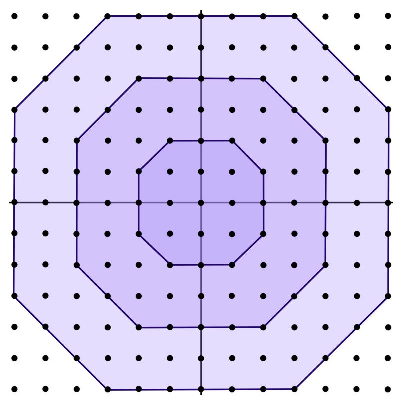
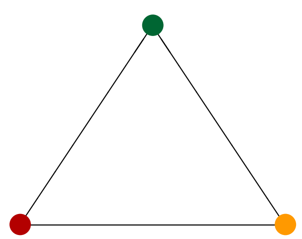
# Chromatic Polynomials, Symmetric Functions & Friends


Matthias Beck
San Francisco State University
matthbeck.github.io




Esme Bajo
Art of Problem Solving

Ben Braun
University of Kentucky

Alvaro Cornejo
University of Kentucky

Thomas Kunze
UC Irvine

Andrés Vindas-Meléndez
Harvey Mudd College

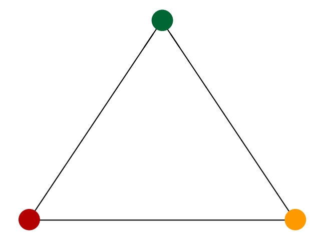
# Chromatic Polynomials

$G = (V, E)$ — graph (without loops)

Proper $n$-coloring — $\kappa : V \to [n] := \{1, 2, \ldots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

Example: $\chi_{K_3}(k) = k(k-1)(k-2)$



(Theorem due to Birkhoff 1912, Whitney 1932)

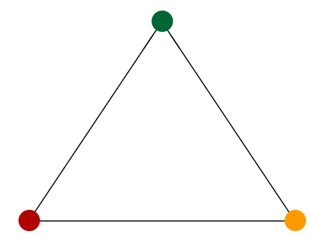
# Chromatic Polynomials

$G = (V, E)$ — graph (without loops)

Proper $n$-coloring — $\kappa : V \to [n] := \{1, 2, \dots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

Example: $\chi_{K_3}(k) = k(k-1)(k-2)$



(Theorem due to Birkhoff 1912, Whitney 1932)

Class of chromatic polynomials $\longrightarrow$ two main research problems:

- Classification — which polynomials are chromatic?
- Detection — does a given polynomial determine the graph?

# Chromatic Polynomials

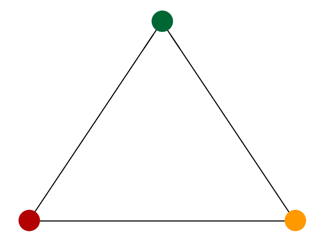
$G = (V, E)$ — graph (without loops)

Proper $n$-coloring — $\kappa : V \to [n] := \{1, 2, \dots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

Example: $\chi_{K_3}(k) \;=\; k(k-1)(k-2)$

(Theorem due to Birkhoff 1912, Whitney 1932)



Polynomial classes in Combinatorics $\longrightarrow$ two main research problems:

- Classification — which polynomials are ...?
- Detection — does a given polynomial determine the ...?

# Chromatic Polynomials

Proper $n$-coloring — $\kappa : V \to [n] := \{1, 2, \dots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

- Classification — which polynomials are chromatic?

... wide open, though we have structural results:

- $\chi_G(n)$ is monic, has constant term $0$ and degree $|V|$.
- The coefficients of $\chi_G(n)$ alternate in sign.
- $|\chi_G(-1)|$ equals $\#$ acyclic orientations of $G$ (Stanley 1973).
- The coefficients of $\chi_G(n)$ are unimodal (Huh 2012).

# Chromatic Polynomials

Proper $n$-coloring — $\kappa : V \to [n] := \{1, 2, \dots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\left(\text{proper } n\text{-colorings of } G\right)$

- Detection — does a given polynomial determine the graph?

... fails spectacularly: If $T$ is a tree with $m$ edges then

$$\chi_T(n) \,=\, n(n-1)^m$$

# Chromatic Symmetric Functions

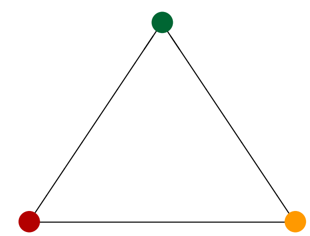
$G = (V, E)$ — graph (without loops)

Proper coloring — $\kappa : V \to \mathbb{Z}_{>0}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic symmetric function

$$X_G(x_1, x_2, \dots) \ := \sum_{\text{proper colorings } \kappa} x_1^{\# \kappa^{-1}(1)} x_2^{\# \kappa^{-1}(2)} \cdots$$

Example: $X_{K_3}(k) \ = \ 6\, x_1\, x_2\, x_3 + 6\, x_1\, x_2\, x_4 + \cdots$



# Chromatic Symmetric Functions

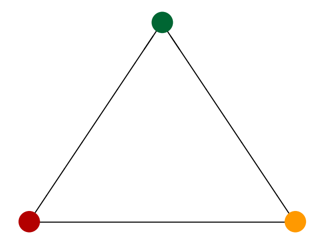
$G = (V, E)$ — graph (without loops)

Proper coloring — $\kappa : V \to \mathbb{Z}_{>0}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic symmetric function

$$X_G(x_1, x_2, \dots) \; := \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

Example: $X_{K_3}(k) \; = \; 6\, x_1\, x_2\, x_3 + 6\, x_1\, x_2\, x_4 + \cdots$



We recover $\chi_G(n) \; = \; X_G(\underbrace{1, \dots, 1}_{n \text{ times}}, 0, 0, \dots)$

# Chromatic Symmetric Functions

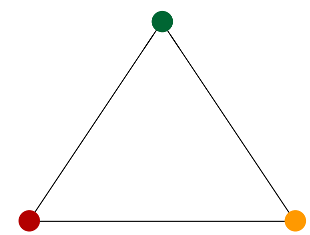
$G = (V, E)$ — graph (without loops)

Proper coloring — $\kappa : V \to \mathbb{Z}_{>0}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic symmetric function

$$X_G(x_1, x_2, \dots) \; := \sum_{\text{proper colorings } \kappa} x_1^{\# \kappa^{-1}(1)} x_2^{\# \kappa^{-1}(2)} \cdots$$

Example: $X_{K_3}(k) \; = \; 6\, x_1\, x_2\, x_3 + 6\, x_1\, x_2\, x_4 + \cdots$



We recover $\chi_G(n) \; = \; X_G(\underbrace{1, \dots, 1}_{n \text{ times}}, 0, 0, \dots)$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \dots)$ distinguishes trees.

(Loehr–Warrington 2024) $X_G(q, q^2, \dots, q^n, 0, 0, \dots)$ distinguishes trees.

# $q$-Chromatic Polynomials

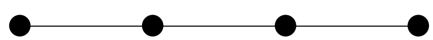
Definition $\chi_G^\lambda(q,n) := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$ where $\lambda \in \mathbb{Z}_{>0}^V$ is fixed

We recover $\chi_G(n) = \chi_G^{\mathbf{1}}(1,n)$ and $\chi_G^{\mathbf{1}}(q,n) = X_G(q, q^2, \dots, q^n, 0, 0, \dots)$

# $q$-Chromatic Polynomials

Definition $\chi_G^\lambda(q,n) := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$ where $\lambda \in \mathbb{Z}_{>0}^V$ is fixed

We recover $\chi_G(n) = \chi_G^{\mathbf{1}}(1,n)$ and $\chi_G^{\mathbf{1}}(q,n) = X_G(q, q^2, \dots, q^n, 0, 0, \dots)$

Example 

$$\begin{aligned} \chi_{P_4}^{\mathbf{1}}(q,n) \quad = \quad & \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times \\ & \Big( 8q^{10}(1-q)^n(1-q)^{n-1}(1-q)^{n-2}(1-q)^{n-3} \\ & +(4q^9+6q^8+4q^7)(1-q)^{n+1}(1-q)^n(1-q)^{n-1}(1-q)^{n-2} \\ & +2q^6(1-q)^{n+2}(1-q)^{n+1}(1-q)^n(1-q)^{n-1} \Big) \end{aligned}$$

# $q$-Chromatic Polynomial Structure

$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings} \\ \kappa:V\to[n]}} q^{\sum_{v\in V} \lambda_v \kappa(v)}$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+) There exists a (unique) polynomial $\widetilde{\chi}_G^\lambda(q,x) \in \mathbb{Z}(q)[x]$ such that

$$\chi_G^\lambda(q,n) \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q) \qquad \text{where} \qquad [n]_q := 1 + q + \cdots + q^{n-1}$$

# $q$-Chromatic Polynomial Structure

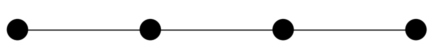
$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+) There exists a (unique) polynomial $\widetilde{\chi}_G^\lambda(q,x) \in \mathbb{Z}(q)[x]$ such that

$$\chi_G^\lambda(q,n) \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q) \qquad \text{where} \qquad [n]_q := 1 + q + \cdots + q^{n-1}$$

Example

$$\widetilde{\chi}_{P_4}^{\mathbf{1}}(q,x) \ = \ \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times$$
$$\Big( \left(2q^8 + 4q^7 + 6q^6 + 4q^5 + 8q^4\right) x^4$$
$$- \left(6q^8 + 10q^7 + 18q^6 + 18q^5 + 20q^4\right) x^3$$
$$+ \left(4q^8 + 10q^7 + 20q^6 + 22q^5 + 16q^4\right) x^2$$
$$- \left(4q^7 + 8q^6 + 8q^5 + 4q^4\right) x \Big)$$



# Why?

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q,n) = \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q,n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q,x)$ distinguishes trees.

# Why?

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^\lambda(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

Remarks $\chi_G^{\mathbf{1}}(q, n)$ was previously studied by Loebl (2007).

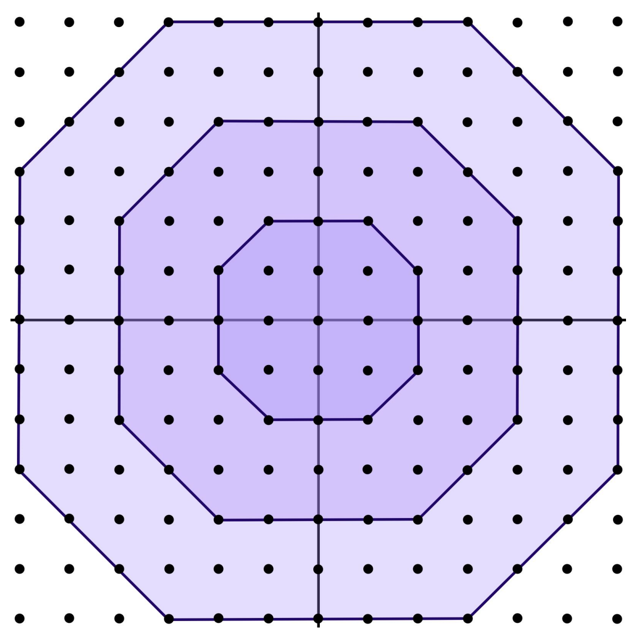
$\chi_G^\lambda(q, n)$ is a special evaluation (with polynomial structure) of Crew–Spirkl's (2020) weighted chromatic symmetric function.

# Where does all this come from?

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $n \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(n) := \# \left( n\mathcal{P} \cap \mathbb{Z}^d \right)$

Theorem (Ehrhart 1962, Macdonald 1971) $L_{\mathcal{P}}(n)$ is a polynomial in $n$. Furthermore, $L_{\mathcal{P}}(-n) = (-1)^{\dim \mathcal{P}} \# \left( n\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$.



# Where does all this come from?

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $n \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(n) := \# \left( n\mathcal{P} \cap \mathbb{Z}^d \right)$

Theorem (Ehrhart 1962, Macdonald 1971) $L_{\mathcal{P}}(n)$ is a polynomial in $n$. Furthermore, $L_{\mathcal{P}}(-n) = (-1)^{\dim \mathcal{P}} \# \left( n\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$.

Example $(\Pi, \preceq)$ — (finite) partially ordered set $\longrightarrow$

$$\Omega_{\Pi}^{(\circ)}(n) := \# \text{ (strictly) order-preserving maps } \Pi \to [n]$$

Observation $\chi_G(n) = \displaystyle\sum_{\rho \in A(G)} \Omega^{\circ}_{\Pi_\rho}(n)$

where $A(G)$ is the set of acyclic orientations of $G$ and $\Pi_\rho$ is the poset corresponding to $\rho$

# Where does all this come from?

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

$$L_{\mathcal{P}}(n) := \# \left( n\mathcal{P} \cap \mathbb{Z}^d \right) \qquad\qquad \chi_G(n) = \sum_{\rho \in A(G)} \Omega^{\circ}_{\Pi_\rho}(n)$$

Theorem (Ehrhart 1962, Macdonald 1971) $L_{\mathcal{P}}(n)$ is a polynomial in $n$. Furthermore, $L_{\mathcal{P}}(-n) = (-1)^{\dim \mathcal{P}} \# \left( n\mathcal{P}^{\circ} \cap \mathbb{Z}^d \right)$.

Now fix a linear form $\lambda$ and let $L^{\lambda}_{\mathcal{P}}(q, n) := \sum_{\mathbf{m} \in n\mathcal{P}} q^{\lambda(\mathbf{m})}$

Theorem (Chapoton 2015) Under some mild assumptions, there exists a polynomial $\widetilde{L}^{\lambda}_{\mathcal{P}}(q, x) \in \mathbb{Z}(q)[x]$ such that $L^{\lambda}_{\mathcal{P}}(q, n) = \widetilde{L}^{\lambda}_{\mathcal{P}}(q, [n]_q)$.

# Where does all this come from?

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

$$L_{\mathcal{P}}(n) := \# \left( n\mathcal{P} \cap \mathbb{Z}^d \right) \qquad\qquad \chi_G(n) = \sum_{\rho \in A(G)} \Omega^\circ_{\Pi_\rho}(n)$$

Theorem (Ehrhart 1962, Macdonald 1971) $L_{\mathcal{P}}(n)$ is a polynomial in $n$. Furthermore, $L_{\mathcal{P}}(-n) = (-1)^{\dim \mathcal{P}} \# \left( n\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$.

Now fix a linear form $\lambda$ and let $L^\lambda_{\mathcal{P}}(q,n) := \sum_{\mathbf{m} \in n\mathcal{P}} q^{\lambda(\mathbf{m})}$

Theorem (Chapoton 2015) Under some mild assumptions, there exists a polynomial $\widetilde{L}^\lambda_{\mathcal{P}}(q,x) \in \mathbb{Z}(q)[x]$ such that $L^\lambda_{\mathcal{P}}(q,n) = \widetilde{L}^\lambda_{\mathcal{P}}(q,[n]_q)$. Furthermore,

$$\widetilde{L}^\lambda_{\mathcal{P}} \left( \tfrac{1}{q}, [-n]_{\frac{1}{q}} \right) = (-1)^{\dim \mathcal{P}} \sum_{\mathbf{m} \in n\mathcal{P}^\circ} q^{\lambda(\mathbf{m})}$$

# Where does all this come from?

Fix a linear form $\lambda$ and let $L_{\mathcal{P}}^{\lambda}(q,n) := \sum_{\mathbf{m} \in n\mathcal{P}} q^{\lambda(\mathbf{m})}$

Theorem (Chapoton 2015) Under some mild assumptions, there exists a polynomial $\widetilde{L}_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Z}(q)[x]$ such that $L_{\mathcal{P}}^{\lambda}(q,n) = \widetilde{L}_{\mathcal{P}}^{\lambda}(q,[n]_q)$. Furthermore,

$$\widetilde{L}_{\mathcal{P}}^{\lambda}\left(\tfrac{1}{q},[-n]_{\frac{1}{q}}\right) = (-1)^{\dim \mathcal{P}} \sum_{\mathbf{m} \in n\mathcal{P}^{\circ}} q^{\lambda(\mathbf{m})}$$

Extensions (MB–Kunze 2025+)

- Explicit formulas in terms of the vertex cones of $\mathcal{P}$
- Bounds on the poles of the cofficients
- Behavior as $n \to \infty$ via $x = \frac{1}{1-q}$
- Quasipolynomials for rational polytopes

# $q$-Chromatic Polynomial Formulas

$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings} \\ \kappa:V\to[n]}} q^{\sum_{v\in V}\lambda_v \kappa(v)} \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q)$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+)

$$\widetilde{\chi}_G^\lambda(q,x) \ = \ q^{\Lambda_V} \sum_{\text{flats } S\subseteq E} \mu(\varnothing,S) \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

where $P(S)$ denotes the collection of vertex sets of the connected components induced by $S$ and $\Lambda_W := \sum_{v\in W}\lambda_v$. In particular, for a tree

$$\widetilde{\chi}_T^\lambda(q,x) \ = \ q^{\Lambda_V} \sum_{S\subseteq E} (-1)^{|S|} \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

$\longrightarrow$ highly-structured formulas for paths, stars, . . .

# The Leading Coefficient for Trees

$$\chi_G^{\lambda}(q,n) \ := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)} \ = \ \widetilde{\chi}_G^{\lambda}(q,[n]_q)$$

**Corollary** Given a tree $T$, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals

$$\begin{aligned} c_T^{\mathbf{1}}(q) &= (q-q^2)^d \sum_{S \subseteq E} \prod_{C \in P(S)} \frac{1}{1-q^{\Lambda_C}} \\ &= \frac{1}{[d]_q!} \sum_{(\rho,\sigma)} q^{d + \operatorname{maj} \sigma} \qquad\qquad d := |V| \end{aligned}$$

where the sum ranges over all pairs of acyclic orientations $\rho$ of $T$ and linear extensions $\sigma$ of the poset induced by $\rho$

**Corollary** $c_T^{\mathbf{1}}(q) \ = \ (-q)^d \, X_T\left(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \dots\right)$

## $G$-Partitions

Given a poset $P = ([d], \preceq)$, a strict $P$-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_j < m_k \ \text{ whenever } \ j \prec k$$

Given a (simple) graph $G = ([d], E)$, a $G$-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \ \text{ whenever } \ vw \in E$$

Let $p_G(n)$ denote the number of $G$-partitions of $n$, with accompanying generating function

$$P_G(q) \ := \ \sum_{n>0} p_G(n)\, q^n \ = \ X_G(q, q^2, q^3, \dots)$$

# $G$-Partitions

Given a (simple) graph $G = ([d], E)$, a $G$-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \ldots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \ \text{ whenever } \ vw \in E$$

Let $p_G(n)$ denote the number of $G$-partitions of $n$, with accompanying generating function $P_G(q) := \sum_{n>0} p_G(n)\, q^n$

Theorem

$$P_G(q) \ = \ \frac{q^{\binom{d+1}{2}} \sum_{(\rho,\sigma)} q^{-\operatorname{maj}\sigma}}{(1-q)(1-q^2)\cdots(1-q^d)}$$

where the sum ranges over all pairs of acyclic orientations $\rho$ of $G$ and linear extensions $\sigma$ of the poset induced by $\rho$

# $G$-Partitions

Given a (simple) graph $G = ([d], E)$, a $G$-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \text{ whenever } vw \in E$$

Let $p_G(n)$ denote the number of $G$-partitions of $n$, with accompanying generating function $P_G(q) := \sum_{n>0} p_G(n)\, q^n$

Collorary Given a tree $T$ on $d$ vertices, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals

$$c_T^{\mathbf{1}}(q) \ = \ (-q)^d \, P_T\left(\frac{1}{q}\right)$$

Conjecture The $G$-partition function $p_G(n)$ distinguishes trees.

# Stanley's Tree Conjecture Revisited

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

Theorem (MB–Braun–Cornejo 2025+) Fix $k \geq d$ and $\lambda_j := k^j$. Then $\widetilde{\chi}_G^{\lambda}(q, x)$ distinguishes graphs on $d$ nodes.