Vector-partition functions

Matthias Beck

San Francisco State University

math.sfsu.edu/beck

Vector partition functions

$\mathbf{A}$ – an $(m \times d)$-integral matrix
$\mathbf{b} \in \mathbb{Z}^m$

Goal: Compute vector partition function $\phi_{\mathbf{A}}(\mathbf{b}) := \# \left\{ \mathbf{x} \in \mathbb{Z}^d_{\geq 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

(defined for $\mathbf{b}$ in the nonnegative linear span of the columns of $\mathbf{A}$)

Vector partition functions

$\mathbf{A}$ – an $(m \times d)$-integral matrix
$\mathbf{b} \in \mathbb{Z}^m$

Goal: Compute vector partition function $\phi_{\mathbf{A}}(\mathbf{b}) := \# \left\{ \mathbf{x} \in \mathbb{Z}^d_{\geq 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

(defined for $\mathbf{b}$ in the nonnegative linear span of the columns of $\mathbf{A}$)

Applications in...

- Number Theory (partitions)
- Discrete Geometry (polyhedra)
- Commutative Algebra (Hilbert series)
- Algebraic Geometry (toric varieties)
- Representation Theory (tensor product multiplicities)
- Optimization (integer programming)
- Chemistry, Biology, Physics, Computer Science, Economics...

An example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

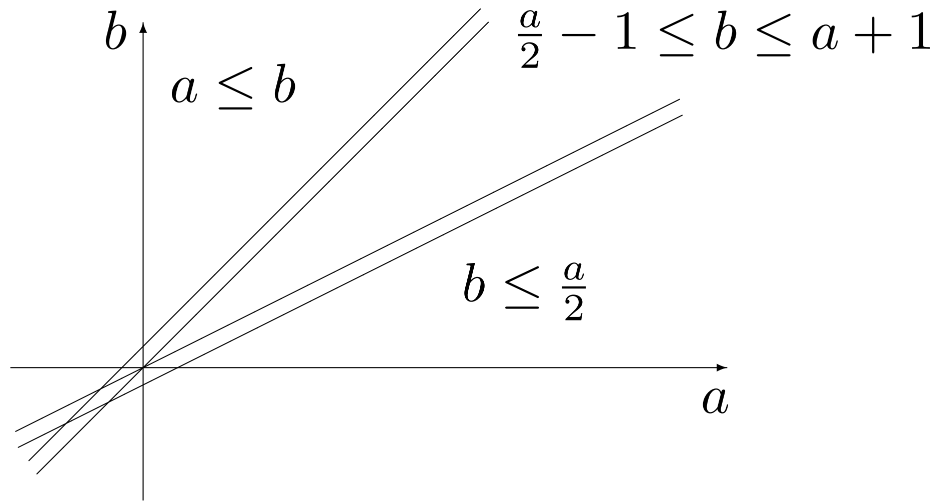
An example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \phi_{\mathbf{A}}(a,b) &= \# \left\{ \mathbf{x} \in \mathbb{Z}^4_{\geq 0} : \ \mathbf{A}\,\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix} \right\} \\ &= \begin{cases} \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} & \text{if } a \leq b \\ ab - \frac{a^2}{4} - \frac{b^2}{2} + \frac{a+b}{2} + \frac{7+(-1)^a}{8} & \text{if } \frac{a}{2} - 1 \leq b \leq a+1 \\ \frac{b^2}{2} + \frac{3b}{2} + 1 & \text{if } b \leq \frac{a}{2} \end{cases} \end{aligned}$$

An example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$



$$\begin{aligned}\phi_{\mathbf{A}}(a,b) &= \#\left\{ \mathbf{x} \in \mathbb{Z}^4_{\ge 0} : \ \mathbf{A}\,\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix} \right\} \\ &= \begin{cases} \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} & \text{if } a \le b \\ ab - \frac{a^2}{4} - \frac{b^2}{2} + \frac{a+b}{2} + \frac{7+(-1)^a}{8} & \text{if } \frac{a}{2} - 1 \le b \le a+1 \\ \frac{b^2}{2} + \frac{3b}{2} + 1 & \text{if } b \le \frac{a}{2} \end{cases}\end{aligned}$$

A “one-dimensional” example

$$\mathbf{A} = (a_1, a_2, \ldots, a_d)$$

A "one-dimensional" example

$$\mathbf{A} = (a_1, a_2, \ldots, a_d)$$

Restricted partition function

$$\phi_{\mathbf{A}}(t) = \# \left\{ (m_1, \ldots, m_d) \in \mathbb{Z}_{\geq 0}^d : \; m_1 a_1 + \cdots + m_d a_d = t \right\},$$

a quasi-polynomial, i.e., $\phi_{\mathbf{A}}(t) = c_{d-1}(t)\, t^{d-1} + c_{d-2}(t)\, t^{d-2} + \cdots + c_0(t)$
where $c_k(t)$ are periodic

A "one-dimensional" example

$$\mathbf{A} = (a_1, a_2, \ldots, a_d)$$

Restricted partition function

$$\phi_{\mathbf{A}}(t) = \# \left\{ (m_1, \ldots, m_d) \in \mathbb{Z}_{\geq 0}^d : \; m_1 a_1 + \cdots + m_d a_d = t \right\},$$

a quasi-polynomial, i.e., $\phi_{\mathbf{A}}(t) = c_{d-1}(t)\, t^{d-1} + c_{d-2}(t)\, t^{d-2} + \cdots + c_0(t)$ where $c_k(t)$ are periodic

Frobenius problem: find the largest value for t such that $\phi_{\mathbf{A}}(t) = 0$

Ehrhart quasi-polynomials

Rational (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Q}^d$

Alternative description: $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}$

For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

Ehrhart quasi-polynomials

Rational (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Q}^d$

Alternative description: $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}$

Translate & introduce slack variables $\longrightarrow \mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\ge 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

Ehrhart quasi-polynomials

Rational (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Q}^d$

Alternative description: $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}$

Translate & introduce slack variables $\longrightarrow \mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\ge 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right) = \phi_{\mathbf{A}}(t\mathbf{b})$ (for fixed $\mathbf{b}$)

Ehrhart quasi-polynomials

Rational (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Q}^d$

Alternative description: $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \; \mathbf{A}\,\mathbf{x} \leq \mathbf{b} \right\}$

Translate & introduce slack variables $\longrightarrow$ $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \; \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right) = \phi_{\mathbf{A}}(t\mathbf{b})$ (for fixed $\mathbf{b}$)

Theorem (Ehrhart 1967) If $\mathcal{P}$ is a rational polytope, then the functions $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are quasi-polynomials in t of degree $\dim \mathcal{P}$. If $\mathcal{P}$ has integer vertices, then $L_{\mathcal{P}}$ and $L_{\mathcal{P}^\circ}$ are polynomials. Furthermore, $L_{\mathcal{P}}(0) = 1$

Theorem (Ehrhart, Macdonald 1970) $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

Corollaries due to Ehrhart theory

The computation of the (Ehrhart-)quasi-polynomial

$$\phi_{\mathbf{A}}(t) = \# \left\{ (m_1, \ldots, m_d) \in \mathbb{Z}^d_{\geq 0} : \ m_1 a_1 + \cdots + m_d a_d = t \right\},$$

gives rise to the Fourier-Dedekind sum (MB–Robins 2003)

$$\sigma_n (a_1, \ldots, a_d; a_0) := \frac{1}{a_0} \sum_{\lambda^{a_0}=1} \frac{\lambda^n}{(1-\lambda^{a_1}) \cdots (1-\lambda^{a_d})} .$$

Corollaries due to Ehrhart theory

The computation of the (Ehrhart-)quasi-polynomial

$$\phi_{\mathbf{A}}(t) = \# \left\{ (m_1, \dots, m_d) \in \mathbb{Z}_{\geq 0}^d : \ m_1 a_1 + \cdots + m_d a_d = t \right\},$$

gives rise to the Fourier-Dedekind sum (MB–Robins 2003)

$$\sigma_n \left(a_1, \dots, a_d; a_0 \right) := \frac{1}{a_0} \sum_{\lambda^{a_0}=1} \frac{\lambda^n}{\left(1 - \lambda^{a_1}\right) \cdots \left(1 - \lambda^{a_d}\right)} .$$

Choosing $d = 2, n = 0, a_1 = a, a_2 = 1, a_0 = b$ gives rise to the classical Dedekind sum

$$s\left(a, b\right) := \frac{1}{4b} \sum_{j=1}^{b-1} \cot\left(\frac{\pi j a}{b}\right) \cot\left(\frac{\pi j}{b}\right)$$

Corollaries due to Ehrhart theory

Ehrhart-Macdonald Reciprocity yields the functional identity

$$\phi_{\mathbf{A}}(-t) = (-1)^{d-1} \, \phi_{\mathbf{A}}(t - (a_1 + \cdots + a_d))$$

Corollaries due to Ehrhart theory

Ehrhart-Macdonald Reciprocity yields the functional identity

$$\phi_{\mathbf{A}}(-t) = (-1)^{d-1} \phi_{\mathbf{A}}(t - (a_1 + \cdots + a_d))$$

The identity $\phi_{\mathbf{A}}(0) = 1$ implies the Reciprocity Law for Zagier's "higher-dimensional Dedekind sums"

$$s\left(a_1, a_2, \ldots, a_d; a_0\right) := \frac{1}{a_0} \sum_{j=1}^{a_0 - 1} \cot\left(\frac{\pi j a_1}{a_0}\right) \cdots \cot\left(\frac{\pi j a_d}{a_0}\right).$$

Corollaries due to Ehrhart theory

Ehrhart-Macdonald Reciprocity yields the functional identity

$$\phi_{\mathbf{A}}(-t) = (-1)^{d-1} \phi_{\mathbf{A}}(t - (a_1 + \cdots + a_d))$$

The identity $\phi_{\mathbf{A}}(0) = 1$ implies the Reciprocity Law for Zagier's "higher-dimensional Dedekind sums"

$$s(a_1, a_2, \ldots, a_d; a_0) := \frac{1}{a_0} \sum_{j=1}^{a_0-1} \cot\left(\frac{\pi j a_1}{a_0}\right) \cdots \cot\left(\frac{\pi j a_d}{a_0}\right).$$

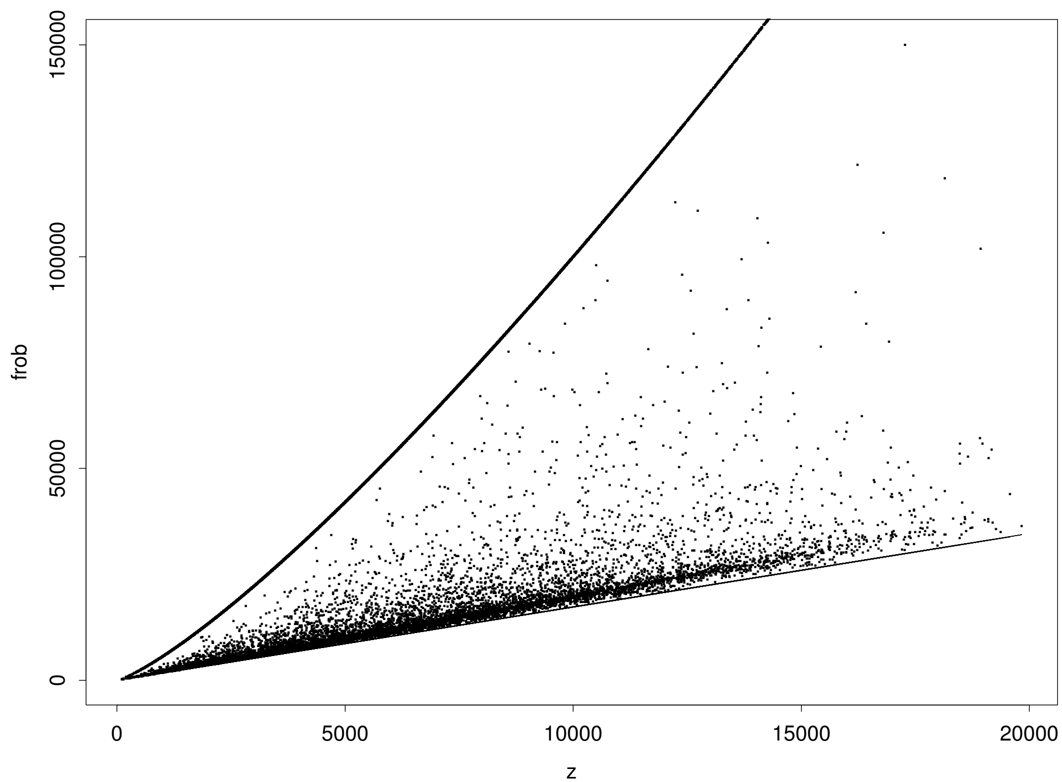
The identity

$$\phi_{\mathbf{A}}(t) = 0 \qquad \text{for} \qquad -(a_1 + \cdots + a_d) < t < 0$$

gives a new reciprocity relation which is a "higher-dimensional" analog of that for the the Dedekind-Rademacher sum.

Corollaries due to Ehrhart theory

Algorithms, bounds, experimental data on Frobenius problem (MB–Einstein–Zacks 2003)



Shameless plug

M. Beck & S. Robins

Computing the continuous discretely
Integer-point enumeration in polyhedra

To appear in Springer Undergraduate Texts in Mathematics

Preprint available at math.sfsu.edu/beck

Vector partition theorems

$\phi_{\mathbf{A}}(\mathbf{b}) := \# \left\{ \mathbf{x} \in \mathbb{Z}_{\geq 0}^{d} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

Quasi-polynomial – a finite sum $\sum_{\mathbf{n}} c_{\mathbf{n}}(\mathbf{b})\, \mathbf{b}^{\mathbf{n}}$ with coefficients $c_{\mathbf{n}}$ that are functions of $\mathbf{b}$ which are periodic in every component of $\mathbf{b}$.

Vector partition theorems

$\phi_{\mathbf{A}}(\mathbf{b}) := \# \left\{ \mathbf{x} \in \mathbb{Z}^d_{\geq 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

Quasi-polynomial – a finite sum $\sum_{\mathbf{n}} c_{\mathbf{n}}(\mathbf{b})\, \mathbf{b}^{\mathbf{n}}$ with coefficients $c_{\mathbf{n}}$ that are functions of $\mathbf{b}$ which are periodic in every component of $\mathbf{b}$.

A matrix is unimodular if every square submatrix has determinant ± 1.

Theorem (Sturmfels 1995) $\phi_{\mathbf{A}}(\mathbf{b})$ is a piecewise-defined quasi-polynomial in $\mathbf{b}$ of degree $d - \operatorname{rank}(\mathbf{A})$. The regions of $\mathbb{R}^m$ in which $\phi_{\mathbf{A}}(\mathbf{b})$ is a single quasi-polynomial are polyhedral. If $\mathbf{A}$ is unimodular then $\phi_{\mathbf{A}}$ is a piecewise-defined polynomial.

Vector partition theorems

$\phi_{\mathbf{A}}(\mathbf{b}) := \# \left\{ \mathbf{x} \in \mathbb{Z}^d_{\geq 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

Quasi-polynomial – a finite sum $\sum_{\mathbf{n}} c_{\mathbf{n}}(\mathbf{b})\,\mathbf{b}^{\mathbf{n}}$ with coefficients $c_{\mathbf{n}}$ that are functions of $\mathbf{b}$ which are periodic in every component of $\mathbf{b}$.

A matrix is unimodular if every square submatrix has determinant ± 1.

Theorem (Sturmfels 1995) $\phi_{\mathbf{A}}(\mathbf{b})$ is a piecewise-defined quasi-polynomial in $\mathbf{b}$ of degree $d - \operatorname{rank}(\mathbf{A})$. The regions of $\mathbb{R}^m$ in which $\phi_{\mathbf{A}}(\mathbf{b})$ is a single quasi-polynomial are polyhedral. If $\mathbf{A}$ is unimodular then $\phi_{\mathbf{A}}$ is a piecewise-defined polynomial.

Theorem (MB 2002) Let r_k denote the sum of the entries in the k^{th} row of $\mathbf{A}$, and let $\mathbf{r} = (r_1, \ldots, r_m)$. Then $\phi_{\mathbf{A}}(\mathbf{b}) = (-1)^{d-\operatorname{rank}\mathbf{A}} \phi_{\mathbf{A}}(-\mathbf{b} - \mathbf{r})$

Issues...

- Compute the regions of (quasi-)polynomiality of $\phi_{\mathbf{A}}(\mathbf{b})$

Issues...

- Compute the regions of (quasi-)polynomiality of $\phi_{\mathbf{A}}(\mathbf{b})$

- Given one such region, compute the (quasi-)polynomial $\phi_{\mathbf{A}}(\mathbf{b})$

Issues...

- Compute the regions of (quasi-)polynomiality of $\phi_{\mathbf{A}}(\mathbf{b})$
- Given one such region, compute the (quasi-)polynomial $\phi_{\mathbf{A}}(\mathbf{b})$
- Barvinok: $\sum_{t \geq 0} \phi_{\mathbf{A}}(t\mathbf{b})\, z^t$ can be computed in polynomial time

Partition functions for root systems

- Columns of $\mathbf{A}$ – vectors of a classical root system

Partition functions for root systems

- Columns of $\mathbf{A}$ – vectors of a classical root system

- Partition function $\phi_{\mathbf{A}}$ arises naturally in computation of the multiplicity of a weight in a finite-dimensional representation or the tensor-product decomposition of two representations

Partition functions for root systems

- Columns of $\mathbf{A}$ – vectors of a classical root system

- Partition function $\phi_{\mathbf{A}}$ arises naturally in computation of the multiplicity of a weight in a finite-dimensional representation or the tensor-product decomposition of two representations

- For root systems of type A, $\phi_{\mathbf{A}}$ has intimate connections to the Birkhoff polytope containing doubly-stochastic matrices

Partition functions for root systems

- Columns of $\mathbf{A}$ – vectors of a classical root system

- Partition function $\phi_{\mathbf{A}}$ arises naturally in computation of the multiplicity of a weight in a finite-dimensional representation or the tensor-product decomposition of two representations

- For root systems of type A, $\phi_{\mathbf{A}}$ has intimate connections to the Birkhoff polytope containing doubly-stochastic matrices

- (Baldoni–MB–Cochet–Vergne 200?) Computational approach using Jeffrey-Kirwan residues and DeConcini-Prochesi's maximal nested sets

Euler's generating function

$$\phi_{\mathbf{A}}(\mathbf{b}) := \#\left\{\mathbf{x} \in \mathbb{Z}_{\geq 0}^{d} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b}\right\} \qquad \mathbf{A} = \begin{pmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_d \\ | & | & & | \end{pmatrix}$$

$\phi_{\mathbf{A}}(\mathbf{b})$ equals the coefficient of $\mathbf{z}^{\mathbf{b}} := z_1^{b_1} \cdots z_m^{b_m}$ of the function

$$\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_d})}$$

expanded as a power series centered at $\mathbf{z} = 0$.

Euler's generating function

$$\phi_{\mathbf{A}}(\mathbf{b}) := \#\left\{\mathbf{x} \in \mathbb{Z}_{\geq 0}^{d} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b}\right\} \qquad \mathbf{A} = \begin{pmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_d \\ | & | & & | \end{pmatrix}$$

$\phi_{\mathbf{A}}(\mathbf{b})$ equals the coefficient of $\mathbf{z}^{\mathbf{b}} := z_1^{b_1} \cdots z_m^{b_m}$ of the function

$$\frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})}$$

expanded as a power series centered at $\mathbf{z} = 0$.

Proof Expand each factor into a geometric series. ⬠

Euler's generating function

$$\phi_{\mathbf{A}}(\mathbf{b}) := \# \left\{ \mathbf{x} \in \mathbb{Z}_{\geq 0}^d : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\} \qquad \mathbf{A} = \begin{pmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_d \\ | & | & & | \end{pmatrix}$$

$\phi_{\mathbf{A}}(\mathbf{b})$ equals the coefficient of $\mathbf{z}^{\mathbf{b}} := z_1^{b_1} \cdots z_m^{b_m}$ of the function

$$\frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})}$$

expanded as a power series centered at $\mathbf{z} = 0$.

Proof Expand each factor into a geometric series. ⬠

Equivalently,

$$\phi_{\mathbf{A}}(\mathbf{b}) = \text{const} \, \frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d}) \, \mathbf{z}^{\mathbf{b}}}$$

Partial fractions

$$\phi_{\mathbf{A}}(\mathbf{b}) = \text{const} \frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\,\mathbf{z}^{\mathbf{b}}}$$

Expand into partial fractions in z_1:

$$\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\,\mathbf{z}^{\mathbf{b}}} = \frac{1}{z_2^{b_2}\cdots z_m^{b_m}} \left(\sum_{k=1}^{d} \frac{A_k(\mathbf{z}, b_1)}{1-\mathbf{z}^{\mathbf{c}_k}} + \sum_{j=1}^{b_1} \frac{B_j(\mathbf{z})}{z_1^j} \right)$$

Here A_k and B_j are polynomials in z_1, rational functions in $z_2, \dots, z_m$, and exponential in b_1.

Partial fractions

$$\phi_{\mathbf{A}}(\mathbf{b}) = \text{const} \frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\,\mathbf{z}^{\mathbf{b}}}$$

Expand into partial fractions in z_1:

$$\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\cdots(1-\mathbf{z}^{\mathbf{c}_d})\,\mathbf{z}^{\mathbf{b}}} = \frac{1}{z_2^{b_2}\cdots z_m^{b_m}} \left(\sum_{k=1}^{d} \frac{A_k(\mathbf{z}, b_1)}{1-\mathbf{z}^{\mathbf{c}_k}} + \sum_{j=1}^{b_1} \frac{B_j(\mathbf{z})}{z_1^j} \right)$$

Here A_k and B_j are polynomials in z_1, rational functions in $z_2, \ldots, z_m$, and exponential in b_1.

$$\phi_{\mathbf{A}}(\mathbf{b}) \quad = \quad \text{const}_{z_2,\ldots,z_m} \frac{1}{z_2^{b_2}\cdots z_m^{b_m}} \, \text{const}_{z_1} \sum_{k=1}^{d} \frac{A_k(\mathbf{z}, b_1)}{1-\mathbf{z}^{\mathbf{c}_k}}$$

Partial fractions

$$\phi_{\mathbf{A}}(\mathbf{b}) = \text{const} \, \frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_d}) \, \mathbf{z}^{\mathbf{b}}}$$

Expand into partial fractions in z_1:

$$\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_d}) \, \mathbf{z}^{\mathbf{b}}} = \frac{1}{z_2^{b_2} \cdots z_m^{b_m}} \left(\sum_{k=1}^{d} \frac{A_k(\mathbf{z}, b_1)}{1-\mathbf{z}^{\mathbf{c}_k}} + \sum_{j=1}^{b_1} \frac{B_j(\mathbf{z})}{z_1^j} \right)$$

Here A_k and B_j are polynomials in z_1, rational functions in $z_2, \ldots, z_m$, and exponential in b_1.

$$\begin{aligned} \phi_{\mathbf{A}}(\mathbf{b}) &= \text{const}_{z_2,\ldots,z_m} \frac{1}{z_2^{b_2} \cdots z_m^{b_m}} \, \text{const}_{z_1} \sum_{k=1}^{d} \frac{A_k(\mathbf{z}, b_1)}{1-\mathbf{z}^{\mathbf{c}_k}} \\ &= \text{const} \, \frac{1}{z_2^{b_2} \cdots z_m^{b_m}} \sum_{k=1}^{d} \frac{A_k(0, z_2, \ldots, z_m, b_1)}{1-(0, z_2, \ldots, z_m)^{\mathbf{c}_k}} \end{aligned}$$

Advantages

- easy to implement

Advantages

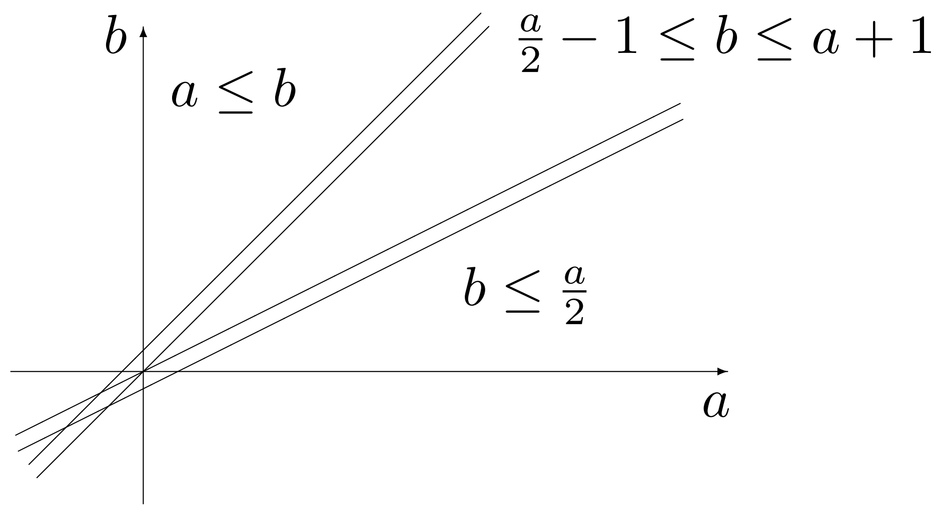
- easy to implement
- allows symbolic computation

Advantages

- easy to implement
- allows symbolic computation
- constraints which define the regions of (quasi-)polynomiality are obtained "automatically"

The first example revisited

$$\phi_{\mathbf{A}}(a,b) = \begin{cases} \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} & \text{if } a \le b \\ ab - \frac{a^2}{4} - \frac{b^2}{2} + \frac{a+b}{2} + \frac{7+(-1)^a}{8} & \text{if } a > b > \frac{a-3}{2} \\ \frac{b^2}{2} + \frac{3b}{2} + 1 & \text{if } b \le \frac{a-3}{2} \end{cases}$$



The first example revisited

$$\phi_{\mathbf{A}}(a,b) = \begin{cases} \frac{a^2}{4} + a + \frac{7+(-1)^a}{8} & \text{if } a \le b \\ ab - \frac{a^2}{4} - \frac{b^2}{2} + \frac{a+b}{2} + \frac{7+(-1)^a}{8} & \text{if } a > b > \frac{a-3}{2} \\ \frac{b^2}{2} + \frac{3b}{2} + 1 & \text{if } b \le \frac{a-3}{2} \end{cases}$$

$$\begin{aligned} x_1, x_2, x_3, x_4 &\ge 0 \\ x_1 + 2x_2 + x_3 &= a \\ x_1 + x_2 + x_4 &= b \end{aligned}$$