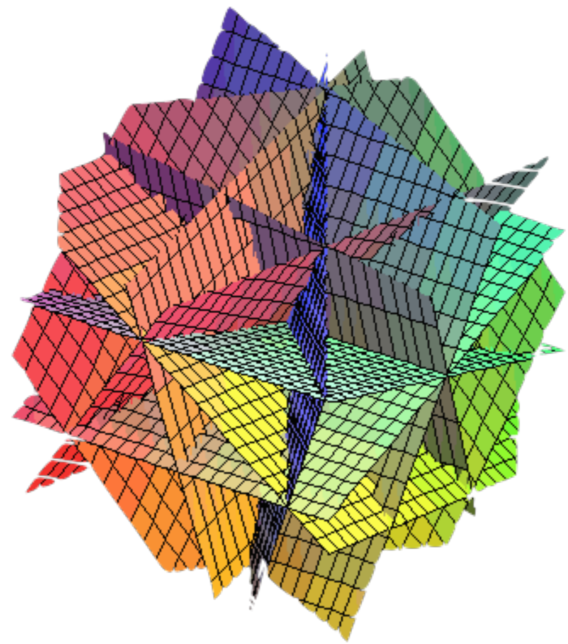
(Enumeration Results for) Signed Graphs



[John Stembridge]

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Why Graphs?

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A [directed] graph $G = (V, E)$ consists of

- a node set V
- an edge set $E \subseteq \binom{V}{2}$ $[V^2]$

So... why?

- Modeling (directional) relations
- Fascinating theorems & conjectures
- ... including of computational nature

Signed Graph Concepts

A signed graph $\Sigma = (G, \sigma)$ consists of:

- a graph $G = (V, E)$ which may have multiple edges, loops (which together form E_*), half edges, and loose edges
- a signature $\sigma : E_* \to \{\pm\}$

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The frustration index of the signed graph Σ is the smallest number of edges whose negation makes Σ balanced.
(Finding the frustration index is NP-hard: for an all-negative signed graph it is equivalent to the maximum cut problem.)

Switching

Switching Σ at $v \in V$ means switching the sign of each edge incident with v. Note that switching does not alter balance.

Exercise A signed graph is balanced if and only if it has no half edges and can be switched to an all-positive signed graph. ($\longrightarrow$ Harary's Theorem)

Other Applications

- Knot theory (positive/negative crossings)
- Biology (perturbed large-scale biological networks
- Chemistry (Möbius systems)
- Physics (spin glasses—mixed Ising model)
- Computer science (correlation clustering)

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* Finding the ground state energy of an Ising model means finding the frustration index of a signed graph.

Incidence Matrices of Graphs

Two versions of incidence matrix (a_{ve}) of a graph $G = (V, E)$:

- $a_{ve} = 1$ if v and e are incident, 0 otherwise
- orient G and define $a_{ve} = \pm 1$ according to whether v points into or away from e and 0 if v and e are not incident

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A matrix is totally unimodular if all its minors are 0 or ± 1. Examples:

- unoriented incidence matrix of a bipartite graph
- oriented incidence matrix of any graph

Incidence Matrices of Signed Graphs

Orienting a signed graph gives rise to a bidirected graph (first introduced by Edmonds–Johnson 1970)

$$\begin{array}{lcl} \sigma_e = + & \multirow{2}{*}{$\longrightarrow$} & e \text{ becomes directed} \\ \sigma_e = - & & e \text{ becomes extra- or introverted} \end{array}$$

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Theorem (Appa–Kotnyek 2006, following Lee 1989) The inverse of any maximal minor of the incidence matrix of a bidirected graph is half integral.

Magic Labelings of Graphs

An edge labeling $E \to \{1, 2, \ldots, k\}$ is magic if each sum of all labels incident to a node is the same.

Theorem (Stanley 1973) The number $m_G(k)$ of all magic k-labelings is a quasipolynomial in k with period ≤ 2. It is a polynomial if G is bipartite.

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The geometry behind this corollary concerns the Birkhoff–von Neumann polytope

$$\mathcal{B}_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{R}_{\ge 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \le k \le n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \le j \le n \end{array} \right\}$$

Ehrhart (Quasi-)Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$.
Equivalently, $\mathcal{P} = \left\{ \boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \right\}$ for some unimodular matrix $\boldsymbol{A}$.

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Theorem (Ehrhart 1962) If $\mathcal{P}$ is a lattice polytope, $\operatorname{ehr}_{\mathcal{P}}(k)$ is a polynomial. If $\mathcal{P}$ is a rational polytope, $\operatorname{ehr}_{\mathcal{P}}(k)$ is a quasipolynomial whose period divides the denominator of $\mathcal{P}$.

Magic Labelings Revisited

Theorem (Stanley 1973) The number $m_G(k)$ of all magic k-labelings is a quasipolynomial in k with period ≤ 2. It is a polynomial if G is bipartite.

(Signed) Graphic Arrangements

$\mathcal{H}_G := \{x_j = x_k : jk \in E\}$ is a hyperplane arrangement in $\mathbb{R}^V$,
a subarrangement of the (real) braid arrangement $\{x_j = x_k : j \neq k\}$

$\mathcal{H}_\Sigma := \{x_j = \sigma_e \, x_k : e = jk \in E\}$ is a subarrangement of the type-B/C arrangement $\{x_j = \pm \, x_k, \ x_j = 0 : j \neq k\}$

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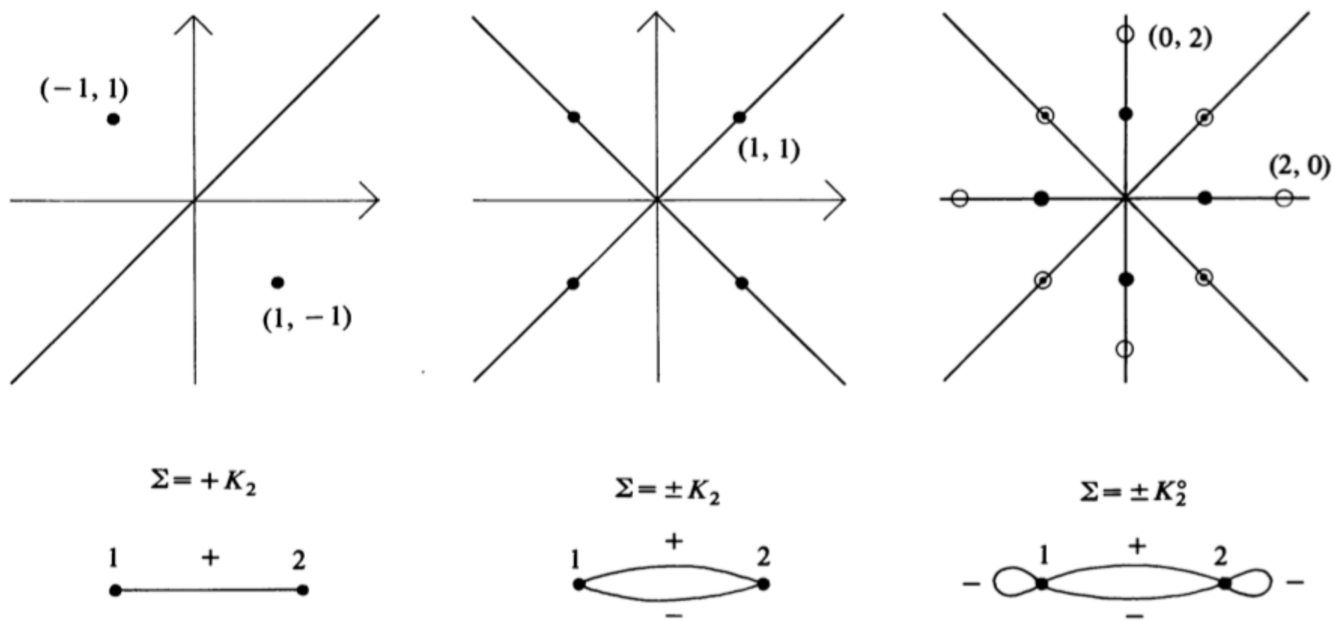
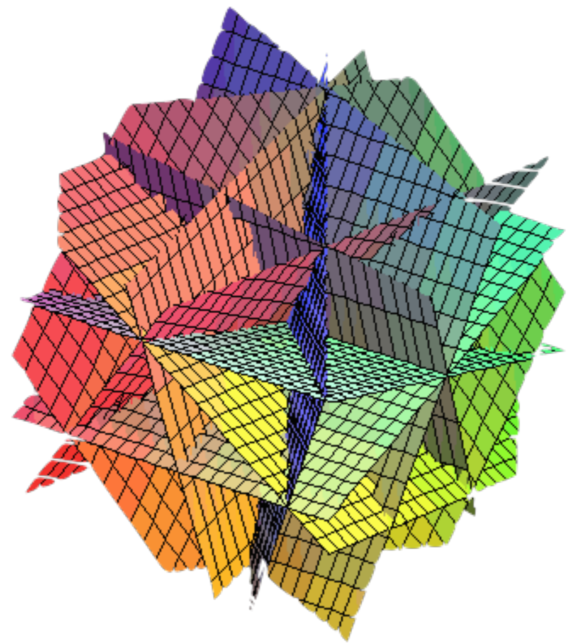


[Thomas Zaslavsky, *Amer. Math. Monthly* 1981]



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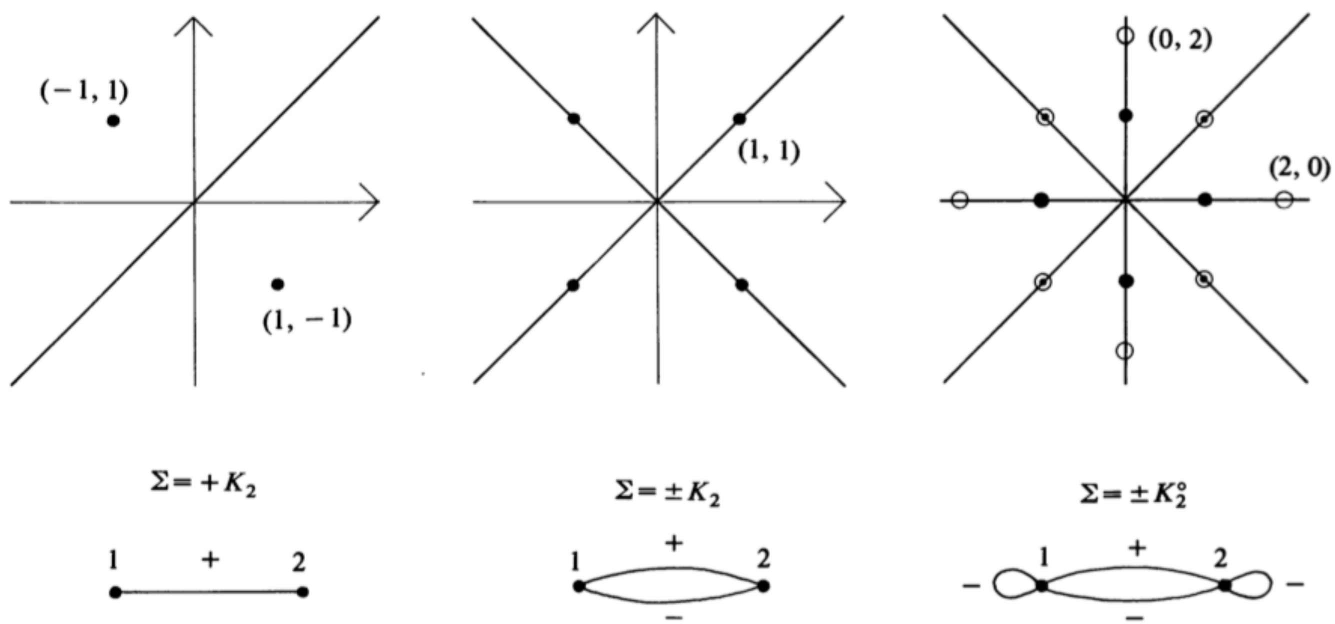

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Signed Graphic Arrangements

A bidirected graph is acyclic if every simple cycle has a source or sink.

Observation (Greene–Zaslavsky 1970s) The regions of

$$\mathcal{H}_\Sigma = \{x_j = \sigma_e\, x_k : \, e = jk \in E\}$$

are in one-to-one correspondence with the acyclic orientations of Σ.

Chromatic Polynomials of Signed Graphs

Proper k-coloring of Σ — mapping $x : V \to \{-k, -k+1, \dots, k\}$ such that for any edge $e = ij$ we have $x_i \neq \sigma_e \, x_j$

$$\chi_\Sigma(2k+1) := \#\,(\text{proper } k\text{-colorings of } \Sigma)$$
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Theorem (Macdonald 1971) $(-1)^{\dim \mathcal{P}} L_{\mathcal{P}}(-k)$ enumerates the interior lattice points in $k\mathcal{P}$.

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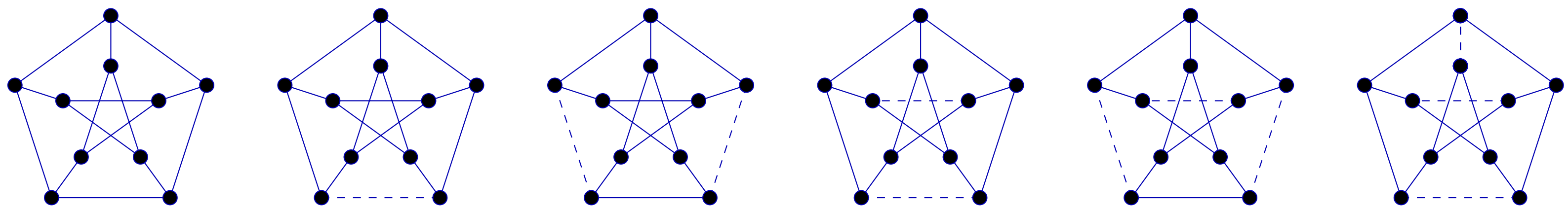
Signed Petersen Graphs

Theorem (Zaslavsky 2012) There are precisely six signed Petersen graphs that are not switching isomorphic:



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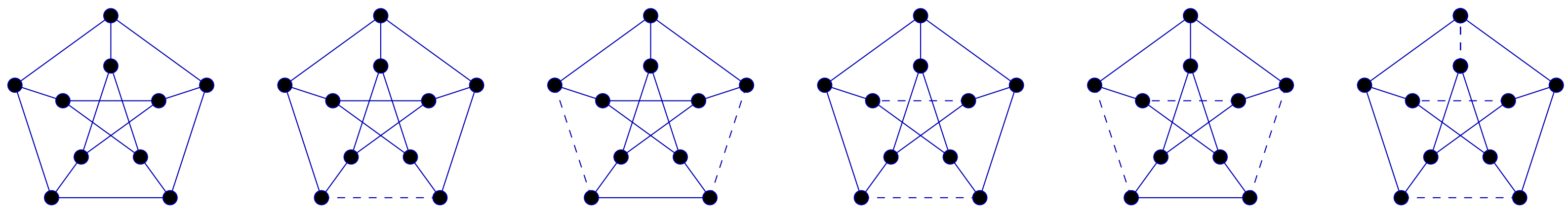

Theorem (MB–Meza–Nevarez–Shine–Young 2015, conjectured and partially proved by Zaslavsky 2012) The six signed Petersen graphs can be told apart by any positive integer evaluation of their (zero-free) chromatic polynomials.

DIY Proof Sage code at math.sfsu.edu/beck/papers/signedpetersen.sage

A Few Open Problems

- Is there a combinatorial interpretation of $\chi_\Sigma^*(-1)$?
- MB–Zaslavsky (2006) introduced a $\mathbb{Z}_{2k+1}$-flow polynomial for signed graphs. Is there any flow polynomial for evaluations at even integers?
- Computations: Birkhoff–von Neumann polytope, flow polytopes, flow polynomials
- Four-Color Theorem? Without computers?
- Five-Flow Conjecture? Antimagic-Graph Conjecture?