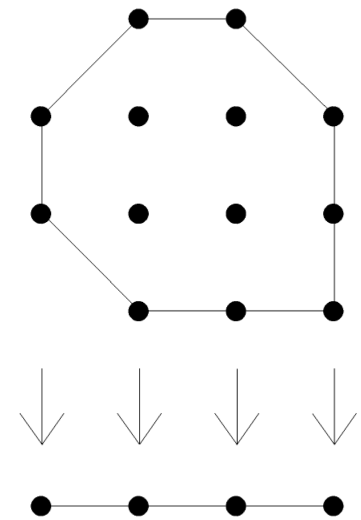
Very Ample and Koszul Segmental Fibrations



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"To many, mathematics is a collection of theorems. For me, mathematics is a collection of examples; a theorem is a statement about a collection of examples and the purpose of proving theorems is to classify and explain the examples..."

John B. Conway

Lattice Polytopes

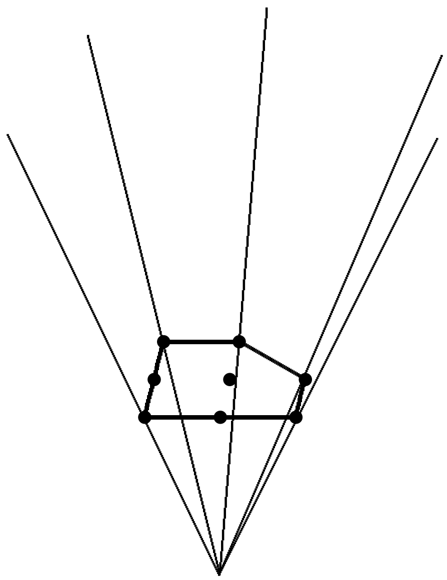
$\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely many points $V \subset \mathbb{Z}^d$

$$R := \mathbb{R}_{\geq 0} \left(\mathcal{P} \times \{1\} \right) \qquad\qquad S := \mathbb{Z}_{\geq 0} \left((\mathcal{P} \cap \mathbb{Z}^d) \times \{1\} \right)$$

$\mathbb{K}[S]$ — monomial algebra associated to S, graded by last coordinate

We say that $\mathcal{P}$ is . . .

- Koszul if the minimal free graded resolution of $\mathbb{K}$ over $\mathbb{K}[S]$ is linear
- normal if $R \cap \mathbb{Z}^{d+1} = S$
- very ample if $R \cap \mathbb{Z}^{d+1} \setminus S$ is finite



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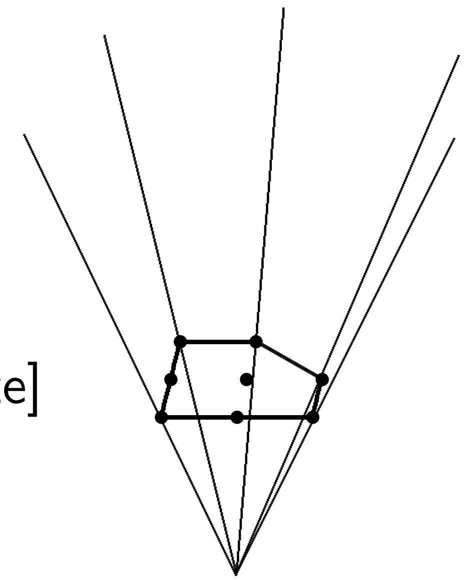

[For much more on this hierarchy, see Bruns–Gubeladze]

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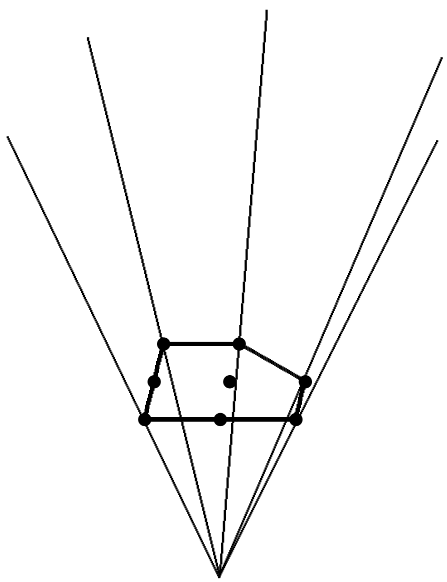

$\mathcal{P}$ is very ample if and only if for every $\mathbf{v} \in V$

$$\mathbb{R}_{\geq 0}(\mathcal{P} - \mathbf{v}) \cap \mathbb{Z}^d = \mathbb{Z}_{\geq 0}(V - \mathbf{v})$$

i.e., $V - \mathbf{v}$ is a Hilbert basis for the cone $\mathbb{R}_{\geq 0}(\mathcal{P} - \mathbf{v})$.

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What can we say about the set $R \cap \mathbb{Z}^{d+1} \setminus S$ of gaps of a very ample polytope? E.g., is there a constraint on their number or their heights?

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Bogart–Haase–Hering–Lorenz–Nill–Paffenholz–Santos–Schenck (2014) constructed very ample polytopes with a prescribed number of gaps.

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Higashitani (2014) constructed very ample polytopes with a prescribed number of gaps in a prescribed dimension ≥ 3.

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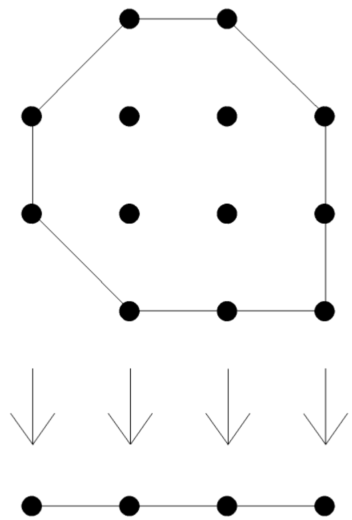
Our goal is to construct very ample polytopes with gaps of arbitrary height (in dimension ≥ 3). Incidentally, the same construction yields a new class of Koszul polytopes in all dimensions.

Lattice Segmental Fibrations

$\mathcal{P} \subset \mathbb{R}^d, \mathcal{Q} \subset \mathbb{R}^e$ — lattice polytopes

An affine map $f : \mathcal{P} \to \mathcal{Q}$ is a lattice segmental fibration if

- $f^{-1}(\mathbf{x})$ is a lattice segment or point for every $\mathbf{x} \in \mathcal{Q} \cap \mathbb{Z}^e$
- $\dim(f^{-1}(\mathbf{x})) = 1$ for at least one $\mathbf{x} \in \mathcal{Q} \cap \mathbb{Z}^e$
- $\mathcal{P} \cap \mathbb{Z}^d \subseteq \displaystyle\bigcup_{\mathbf{x} \in \mathcal{Q} \cap \mathbb{Z}^e} f^{-1}(\mathbf{x})$



Note that our definition implies that f is surjective and $d = e + 1$ if $\mathcal{P}$ and $\mathcal{Q}$ are full dimensional.

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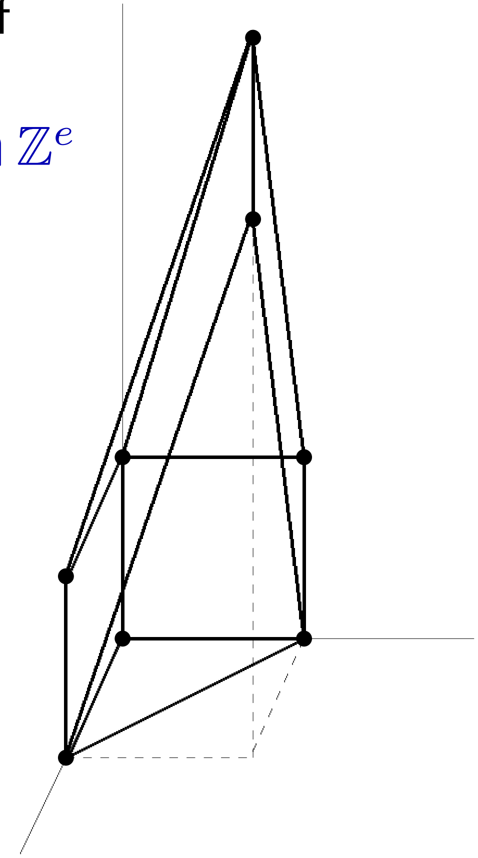

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Mother of all examples

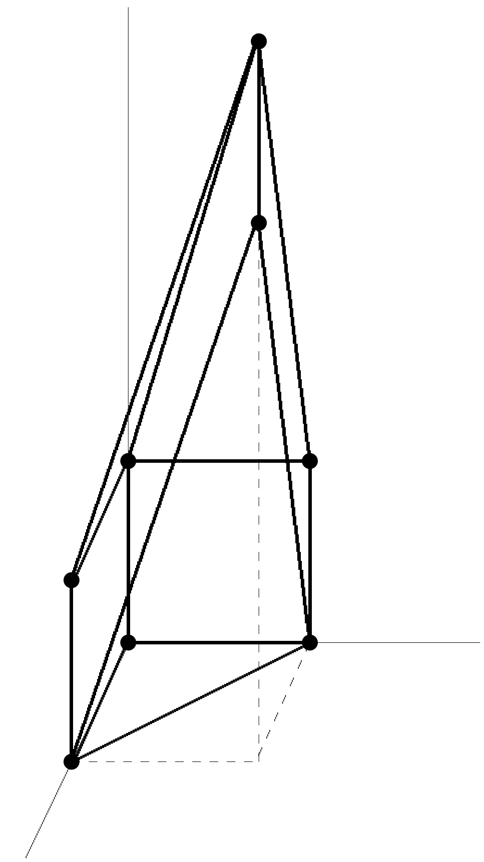
$\mathcal{P}_m := \operatorname{conv}\big\{(0,0,[0,1]),(1,0,[0,1]),(0,1,[0,1]),(1,1,[m,m+1])\big\}$

Gaps At Arbitrary Heights

$$\mathcal{P}_m := \operatorname{conv}\left\{(0,0,[0,1]),(1,0,[0,1]),(0,1,[0,1]),(1,1,[m,m+1])\right\}$$

Theorem For $m \geq 3$ the gap vector of $\mathcal{P}_m$ has entries

$$\operatorname{gap}_k(\mathcal{P}_m) = \binom{k+1}{3}(m-k-1)$$



In particular,

$$\operatorname{gap}_1(\mathcal{P}_m) \leq \cdots \leq \operatorname{gap}_{\lceil \frac{3m-5}{4} \rceil}(\mathcal{P}_m) \geq \operatorname{gap}_{\lceil \frac{3m-5}{4} \rceil + 1}(\mathcal{P}_m)$$

$$\geq \cdots \geq \operatorname{gap}_{m-2}(\mathcal{P}_m)$$

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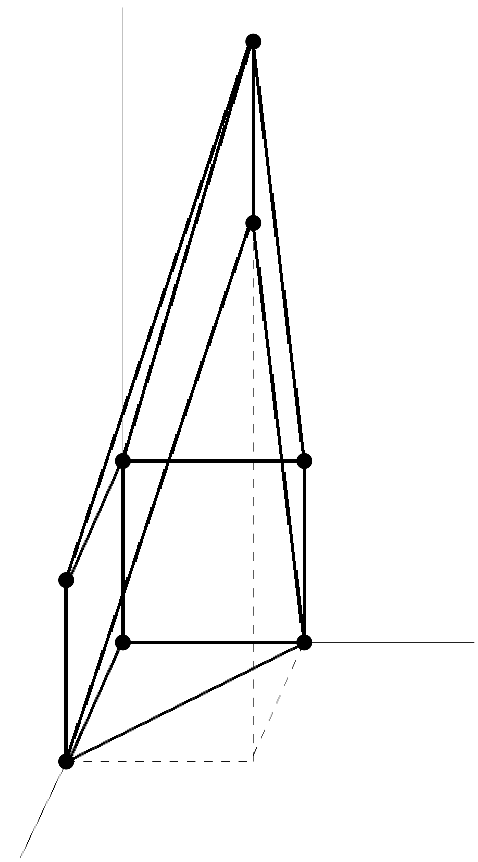

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Note that $\mathcal{P}_m \times [0,1]$ is again very ample, which implies the existence of non-normal very ample polytopes in all dimensions ≥ 3 with an arbitrarily large number of gaps with arbitrary heights.

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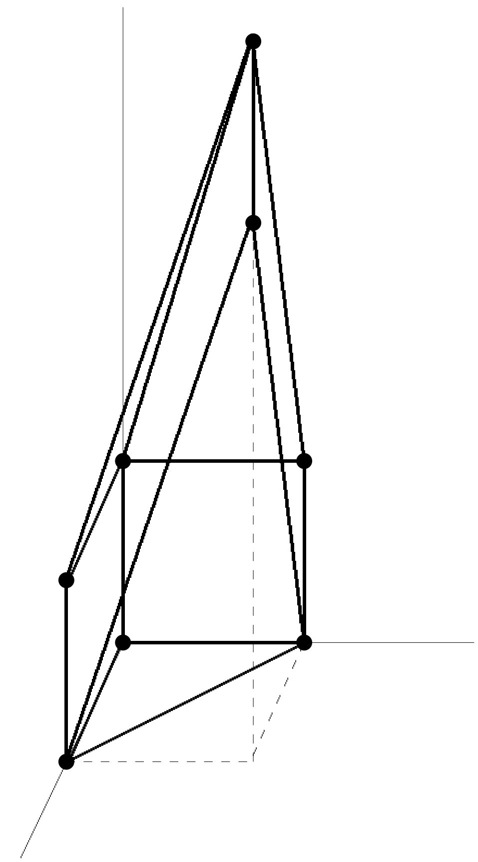

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Corollary For $m \geq 3$ the polytopes $\mathcal{P}_m$ have gaps at arbitrary heights.

Alternative proof: Check that $\#\left(k\mathcal{P}_m \cap \mathbb{Z}^3\right) \geq \frac{m}{2}$ (independent of k). But if $k \geq$ the highest gap,

$$\frac{m}{2} \leq \#\left(k\mathcal{P}_m \cap \mathbb{Z}^3\right) \leq 8^k$$

A 1-dimensional Analogue (well, sort of...)

Recall that $\mathcal{P}$ is very ample if its set of gaps

$$\left(\mathbb{R}_{\geq 0}\left(\mathcal{P} \times \{1\}\right) \cap \mathbb{Z}^{d+1}\right) \setminus \mathbb{Z}_{\geq 0}\left(\left(\mathcal{P} \cap \mathbb{Z}^{d}\right) \times \{1\}\right)$$

is finite.

Given a finite set $A \subset \mathbb{Z}_{>0}$ with $\gcd(A) = 1$ one can prove (try it—it's fun!) that $\mathbb{Z}_{\geq 0} \setminus \mathbb{Z}_{\geq 0} A$ is finite.

Frobenius Problem What is the largest gap in $\mathbb{Z}_{\geq 0} \setminus \mathbb{Z}_{\geq 0} A$?

[open for $|A| = 3$, wide open for $|A| \geq 4$]

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The lattice polytope $\mathcal{P}$ is Koszul if the minimal free graded resolution

$$\cdots \longrightarrow \mathbb{K}[S]^{\beta_2} \xrightarrow{\partial_2} \mathbb{K}[S]^{\beta_1} \xrightarrow{\partial_1} \mathbb{K}[S] \xrightarrow{\partial_0} \mathbb{K} \longrightarrow 0$$

is linear, that is, $\deg(\partial_j) = 1$ for $j > 0$.

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- $\deg(\partial_1) = 1$ means $\mathbb{K}[S]$ is homogeneous
- $\deg(\partial_1) = \deg(\partial_2) = 1$ means $\mathbb{K}[S]$ is quadratically defined, that is,

$$\mathbb{K}[S] = \mathbb{K}[x_1, x_2, \ldots, x_{d+1}] \,/\, \langle f_1, f_2, \ldots, f_n \rangle$$

for some homogeneous quadratic polynomials $f_1, f_2, \ldots, f_n$.

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- [Priddy 1970] $\mathcal{P}$ is Koszul if $\mathbb{K}[S] = \mathbb{K}[x_1, x_2, \dots, x_{d+1}] \,/\, I$ for a quadratic Gröbner basis I.

Unimodular Triangulations

A triangulation is unimodular if for any simplex in the triangulation, with vertices $v_0, v_1, \dots, v_d$, the vectors $v_1 - v_0, \dots, v_d - v_0$ form a basis of $\mathbb{Z}^d$. (That is, each simplex is smooth.)

We'll call a regular unimodular flag triangulation good.

Sturmfels Correspondence $\mathcal{P} = \operatorname{conv}(V)$ admits a good triangulation if and only if the toric ideal corresponding to V admits a square-free quadratic Gröbner basis.

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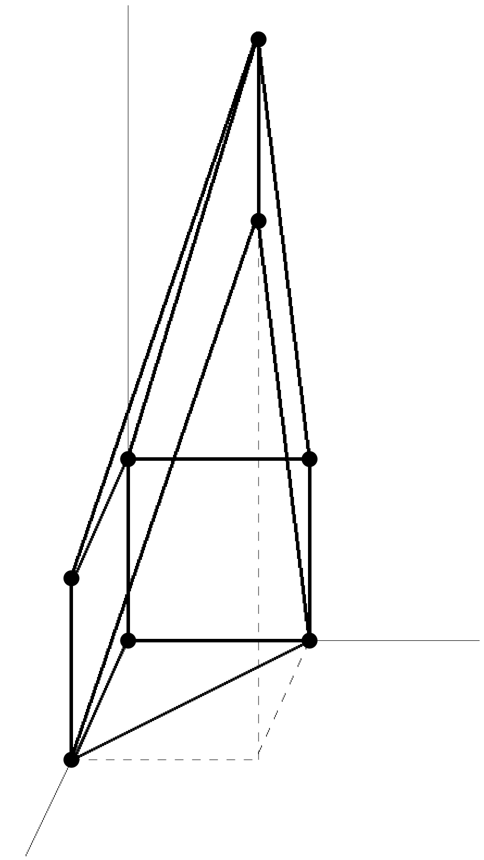
Example [Dais–Haase–Ziegler 2001] Let $\mathcal{Q} \subset \mathbb{R}^d$ be a lattice polytope and $\alpha, \beta : \mathcal{Q} \to \mathbb{R}$ affine maps such that $\alpha(\mathbf{x}), \beta(\mathbf{x}) \in \mathbb{Z}$ for all $\mathbf{x} \in \mathcal{Q} \cap \mathbb{Z}^d$ and $\alpha \le \beta$ on $\mathcal{Q}$. If $\mathcal{Q}$ has a good triangulation, so does the Nakajima polytope

$$\mathcal{Q}(\alpha, \beta) := \operatorname{conv}\left\{(\mathbf{x}, y) : \mathbf{x} \in \mathcal{Q},\ \alpha(\mathbf{x}) \le y \le \beta(\mathbf{x})\right\} \subset \mathbb{R}^{d+1}$$

Good Fibrations

Theorem Let $f : \mathcal{P} \to \mathcal{Q}$ be a lattice segmental fibration. If Δ is a good triangulation of $\mathcal{Q}$ such that the image of every face of $\mathcal{P}$ is a union of faces of Δ then $\mathcal{P}$ admits a good triangulation; in particular, $\mathcal{P}$ is Koszul.

Example $\operatorname{conv}\{(0,0,I_1),(1,0,I_2),(0,1,I_3),(1,1,I_4)\}$ for some lattice segments I_1, I_2, I_3, I_4 [Bruns 2007]



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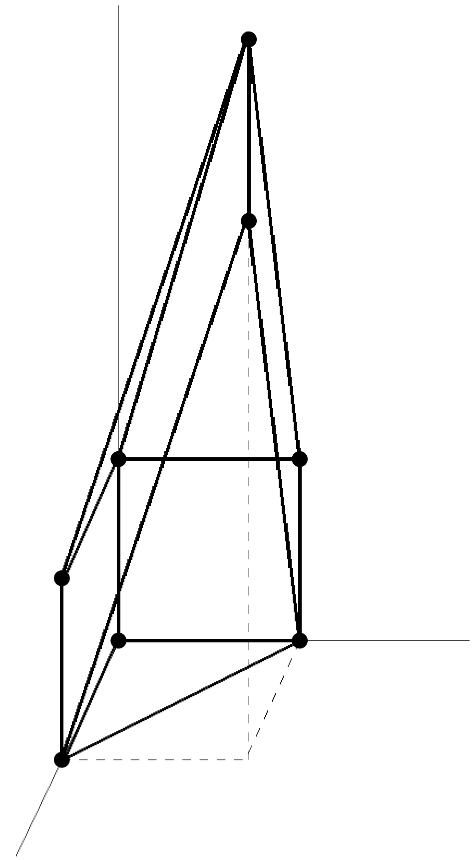

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If this lattice segmental fibration is smooth, it admits a good triangulation, and thus we can construct infinite classes of Koszul polytopes.

Example [Lattice A-fibrations] A lattice polytope bounded by hyperplanes parallel to hyperplanes of the form $x_j = 0$ and $x_j = x_k$ comes with a canonical good triangulation. [Bruns–Gubeladze–Trung 1997]

Open Problems

- Conjecture If $\mathcal{P}$ is very ample then $\operatorname{gap}(\mathcal{P})$ contains no internal zeros. [true for $\dim \mathcal{P} = 3$]

- Conjecture If $\mathcal{P}$ is very ample with normal facets, $\operatorname{gap}(\mathcal{P})$ is unimodal.

- Oda's Question Is every smooth polytope normal?

- Bøgvad's Conjecture If $\mathcal{P}$ is smooth then $\mathbb{K}[\mathbb{R}_{\geq 0}\,(\mathcal{P} \times \{1\}) \cap \mathbb{Z}^{d+1}]$ is Koszul.

- Is there a lower bound for c depending only on $\dim(\mathcal{P})$ such that $c\mathcal{P}$ has a unimodular triangulation? [$c\mathcal{P}$ is Koszul for $c \geq \dim(\mathcal{P})$]