Facing up to Tom Zaslavsky: Making arrangements count (lattice points)

Dedicated to my friend & mentor @ 75

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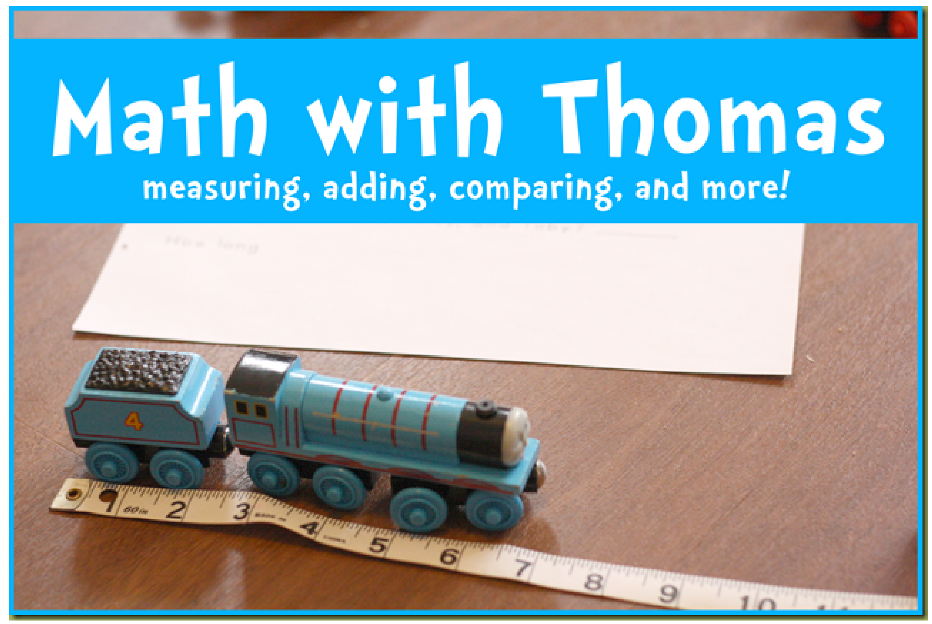

[royalbaloo.com]

Reminiscences

cycle (in a graph)

In class I gave an older, obsolete rule (sorry!), namely the opposite of the "best rule" rule just stated. You should use the best rule.

BibTEX

Before Prop. 9: "Fortunately for grid graphs," -- ah, those fortunate grid graphs! That comma ought to be moved to "Fortunately, for"

$\mathbb{N}$

Okay, I'll try around noon my time, probably. Don't want to wake you up! (That would mean calling before 10 my time, probably.)

$\mathbb{Z}_+$

bridge (of a graph)

On Wed, Sep 3, 2014 at 8:13 PM, <zaslav@math.binghamton.edu> wrote:
> I don't know Jim's even-odd stuff.
That's funny, because he credits you for a (very nice) notation--see attached.
:) m

Reminiscences

NOTE. The basic results here date from 1975 and were announced in **[13]** and **[27]**. For the delay in preparing this article the authors apologize to their readers and to each other.

M. Beck, T. Zaslavsky/Advances in Mathematics 205 (2006) 134–162 135

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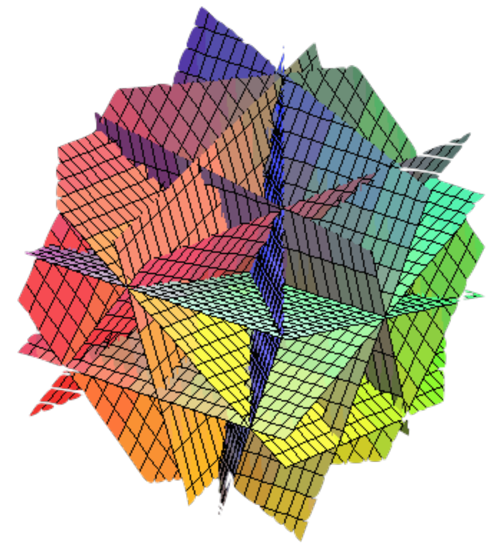
Famous 20th Century Results: Hyperplane Arrangements

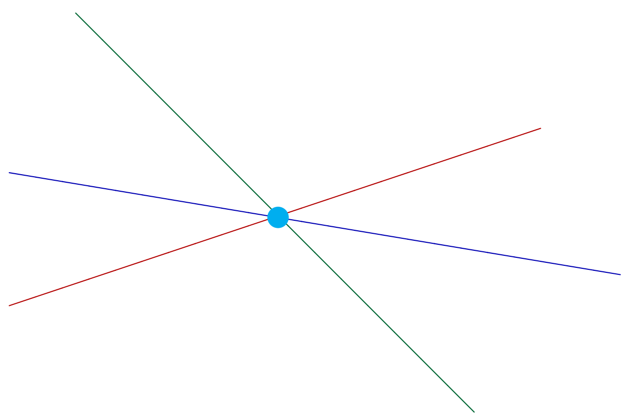
$\mathcal{H} \subset \mathbb{R}^d$ — arrangement of affine hyperplanes
$\mathcal{L}(\mathcal{H})$ — all nonempty intersections of hyperplanes in $\mathcal{H}$

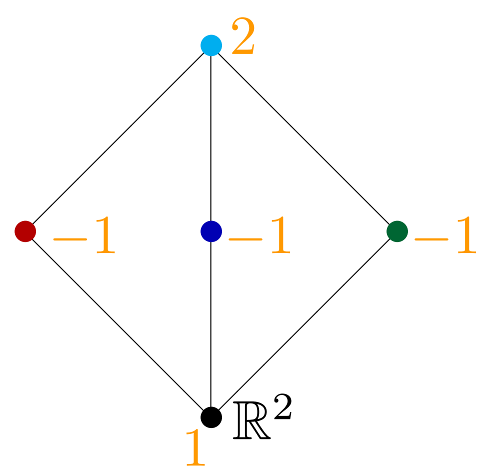


[John Stembridge]

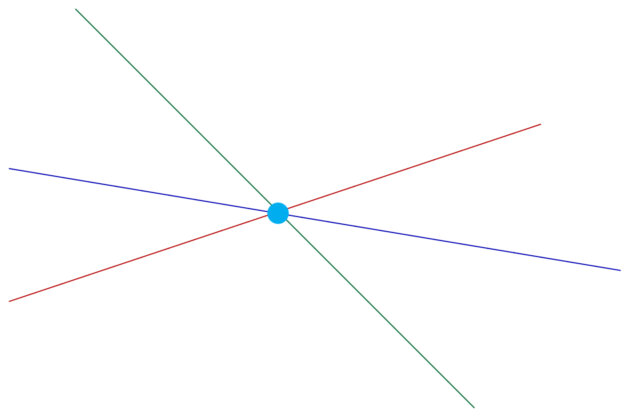
Möbius function $\mu(F) := \begin{cases} 1 & \text{if } F = \mathbb{R}^d \\ -\sum_{G \supsetneq F} \mu(F) & \text{otherwise} \end{cases}$

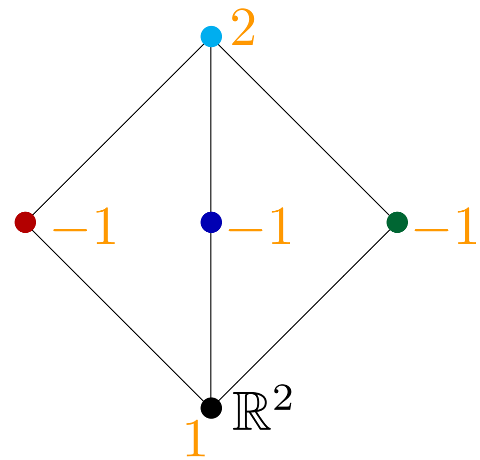
Characteristic polynomial $p_{\mathcal{H}}(k) := \sum_{F \in \mathcal{L}(\mathcal{H})} \mu(F)\, k^{\dim F}$





Famous 20th Century Results: Hyperplane Arrangements





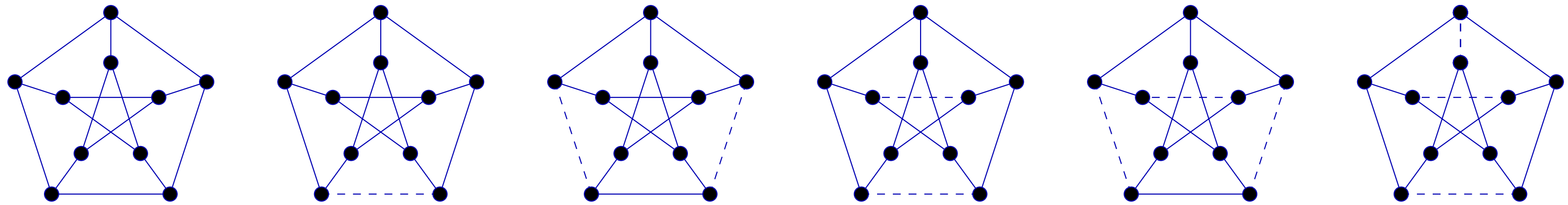
$$p_{\mathcal{H}}(k) \ = \sum_{F \in \mathcal{L}(\mathcal{H})} \mu(F)\, k^{\dim F} \ = \ k^2 - 3k + 2$$

Note that $\mathcal{H}$ divides $\mathbb{R}^2$ into $p_{\mathcal{H}}(-1) = 6$ regions...

Theorem (Zaslavsky 1975) $(-1)^d \, p_{\mathcal{H}}(-1)$ equals the number of regions into which a hyperplane arrangement $\mathcal{H}$ divides $\mathbb{R}^d$.

Philosophy Euler characteristics . . .

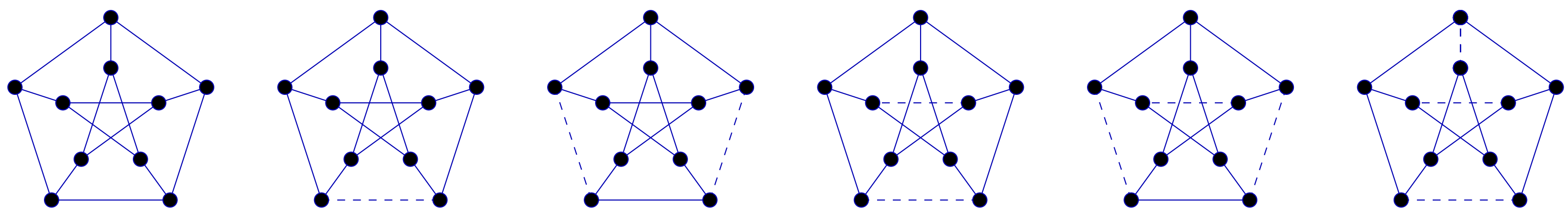
Famous 20th Century Results: Signed Graph Coloring



$\Sigma = (V, E, \epsilon)$ — signed graph: each edge is labelled $+$ or $-$

"Type B" graphs, originated in social psychology (the enemy of my enemy...)

Famous 20th Century Results: Signed Graph Coloring



$\Sigma = (V, E, \epsilon)$ — signed graph: each edge is labelled $+$ or $-$

"Type B" graphs, originated in social psychology (the enemy of my enemy...)

Proper k-coloring of Σ : mapping $x : V \to \{-k, -k+1, \ldots, k\}$ such that $x_i \neq \epsilon_{ij} x_j$ if the edge ij has sign ϵ_{ij}

Theorem (Zaslavsky 1982) $\chi_\Sigma(2k+1) := \#\,(\text{proper } k\text{-colorings of } \Sigma)$ and $\chi^*_\Sigma(2k) := \#\,(\text{proper zero-free } k\text{-colorings of } \Sigma)$ are monic polynomials of degree $|V|$. Mirroring Stanley's work on the "usual" chromatic polynomial, there are meaningful reciprocity interpretations for $\chi_\Sigma(-(2k+1))$ and $\chi^*_\Sigma(-2k)$, e.g., $(-1)^{|V|}\chi_\Sigma(-1)$ is the number of acyclic orientations of Σ.

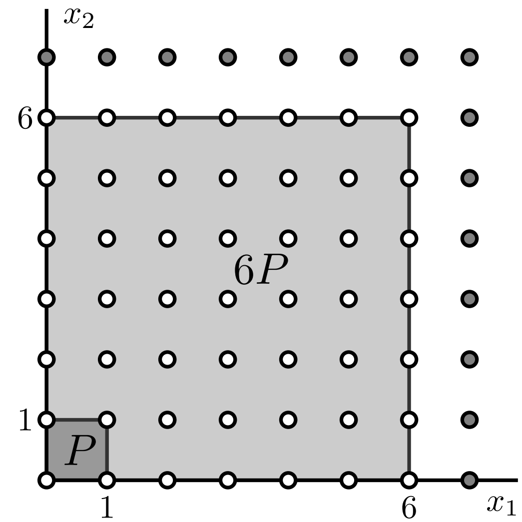
Famous 20th Century Results: Ehrhart Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $k \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(k) := \# \left(k\mathcal{P} \cap \mathbb{Z}^d \right)$

Theorem (Ehrhart 1962, Macdonald 1971) $L_{\mathcal{P}}(k)$ is a polynomial in k with constant term 1. The evaluation $(-1)^{\dim \mathcal{P}} L_{\mathcal{P}}(-k)$ enumerates the interior lattice points in $k\mathcal{P}$.

Philosophy Euler characteristics . . .



$$L_{\mathcal{P}}(k) = (k+1)^2$$

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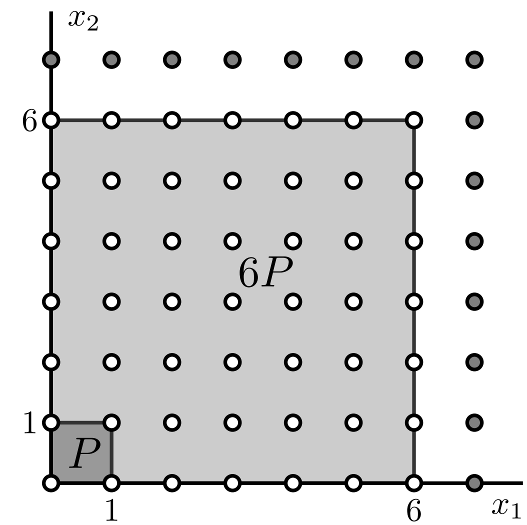

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$$L_{\mathcal{P}}(-k) = (k-1)^2$$

Philosophy Euler characteristics . . .

Everything works for rational polytopes, but now $L_{\mathcal{P}}(k)$ is a quasipolynomial

$$L_{\mathcal{P}}(k) \;=\; c_d(k)\,k^d + c_{d-1}(k)\,k^{d-1} + \cdots + c_0(k)$$

where $c_0(k), \ldots, c_d(k)$ are periodic functions in k

How Inside-Out Polytopes Got Conceived

Journal of Algebraic Combinatorics **7** (1998), 115–126

Characteristic and Ehrhart Polynomials*

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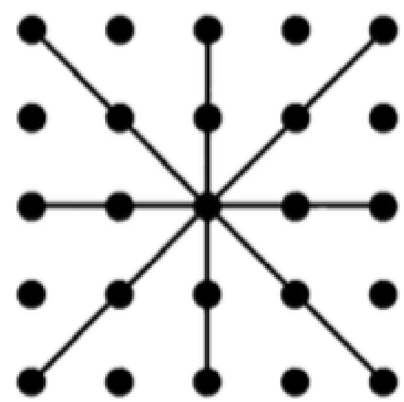
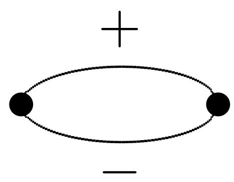
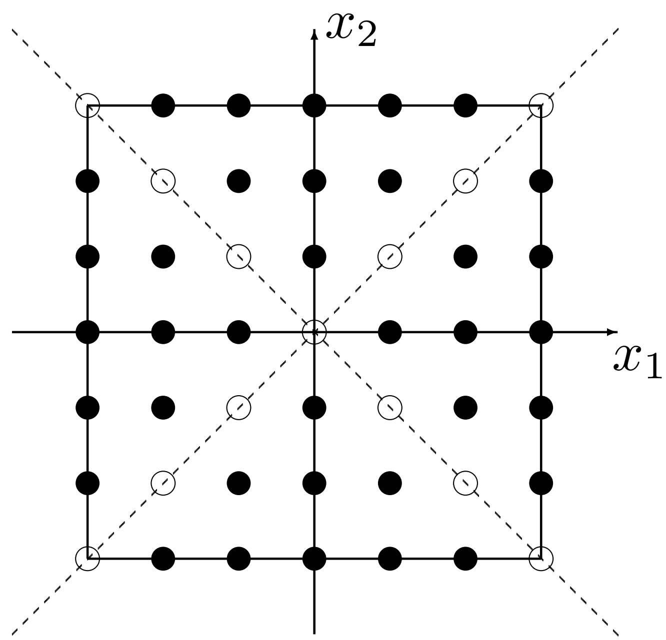


Figure 1. The lattice points of $[-2, 2]^2 \setminus \bigcup \mathcal{B}_2$.

How To Color a Signed Graph, Geometrically



$$\chi_\Sigma(2k+1) = 4k^2$$



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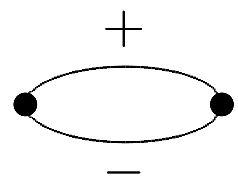
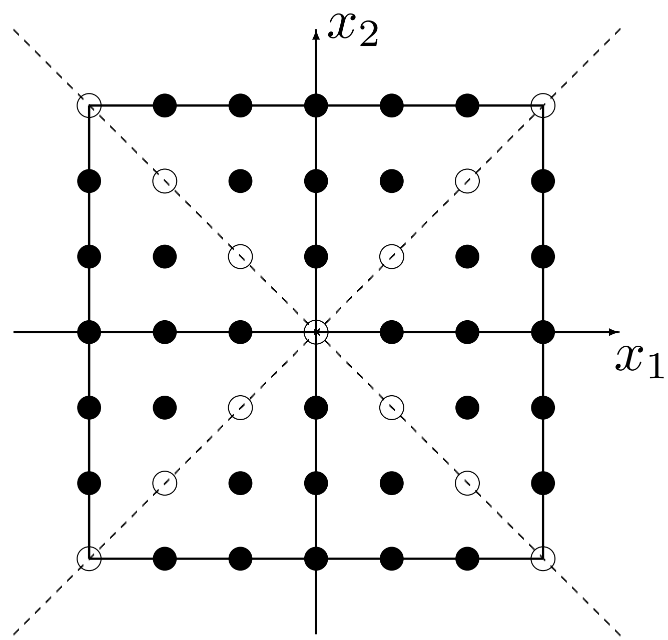
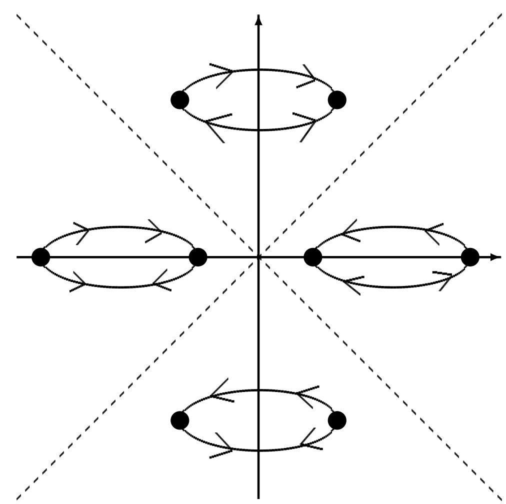




$\chi_\Sigma(2k+1) = 4k^2$

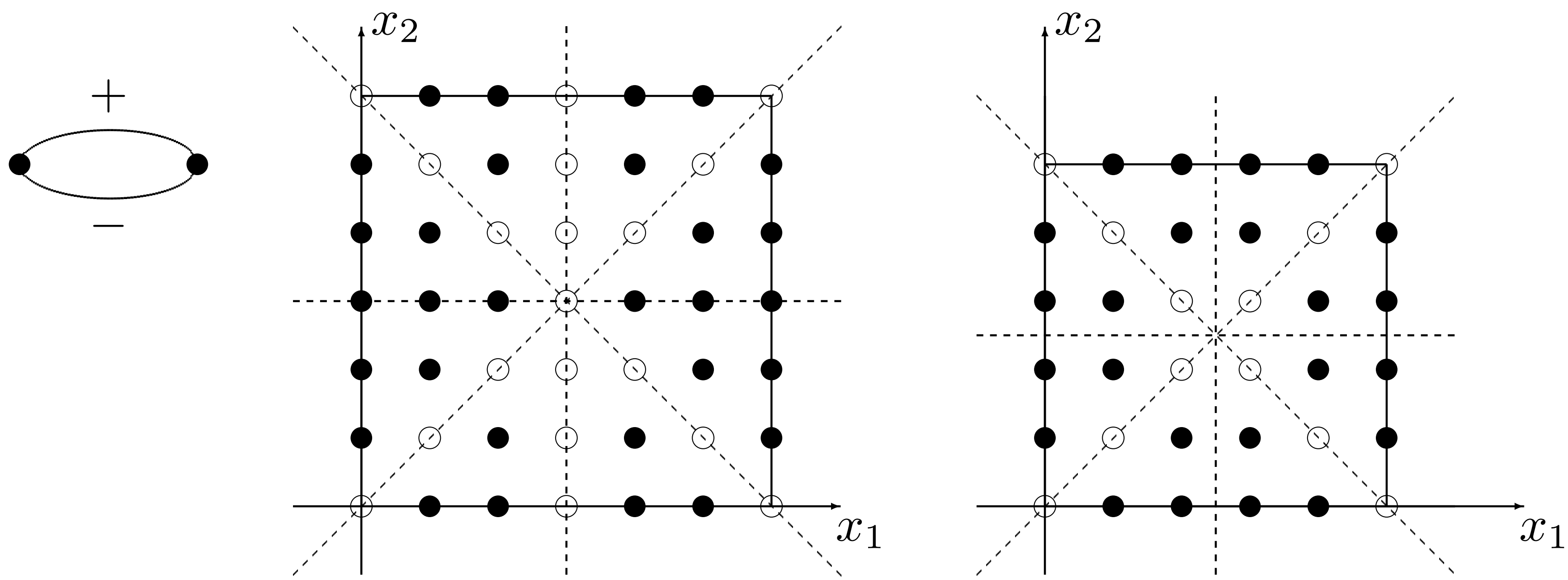
Note that $\chi_\Sigma(-1) = 4$ . . .



Observation (Greene–Zaslavsky 1970s)
Regions of $\mathcal{H}_\Sigma := \{x_j = \epsilon_{jk}\, x_k : \; jk \in E\}$ are in one-to-one correspondence with the acyclic orientations of Σ.

How To Color a Signed Graph, Geometrically

To also capture zero-free colorings, replace $[-1,1]^V$ by $[0,1]^V$, add the coordinate hyperplanes to $\mathcal{H}_\Sigma$, and shift this new arrangement by $(\frac{1}{2},\dots,\frac{1}{2})$

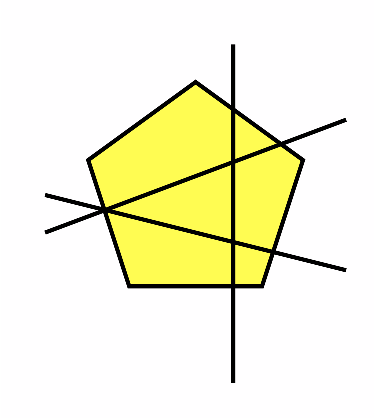


$\longrightarrow$ $\chi_\Sigma(2k+1)$ and $\chi^*_\Sigma(2k)$ are two halves of one inside-out quasipolynomial.

Inside-Out Polytopes

Underlying setup of our treatment of signed graph colorings:

$\mathcal{P}$ — (rational) polytope in $\mathbb{R}^d$



$\mathcal{H}$ — (rational) hyperplane arrangement

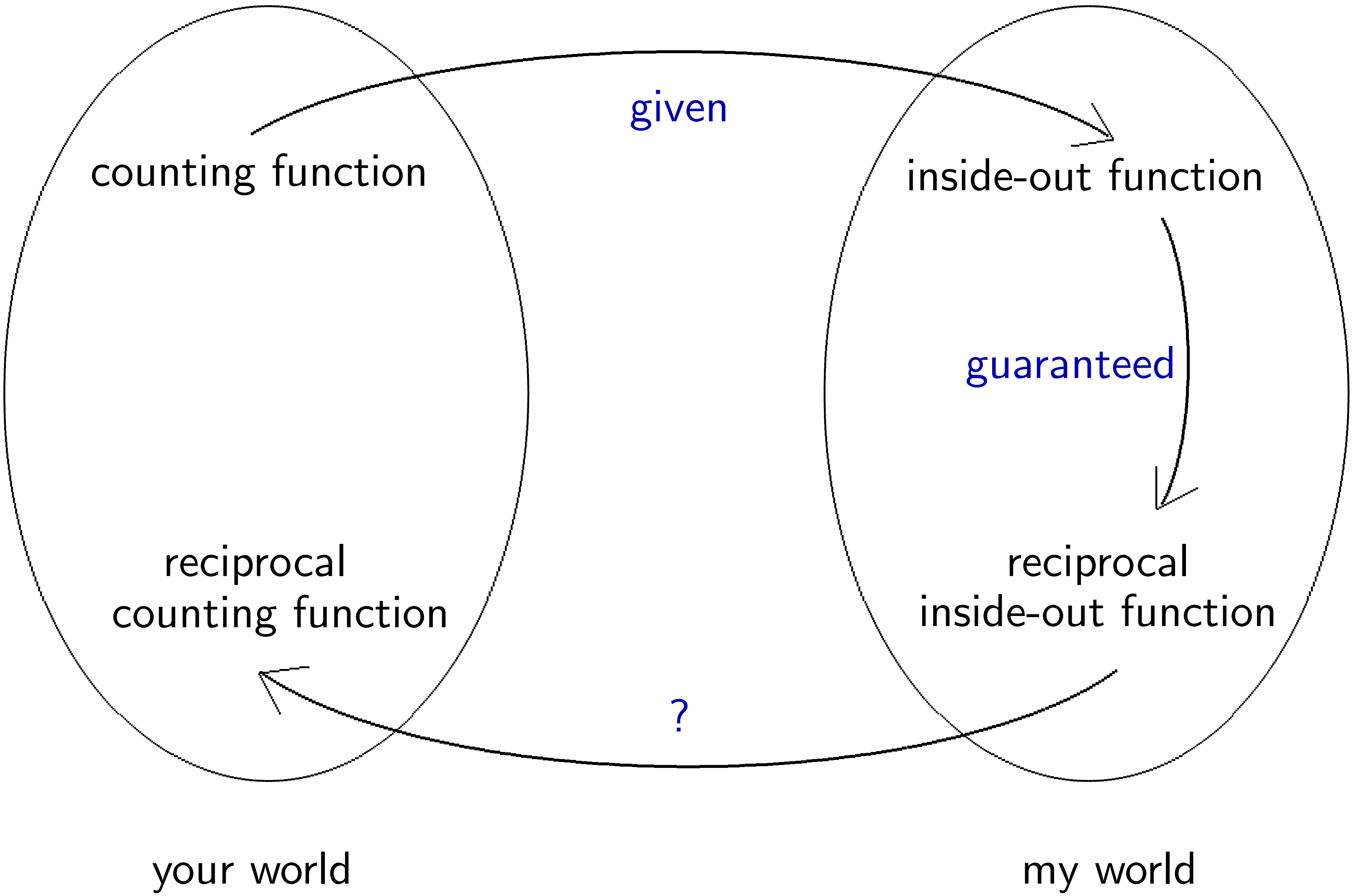
Say we're interested in the counting function

$$f(k) := \# \left(k \left(\mathcal{P} \setminus \mathcal{H} \right) \cap \mathbb{Z}^d \right) .$$

Ehrhart says that this is a (quasi-)polynomial, and Ehrhart–Macdonald reciprocity that $(-1)^d f(-k)$ counts lattice points with multiplicity #regions

Philosophy Euler characteristics . . . variation of the themes polyhedral dissections, triangulations, half-open decompositions

Inside-Out Theme I: Combinatorial Reciprocity Theorems



More Applications

- Nowhere-zero flow polynomials (MB–Zaslavsky 2006, Breuer–Dall 2011, Breuer–Sanyal 2011)
- Nowhere-harmonic & bivariable graph colorings (MB–Braun 2011, MB–Hardin 2015)
- Golomb rulers (MB–Bogart–Pham 2012)
- Colorings & flows on cell complexes (MB–Breuer–Godkin–Martin 2014)
- Transfer matrix method (Engstrom–Kohl 2018)

Inside-Out Theme II: Computing Quasipolynomials

The period p of the quasipolynomial

$$q(k) \ = \ c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$$

is the lcm of the periods of $c_0(k), \ldots, c_d(k)$

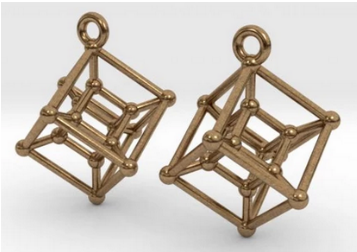
Computational Complexity Philosphy We need $(d+1)p$ pieces of data to understand $q(k)$

- Natural classification questions, unimodality, periods, . . .
- at most dp roots ($\longrightarrow$ answer existence questions)

Even More Applications

- Magic and Latin squares (MB–Zaslavsky 2006 & 2010, MB–Van Herick 2011)
- Antimagic graphs (MB–Zaslavsky 2006, MB–Farahmand 2017)
- Non-attacking q-queens (Chaiken–Hanusa–Zaslavsky 2014–present)

. . . with lots of open questions.



Exciting new research in the signed graph vicinity:

- Signed Birkhoff polytope & relatives (Kohl–Olson–Sanyal 2019+)
- Signed order polytopes (Hlavacek 2020+)

Happy 75.736986...th Birthday, Tom!