q-Chromatic Polynomials

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Chromatic Polynomials and Symmetric Functions

$G = (V, E)$ — graph (without loops)

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \ldots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

Example $\chi_{P_4}(n) = n\,(n-1)^3$

Chromatic symmetric function

$$X_G(x_1, x_2, \ldots) \;:=\; \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

We recover $\chi_G(n) = X_G(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$

q-Chromatic Polynomials

Definition $\chi_G^{\lambda}(q,n) := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$ where $\lambda \in \mathbb{Z}_{>0}^V$ is fixed

We recover $\chi_G(n) = \chi_G^{\mathbf{1}}(1,n)$ and $\chi_G^{\mathbf{1}}(q,n) = X_G(q, q^2, \dots, q^n, 0, 0, \dots)$

Example

$$\begin{aligned} \chi_{P_4}^{\mathbf{1}}(q,n) \quad = \quad & \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times \\ & \Big(8q^{10}(1-q)^n(1-q)^{n-1}(1-q)^{n-2}(1-q)^{n-3} \\ & +(4q^9+6q^8+4q^7)(1-q)^{n+1}(1-q)^n(1-q)^{n-1}(1-q)^{n-2} \\ & +2q^6(1-q)^{n+2}(1-q)^{n+1}(1-q)^n(1-q)^{n-1} \Big) \end{aligned}$$

q-Chromatic Polynomial Structure

$$\chi_G^\lambda(q,n) \;:=\; \sum_{\substack{\text{proper colorings}\\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$$

Theorem There exists a (unique) polynomial $\widetilde{\chi}_G^\lambda(q,x) \in \mathbb{Z}(q)[x]$ such that

$$\chi_G^\lambda(q,n) \;=\; \widetilde{\chi}_G^\lambda(q,[n]_q) \qquad \text{where} \qquad [n]_q := 1 + q + \cdots + q^{n-1}$$

Example

$$\begin{aligned}\widetilde{\chi}_{P_4}^{\mathbf{1}}(q,x) \;=\;& \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times \\ & \Big(\left(2q^8+4q^7+6q^6+4q^5+8q^4\right)x^4 \\ & -\left(6q^8+10q^7+18q^6+18q^5+20q^4\right)x^3 \\ & +\left(4q^8+10q^7+20q^6+22q^5+16q^4\right)x^2 \\ & -\left(4q^7+8q^6+8q^5+4q^4\right)x\Big)\end{aligned}$$

Why?

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q,n) = \sum_{\substack{\text{proper colorings} \\ \kappa:V\to[n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

$$\chi_G^{\mathbf{1}}(q,n) = \sum_{\substack{\text{proper colorings} \\ \kappa:V\to[n]}} q^{\sum_{v\in V} \kappa(v)}$$

$$\chi_G(n) = \#\left(\text{proper } n\text{-colorings of } G\right)$$

More Why?

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Remarks $\chi_G^{\mathbf{1}}(q, n)$ was previously studied by Loebl (2007).

$\chi_G^{\lambda}(q, n)$ is a special evaluation (with polynomial structure) of Crew–Spirkl's (2020) weighted chromatic symmetric function.

José Aliste-Prieto will talk about related (very cool) things in four hours.

Where does all this come from?

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $n \in \mathbb{Z}_{>0}$ let $L_\mathcal{P}(n) := \# \left(n\mathcal{P} \cap \mathbb{Z}^d \right)$

Theorem (Ehrhart 1962, Macdonald 1971) $L_\mathcal{P}(n)$ is a polynomial in n. Furthermore, $L_\mathcal{P}(-n) = (-1)^{\dim \mathcal{P}} \# \left(n\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$.

Example $(\Pi, \preceq)$ — (finite) partially ordered set $\longrightarrow$

$$\Omega_\Pi^{(\circ)}(n) := \# \text{ (strictly) order-preserving maps } \Pi \to [n]$$

Observation $\chi_G(n) = \displaystyle\sum_{\rho \in A(G)} \Omega^\circ_{\Pi_\rho}(n)$

where $A(G)$ is the set of acyclic orientations of G and Π_ρ is the poset corresponding to ρ

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Now fix a linear form λ and let $L^{\lambda}_{\mathcal{P}}(q,n) := \sum_{\mathbf{m} \in n\mathcal{P}} q^{\lambda(\mathbf{m})}$

Theorem (Chapoton 2015) Under some mild assumptions, there exists a polynomial $\widetilde{L}^{\lambda}_{\mathcal{P}}(q,x) \in \mathbb{Z}(q)[x]$ such that $L^{\lambda}_{\mathcal{P}}(q,n) = \widetilde{L}^{\lambda}_{\mathcal{P}}(q,[n]_q)$. Furthermore,

$$\widetilde{L}^{\lambda}_{\mathcal{P}}\left(\tfrac{1}{q}, [-n]_{\frac{1}{q}}\right) = (-1)^{\dim \mathcal{P}} \sum_{\mathbf{m} \in n\mathcal{P}^\circ} q^{\lambda(\mathbf{m})}$$

q-Chromatic Structures

$$\chi_G^\lambda(q,n) \;:=\; \sum_{\substack{\text{proper colorings} \\ \kappa:V\to[n]}} q^{\sum_{v\in V}\lambda_v\kappa(v)} \;=\; \widetilde{\chi}_G^\lambda(q,[n]_q)$$

Deletion–Contraction (Crew–Spirkl 2020)

$$\chi_G^\lambda(q,n) \;=\; \chi_{G\setminus 12}^\lambda(q,n) - \chi_{G/12}^{(\lambda_1+\lambda_2,\lambda_3,\dots,\lambda_d)}(q,n)$$

$\longrightarrow$ naturally extends to the coefficients of $\widetilde{\chi}_G^\lambda(q,[n]_q)$

Reciprocity $\quad (-1)^{|V|}\, q^{\sum_{v\in V}\lambda_v}\, \widetilde{\chi}_G^\lambda\left(\frac{1}{q},[-n]_{\frac{1}{q}}\right) \;=\; \sum_{(c,\rho)} q^{\sum_{v\in V(G)}\lambda_v c(v)}$

where the sum is over all pairs of an n-coloring c and a compatible acyclic orientation ρ

q-Chromatic Polynomial Formulas

$$\chi_G^\lambda(q,n) \; := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)} \; = \; \widetilde{\chi}_G^\lambda(q,[n]_q)$$

Theorem $$\widetilde{\chi}_G^\lambda(q,x) \; = \; q^{\Lambda_V} \sum_{\text{flats } S \subseteq E} \mu(\varnothing, S) \prod_{C \in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

where $P(S)$ denotes the collection of vertex sets of the connected components induced by S and $\Lambda_W := \sum_{v \in W} \lambda_v$

In particular, for a tree

$$\widetilde{\chi}_T^\lambda(q,x) \; = \; q^{\Lambda_V} \sum_{S \subseteq E} (-1)^{|S|} \prod_{C \in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

$\longrightarrow$ highly-structured formulas for paths, stars, . . .

The Leading Coefficient for Trees

$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings} \\ \kappa:V\to[n]}} q^{\sum_{v\in V} \lambda_v \kappa(v)} \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q)$$

Theorem Given a tree T, the leading coefficient of $\widetilde{\chi}_T^\lambda(q,n)$ equals

$$c_T^\lambda(q) \ = \ (-1)^{|V|}(q^2-q)^{\Lambda_V} \sum_{S\subseteq E} \prod_{C\in P(S)} \frac{1}{1-q^{\Lambda_C}}$$

In particular,

$$[\Lambda_V]_q! \ c_T^\lambda(q) \ = \ q^{\Lambda_V}(-1)^{|V|+\Lambda_V} \sum_{S\subseteq E} (1-q)^{\Lambda_V-\kappa(S)} \frac{[\Lambda_V]_q!}{\prod_{C\in P(S)} [\Lambda_C]_q}$$

where $\kappa(S)$ is the number of components of the subgraph induced by S

The Leading Coefficient for Trees

$$\chi_G^{\lambda}(q,n) \;:=\; \sum_{\substack{\text{proper colorings}\\ \kappa:V\to[n]}} q^{\sum_{v\in V}\lambda_v\kappa(v)} \;=\; \widetilde{\chi}_G^{\lambda}(q,[n]_q)$$

Corollary Given a tree T, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals

$$c_T^{\mathbf{1}}(q) \;=\; (q-q^2)^d \sum_{S\subseteq E}\prod_{C\in P(S)} \frac{1}{1-q^{\Lambda_C}}$$

$$= \frac{1}{[d]_q!}\sum_{(\rho,\sigma)} q^{d+\operatorname{maj}\sigma} \qquad\qquad d:=|V|$$

where the sum ranges over all pairs of acyclic orientations ρ of T and linear extensions σ of the poset induced by ρ

Corollary^2 (via the following slides) $\quad c_T^{\mathbf{1}}(q) \;=\; (-q)^d\, X_T\left(\frac{1}{q},\frac{1}{q^2},\frac{1}{q^3},\dots\right)$

G-Partitions

Given a poset $P = ([d], \preceq)$, a strict P-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_j < m_k \ \text{ whenever } \ j \prec k$$

Given a (simple) graph $G = ([d], E)$, a G-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \ \text{ whenever } \ vw \in E$$

Let $p_G(n)$ denote the number of G-partitions of n, with accompanying generating function

$$P_G(q) \ := \ \sum_{n>0} p_G(n)\, q^n \ = \ X_G(q, q^2, q^3, \dots)$$

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Theorem

$$P_G(q) \ = \ \frac{q^{\binom{d+1}{2}} \sum_{(\rho,\sigma)} q^{-\operatorname{maj}\sigma}}{(1-q)(1-q^2)\cdots(1-q^d)}$$

where the sum ranges over all pairs of acyclic orientations ρ of G and linear extensions σ of the poset induced by ρ

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Let $p_G(n)$ denote the number of G-partitions of n, with accompanying generating function $P_G(q) := \sum_{n>0} p_G(n)\, q^n$

Collorary Given a tree T on d vertices, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals

$$c_T^{\mathbf{1}}(q) \;=\; (-q)^d \, P_T\left(\frac{1}{q}\right)$$

Conjecture The G-partition function $p_G(n)$ distinguishes trees.

There's more...

- Computations
- Formulas for order polytopes
- Play with different polynomial bases
- Number-theoretic properties
- Applications...

arXiv:2403.19573