

Rational Ehrhart Theory

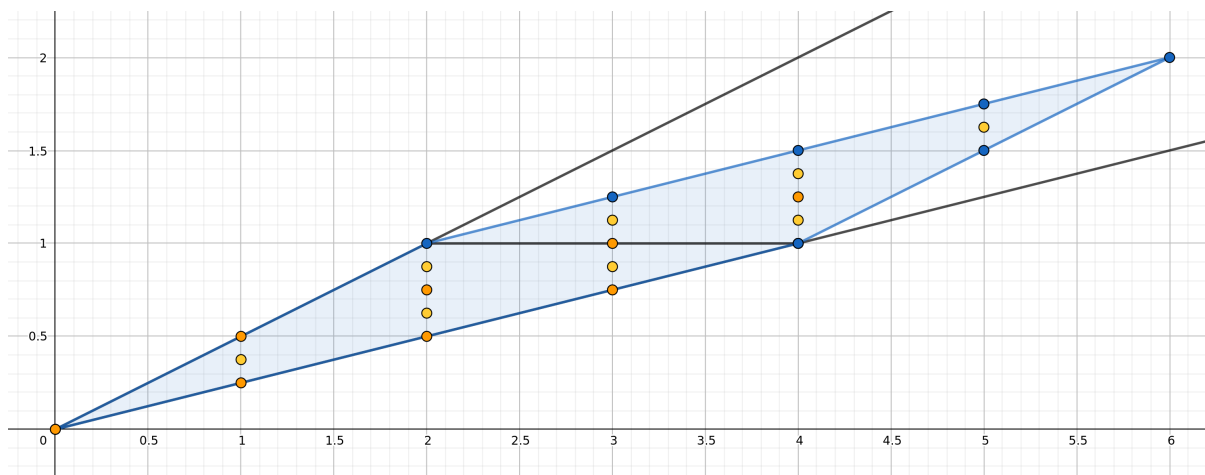
Matthias Beck

San Francisco State University

Sophia Elia

Sophie Rehberg

Freie Universität Berlin



Characteristic Polynomials of Hyperplane Arrangements and Ehrhart
Polynomials of Convex Polytopes @ RIMS 15 February 2023

The Menu

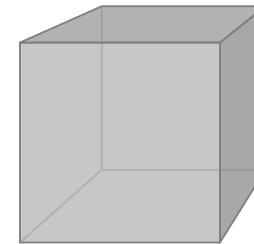
- ▶ Ehrhart (quasi-)polynomials
- ▶ Motivation
- ▶ History
- ▶ Rational Ehrhart series
- ▶ Rational Gorenstein polytopes
- ▶ Symmetric h^* -decompositions
- ▶ Open problems

Measuring Polytopes

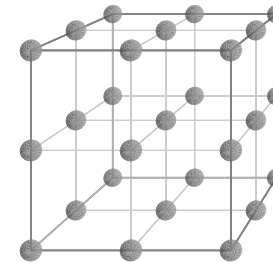
Rational polytope — convex hull of finitely many points in \mathbb{Q}^d
— solution set of a system of linear (in-)equalities with integer coefficients

Goal: measuring...

► volume $\text{vol}(\mathcal{P}) = \lim_{n \rightarrow \infty} \frac{1}{n^d} \left| \mathcal{P} \cap \frac{1}{n} \mathbb{Z}^d \right|$



► discrete volume $|\mathcal{P} \cap \mathbb{Z}^d|$



Ehrhart function $\text{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := \left| \mathcal{P} \cap \frac{1}{n} \mathbb{Z}^d \right| = |n\mathcal{P} \cap \mathbb{Z}^d|$ for $n \in \mathbb{Z}_{>0}$

Discrete Volumes & Ehrhart Quasipolynomials

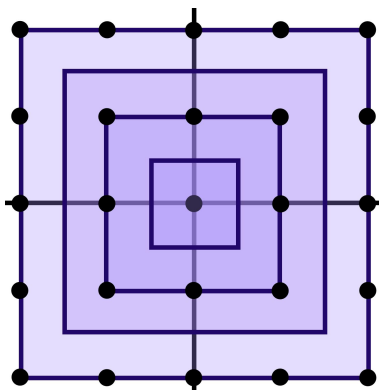
Rational polytope — convex hull of finitely many points in \mathbb{Q}^d

$q(n) = c_d(n)n^d + \dots + c_0(n)$ is a **quasipolynomial** if $c_0(n), \dots, c_d(n)$ are periodic functions; the lcm of their periods is the **period** of $q(n)$.

Theorem (Ehrhart 1962) For any rational polytope $\mathcal{P} \subset \mathbb{R}^d$, $\text{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := |n\mathcal{P} \cap \mathbb{Z}^d|$ is a quasipolynomial in the integer variable n whose period divides the lcm of the denominators of the vertex coordinates of \mathcal{P} (the **denominator** of \mathcal{P}).



Example $\mathcal{P} = [-\frac{1}{2}, \frac{1}{2}]^2$



Discrete Volumes & Ehrhart Quasipolynomials

Rational polytope — convex hull of finitely many points in \mathbb{Q}^d

$q(n) = c_d(n)n^d + \dots + c_0(n)$ is a **quasipolynomial** if $c_0(n), \dots, c_d(n)$ are periodic functions; the lcm of their periods is the **period** of $q(n)$.

Theorem (Ehrhart 1962) For any rational polytope $\mathcal{P} \subset \mathbb{R}^d$, $\text{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := |n\mathcal{P} \cap \mathbb{Z}^d|$ is a quasipolynomial in the integer variable n whose period divides the lcm of the denominators of the vertex coordinates of \mathcal{P} (the **denominator** q of \mathcal{P}).



EF
1959

Equivalently, the **Ehrhart series** can be written as

$$\text{Ehr}_{\mathbb{Z}}(\mathcal{P}; t) := 1 + \sum_{n \geq 1} \text{ehr}_{\mathbb{Z}}(\mathcal{P}; n) t^n = \frac{h_{\mathbb{Z}}^*(\mathcal{P}; t)}{(1 - tq)^{\dim \mathcal{P} + 1}}$$

Why care about... Ehrhart (Quasi-)Polynomials

- ▶ Linear systems are **everywhere**, and so polyhedra are everywhere.
- ▶ In applications, the **volume** of the polytope represented by a linear system measures some fundamental data of this system (“average”).
- ▶ Polytopes are basic geometric objects, yet even for these basic objects volume computation is **hard** and there remain many open problems.
- ▶ Many **discrete problems** in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.
- ▶ Much discrete geometry can be modeled using **polynomials** and, conversely, many combinatorial polynomials can be modeled geometrically.

Motivation I: Let's be rational

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope

Ehrhart functions $\text{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right| = |n\mathcal{P} \cap \mathbb{Z}^d|$ for $n \in \mathbb{Z}_{>0}$

$$\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) := |\lambda\mathcal{P} \cap \mathbb{Z}^d| \text{ for } \lambda \in \mathbb{Q}_{>0}$$

Fun Fact (Linke 2011, Baldoni–Berline–Köppe–Vergne 2013, Stapledon 2017). There is an Ehrhart theory for the quasipolynomial $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ in the rational (equivalently, real) variable λ .

Motivation I: Let's be rational

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope

Ehrhart functions $\text{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right| = |n\mathcal{P} \cap \mathbb{Z}^d|$ for $n \in \mathbb{Z}_{>0}$

$$\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) := |\lambda\mathcal{P} \cap \mathbb{Z}^d| \text{ for } \lambda \in \mathbb{Q}_{>0}$$

Fun Fact (Linke 2011, Baldoni–Berline–Köppe–Vergne 2013, Stapledon 2017). There is an Ehrhart theory for the quasipolynomial $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ in the rational (equivalently, real) variable λ .

Examples $\text{ehr}_{\mathbb{Q}}([1, 2]; \lambda) = \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1 = \lambda + 1 - \{2\lambda\} - \{-\lambda\}$

$$\text{ehr}_{\mathbb{Q}}\left([-1, \frac{2}{3}]; \lambda\right) = \frac{5}{3}\lambda + 1 - \left\{\frac{2}{3}\lambda\right\} - \{\lambda\}$$

Motivation II: Ehrhart Veronese

$\mathcal{P} \subset \mathbb{R}^d$ — lattice polytope

Ehrhart function $\text{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := \left| \mathcal{P} \cap \frac{1}{n} \mathbb{Z}^d \right| = |n\mathcal{P} \cap \mathbb{Z}^d|$ for $n \in \mathbb{Z}_{>0}$

Ehrhart series $\text{Ehr}_{\mathbb{Z}}(\mathcal{P}; z) := 1 + \sum_{n \geq 1} \text{ehr}_{\mathbb{Z}}(\mathcal{P}; n) z^n = \frac{h_{\mathbb{Z}}^*(\mathcal{P}; z)}{(1 - z)^{\dim \mathcal{P} + 1}}$

Fun Fact (Brenti–Welker 2009, MB–Stapledon 2010, Jochemko 2018).
The Ehrhart series of $k\mathcal{P}$ becomes nicer as k increases.

A Bit of Real Ehrhart History

Theorem (Linke 2011) Let $\mathcal{P} \subset \mathbb{R}^d$ be a rational polytope. Then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) = |\lambda\mathcal{P} \cap \mathbb{Z}^d|$ is a quasipolynomial in the rational variable $\lambda > 0$ whose period divides the smallest $q \in \mathbb{Q}_{>0}$ such that $q\mathcal{P}$ is a lattice polytope.

Linke also proved rational analogs of the Ehrhart–Macdonald reciprocity theorem and McMullen’s (1978) theorem about the periods of the coefficient functions when writing

$$\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) = c_d(\lambda) \lambda^d + c_{d-1}(\lambda) \lambda^{d-1} + \cdots + c_0(\lambda)$$

She views these coefficient functions as piecewise polynomials and proved a differential equation for them.

A Bit of Real Ehrhart History

Theorem (Linke 2011) Let $\mathcal{P} \subset \mathbb{R}^d$ be a rational polytope. Then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) = |\lambda\mathcal{P} \cap \mathbb{Z}^d|$ is a quasipolynomial in the rational variable $\lambda > 0$ whose period divides the smallest $q \in \mathbb{Q}_{>0}$ such that $q\mathcal{P}$ is a lattice polytope.

Baldoni–Berline–Köppe–Vergne (2013): algorithmic theory of **intermediate sums** on polyhedra, with $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ as a special case.

Royer (2017+): study of rational Gorenstein polytopes (and much more).

A Bit of Real Ehrhart History

Motivated by motivic integration, Stapledon (2008) introduced the **weighted h^* -polynomial** of a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, which he later (2017) realizes via

$$1 + \sum_{\lambda \in \mathbb{Q}_{>0}} |\partial_{\neq 0}(\lambda \mathcal{P}) \cap \mathbb{Z}^d| t^\lambda = \frac{\tilde{h}(\mathcal{P}; t)}{(1-t)^{\dim \mathcal{P}}}$$

and uses them to compute Ehrhart polynomials of free sums, generalizing work by Braun (2006) and MB–Jayawant–McAllister (2013).

He realizes that $\tilde{h}(\mathcal{P}; t)$ is a polynomial in certain fractional powers of t with nonnegative coefficients. In the case that $\mathbf{0} \in \mathcal{P}^\circ$ he proves that $\tilde{h}(\mathcal{P}; t)$ is symmetric.

The Setup

$$\mathcal{P} \subset \mathbb{R}^d \text{ — rational polytope} \quad \longrightarrow \quad \mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A} \mathbf{x} \leq \mathbf{b}\}$$

The **codemoninator** of \mathcal{P} is $r := \text{lcm}(\mathbf{b})$

Lemma (1) $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) = |\lambda \mathcal{P} \cap \mathbb{Z}^d|$ is constant for $\lambda \in (\frac{n}{r}, \frac{n+1}{r})$, $n \in \mathbb{Z}_{\geq 0}$

(2) If $\mathbf{0} \in \mathcal{P}$ then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ is monotone

Examples $\text{ehr}_{\mathbb{Q}}([1, 2]; \lambda) = \lambda + 1 - \{2\lambda\} - \{-\lambda\}$

$$\text{ehr}_{\mathbb{Q}}([-1, \frac{2}{3}]; \lambda) = \frac{5}{3}\lambda + 1 - \{\frac{2}{3}\lambda\} - \{\lambda\}$$

The Setup

$$\mathcal{P} \subset \mathbb{R}^d \text{ — rational polytope} \quad \longrightarrow \quad \mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A} \mathbf{x} \leq \mathbf{b}\}$$

The **codemoninator** of \mathcal{P} is $r := \text{lcm}(\mathbf{b})$

Lemma (1) $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) = |\lambda \mathcal{P} \cap \mathbb{Z}^d|$ is constant for $\lambda \in (\frac{n}{r}, \frac{n+1}{r})$, $n \in \mathbb{Z}_{\geq 0}$

(2) If $\mathbf{0} \in \mathcal{P}$ then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ is monotone

Corollary If $\lambda \notin \frac{1}{r}\mathbb{Z}_{\geq 0}$ then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) = \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lfloor \lambda \rfloor)$

where $\lfloor \lambda \rfloor := \frac{2j+1}{2r}$ for $|\lambda - \frac{2j+1}{2r}| < \frac{1}{2r}$

If $\mathbf{0} \in \mathcal{P}$ then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) = \text{ehr}_{\mathbb{Q}}\left(\mathcal{P}; \frac{\lfloor r\lambda \rfloor}{r}\right)$

The Setup

$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A} \mathbf{x} \leq \mathbf{b}\}$ with codemoninator $r := \text{lcm}(\mathbf{b})$

If $\mathbf{0} \in \mathcal{P}$ then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) = |\lambda \mathcal{P} \cap \mathbb{Z}^d| = \text{ehr}_{\mathbb{Q}}\left(\mathcal{P}; \frac{\lfloor r\lambda \rfloor}{r}\right)$

If $\mathbf{0} \notin \mathcal{P}$ then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) = \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lfloor \lambda \rfloor)$ for $\lambda \notin \frac{1}{r}\mathbb{Z}_{\geq 0}$

where $\lfloor \lambda \rfloor := \frac{2j+1}{2r}$ for $|\lambda - \frac{2j+1}{2r}| < \frac{1}{2r}$

The Upshot $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ is determined by $\{\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) : \lambda \in \frac{1}{2r}\mathbb{Z}_{\geq 0}\}$

If $\mathbf{0} \in \mathcal{P}$ then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ is determined by $\{\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) : \lambda \in \frac{1}{r}\mathbb{Z}_{\geq 0}\}$

Rational Ehrhart Series

$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A} \mathbf{x} \leq \mathbf{b}\}$ with codemoninator $r := \text{lcm}(\mathbf{b})$

(Refined) rational Ehrhart series

$$\begin{aligned}\text{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) &:= 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) t^{\lambda} \\ \text{Ehr}_{\mathbb{Q}}^{\text{ref}}(\mathcal{P}; t) &:= 1 + \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) t^{\lambda}\end{aligned}$$

Examples

$$\begin{aligned}\text{Ehr}_{\mathbb{Q}}\left(\left[-1, \frac{2}{3}\right]; t\right) &= \frac{1 + t^{\frac{1}{2}} + t + t^{\frac{3}{2}} + t^2}{(1-t)(1-t^{\frac{3}{2}})} \\ \text{Ehr}_{\mathbb{Q}}^{\text{ref}}([1, 2]; t) &= \frac{1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}}{(1-t)^2}\end{aligned}$$

Rational Ehrhart Series

$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A} \mathbf{x} \leq \mathbf{b}\}$ with codemoninator $r := \text{lcm}(\mathbf{b})$

(Refined) rational Ehrhart series

$$\begin{aligned}\text{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) &:= 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) t^{\lambda} \\ \text{Ehr}_{\mathbb{Q}}^{\text{ref}}(\mathcal{P}; t) &:= 1 + \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) t^{\lambda}\end{aligned}$$

Theorem Let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathcal{P}$ is a lattice polytope. Then

$$\text{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) = \frac{h_{\mathbb{Q}}^*(\mathcal{P}; t)}{(1 - t^{\frac{m}{r}})^{d+1}}$$

where $h_{\mathbb{Q}}^*$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Rational Ehrhart Series

$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A} \mathbf{x} \leq \mathbf{b}\}$ with codemoninator $r := \text{lcm}(\mathbf{b})$

Theorem Let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathcal{P}$ is a lattice polytope. Then

$$\text{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) = 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) t^\lambda = \frac{h_{\mathbb{Q}}^*(\mathcal{P}; t)}{(1 - t^{\frac{m}{r}})^{d+1}}$$

where $h_{\mathbb{Q}}^*(t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Proof Idea: $\text{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) = \text{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathcal{P}; t^{\frac{1}{r}}\right)$

Rational Ehrhart Series

$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A} \mathbf{x} \leq \mathbf{b}\}$ with codemoninator $r := \text{lcm}(\mathbf{b})$

Theorem Let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathcal{P}$ is a lattice polytope. Then

$$\text{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) = 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) t^\lambda = \frac{h_{\mathbb{Q}}^*(\mathcal{P}; t)}{(1 - t^{\frac{m}{r}})^{d+1}}$$

where $h_{\mathbb{Q}}^*(\mathcal{P}; t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Similar results hold for $\text{Ehr}_{\mathbb{Q}}^{\text{ref}}(\mathcal{P}; t)$, with r replaced by $2r$.

Corollary (Linke 2011) The period of the quasipolynomial $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ divides $\frac{m}{r}$. In particular, if \mathcal{P} is a lattice polytope then $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ is a quasipolynomial of period 1.

Rational Ehrhart Series

$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A} \mathbf{x} \leq \mathbf{b}\}$ with codemoninator $r := \text{lcm}(\mathbf{b})$

Theorem Let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathcal{P}$ is a lattice polytope. Then

$$\text{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) = 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) t^{\lambda} = \frac{h_{\mathbb{Q}}^*(\mathcal{P}; t)}{(1 - t^{\frac{m}{r}})^{d+1}}$$

where $h_{\mathbb{Q}}^*(\mathcal{P}; t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Corollary (Linke 2011) For a rational polytope, $(-1)^{\dim \mathcal{P}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; -\lambda)$ equals the number of interior lattice points in $\lambda\mathcal{P}$, for any $\lambda > 0$.

Remark If $\frac{m}{r} \in \mathbb{Z}$ then

$$h_{\mathbb{Z}}^*(\mathcal{P}; t) = \text{Int} [h_{\mathbb{Q}}^*(\mathcal{P}; t)].$$

$$\text{Ehr}_{\mathbb{Q}}^{\text{ref}}([1, 2]; t) = \frac{1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}}{(1 - t)^2}$$

Gorenstein Musings

Theorem (Hibi 1991, Fiset–Kasprzyk 2008) Let \mathcal{P} be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. If the polar dual of \mathcal{P} is a lattice polytope then $h_{\mathbb{Z}}^*(\mathcal{P}; t)$ is symmetric.

(This fits, more generally, with Stanley's theory of Gorenstein rings.)

Theorem Let \mathcal{P} be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $h_{\mathbb{Q}}^*(\mathcal{P}; t)$ is symmetric.

Proof Idea: $(1, \mathbf{0})$ is the Gorenstein point for $\text{cone}(\{1\} \times \frac{1}{r}\mathcal{P})$.

Remark This is a corollary of a more general, technical result, with similar musings for the h^* -polynomial of the refined Ehrhart series.

Gorenstein Musings

Theorem Let \mathcal{P} be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $h_{\mathbb{Q}}^*(\mathcal{P}; t)$ is symmetric.

Proof Idea: $(1, \mathbf{0})$ is the Gorenstein point for $\text{cone}(\{1\} \times \frac{1}{r}\mathcal{P})$.

Remark This is a corollary of a more general, technical result, with similar musings for the h^* -polynomial of the refined Ehrhart series.

Corollary Let \mathcal{P} be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $h_{\mathbb{Z}}^*(\mathcal{P}; t)$ is the Veronese of a symmetric polynomial.

Example $\text{Ehr}_{\mathbb{Q}}([-1, \frac{2}{3}]; t) =$

$$\frac{1 + t^{\frac{1}{2}} + 2t + 3t^{\frac{3}{2}} + 4t^2 + 4t^{\frac{5}{2}} + 4t^3 + 4t^{\frac{7}{2}} + 3t^4 + 2t^{\frac{9}{2}} + t^5 + t^{\frac{11}{2}}}{(1 - t^3)^2}$$

Symmetric Decompositions

Theorem Let \mathcal{P} be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $h_{\mathbb{Q}}^*(\mathcal{P}; t)$ is symmetric.

Corollary (Betke–McMullen 1985, MB–Braun–Vindas–Meléndez 2022) Let \mathcal{P} be a rational polytope with denominator k and $\mathbf{0} \in \mathcal{P}^\circ$. Then there exist polynomials $a(t)$ and $b(t)$ with nonnegative coefficients such that

$$h_{\mathbb{Z}}^*(\mathcal{P}; t) = a(t) + t b(t), \quad t^{k(d+1)-1} a\left(\frac{1}{t}\right) = a(t), \quad t^{k(d+1)-2} b\left(\frac{1}{t}\right) = b(t).$$

Symmetric Decompositions

Theorem Let \mathcal{P} be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $h_{\mathbb{Q}}^*(\mathcal{P}; t)$ is symmetric.

Corollary (Betke–McMullen 1985, MB–Braun–Vindas–Meléndez 2022) Let \mathcal{P} be a rational polytope with denominator k and $\mathbf{0} \in \mathcal{P}^\circ$. Then there exist polynomials $a(t)$ and $b(t)$ with nonnegative coefficients such that

$$h_{\mathbb{Z}}^*(\mathcal{P}; t) = a(t) + t b(t), \quad t^{k(d+1)-1} a\left(\frac{1}{t}\right) = a(t), \quad t^{k(d+1)-2} b\left(\frac{1}{t}\right) = b(t).$$

Proof Idea ($k = 1$):

$$\text{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) = \frac{h_{\mathbb{Z}}^*\left(\partial\left(\frac{1}{r}\mathcal{P}\right); t^{\frac{1}{r}}\right)}{\left(1 - t^{\frac{1}{r}}\right) (1 - t)^d}$$

Note that $h_{\mathbb{Z}}^*\left(\partial\left(\frac{1}{r}\mathcal{P}\right); t\right)$ is symmetric and nonnegative, and

$$\begin{aligned} h_{\mathbb{Q}}^*(\mathcal{P}; t) &= \text{Int} \left[\left(1 + t^{\frac{1}{r}} + \dots + t^{\frac{r-1}{r}}\right) h_{\mathbb{Z}}^*\left(\partial\left(\frac{1}{r}\mathcal{P}\right); t^{\frac{1}{r}}\right) \right] \\ &= \text{Int} \left[h_{\mathbb{Z}}^*\left(\partial\left(\frac{1}{r}\mathcal{P}\right); t^{\frac{1}{r}}\right) \right] + \text{Int} \left[\left(t^{\frac{1}{r}} + t^{\frac{2}{r}} + \dots + t^{\frac{r-1}{r}}\right) h_{\mathbb{Z}}^*\left(\partial\left(\frac{1}{r}\mathcal{P}\right); t^{\frac{1}{r}}\right) \right] \end{aligned}$$

A Remark on Complexity

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope with codemoninator r

To capture $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ (or $\text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$), we need to compute...

$$\mathbf{0} \notin \mathcal{P} \quad \longrightarrow \quad \text{Ehr}_{\mathbb{Q}}^{\text{ref}}(\mathcal{P}; t) = 1 + \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) t^{\lambda} = \frac{h_{\mathbb{Q}}^{*\text{ref}}(\mathcal{P}; t)}{(1 - t^q)^{d+1}}$$

$$\mathbf{0} \in \partial\mathcal{P} \quad \longrightarrow \quad \text{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) = 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \text{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) t^{\lambda} = \frac{h_{\mathbb{Q}}^*(\mathcal{P}; t)}{(1 - t^q)^{d+1}}$$

$$\mathbf{0} \in \mathcal{P}^{\circ} \quad \longrightarrow \quad h_{\mathbb{Q}}^*(\mathcal{P}; t) \text{ symmetric}$$

A Few Open Problems

- ▶ Periods of rational Ehrhart quasipolynomials
- ▶ Classification questions a la Kasprzyk–Nill (“reflexive polytopes of higher index” 2012)
- ▶ Stapledon’s (2009) symmetric decomposition of $h_{\mathbb{Z}}^*(\mathcal{P}; t)$
- ▶ Connections to the Fine (1983) interior of a lattice polytope (Batyrev 2017, Batyrev–Kasprzyk–Schaller 2022)
- ▶ Symmetric polytopes, equivariant Ehrhart theory
- ▶ Further applications

