Quasipolynomials in Discrete Geometry & Combinatorial Commutative Algebra

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based on joint work with

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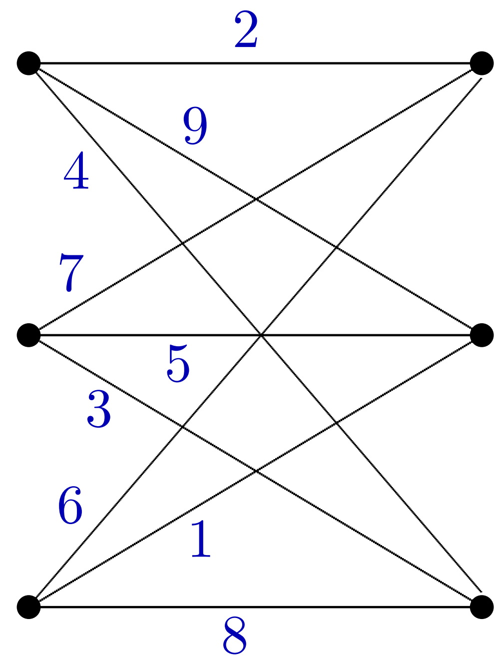
Motivation: Magic Squares & Graphs

2	9	4
7	5	3
6	1	8



A magic labeling of $G = (V, E)$ is an assignment of positive integers to the edges of G such that

- each edge label $1, 2, \dots, |E|$ is used exactly once;
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Conjecture [Hartsfield & Ringel 1990] Every connected graph except K_2 has an antimagic labeling.

- [Alon et al 2004] connected graphs with minimum degree $\geq c \log |V|$
- [Bérczi et al 2017] connected regular graphs
- open for trees

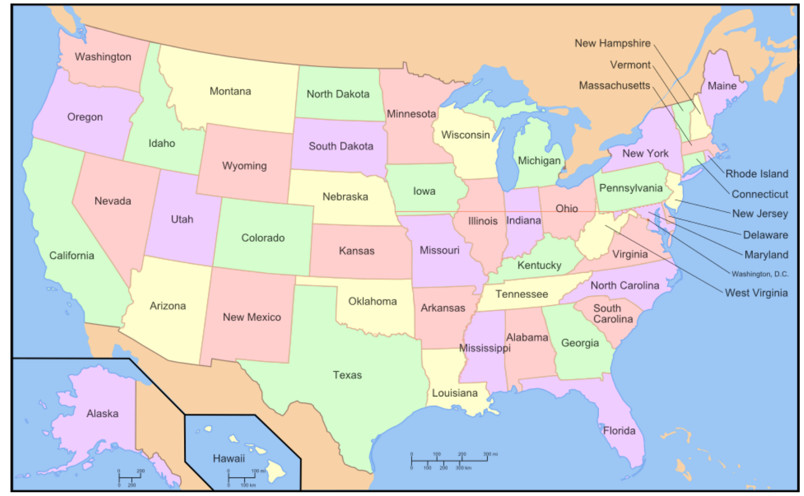
[bart.gov]

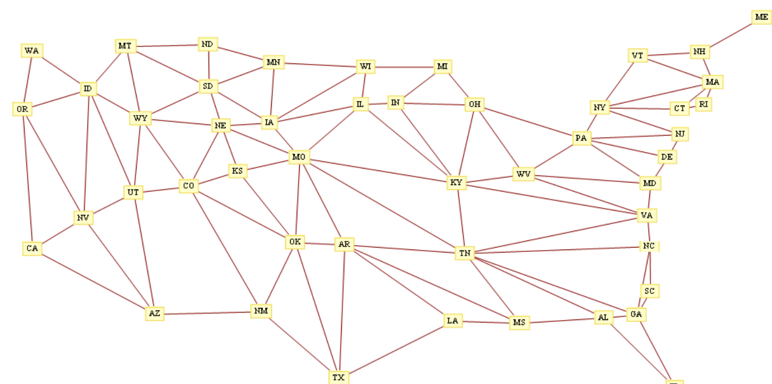
Motivation: Graph Coloring

Theorem [Appel & Haken 1976]
The chromatic number of any planar graph is at most 4.

This theorem had been a conjecture (conceived by Guthrie when trying to color maps) for 124 years.

Birkhoff [1912] says:
Try polynomials!





[mathforum.org]

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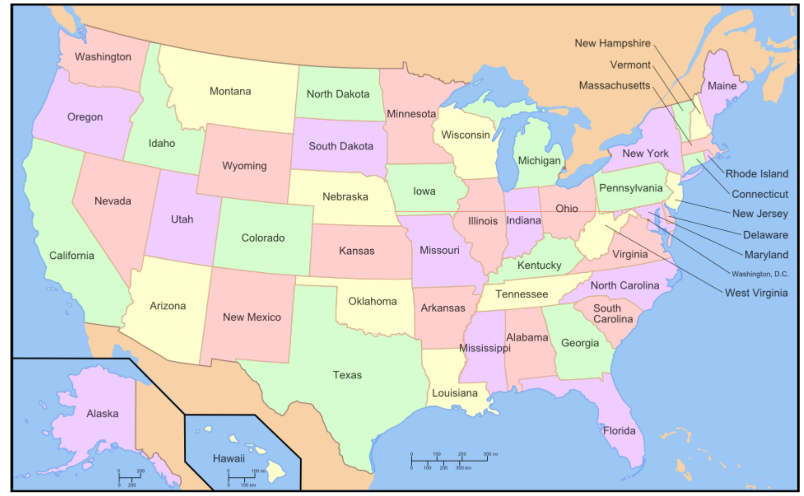

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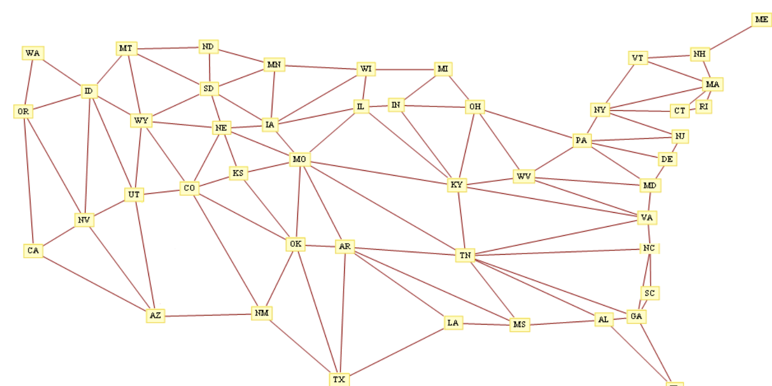

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Four-Color Theorem Rephrased For a planar graph G, we have $\chi_G(4) > 0$, that is, 4 is not a root of the polynomial $\chi_G(k)$.

Motivation: Antimagic Counting

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Idea Introduce a counting function: let $A^*_G(k)$ be the number of assignments of positive integers to the edges of G such that

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Then G has an antimagic labeling if and only if $A^*_G(|E|) > 0$.

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Then G has an antimagic labeling if and only if $A_G^*(|E|) > 0$.

Bad (?) News The counting function $A_G^*(k)$ is in general not a polynomial:

$$A_{C_4}^*(k) = k^4 - \tfrac{22}{3}k^3 + 17k^2 - \tfrac{38}{3}k + \begin{cases} 0 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd.} \end{cases}$$

Quasipolynomials

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$$q(k) \ = \ c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$$

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Example $\sum_{k \ge 0} A^*_{C_4}(k)\, z^k = \frac{40z^9 + 168z^8 + 272z^7 + 208z^6 + 72z^5 + 8z^4}{(1 - z^2)^5}$

Computational Complexity Philosphy We need $(d+1)p$ pieces of data to understand $q(k)$

Quasipolynomials in Nature

Very Basic Problem Given $\Phi \in \mathbb{Z}^{r \times m}$ (of rank r), enumerate all solutions $\mathbf{x} \in \mathbb{Z}_{\geq 0}^{m}$ to the system of eqations $\Phi\,\mathbf{x} = \mathbf{0}$.

These solutions form a semigroup S. If $\mathbf{x} \in S$ satisfies

$$n\,\mathbf{x} = \mathbf{y} + \mathbf{y}' \qquad \Longrightarrow \qquad \mathbf{y} = j\,\mathbf{x}, \ \ \mathbf{y} = (n-j)\,\mathbf{x}$$

for any $n \in \mathbb{Z}_{>0}$ and $\mathbf{y}, \mathbf{y}' \in S$ then $\mathbf{x}$ is completely fundamental. We collect the completely fundamental elements of S in the set $\mathrm{CF}(S)$.

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Theorem (Stanley 1973) The generating function $\displaystyle\sum_{\mathbf{x} \in S} \mathbf{z}^{\mathbf{x}} = \sum_{\mathbf{x} \in S} z_1^{x_1} \cdots z_m^{x_m}$ can be written as a rational function with denominator $\displaystyle\prod_{\mathbf{x} \in \mathrm{CF}(S)} \left(1 - \mathbf{z}^{\mathbf{x}}\right)$.

Terminology The algebra $\langle \mathbf{z}^{\mathbf{x}} : \mathbf{x} \in S \rangle$ is the monomial algebra associated with S and (evaluations of) $\displaystyle\sum_{\mathbf{x} \in S} \mathbf{z}^{\mathbf{x}}$ is its Hilbert series.

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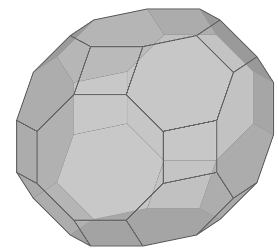

My Favorite Example $\mathcal{P} := \left\{ \mathbf{x} \in \mathbb{R}_{\geq 0}^{d} : \; \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$ is a rational polytope if we can choose the entries of $\mathbf{A}$ and $\mathbf{b}$ to be integral.

Here $\Phi = [\mathbf{A} \;\; -\mathbf{b}]$ and S consists of all $(\mathbf{x}, t) \in \mathbb{Z}_{\geq 0}^{d+1}$ with $\mathbf{x} \in t\mathcal{P}$. A vertex $\mathbf{v}$ of $\mathcal{P}$ gives rise to a completely fundamental element $(\mathrm{den}(\mathbf{v})\,\mathbf{v}, \mathrm{den}(\mathbf{v}))$ where $\mathrm{den}(\mathbf{v})$ is the lowest common denominator of $v_1, \dots, v_d$.

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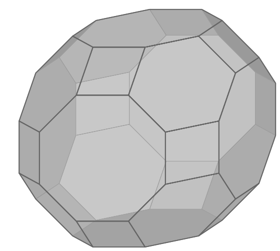

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$$\mathrm{Ehr}_{\mathcal{P}}(z) := \sum_{(\mathbf{x},t) \in S} z^t = \frac{h_{\mathcal{P}}^*(z)}{\left(1 - z^{\mathrm{den}(\mathcal{P})}\right)^{\dim(\mathcal{P})+1}}$$

Ehrhart Quasipolynomials

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope $\qquad S := \left\{ (\mathbf{x}, t) \in \mathbb{Z}_{\geq 0}^{d+1} : \, \mathbf{x} \in t\mathcal{P} \right\}$

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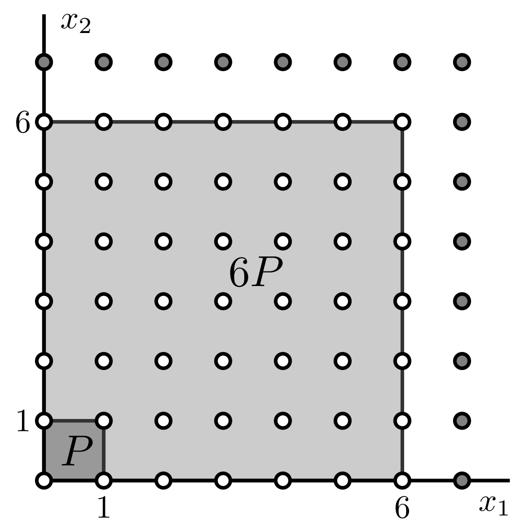
Theorem (Ehrhart 1962) Given a rational polytope $\mathcal{P}$ the counting function $\left| t\mathcal{P} \cap \mathbb{Z}^d \right|$ is a quasipolynomial in t of degree $\dim(\mathcal{P})$ and period dividing $\mathrm{den}(\mathcal{P})$.



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Sample Open Problems

- Classify $h^*_{\mathcal{P}}(z)$ (open for $(\dim(\mathcal{P}), \mathrm{den}(\mathcal{P})) = (2,2), (3,1), \ldots$)

- Find families of polytopes for which $h^*_{\mathcal{P}}(z)$ is unimodal (conjectured, e.g., for hypersimplices $\Delta(d,k) := \left\{ \mathbf{x} \in [0,1]^d : \sum x_j = k \right\}$)

The Message

Your linear enumeration problem here

$$\swarrow \qquad\qquad \searrow$$

Combinatorial Commutative Algebra $\longleftrightarrow$ Ehrhart Theory

$$\downarrow$$

quasipolynomials

$$\downarrow$$

...

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Then G has an antimagic labeling if and only if $A_G^*(|E|) > 0$.

Bad News We do not seem to be able to control the period of the quasipolynomial $A_G^*(k)$.

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Corollary If G has no K_2 component and at most one isolated vertex, then $A_G(2|E|) > 0$. If, in addition, G is bipartite, then $A_G(|E|) > 0$.

Partially Magic Labelings

A partially magic k-labeling of G over $S \subseteq V$ is an assignment of positive integers to the edges of G such that

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Let $M_S(k)$ be the number of partially magic k-labeling of G over S.

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For $S = V$ this theorem is due to Stanley [1973].

Extensions & Open Problems

- Directed Antimagic Graph Conjecture [Hefetz–Mütze–Schwartz 2010]
- Distinct Antimagic Counting
- Harmonious Tree Conjecture [Graham–Sloane 1980]