Parking Functions & Friends

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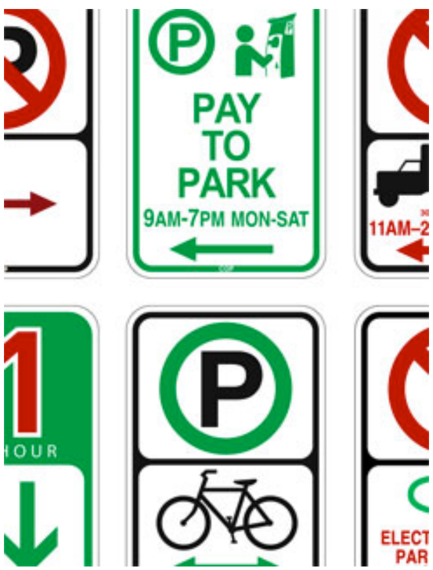

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Parking Functions

$(x_1, x_2, \ldots, x_n)$ is a parking function if, when re-indexed from smallest to largest value, satisfies

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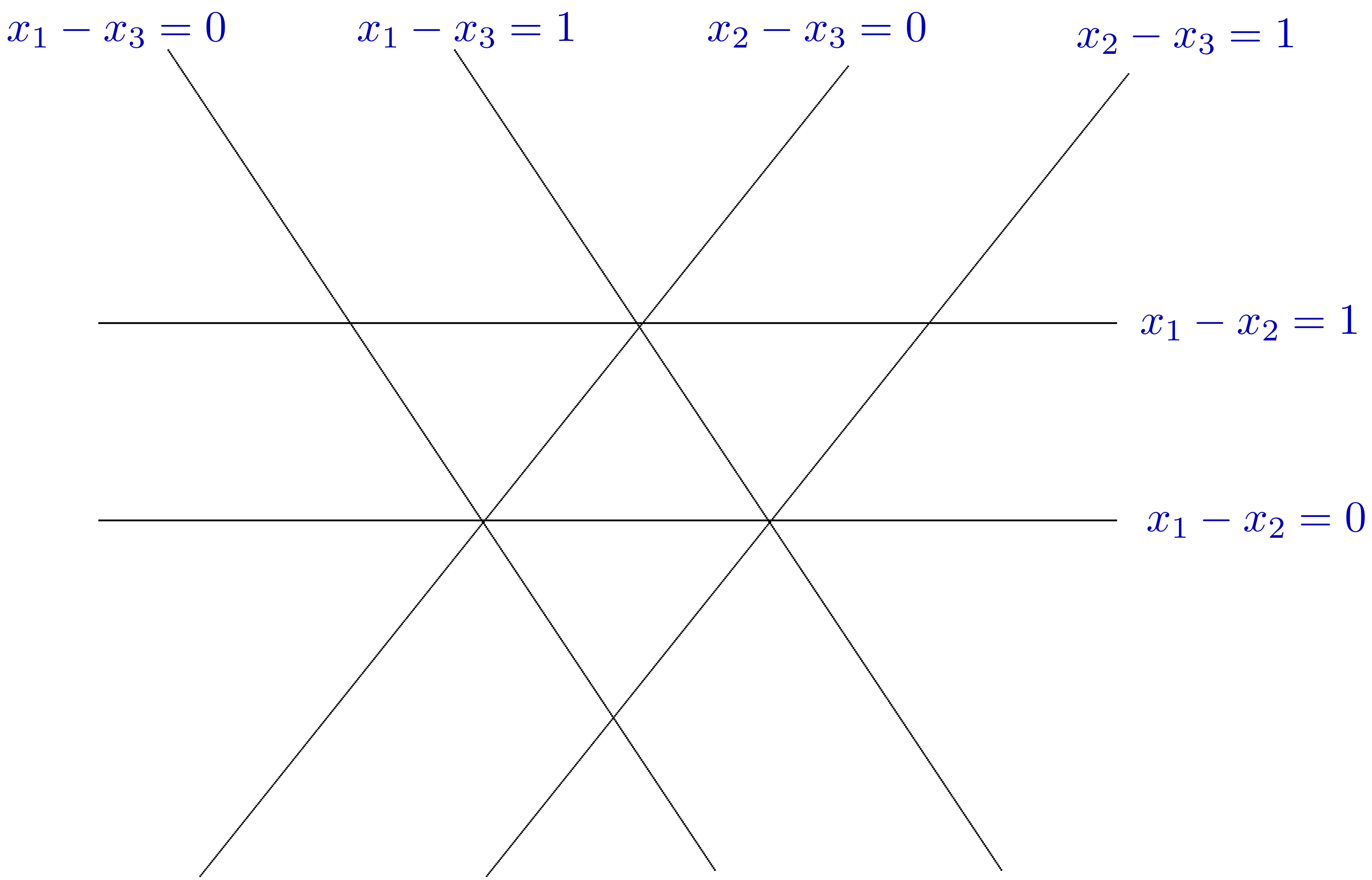
- First appearance: Pyke's work [1959] on queuing theory in probability.
- Konheim and Weiss introduced parking functions to illustrate computer storage procedures.
- Modern applications to polyhedra, vertex operators, Hopf algebras, etc.

The 3-dimensional Shi Arrangement



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If you're bored. . . The n-dimensional braid arrangment

$$\{x_j - x_k = 0 : \, 1 \le j < k \le n\}$$

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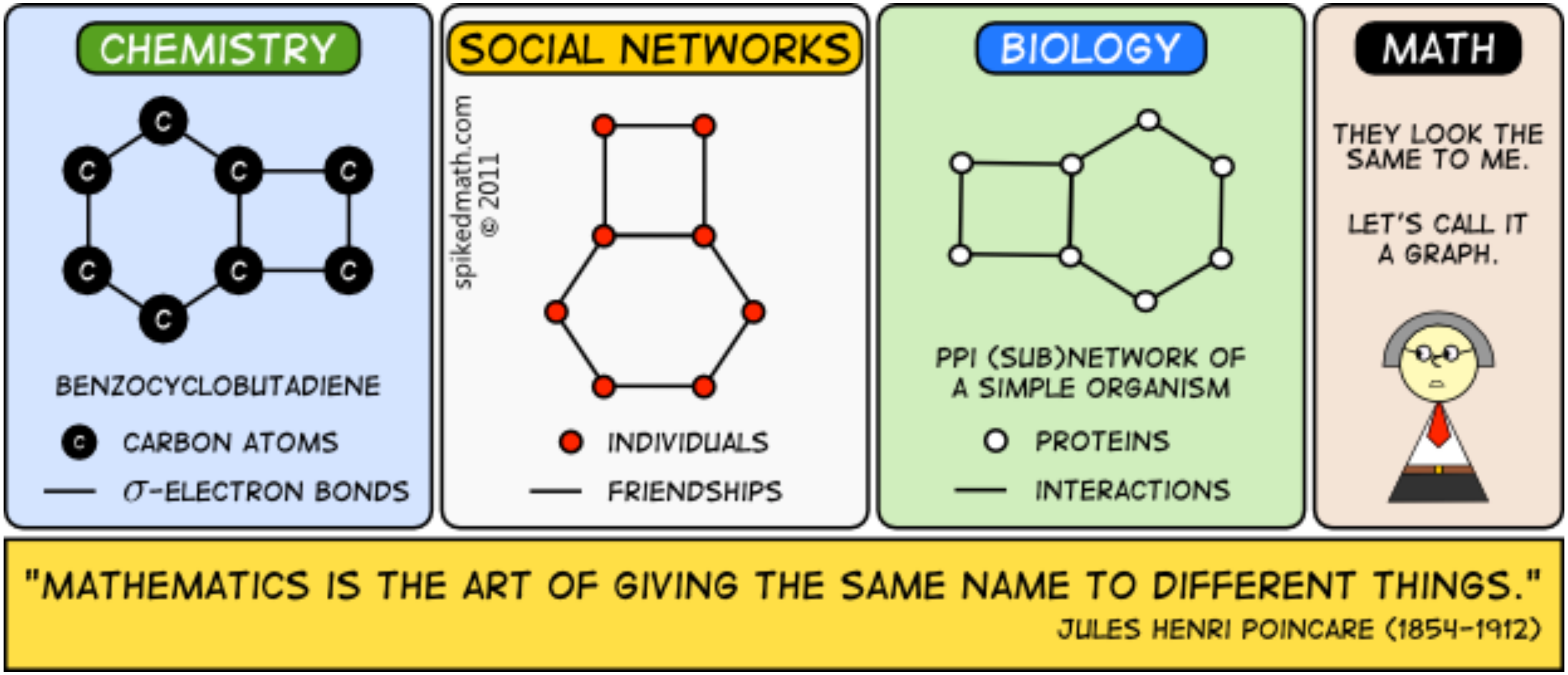
If you're still bored... There are precisely $(n+1)^{n-1}$ labeled trees with $n+1$ vertices. Find a bijection between labeled trees and parking functions.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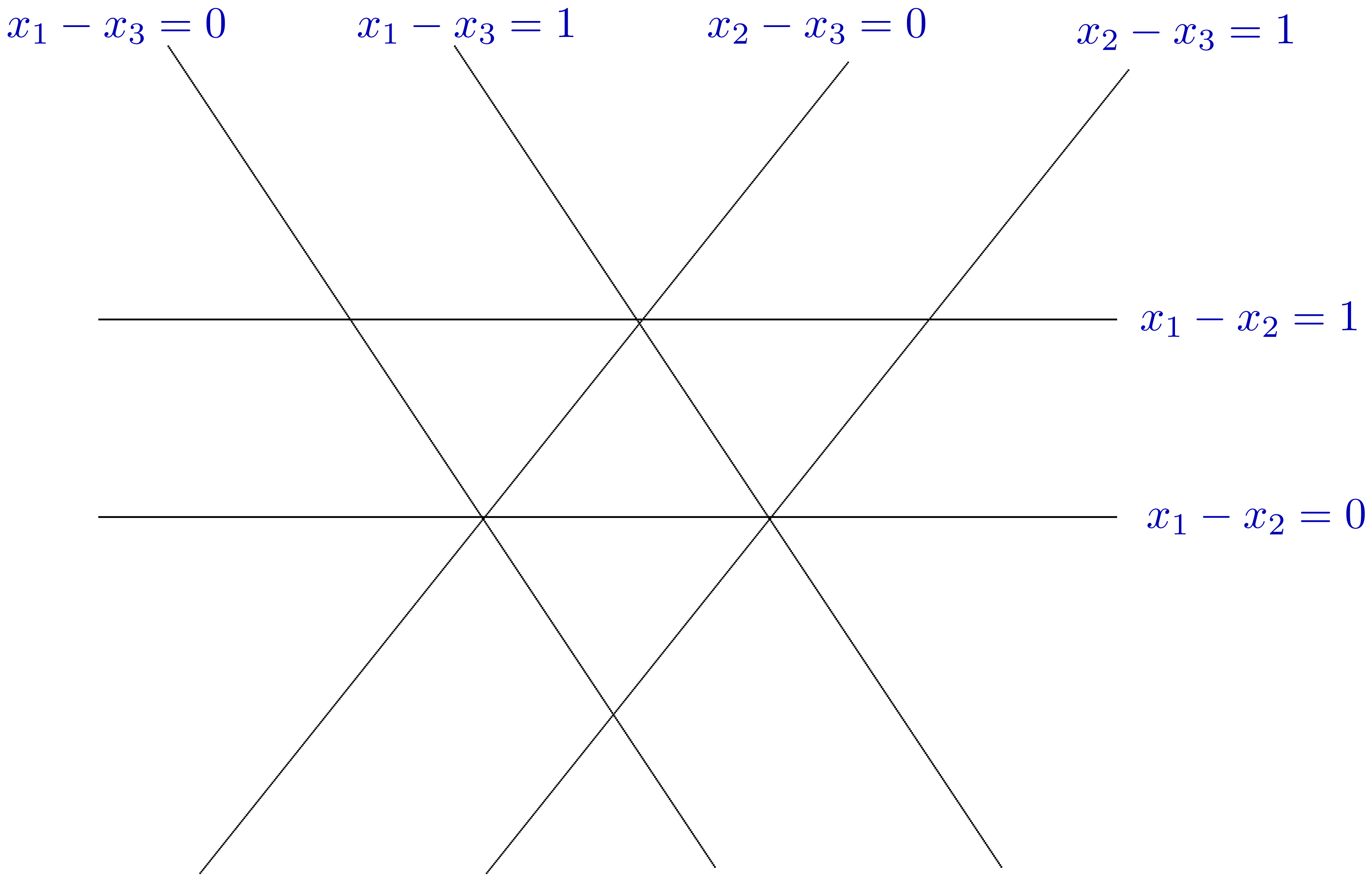
FRIENDSHIPS
BIOLOGY
PPI (SUB)NETWORK OF
A SIMPLE ORGANISM
PROTEINS
INTERACTIONS
MATH
THEY LOOK THE SAME TO ME.
LET'S CALL IT A GRAPH.
"MATHEMATICS IS THE ART OF GIVING THE SAME NAME TO DIFFERENT THINGS."
JULES HENRI POINCARE (1854-1912)

Shi Arrangements & Parking Graphs



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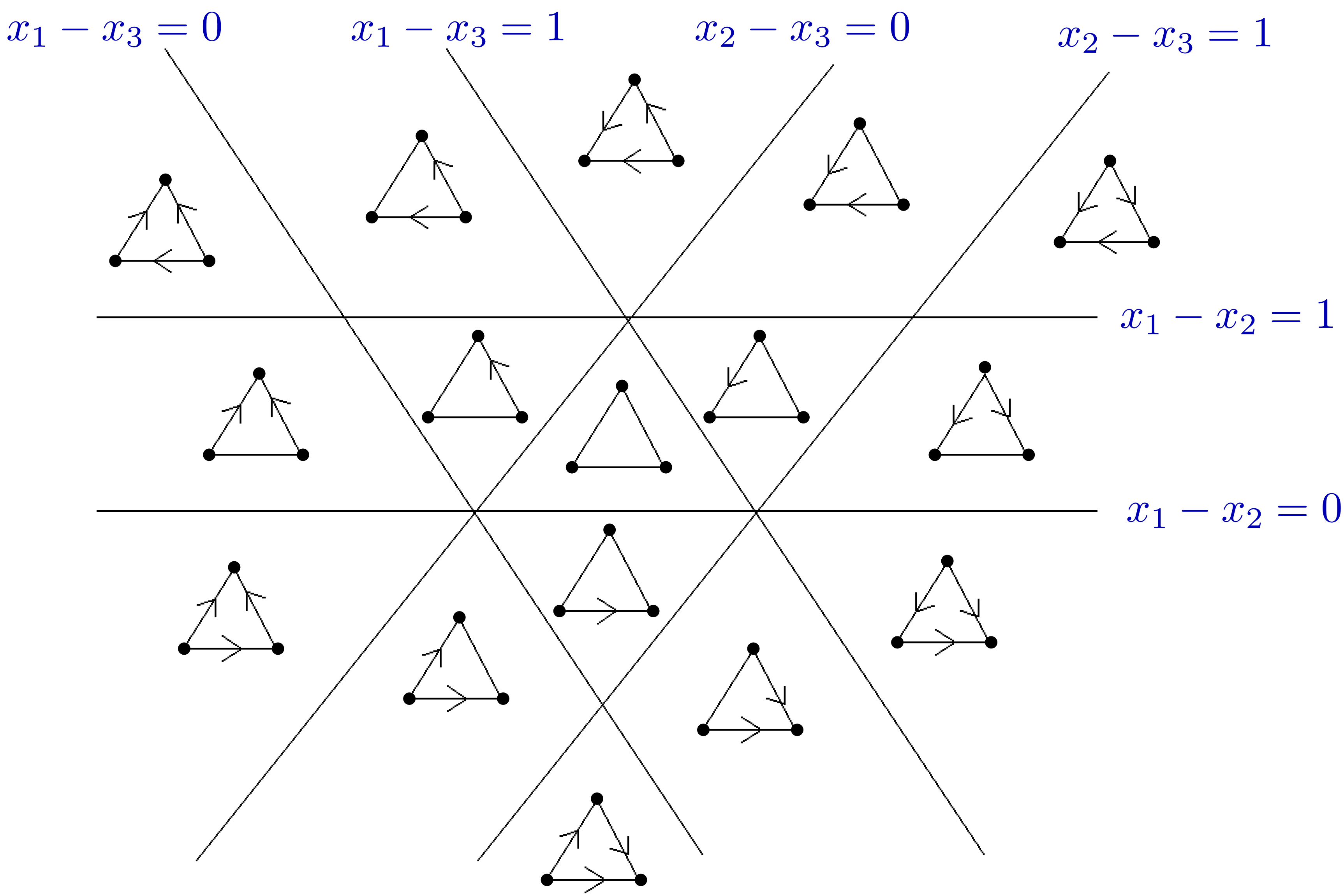

From Parking Functions to Parking Graphs

Input: parking function $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$

$\mathbf{y} := \mathbf{x} - (1, 1, \dots, 1)$

$(\star)$ If there exists $y_k = 0$ then

$$j := \max \{ k : \, y_k = 0 \}$$

$$\left. \begin{array}{l} \text{introduce } j \to k \\ y_k := y_k - 1 \end{array} \right\} \text{ for all } k > j \text{ with } y_k > 0$$

$$y_j := y_j - 1$$

$y_k := y_k - 1$ for all $y_k < 0$

go to $(\star)$

Else if there exists $y_k > 0$ then

choose minimal y_j such that there exists $k < j$ with $y_k > 0$ and $k \not\leftarrow j$

$$\left. \begin{array}{l} \text{introduce } k \leftarrow j \\ y_k := y_k - 1 \end{array} \right\} \text{ for all } k < j \text{ with } y_k > 0$$

go to $(\star)$

Else [all $y_k < 0$] introduce remaining edges as undirected and stop.

A Few Open Questions

- How do parking graphs interact with the Athanasiadis–Linusson bijection?

- Can parking graphs shed further light upon the arrangement

$$\begin{aligned} x_j - x_k &= 0 \qquad \text{for all } 1 \le j < k \le n \\ x_j - x_k &= 1 \qquad \text{for all } 1 \le j < k \le n \text{ with } jk \in E \end{aligned}$$

  where E is the edge set of a given fixed graph on n vertices?

- Are the any connections with chromatic theory for gain graphs (Berthomé–Cordovil–Forge–Ventos–Zaslavsky 2009)?