q-polynomials

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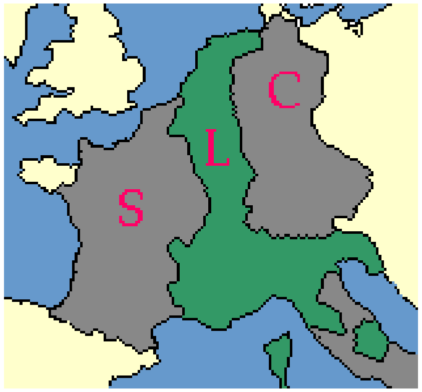
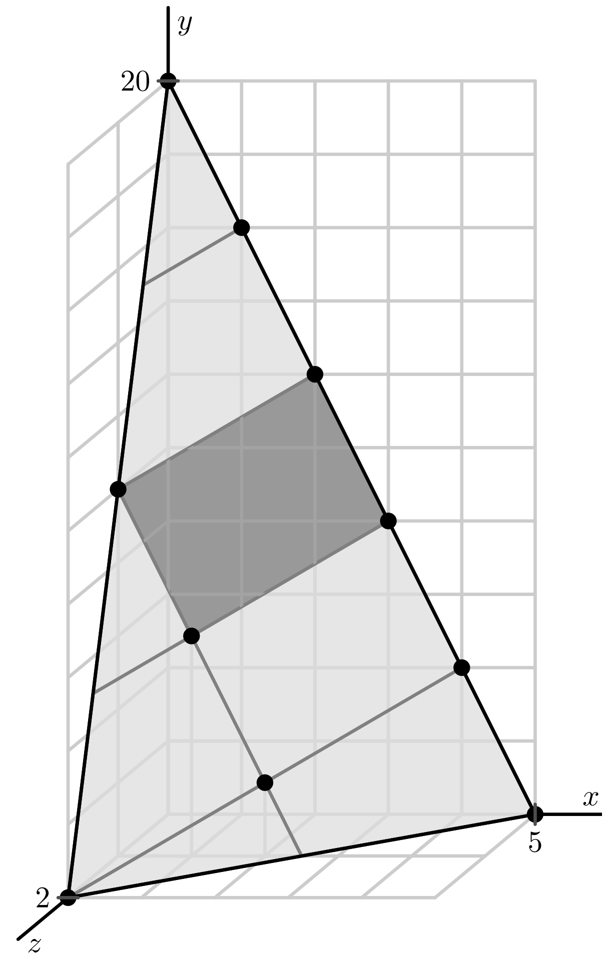


September 2025

Menu I: Ehrhart Polynomials

- Polytopes, integer points, and their polynomials
- Polynomial classification and detection
- Examples
- Central theorems in Ehrhart theory
- (Unimodular) triangulations
- Symmetric decompositions
- Brion's theorem
- Open problems

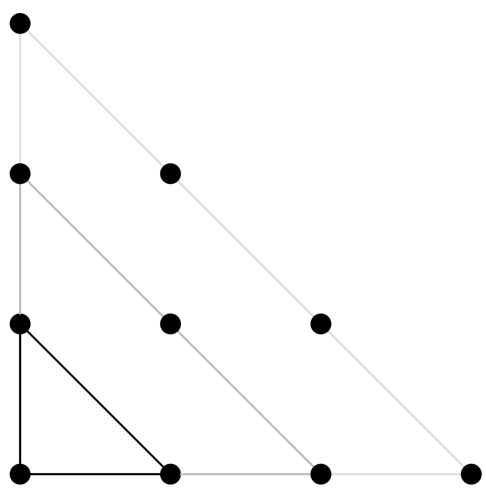


Ehrhart Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $\mathrm{ehr}_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$

Theorem (Ehrhart 1962, Macdonald 1971) $\mathrm{ehr}_{\mathcal{P}}(t)$ is a polynomial in t. Furthermore, $\mathrm{ehr}_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} \#\left(t\mathcal{P}^\circ \cap \mathbb{Z}^d\right)$.



Example $\Delta = \mathrm{conv}\{(0,0),\ (1,0),\ (0,1)\}$

$$\mathrm{ehr}_{\Delta}(t) = \binom{t+2}{2} = \frac{1}{2}t^2 + \frac{3}{2}t + 1$$

$$\mathrm{ehr}_{\Delta^\circ}(t) = \binom{t-1}{2} = \frac{1}{2}t^2 - \frac{3}{2}t + 1$$

Ehrhart polynomials

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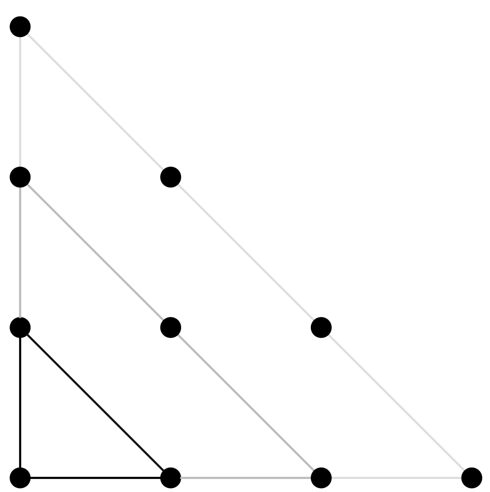


Example $\Delta = \mathrm{conv}\{(0,0),\ (1,0),\ (0,1)\}$

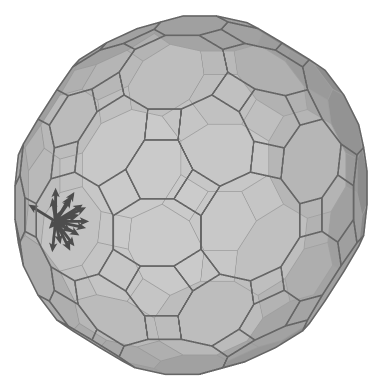
$$\mathrm{ehr}_{\Delta}(t) \;=\; \frac{1}{2}t^2 + \frac{3}{2}t + 1$$

Philosophy We do not need limits for

$$\mathrm{vol}(\mathcal{P}) = \lim_{t\to\infty} \frac{1}{t^d}\,\mathrm{ehr}_{\mathcal{P}}(t)$$

Some Motivation

- Linear systems are everywhere, and so polyhedra are everywhere.
- In applications, the volume of the polytope represented by a linear system measures some fundamental data of this system ("average").
- Many discrete problems in various areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.
- Much discrete geometry can be modeled using polynomials and, conversely, many combinatorial polynomials can be modeled geometrically.
- Volume computation is hard.
- Also, polytopes are cool.



♡ Polynomials

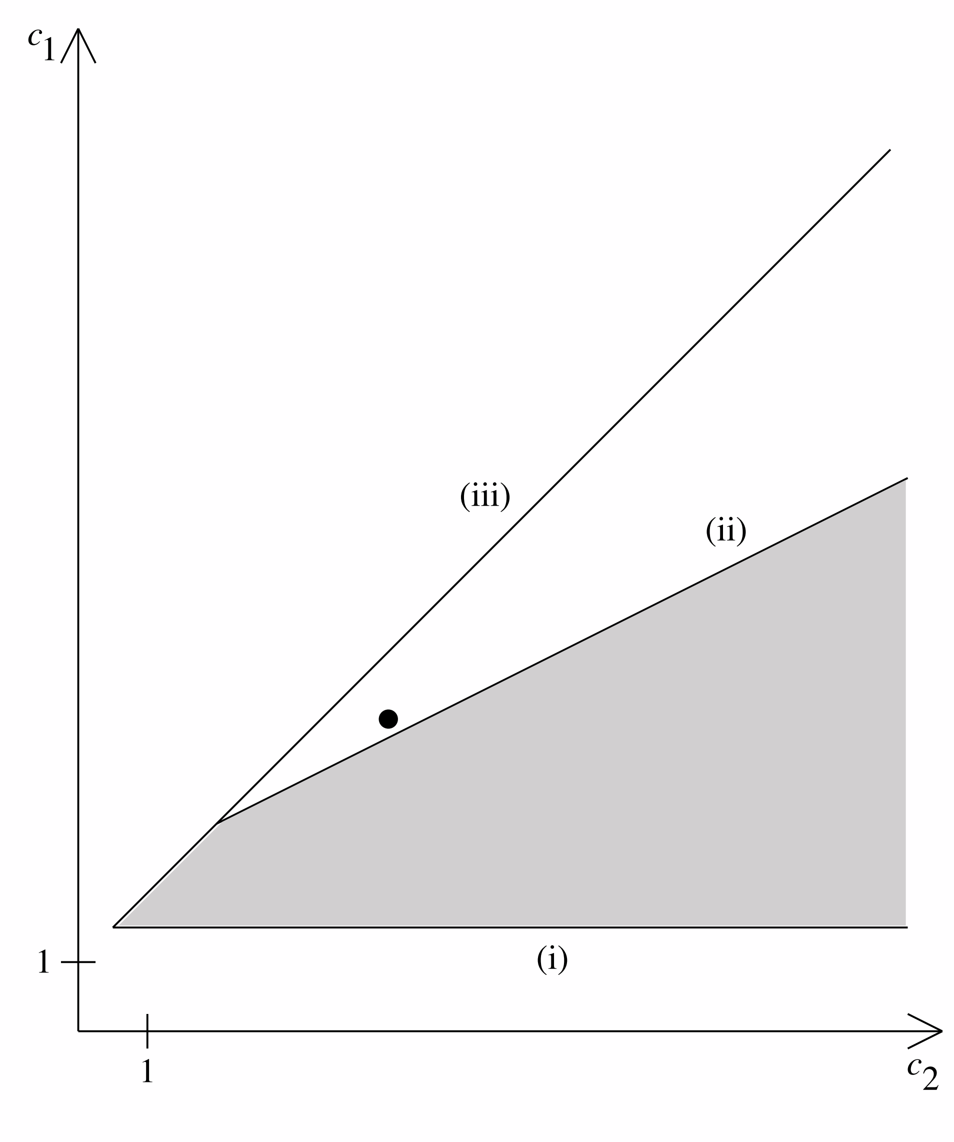
- Computation

Class of Ehrhart polynomials $\longrightarrow$ two main research problems:

- Classification — which polynomials are Ehrhart polynomials?
  (open in dimension 3)

- Detection — does a given polynomial determine the polytope?
  (fails somewhwat spectacularly)

Ehrhart Polynomials in Dimension 2



$\mathcal{P}$ — lattice polygon

$\longrightarrow \ \mathrm{ehr}_{\mathcal{P}}(t) = c_2\, t^2 + c_1\, t + 1$

Ehrhart Series

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $\mathrm{ehr}_\mathcal{P}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$ and

$$\mathrm{Ehr}_\mathcal{P}(z) \ := \ 1 + \sum_{t \ge 1} \mathrm{ehr}_\mathcal{P}(t) \, z^t$$

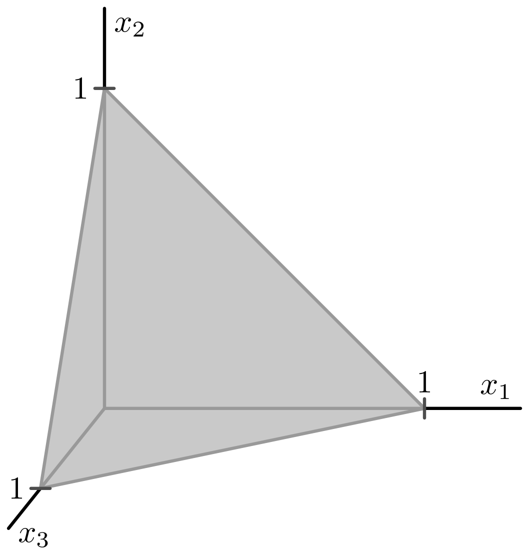
Theorem (Ehrhart 1962, Macdonald 1971) $\mathrm{Ehr}_\mathcal{P}(z) = \dfrac{h^*_\mathcal{P}(z)}{(1-z)^{d+1}}$.

Furthermore, $(-1)^{\dim \mathcal{P}+1} \, \mathrm{Ehr}_\mathcal{P}(\frac{1}{z}) = \sum_{t \ge 1} \mathrm{ehr}_{\mathcal{P}^\circ}(t) \, z^t$.

Philosophy:

Change of basis $\mathrm{ehr}_\mathcal{P}(t) \ = \ h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_d^* \binom{t}{d}$

Familiar Faces

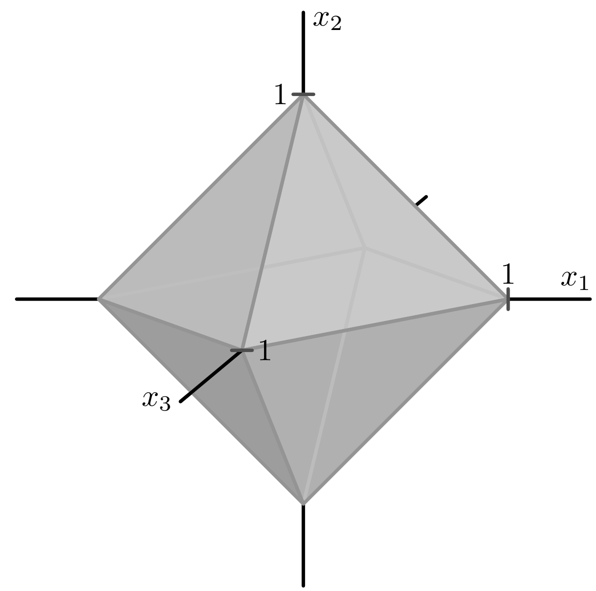


$$\Delta = \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \, x_1 + x_2 + \cdots + x_d \leq 1 \right\}$$

$$\mathrm{ehr}_\Delta(t) = \binom{d+t}{d} \qquad h^*_\Delta(z) = 1$$

$$\diamond = \left\{ \mathbf{x} \in \mathbb{R}^d : \, |x_1| + |x_2| + \cdots + |x_d| \leq 1 \right\}$$

$$h^*_\diamond(z) = (1+z)^d$$



$$\square = [0,1]^d$$

$\mathrm{ehr}_\square(t) = (t+1)^d$ $\qquad$ $h^*_\square(z)$ — Eulerian polynomial

Ehrhart Positivity & Friends

Theorem (Ehrhart 1962, Macdonald 1971)

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \ge 1} \mathrm{ehr}_{\mathcal{P}}(t)\, z^t = \frac{h^*_{\mathcal{P}}(z)}{(1-z)^{d+1}}$$

Theorem (Stanley 1980) The coefficients of $h^*_{\mathcal{P}}(z)$ are nonnegative integers.

Theorem (Hibi–Stanley–Folklore) $h^*_{\mathcal{P}}(z)$ is palindromic $\iff$ $\mathcal{P}$ is Gorenstein.

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Open Problem Prove that the h^*-polynomial of
- hypersimplices
- polytopes admitting a unimodular triangulation (next slides)
- polytope with the integer decomposition property are unimodal

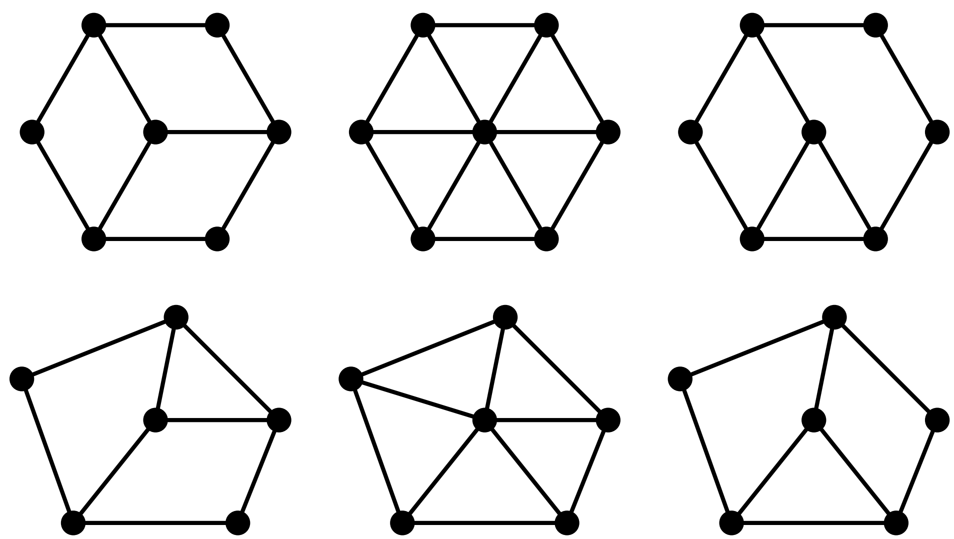
✓ Gorenstein polytopes with regular unimodular triangulation (Bruns–Römer 2007)

✓ Zonotopes (MB–Jochemko–McCullough 2019)

Trials & Triangulations

Subdivision of a polyhedron $\mathcal{P}$ — finite collection S of polyhedra such that

- if $\mathcal{F}$ is a face of $\mathcal{G} \in S$ then $\mathcal{F} \in S$
- if $\mathcal{F}, \mathcal{G} \in S$ then $\mathcal{F} \cap \mathcal{G}$ is a face of both
- $\mathcal{P} = \bigcup_{\mathcal{F} \in S} \mathcal{F}$



If each $\mathcal{F}$ is a simplex $\longrightarrow$ triangulation of a polytope

Unimodular Triangulations

A lattice d-simplex with volume $\frac{1}{d!}$ is unimodular

Alternative description: if the simplex has vertices $v_0, v_1, \ldots, v_d$, the vectors $v_1 - v_0, \ldots, v_d - v_0$ form a basis of $\mathbb{Z}^d$.

Every lattice polygon admits a unimodular triangulation, the regular tetrahedron with vertices $(0,0,0), (1,1,0), (1,0,1), (0,1,1)$ does not.

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Theorem (Kempf–Knudsen–Mumford–Saint-Donat–Waterman 1970's)
For every lattice polytope $\mathcal{P}$ there exists an integer m such that $m\mathcal{P}$ admits a regular unimodular triangulation.

Theorem (Liu 2025+) For every lattice polytope $\mathcal{P}$ there exists an integer m such that $k\mathcal{P}$ admits a regular unimodular triangulation for $k \geq m$.

Conjecture There exists an integer m_d such that, if $\mathcal{P}$ is a d-dimensional lattice polytope, then $m_d\mathcal{P}$ admits a regular unimodular triangulation.

f- and h-vectors of triangulation

f_k — number of k-simplices in a given triangulation T of a polytope

$f_{-1} := 1$

h-polynomial of T

$$h_T(z) := \sum_{k=-1}^{d} f_k \, z^{k+1} \, (1-z)^{d-k}$$

f- and h-vectors of triangulation

f_k — number of k-simplices in a given triangulation T of a polytope

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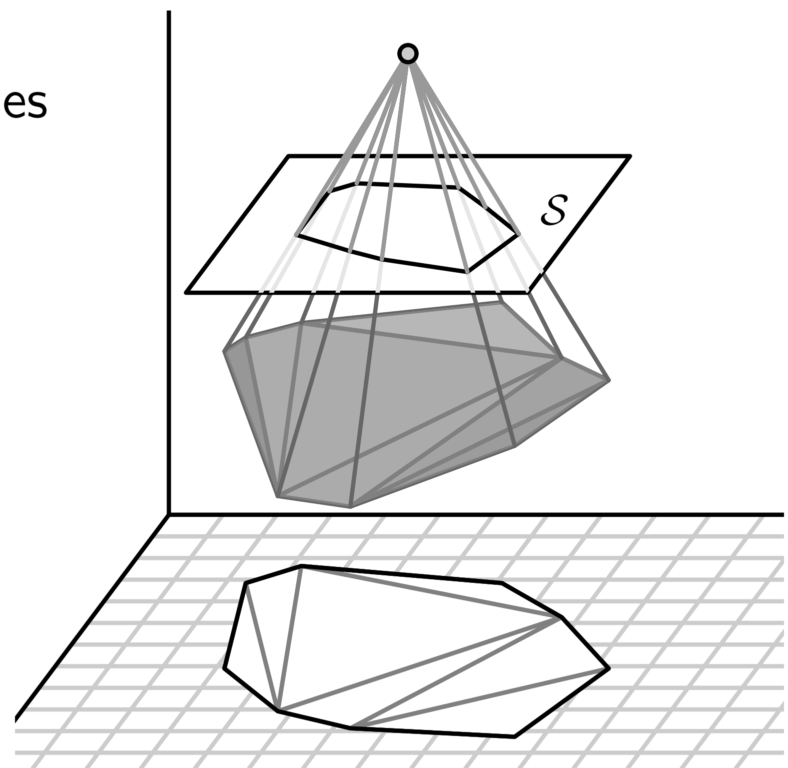
h-polynomial of T $$h_T(z) := \sum_{k=-1}^{d} f_k \, z^{k+1} \, (1-z)^{d-k}$$

For a boundary triangulation T one defines

$$h_T(z) := \sum_{k=-1}^{d-1} f_k \, z^{k+1} \, (1-z)^{d-1-k}$$

and if this triangulation is regular, Dehn–Sommerville holds.



Unimodular Triangulations and h^*

A lattice d-simplex with volume $\frac{1}{d!}$ is unimodular

Alternative description: if the simplex has vertices $v_0, v_1, \dots, v_d$, the vectors $v_1 - v_0, \dots, v_d - v_0$ form a basis of $\mathbb{Z}^d$.

If Δ is a unimodular k-simplex then $\quad \operatorname{Ehr}_\Delta(z) = \frac{1}{(1-z)^{k+1}}$

Ehrhart–Macdonald Reciprocity $\longrightarrow$ $\operatorname{Ehr}_{\Delta^\circ}(z) = \left(\frac{z}{1-z}\right)^{k+1}$

The Point These Ehrhart series can help us count things.

$\longrightarrow$ If $\mathcal{P}$ admits a unimodular triangulation T then $h^*_{\mathcal{P}}(z) = h_T(z)$.

Stapledon Decompositions

If $\mathcal{P}$ admits a unimodular triangulation T then $h^*_{\mathcal{P}}(z) = h_T(z)$

What if not?

Stapledon Decompositions

If $\mathcal{P}$ admits a unimodular triangulation T then $h^*_{\mathcal{P}}(z) = h_T(z)$

What if not?

The degree s of a lattice polytope $\mathcal{P}$ is the degree of $h^*_{\mathcal{P}}(z)$

Codegree $d+1-s$ $\longleftarrow$ smallest integer ℓ such that $\ell\mathcal{P}^\circ \cap \mathbb{Z}^d \neq \varnothing$

Theorem (Stapledon 2009) If $\mathcal{P}$ is a lattice d-polytope with codegree ℓ then

$$\left(1+z+\cdots+z^{\ell-1}\right) h^*_{\mathcal{P}}(z) \;=\; a(z) + z^{\ell}\, b(z)$$

where $a(z) = z^d\, a(\frac{1}{z})\,,\; b(z) = z^{d-\ell}\, b(\frac{1}{z})$ and $a(z)$ and $b(z)$ are nonnegative.

The case $\ell = 1$ was proved by Betke & McMullen (1985). There is a version for rational polytopes (MB–Braun–Vindas-Meléndez 2022).

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Topological story $a(z)$ and $b(z)$ can be written in terms of h-polynomials of links of a given triangulation of $\mathcal{P}$ and associated arithmetic datat ("box polynomials").

Arithmetic story (Bajo–MB 2023) $a(z) = h^*_{\partial\mathcal{P}}(z)$ . . .

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where $a(z) = z^d \, a(\frac{1}{z})$, $b(z) = z^{d-\ell} \, b(\frac{1}{z})$ and $a(z)$ and $b(z)$ are nonnegative.

Corollary Inequalities for h^*-coefficients

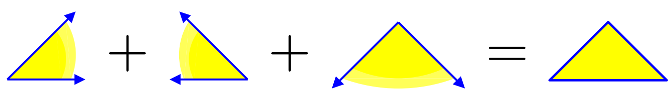
Open Problem Try to prove an analogous theorem for your favorite combinatorial polynomial with nonnegative coefficients.

Brion Magic

Integer point transform $\sigma_S(\mathbf{z}) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

When S is a rational polyhedron, $\sigma_S(\mathbf{z})$ evaluates to a rational function.

Given a vertex $\mathbf{v}$ of P, let $\mathcal{K}_{\mathbf{v}} := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\geq 0}(\mathbf{w} - \mathbf{v})$



Theorem (Brion 1988) If $\mathcal{P}$ is a rational polytope, then

$$\sigma_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \sigma_{\mathbf{v}+\mathcal{K}_{\mathbf{v}}}(\mathbf{z}) \, .$$

Recap Day I: Ehrhart Polynomials

- Polytopes ♡ polynomials
- Classification of Ehrhart polynomials is hard
- Partial classification is possible & interesting
- Unimodular triangulations
- Symmetric decompositions
- Tomorrow: where's q?

q-polynomials

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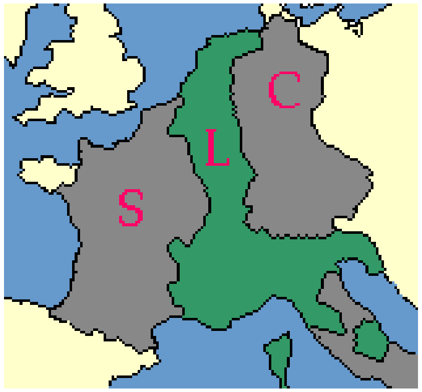

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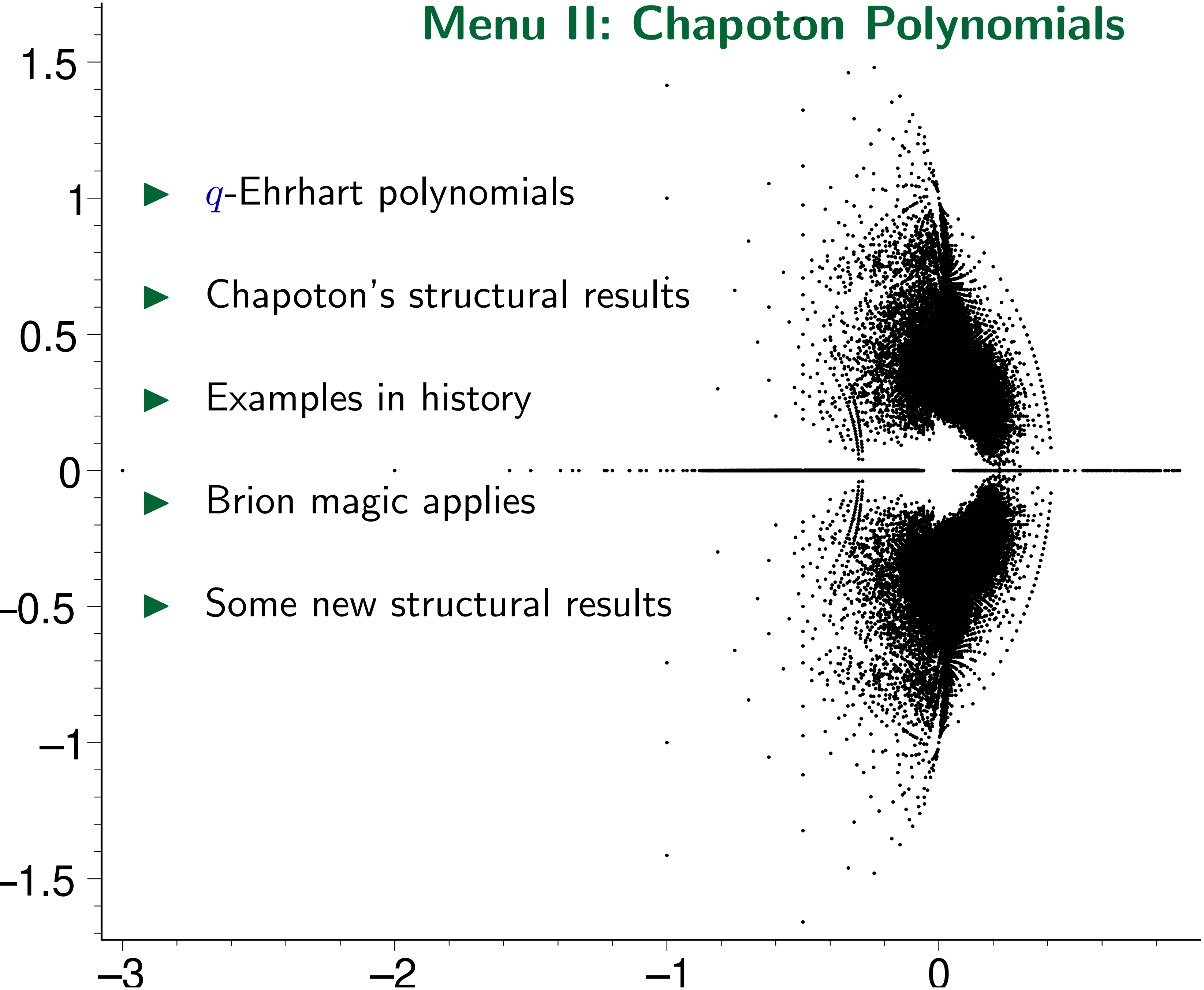
Menu II: Chapoton Polynomials



- q-Ehrhart polynomials
- Chapoton's structural results
- Examples in history
- Brion magic applies
- Some new structural results

q-Ehrhart Polynomials

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Theorem (Ehrhart 1962, Macdonald 1971) $\operatorname{ehr}_{\mathcal{P}}(t)$ is a polynomial in t. Furthermore, $\operatorname{ehr}_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} \# \left(t\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$.

Now fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})}$$

Philosophy (Sanyal) Tomography Ehrhart counting

q-Ehrhart Polynomials

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Theorem (Chapoton 2015) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form, there exists a polynomial $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Z}(q)[x]$ such that $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q)$, where $[t]_q := 1 + q + \cdots + q^{t-1}$.

q-Ehrhart Polynomials

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

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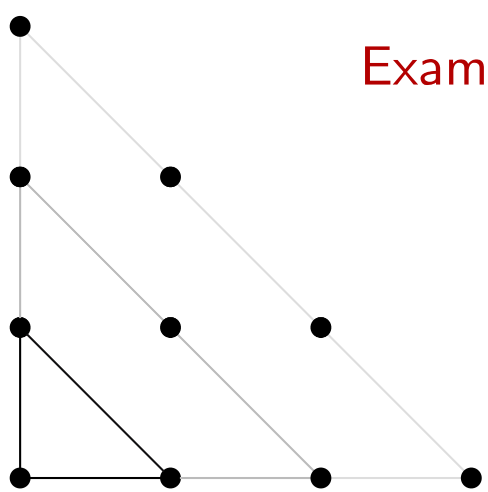

Example $\Delta = \operatorname{conv}\{(0,0),\ (1,0),\ (0,1)\}$ and $\lambda = (1,2)$

$$\operatorname{cha}_{\Delta}^{\lambda}(q,x) = \frac{q^3}{q+1}x^2 + \frac{q(2q+1)}{q+1}x + 1$$

Chapoton Polynomials

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

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The degree of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ is $m := \max\{\lambda(\mathbf{v}) : \mathbf{v} \text{ vertex of } \mathcal{P}\}$ and all the poles of the coefficients of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ are roots of unity of order $\leq m$.

Furthermore, $(-1)^{\dim \mathcal{P}} \, \mathrm{cha}_{\mathcal{P}}^{\lambda}\left(\frac{1}{q}, -qx\right) = \mathrm{cha}_{\mathcal{P}^\circ}^{\lambda}(q,x)$.

Chapoton Polynomials

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

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The degree of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ is $m := \max\{\lambda(\mathbf{v}) : \mathbf{v} \text{ vertex of } \mathcal{P}\}$ and all the poles of the coefficients of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ are roots of unity of order $\leq m$.

Furthermore, $(-1)^{\dim \mathcal{P}} \, \mathrm{cha}_{\mathcal{P}}^{\lambda}\left(\frac{1}{q}, -qx\right) = \mathrm{cha}_{\mathcal{P}^\circ}^{\lambda}(q,x)$.

Theorem (Robins 2023) The set of all $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$, where λ ranges over all generic and positive integral forms, determines $\mathcal{P}$.

Some More Motivation

- $\mathrm{ehr}_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$ has polynomial structure, but sometimes we need to understand the integer point transform

$$\sigma_{\mathcal{P}}(\mathbf{z}) := \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$$

- For fixed λ,

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})} = \sigma_{t\mathcal{P}}\left(q^{\lambda_1}, q^{\lambda_2}, \dots, q^{\lambda_d}\right)$$

still has polynomial structure.

- Chapoton polynomials contain interesting number theory, connection to partition functions, . . .

Familiar Faces

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})}$$

- $\Box = [0,1]^d$ and $\lambda = \mathbf{1} := (1,1,\dots,1)$

$$\mathrm{ehr}_{\Box}^{\mathbf{1}}(q,t) \ = \ [t+1]_q^d \qquad \longrightarrow \qquad \mathrm{cha}_{\Box}^{\mathbf{1}}(q,x) \ = \ (1+qx)^d$$

Carlitz identity (really due to MacMahon)

$$\sum_{t \geq 0} [t+1]_q^n \, x^t \ = \ \frac{\sum_{\pi \in S_n} x^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}}{\prod_{j=0}^{n} (1 - xq^j)}$$

$$\mathrm{des}(\pi) := |\{ j : \pi(j+1) < \pi(j) \}| \qquad\qquad \mathrm{maj}(\pi) := \sum_{\pi(j+1) < \pi(j)} j$$

Familiar Faces

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})}$$

- $\Delta = \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \, x_1 + x_2 + \cdots + x_d = 1 \right\}$

$$\mathrm{ehr}_{\Delta}^{\lambda}(q,t) \; = \sum_{\mathbf{m} \in t\Delta \cap \mathbb{Z}^d} q^{\lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_d m_d}$$

is the generating function for partitions with exactly t parts in the set $\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$

$$\mathrm{cha}_{\Delta}^{\lambda}(q,x) \; = \sum_{j=1}^{d} \frac{1}{\prod_{k \neq j} \left(1 - q^{\lambda_k - \lambda_j}\right)} \big((q-1)x + 1\big)^{\lambda_j}$$

Familiar Faces

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$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})}$$

- $\Delta = \left\{ \mathbf{x} \in \mathbb{R}^d : \, 0 \leq x_1 \leq x_2 \leq \cdots \leq x_d \leq 1 \right\}$ and $\lambda = \mathbf{1}$

$$\mathrm{ehr}_{\Delta}^{\mathbf{1}}(q,t) \; = \sum_{\mathbf{m} \in t\Delta \cap \mathbb{Z}^d} q^{m_1 + m_2 + \cdots + m_d} \; = \begin{bmatrix} t+d \\ d \end{bmatrix}_q$$

is the generating function for partitions with $\leq d$ parts, each of which $\leq t$

$$\mathrm{cha}_{\Delta}^{\mathbf{1}}(q,x) \; = \sum_{j=0}^{d} \frac{1}{\prod_{k \neq j} \left(1 - q^{k-j}\right)} \big((q-1)x + 1\big)^j$$

Familiar Faces

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$(\Pi, \preceq)$ — poset on d elements

Order polytope $\mathcal{O}(\Pi) := \left\{ \mathbf{x} \in [0,1]^d : \, j \preceq k \implies x_j \leq x_k \right\}$

MacMahon (1909) $\mathrm{cha}_{\mathcal{O}([m]\times[n])}^{\mathbf{1}}(q,x) = \displaystyle\prod_{i=1}^{m} \prod_{j=1}^{n} \frac{[i+j-1]_q + x\, q^{i+j-1}}{[i+j-1]_q}$

Familiar Faces

- Lecture hall simplex $\Delta_n := \left\{ \mathbf{x} \in [0,1]^n : \, x_1 \le \frac{x_2}{2} \le \frac{x_3}{3} \le \cdots \le \frac{x_n}{n} \right\}$

$\mathrm{ehr}^{\mathbf{1}}_{\Delta_n}(q,t) = \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{m_1 + \cdots + m_n}$ enumerates lecture hall partitions with $m_j \le t$

Corteel–Lee–Savage (2005) For any $j \ge 0$ and $1 \le i \le n$

$$\mathrm{ehr}^{\mathbf{1}}_{\Delta_n}(q, jn+i) \; = \; \mathrm{ehr}^{\mathbf{1}}_{\Delta_n}(q, jn+i-1) + q^{jn+i} \, \mathrm{ehr}^{\mathbf{1}}_{\Delta_{n-1}}(q, j(n-1)+i-1)$$

Familiar Faces

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Chapoton polynomials, anyone?

$$\mathrm{cha}_{1,0}(x) \; := \; 1 + qx \qquad \text{and} \qquad \mathrm{cha}_{1,1}(x) \; := \; 1 + q + q^2 x$$

and for $j \ge 0$ and $1 \le i \le n$

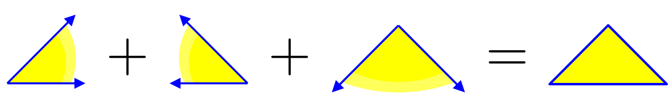
$$\mathrm{cha}_{n,i}(x) \; = \; \mathrm{cha}_{n,i-1}(x) + q^i \big((q-1)x + 1 \big)^n \, \mathrm{cha}_{n-1,i-1}(x)$$

Brion Magic

Integer point transform $\sigma_S(\mathbf{z}) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

When S is a rational polyhedron, $\sigma_S(\mathbf{z})$ evaluates to a rational function.

Given a vertex $\mathbf{v}$ of P, let $\mathcal{K}_\mathbf{v} := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\geq 0}(\mathbf{w} - \mathbf{v})$



Theorem (Brion 1988) If $\mathcal{P}$ is a rational polytope, then

$$\sigma_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \sigma_{\mathbf{v}+\mathcal{K}_\mathbf{v}}(\mathbf{z}) \, .$$

Brion Magic

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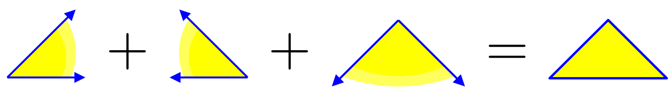

Theorem (Brion 1988) If $\mathcal{P}$ is a lattice polytope, then

$$\sigma_\mathcal{P}(\mathbf{z}) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \sigma_{\mathbf{v}+\mathcal{K}_\mathbf{v}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \mathbf{z}^\mathbf{v} \sigma_{\mathcal{K}_\mathbf{v}}(\mathbf{z}) \, .$$

Brion Magic

Integer point transform $\sigma_S(\mathbf{z}) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

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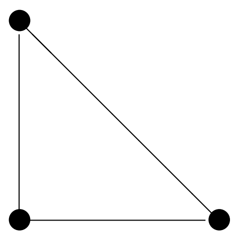
Theorem (Brion 1988) If $\mathcal{P}$ is a lattice polytope, then

$$\sigma_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \mathbf{z}^{\mathbf{v}} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z}) \, .$$

Example $\sigma_{\mathcal{K}_{(0,0)}}(\mathbf{z}) = \frac{1}{(1-z_1)(1-z_2)}$ $\qquad$ $\sigma_{\mathcal{K}_{(0,1)}}(\mathbf{z}) = \frac{z_2^2}{(z_2-1)(z_2-z_1)}$



$$\sigma_{\mathcal{K}_{(1,0)}}(\mathbf{z}) = \frac{z_1^2}{(z_1-1)(z_1-z_2)}$$

Brion $\longrightarrow$ Chapoton

Integer point transform $\sigma_S(\mathbf{z}) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

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$$\begin{aligned} \operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t) &= \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})} = \sigma_{t\mathcal{P}}\left(q^{\lambda_1}, q^{\lambda_2}, \dots, q^{\lambda_d}\right) \\ &= \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} q^{t\lambda(\mathbf{v})}\, \sigma_{\mathcal{K}_{\mathbf{v}}}\left(q^{\lambda_1}, q^{\lambda_2}, \dots, q^{\lambda_d}\right) \end{aligned}$$

Chapoton Polynomials Revisited

Theorem (Chapoton 2015) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form, there exists a polynomial $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Z}(q)[x]$ such that $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q)$, where $[t]_q := 1 + q + \cdots + q^{t-1}$.

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) \ = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} q^{t\lambda(\mathbf{v})} \, \sigma_{\mathcal{K}_{\mathbf{v}}} \left(q^{\lambda_1}, q^{\lambda_2}, \dots, q^{\lambda_d}\right)$$

Now use $q^{kt} \ = \ \left((q-1)[t]_q + 1\right)^k$ . . .

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form,

$$\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) \ = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \rho_{\mathbf{v}}^{\lambda}(q) \big((q-1)x + 1\big)^{\lambda(\mathbf{v})}$$

where $\rho_{\mathbf{v}}^{\lambda}(q) \ := \ \sigma_{\mathcal{K}_{\mathbf{v}}} \left(q^{\lambda_1}, \ q^{\lambda_2}, \ \dots, \ q^{\lambda_d}\right)$

Chapoton Polynomials Revisited

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})} = \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q) \qquad \mathcal{K}_{\mathbf{v}} := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\geq 0}(\mathbf{w}-\mathbf{v})$$

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form,

$$\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \rho_{\mathbf{v}}^{\lambda}(q) \big((q-1)x+1\big)^{\lambda(\mathbf{v})}$$

where $\rho_{\mathbf{v}}^{\lambda}(q) := \sigma_{\mathcal{K}_{\mathbf{v}}}\left(q^{\lambda_1},\ q^{\lambda_2},\ \ldots,\ q^{\lambda_d}\right)$.

Corollary Each pole of $\rho_{\mathbf{v}}^{\lambda}(q)$ is an nth root of unity where $n = |\lambda(g(\mathbf{w}-\mathbf{v}))|$ for some adjacent vertex $\mathbf{w}$, where $g(\mathbf{w}-\mathbf{v})$ is primitive.

Corollary The leading coefficient of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ is $(q-1)^{\lambda(\mathbf{v})}\rho_{\mathbf{v}}^{\lambda}(q)$ where $\mathbf{v}$ is the vertex of $\mathcal{P}$ that maximizes $\lambda(\mathbf{v})$.

Chapoton Polynomials Revisited

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})} = \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q) \qquad \mathcal{K}_{\mathbf{v}} := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\geq 0}(\mathbf{w}-\mathbf{v})$$

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form,

$$\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \rho_{\mathbf{v}}^{\lambda}(q) \big((q-1)x+1\big)^{\lambda(\mathbf{v})}$$

where $\rho_{\mathbf{v}}^{\lambda}(q) := \sigma_{\mathcal{K}_{\mathbf{v}}}\left(q^{\lambda_1},\ q^{\lambda_2},\ \ldots,\ q^{\lambda_d}\right)$.

Chapoton: compute $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ in the limit as $t \to \infty$ . . .

Corollary

$$\mathrm{cha}_{\mathcal{P}}^{\lambda}\left(q, \frac{1}{1-q}\right) = \begin{cases} \rho_{\mathbf{0}}^{\lambda}(q) & \text{if } \mathbf{0} \text{ is a vertex of } \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

Chapoton Polynomials Revisited

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) \ := \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})} \ = \ \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q) \qquad \mathcal{K}_{\mathbf{v}} \ := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\geq 0}(\mathbf{w}-\mathbf{v})$$

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form,

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where $\rho_{\mathbf{v}}^{\lambda}(q) \ := \ \sigma_{\mathcal{K}_{\mathbf{v}}}\left(q^{\lambda_1},\ q^{\lambda_2},\ \dots,\ q^{\lambda_d}\right)$.

Corollary The constant term of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ is 1.

Chapoton Quasipolynomials

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a rational polytope with denominator p and λ is an integral form that is generic and positive, then there exist polynomials $\mathrm{cha}_{\mathcal{P}}^{\lambda,r}(q,x) \in \mathbb{Q}(q)[x]$ such that

$$\mathrm{cha}_{\mathcal{P}}^{\lambda,r}\left(q,[k]_q\right) \;=\; \mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,kp+r)$$

for all integers $k \ge 0$ and all $0 \le r < p$.

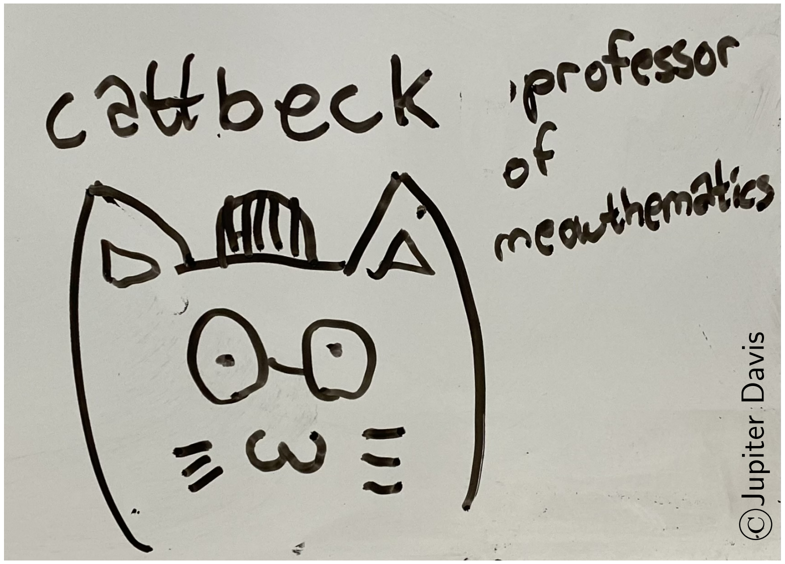
The degree of $\mathrm{cha}_{\mathcal{P}}^{\lambda,r}(q,x)$ is $\max\{\lambda(p\mathbf{v}) : \mathbf{v} \text{ vertex of } \mathcal{P}\}$. Each pole of a coefficient of $\mathrm{cha}_{\mathcal{P}}^{\lambda,r}(q,x)$ is an nth root of unity where $n = |\lambda(g(p(\mathbf{w}-\mathbf{v})))|$ for some adjacent vertices $\mathbf{v}$ and $\mathbf{w}$.

For any $0 \le r < p$ and $k > 0$

$$(-1)^{\dim \mathcal{P}}\,\mathrm{cha}_{\mathcal{P}}^{\lambda,r}\left(\tfrac{1}{q},[-k]_{\frac{1}{q}}\right) \;=\; \mathrm{ehr}_{\mathcal{P}^\circ}^{\lambda}(q,kp-r)\,.$$

Recap Day II: Chapoton Polynomials

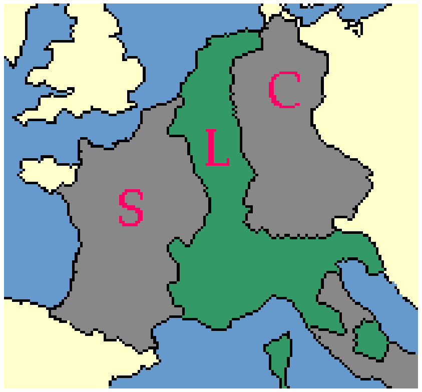
- q-counting, Ehrhart style
- Polynomials in $[t]_q$
- Partition functions know Chapoton (and vice versa)
- Brion's theorem gives a computational edge & more structure
- Tomorrow: let's try this for chromatic polynomials for graphs



q-polynomials

Matthias Beck
San Francisco State University
matthbeck.github.io

slides



September 2025

Menu III: q-Graph Coloring

- Chromatic polynomials
- Chromatic symmetric functions
- Stanley's tree conjecture
- q-chromatic polynomials
- Structural q-chromatic results
- G-partitions

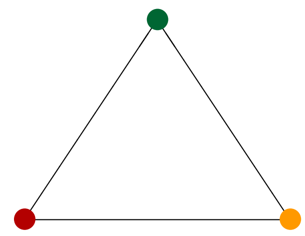
Chromatic Polynomials

$G = (V, E)$ — graph (without loops)

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \ldots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#$ (proper n-colorings of G)

Example: $\chi_{K_3}(n) = n(n-1)(n-2)$



Theorem (Birkhoff 1912, Whitney 1932)
$\chi_G(n)$ is a polynomial.

Chromatic Polynomials

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \ldots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

- Classification — which polynomials are chromatic?

... wide open, though we have structural results:

- $\chi_G(n)$ is monic, has constant term 0 and degree $|V|$.
- The coefficients of $\chi_G(n)$ alternate in sign.
- $|\chi_G(-1)|$ equals $\#$ acyclic orientations of G (Stanley 1973).
- The coefficients of $\chi_G(n)$ are unimodal (Huh 2012).

Chromatic Polynomials

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \dots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

- Detection — does a given polynomial determine the graph?

... fails badly: If T is a tree with m edges then

$$\chi_T(n) \;=\; n(n-1)^m$$

Chromatic Symmetric Functions

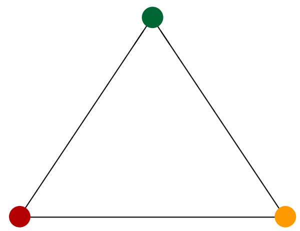
$G = (V, E)$ — graph (without loops)

Proper coloring — $\kappa : V \to \mathbb{Z}_{>0}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic symmetric function

$$X_G(x_1, x_2, \dots) \; := \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

Example: $X_{K_3}(k) \; = \; 6\, x_1\, x_2\, x_3 + 6\, x_1\, x_2\, x_4 + \cdots$



Chromatic Symmetric Functions

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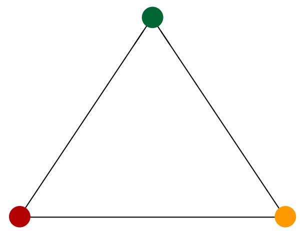

We recover $\chi_G(n) \ = \ X_G(\underbrace{1, \dots, 1}_{n \text{ times}}, 0, 0, \dots)$

Chromatic Symmetric Functions

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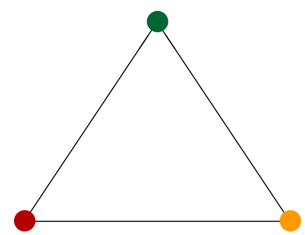


We recover $\chi_G(n) \ = \ X_G(\underbrace{1, \dots, 1}_{n \text{ times}}, 0, 0, \dots)$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \dots)$ distinguishes trees.

(Loehr–Warrington 2024) $X_G(q, q^2, \dots, q^n, 0, 0, \dots)$ distinguishes trees.

q-Chromatic Polynomials

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

Chromatic symmetric function

$$X_G(x_1, x_2, \ldots) \ := \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

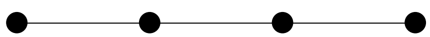
Definition $\chi_G^\lambda(q,n) := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$ where $\lambda \in \mathbb{Z}_{>0}^V$ is fixed

We recover $\chi_G(n) = \chi_G^{\mathbf{1}}(1,n)$ and $\chi_G^{\mathbf{1}}(q,n) = X_G(q, q^2, \ldots, q^n, 0, 0, \ldots)$

q-Chromatic Polynomials

Definition $\chi_G^{\lambda}(q,n) := \sum_{\substack{\text{proper colorings} \\ \kappa:V\to[n]}} q^{\sum_{v\in V} \lambda_v \kappa(v)}$ where $\lambda \in \mathbb{Z}_{>0}^{V}$ is fixed

We recover $\chi_G(n) = \chi_G^{\mathbf{1}}(1,n)$ and $\chi_G^{\mathbf{1}}(q,n) = X_G(q,q^2,\dots,q^n,0,0,\dots)$

Example 

$$\chi_{P_4}^{\mathbf{1}}(q,n) = \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times \Big(8q^{10}(1-q)^n(1-q)^{n-1}(1-q)^{n-2}(1-q)^{n-3}$$
$$+(4q^9+6q^8+4q^7)(1-q)^{n+1}(1-q)^n(1-q)^{n-1}(1-q)^{n-2}$$
$$+2q^6(1-q)^{n+2}(1-q)^{n+1}(1-q)^n(1-q)^{n-1}\Big)$$

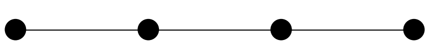
q-Chromatic Polynomial Structure

$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings}\\ \kappa:V\to[n]}} q^{\sum_{v\in V}\lambda_v\kappa(v)}$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+) There exists a (unique) polynomial $\widetilde{\chi}_G^\lambda(q,x) \in \mathbb{Z}(q)[x]$ such that

$$\chi_G^\lambda(q,n) \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q) \qquad \text{where} \qquad [n]_q := 1+q+\cdots+q^{n-1}$$

Example $$\widetilde{\chi}_{P_4}^{\mathbf{1}}(q,x) \ = \ \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times \Big(\left(2q^8+4q^7+6q^6+4q^5+8q^4\right)x^4 - \left(6q^8+10q^7+18q^6+18q^5+20q^4\right)x^3 + \left(4q^8+10q^7+20q^6+22q^5+16q^4\right)x^2 - \left(4q^7+8q^6+8q^5+4q^4\right)x \Big)$$



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$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings} \\ \kappa:V\to[n]}} q^{\sum_{v\in V} \lambda_v \kappa(v)}$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+) There exists a (unique) polynomial $\widetilde{\chi}_G^\lambda(q,x) \in \mathbb{Z}(q)[x]$ such that

$$\chi_G^\lambda(q,n) \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q) \qquad \text{where} \qquad [n]_q := 1+q+\cdots+q^{n-1}$$

$$\uparrow$$

$$\chi_G^\lambda(q,n) \ = \sum_{\rho\in A(G)} \operatorname{ehr}_{\mathcal{O}(\Pi_\rho)^\circ}^\lambda(q,n+1)$$

$A(G)$ — set of acyclic orientations of G

Π_ρ — poset corresonponding to the acyclic orientation ρ

Motivation

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q,n) = \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

$$\chi_G^{\mathbf{1}}(q,n) = \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} q^{\sum_{v \in V} \kappa(v)}$$

$$\chi_G(n) = \#\left(\text{proper } n\text{-colorings of } G\right)$$

More Motivation

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

More Motivation

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

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Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

There are more coefficients of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ . . .

More Motivation

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

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Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

Remarks $\chi_G^{\mathbf{1}}(q, n)$ was previously studied by Loebl (2007).

$\chi_G^{\lambda}(q, n)$ is a special evaluation (with polynomial structure) of Crew–Spirkl's (2020) weighted chromatic symmetric function.

q-Chromatic Structures

$$\chi_G^\lambda(q,n) \;:=\; \sum_{\substack{\text{proper colorings}\\ \kappa:V\to[n]}} q^{\sum_{v\in V}\lambda_v\kappa(v)} \;=\; \widetilde{\chi}_G^\lambda(q,[n]_q)$$

Deletion–Contraction (Crew–Spirkl 2020)

$$\chi_G^\lambda(q,n) \;=\; \chi_{G\setminus 12}^\lambda(q,n) - \chi_{G/12}^{(\lambda_1+\lambda_2,\lambda_3,\dots,\lambda_d)}(q,n)$$

$\longrightarrow$ naturally extends to the coefficients of $\widetilde{\chi}_G^\lambda(q,[n]_q)$

Reciprocity $\quad (-1)^{|V|}\, q^{\sum_{v\in V}\lambda_v}\, \widetilde{\chi}_G^\lambda\left(\frac{1}{q},[-n]_{\frac{1}{q}}\right) \;=\; \sum_{(c,\rho)} q^{\sum_{v\in V(G)}\lambda_v c(v)}$

where the sum is over all pairs of an n-coloring c and a compatible acyclic orientation ρ

q-Chromatic Polynomial Formulas

$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings}\\ \kappa:V\to[n]}} q^{\sum_{v\in V}\lambda_v\kappa(v)} \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q)$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+)

$$\widetilde{\chi}_G^\lambda(q,x) \ = \ q^{\Lambda_V} \sum_{\text{flats } S\subseteq E} \mu(\varnothing,S) \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

where $P(S)$ denotes the collection of vertex sets of the connected components induced by S and $\Lambda_W := \sum_{v\in W}\lambda_v$. In particular, for a tree

$$\widetilde{\chi}_T^\lambda(q,x) \ = \ q^{\Lambda_V} \sum_{S\subseteq E} (-1)^{|S|} \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

$\longrightarrow$ highly-structured formulas for paths, stars, . . .

The Leading Coefficient for Trees

$$\chi_G^{\lambda}(q,n) \ := \sum_{\substack{\text{proper colorings}\\ \kappa:V\to[n]}} q^{\sum_{v\in V}\lambda_v\kappa(v)} \ = \ \widetilde{\chi}_G^{\lambda}(q,[n]_q)$$

Corollary Given a tree T, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals

$$c_T^{\mathbf{1}}(q) \ = \ (q-q^2)^d \sum_{S\subseteq E}\prod_{C\in P(S)} \frac{1}{1-q^{\Lambda_C}}$$

$$= \ \frac{1}{[d]_q!}\sum_{(\rho,\sigma)} q^{d+\operatorname{maj}\sigma} \qquad\qquad d := |V|$$

where the sum ranges over all pairs of acyclic orientations ρ of T and linear extensions σ of the poset induced by ρ

Corollary^2 (via the following slides) $\quad c_T^{\mathbf{1}}(q) \ = \ (-q)^d \, X_T\left(\frac{1}{q},\frac{1}{q^2},\frac{1}{q^3},\dots\right)$

G-Partitions

Given a poset $P = ([d], \preceq)$, a strict P-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_j < m_k \ \text{ whenever } \ j \prec k$$

Given a (simple) graph $G = ([d], E)$, a G-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \ \text{ whenever } \ vw \in E$$

Let $p_G(n)$ denote the number of G-partitions of n, with accompanying generating function

$$P_G(q) \ := \ \sum_{n>0} p_G(n)\, q^n \ = \ X_G(q, q^2, q^3, \dots)$$

G-Partitions

Given a (simple) graph $G = ([d], E)$, a G-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \text{ whenever } vw \in E$$

Let $p_G(n)$ denote the number of G-partitions of n, with accompanying generating function $P_G(q) := \sum_{n>0} p_G(n)\, q^n$

Theorem

$$P_G(q) \;=\; \frac{q^{\binom{d+1}{2}} \sum_{(\rho,\sigma)} q^{-\operatorname{maj} \sigma}}{(1-q)(1-q^2)\cdots(1-q^d)}$$

where the sum ranges over all pairs of acyclic orientations ρ of G and linear extensions σ of the poset induced by ρ

G-Partitions

Given a (simple) graph $G = ([d], E)$, a G-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \text{ whenever } vw \in E$$

Let $p_G(n)$ denote the number of G-partitions of n, with accompanying generating function $P_G(q) := \sum_{n>0} p_G(n)\, q^n$

Collorary Given a tree T on d vertices, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals

$$c_T^{\mathbf{1}}(q) \;=\; (-q)^d \, P_T\left(\frac{1}{q}\right)$$

Conjecture The G-partition function $p_G(n)$ distinguishes trees.

One Last Theorem

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

Equivalent Conjecture The G-partition function $p_G(n)$ distinguishes trees.

Theorem (MB–Braun–Cornejo 2026+) Fix $k \geq d$ and $\lambda_j := k^j$. Then $\widetilde{\chi}_G^{\lambda}(q, x)$ distinguishes graphs on d nodes.

Recap Day III: q-Graph Coloring

- Chromatic polynomials, symmetric functions, and q
- Stanley's tree conjecture & refinements
- More q-polynomial structure
- G-partitions