$
$\#$ partitions of k into distinct parts

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Example (triangle partitions) $T := \{\lambda : \, 1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3, \; \lambda_1 + \lambda_2 > \lambda_3\}$

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$\longrightarrow$ $\#$ partitions of k in T equals $\left\lfloor \frac{k^3}{12} \right\rceil - \left\lfloor \frac{k}{4} \right\rfloor \left\lfloor \frac{k+2}{4} \right\rfloor$

n-gon Partitions

$$P_n := \{\lambda : 1 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n, \ \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n\}$$

(Sample) Theorem 1 (Andrews, Paule & Riese 2001)

$$\sum_{\lambda \in P_n} q^{\lambda_1 + \cdots + \lambda_n} = \frac{q}{(1-q)(1-q^2)\cdots(1-q^n)} - \frac{q^{2n-2}}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2n-2})}$$

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Natural extension: symmetrize, e.g., the triangle condition to

$$\lambda_{\pi(1)} + \lambda_{\pi(2)} > \lambda_{\pi(3)} \qquad \forall \ \pi \in S_3$$

and enumerate compositions λ with this condition.

Symmetrically Constrained Compositions

(Sample) Theorem 2 (Andrews, Paule & Riese 2001) Given positive integers b and $n \geq 2$, let K consist of all nonnegative integer sequences λ satisfying

$$b(\lambda_{\pi(1)} + \cdots + \lambda_{\pi(n-1)}) \geq (nb - b - 1)\lambda_{\pi(n)} \qquad \forall\ \pi \in S_n$$

Then
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Andrews, Paule & Riese found several identities of this form; all of them concerned symmetric constraints of the form

$$a_1\lambda_{\pi(1)} + a_2\lambda_{\pi(2)} + \cdots + a_n\lambda_{\pi(n)} \geq 0 \qquad \forall\ \pi \in S_n$$

with the condition $a_1 + \cdots + a_n = 1$.

Enter Geometry

We view a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ as an integer lattice point in (a subcone of) $\{\mathbf{x} \in \mathbb{R}^n : \, 0 \le x_1 \le x_2 \le \cdots \le x_n\}$

$C = \sum_{j=1}^{n} \mathbb{R}_{\ge 0} \, \mathbf{v}_j$ is unimodular if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a lattice basis of $\mathbb{Z}^n$

$$\longrightarrow \quad \sigma_C(\mathbf{x}) := \sum_{\mathbf{m} \in C \cap \mathbb{Z}^n} \mathbf{x}^{\mathbf{m}} \; = \; \frac{1}{\prod_{j=1}^{n} \left(1 - \mathbf{x}^{\mathbf{v}_j}\right)}$$

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Example $P \; := \; \{\mathbf{x} \in \mathbb{R}^n : \, 0 \le x_1 \le x_2 \le \cdots \le x_n\}$ is unimodular with generators $\mathbf{e}_n, \, \mathbf{e}_{n-1} + \mathbf{e}_n, \, \dots, \, \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n$

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Remark This geometric viewpoint is not new:

Pak (Proceedings AMS 2004, Ramanujan Journal 2006) realized that several partition identities can be interpreted as bijections of lattice points in two unimodular cones.

Corteel, Savage & Wilf (Integers 2005) discussed several families of partitions/compositions giving rise to unimodular cones (and thus a nice product description of their generating function).

n-gon Partitions Revisited

Theorem 1 (Andrews, Paule & Riese 2001)

$$\sum_{\lambda \in P_n} q^{\lambda_1 + \cdots + \lambda_n} = \frac{q}{(1-q)(1-q^2)\cdots(1-q^n)} - \frac{q^{2n-2}}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2n-2})}$$

An n-gon partition $\lambda \in P_n$ lies in the "fat" cone

$$C_1 := \{\mathbf{x} \in \mathbb{R}^n : \ 0 < x_1 \le x_2 \le \cdots \le x_n, \ x_1 + \cdots + x_{n-1} > x_n\}$$

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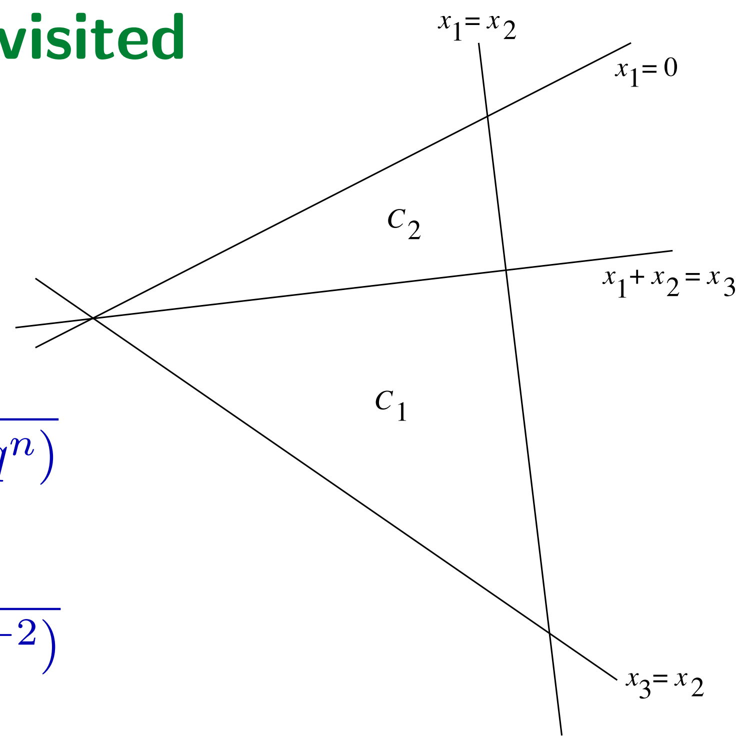

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However, $C_1 = P \setminus C_2$ for the unimodular cone

$$C_2 := \{\mathbf{x} \in \mathbb{R}^n : \, 0 < x_1 \le x_2 \le \cdots \le x_n, \; x_1 + \cdots + x_{n-1} \le x_n\}$$

Theorem 1 is the statement $\sigma_{C_1}(q, \ldots, q) = \sigma_P(q, \ldots, q) - \sigma_{C_2}(q, \ldots, q)$

Symmetrically Constrained Compositions Revisited

Theorem 2 (Andrews, Paule & Riese 2001) Given positive integers b and $n \geq 2$ let K consist of all nonnegative integer sequences λ satisfying

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Then
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General Setup Fix integers $a_1 \leq a_2 \leq \cdots \leq a_n$ and consider all compositions $\lambda \in \mathbb{Z}_{\geq 0}^n$ satisfying

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(Andrews, Paule & Riese: the case $a_1 + \cdots + a_n = 1$ seems special)

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These cones are unimodular if $a_1 + \cdots + a_n = 1$.

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where the union is disjoint and

$$K_\pi = \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{l} x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots \geq x_{\pi(n)} \,, \; x_{\pi(j)} > x_{\pi(j+1)} \text{ if } j \in \mathrm{Des}(\pi) \\ \displaystyle\sum_{j=1}^{n} a_j x_{\pi(j)} \geq 0 \end{array} \right\}$$

Here $\mathrm{Des}(\pi) := \{ j : \, \pi(j) > \pi(j+1) \}$ is the descent set of π

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Theorem (MB, Gessel, Lee & Savage 2010) If $a_1 + \cdots + a_n = 1$ then

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Note that $n \notin \mathrm{Des}(\pi)$ and so $a_1 = \cdots = a_{n-1} = b$ could be interesting...

$$\sum_{\lambda} q^{\lambda_1 + \cdots + \lambda_n} = \frac{\sum_{\pi \in S_n} \prod_{j \in \mathrm{Des}(\pi)} q^{j(1-nb)}}{(1 - q^n) \prod_{j=1}^{n-1} \left(1 - q^{j(1-nb)}\right)}$$

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$$\sum_{\lambda} q^{\lambda_1 + \cdots + \lambda_n} = \frac{\sum_{\pi \in S_n} (q^{1-nb})^{\mathrm{maj}(\pi)}}{(1 - q^n) \prod_{j=1}^{n-1} \left(1 - q^{j(1-nb)}\right)}$$

where $\mathrm{maj}(\pi) := \sum_{j \in \mathrm{Des}(\pi)} j$. Now use $\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = \prod_{j=1}^{n} \frac{1 - q^j}{1 - q} = [\, n \,]_q!$

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There are analogues of this theorem for composition cones that are invariant under the action of other finite reflection groups. Specifically, for symmetry groups of types B and D, our formulas involve signed permutation statistics (MB, Bliem, Braun & Savage 2013).

Lecture Hall Partitions

$$L_n := \left\{ \lambda : \, 0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \cdots \le \frac{\lambda_n}{n} \right\}$$

Lecture Hall Theorem (Bousquet–Mélou & Eriksson 1997)

$$\sum_{\lambda \in L_n} q^{\lambda_1 + \cdots + \lambda_n} = \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}$$

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Remark Euler läßt grüßen...

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Note that the cone is not unimodular...

$$\mathbb{R}_{\ge 0} \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{pmatrix} + \mathbb{R}_{\ge 0} \begin{pmatrix} 0 \\ 2 \\ 3 \\ \vdots \\ n \end{pmatrix} + \cdots + \mathbb{R}_{\ge 0} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ n \end{pmatrix}$$

Lecture Hall Partitions

$$L_{a_1,\dots,a_n} := \left\{ \lambda : \, 0 \le \frac{\lambda_1}{a_1} \le \frac{\lambda_2}{a_2} \le \cdots \le \frac{\lambda_n}{a_n} \right\}$$

Theorem (Bousquet–Mélou & Eriksson 1997) Given $\ell \in \mathbb{Z}_{\ge 2}$ define $a_0 = 0$, $a_1 = 1$, and $a_j = \ell \, a_{j-1} - a_{j-2}$ for $j \ge 2$. Then

$$\sum_{\lambda \in L_{a_1,\dots,a_n}} q^{\lambda_1 + \cdots + \lambda_n} \;=\; \frac{1}{(1-q^{a_1+a_0})(1-q^{a_2+a_1})\cdots(1-q^{a_n+a_{n-1}})}$$

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Question (Bousquet–Mélou & Eriksson 1997) For which sequences (a_j) is $\sum_{\lambda \in L_{a_1,\ldots,a_n}} q^{\lambda_1 + \cdots + \lambda_n}$ the reciprocal of a polynomial?

(Bousquet–Mélou & Eriksson give a complete characterization for the case that (a_j) is increasing and $\gcd(a_j, a_{j+1}) = 1$.)

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Theorem (MB, Braun, Köppe, Savage & Zafeirakopoulos 2014)
Given integers $\ell > 0$ and $b \neq 0$ with $\ell^2 + 4b \ge 0$, let $a_0 = 0$, $a_1 = 1$, and $a_j = \ell \, a_{j-1} + b \, a_{j-2}$ for $j \ge 2$. Then $\sum_{\lambda \in L_{a_1,\ldots,a_n}} q^{\lambda_1 + \cdots + \lambda_n}$ is the reciprocal of a polynomial for all n if and only if $b = -1$.

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$f(q) := \sum_{\lambda \in L_{a_1,\dots,a_n}} q^{\lambda_1 + \cdots + \lambda_n}$ is self-reciprocal if $f(\frac{1}{q}) = \pm\, q^m f(q)$ for some m

$$f(q) = \frac{1}{(1-q^{e_1})(1-q^{e_2})\cdots(1-q^{e_n})} \quad \longrightarrow \quad f(q) \text{ is self-reciprocal}$$

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A pointed rational cone $K \subset \mathbb{R}^n$ is Gorenstein if there exists $\mathbf{c} \in \mathbb{Z}^n$ such that

$$K^\circ \cap \mathbb{Z}^n = \mathbf{c} + (K \cap \mathbb{Z}^n)$$

This translates (by a theorem of Stanley) to $\sigma_K(\frac{1}{\mathbf{x}}) = \pm \, \mathbf{x}^{\mathbf{c}} \, \sigma_K(\mathbf{x})$

Lecture Hall Cones

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Coincidence? Recall that for an ℓ-sequence,

$$\sum_{\lambda \in L_{a_1,\dots,a_n}} q^{\lambda_1 + \cdots + \lambda_n} \;=\; \frac{1}{(1-q^{a_1})(1-q^{a_2+a_1})\cdots(1-q^{a_n+a_{n-1}})}$$

The accompanying cone $K_{a_1,\dots,a_n}$ has Gorenstein point

$$\mathbf{c} = (a_1, a_2 + a_1, \dots, a_n + a_{n-1})$$

Take-Home Message

Many "finite-dimensional" partition/composition identities have a life in polyhedral geometry:

- Bijections between two unimodular cones (Pak)
- Generator descriptions of unimodular cones (Corteel, Savage & Wilf)
- Differences between (unimodular) cones
- Triangulations into (unimodular) cones
- Natural connections to permutation statistics
- Interesting discrete-geometric questions