THE FLOW AND TENSION COMPLEXES

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Chapter 1

Introduction

The main objects in this thesis are polynomials. In particular, we are interested in two types of polynomials that come to us via graph theory−tension polynomials and flow polynomials. Our major goal is to realize that these polynomials have alternate lives in the world of commutative algebra. The motivation for this work were the papers [18] and [9] in which a similar realization was made for the chromatic polynomial of a graph. This paper relies heavily on notions from three related fields of mathematics: graph theory, polyhedral geometry, and combinatorial commutative algebra. To streamline the introduction we refer the reader to Sections 2.1, 3.1, and 4.1, respectively, for the relevant definitions. Throughout this paper we let $\mathbb{N}, \mathbb{Z}_{>0}, \mathbb{Z}$, and $\mathbb{R}$ refer to the sets of nonnegative integers, positive integers, integers, and real numbers, respectively. We also let $[d] := \{1, 2, \ldots, d\}$.

We begin with a finite, directed graph $G = (V, E)$. A *(proper) coloring* on G is a function $f : V \to \mathbb{Z}_{>0}$ that labels the vertices of G such that $f(v) \neq f(v')$ if $vv' \in E$. We call f a *k-coloring* if f is a coloring and $f(v) \leq k$ for every vertex v. A $\mathbb{Z}$*-tension* on G is a function $\tau : E \to \mathbb{Z}$ that labels the edges of G such that the sum around any cycle, taken with respect to some fixed orientation of the edges, is 0. A *k-tension* is a $\mathbb{Z}$-tension τ such that $|\tau(e)| < k$ for every edge e. A $\mathbb{Z}$*-flow* on G is a function $\phi : E \to \mathbb{Z}$, also an edge labelling, such that the sum at each vertex (again, taken with respect to some fixed orientation of the edges) is 0. A *k-flow* is a $\mathbb{Z}$-flow ϕ such that $|\phi(e)| < k$ for every edge e. We call a tension (or a flow) *nowhere-zero* (nwz) if no edge is sent to zero.

Three theorems from graph theory (see [20], [4], [12]) tell us that there are polynomials $\chi_G(k)$, $\tau_{G,nwz}(k)$, and $\phi_{G,nwz}(k)$ that count the number of k-colorings, nowhere-zero k-tensions, and nowhere-zero k-flows of G, respectively. They are called the *chromatic polynomial*, *tension polynomial*, and *flow polynomial*, respectively. A great deal of literature exists on (especially the first of) these polynomials but interesting question remain. For example, one might ask for good criteria when a given polynomial is the chromatic (or tension, or flow) polynomial of some graph G.

In [18], Steingrimsson introduced, for each simple graph G, a simplicial complex Δ_G whose nonfaces correspond to proper colorings of G. These non-faces generate the *Stanley-Reisner ideal*, I_Δ, of this complex. This ideal is a finitely-generated $\mathbb{Z}$-module over a graded polynomial ring. It follows that the *Hilbert function* of I_Δ, $H(I_\Delta, k)$, is a polynomial and this polynomial is $\chi_G(k + 1)$, i.e., the chromatic polynomial of G shifted by 1.

In [9], Hersh and Swartz gave a description of Δ_G using hyperplane arrangements. Via this approach one obtains certain bounds on the h-vector of Δ_G. These, in turn, yield bounds on the coefficients of $\chi_G(k)$. Thus, Δ_G provides a tool to help understand which polynomials are chromatic.

The approach in [9] is closely related to the notion of an *inside-out polytope*. An inside-out polytope is a pair $(\mathcal{P}, \mathcal{H})$ consisting of a polytope $\mathcal{P}$ and a hyperplane arrangement $\mathcal{H}$ such that each hyperplane $H \in \mathcal{H}$ intersects the interior of $\mathcal{P}$. In [2], a generalization of *Ehrhart theory* was developed using inside-out polytopes. As applications of this generalization Beck and Zaslavsky showed that $\chi_G(k)$ and $\phi_{G,nwz}(k)$ count the number lattice points in the k^{th} dilates of certain inside-out polytopes. For $\chi_G(k)$ the inside-out polytope is the unit cube of dimension $|V|$ together with a subarrangement of the braid arrangement. For $\phi_{G,nwz}(k)$, the inside-out polytope is $(\mathcal{F}, \mathcal{B})$ where $\mathcal{F}$ is obtained by intersecting the $|E|$-dimensional $\{-1, 1\}$-cube with the *cycle space* of G, while the arrangement $\mathcal{B}$ is the Boolean arrangement consisting of all coordinate hyperplanes.

The first goal of this thesis is to prove that $\tau_{G,nwz}(k)$ can be realized as the polynomial that counts the integral points in the inside-out polytope $(\mathcal{T}, \mathcal{B})$ where $\mathcal{T}$ is the polytope obtained by intersecting the $|E|$-dimensional $\{-1, 1\}$-cube with the *cut space* of G, while the arrangement $\mathcal{B}$ is again the Boolean arrangement (see Theorem 3.11). We note that this is essentially contained in [4].

To this end, we begin in Chapter 2 by reviewing the requisite notions from graph theory and then proceed to study the linear algebra involved with tensions. Then in Chapter 3 we prove the above statement after recalling the terminology and theorems from the world of polytopes and, in particular, Ehrhart theory and its generalizations. Finally, in Chapter 4 we prove a general statement that associates to certain polytopal complexes with vertices in the $\{-1, 1\}$-cube a simplicial complex Δ with the property that the Hilbert function of the Stanley-Reisner ideal I_Δ is precisely the inside-out polynomial (see Theorem 4.1). We then apply this construction to our motivating examples: the flow and tension polynomials of a graph G.

Chapter 2

The Basics

2.1 Preliminaries

We begin with some definitions from graph theory. For explanations of unexplained terminology see [5].

A (finite) **graph** is a pair $G = (V, E)$ where V is a (finite) set and E is a (finite) multiset such that every element of E is contained in $V \times V$. (All of the graphs we consider will be finite.) We will often write uv for an edge $e = (u, v)$. The set V is called the **vertex set** of G and the multiset E is the **edge set**. We denote by $\mathbb{R}^E$ the $|E|$-dimensional real vector space which we call the **edge space** of G. We identify the standard unit vector $\mathbf{e}_i$ with $e_i \in E$. We define the **vertex space** of G, $\mathbb{R}^V$, similarly and we identify the standard unit vector $\mathbf{v}_i$ $\in \mathbb{R}^V$ with the vertex $v_i \in V$.

A **loop** in G is an edge of the form vv (see edge e_6 in Figure 2.1). A **multiple edge** in G is an element $uv \in E$ with multiplicity greater than 1 (i.e. the element uv appears in E more than once). For example, see edges e_4, e_5 in Figure 2.1.

Definition 2.1. Suppose G is loopless and let A be an abelian group. A **coloring** of the vertices of G is a function $f : V \to A$. We say that f is a **proper coloring** if $f(v) \neq f(u)$ whenever $uv \in E$.

Notice that real-valued colorings f (i.e., take $A = \mathbb{R}$ in the above definition) are in bijection with vectors $\mathbf{x} = (x_1, \ldots, x_{|V|}) \in \mathbb{R}^V$ in the vertex space of G where $x_i = f(v_i)$ for each vertex $v_i \in V$. In this context a proper coloring is a vector $\mathbf{x} = (x_1, \ldots, x_{|V|})$ in $\mathbb{R}^V$ such that $x_i \neq x_j$ whenever $v_i v_j$ form an edge in G.

A **path** in G is a sequence $v_0 e_1 v_1 e_2 \ldots v_{n-1} e_n v_n$ of vertices and edges in G such that $e_i = v_{i-1} v_i \in E$ for $i = 1, 2, \ldots, n$ and no vertex nor edge is repeated. G is **connected** if there is a path between any two vertices; otherwise G is said to be disconnected. An edge e of a connected graph G is called a **bridge** if removing e from the edge set disconnects G.

A **cycle** in G is a sequence $v_0 e_1 v_1 e_2 \ldots v_{n-1} e_n v_n = v_0$ with $e_i = v_{i-1} v_i$ such that no edge nor vertex (except for v_0) is repeated.

An **orientation** on a graph G is a pair of maps, $h : E \to V$ and $t : E \to V$, such that for every edge $e = uv$ either $h(e) = u$ and $t(e) = v$ or $h(e) = v$ and $t(e) = u$. We call a graph with an orientation an **oriented graph** or a **directed graph** (a **digraph** for short). We

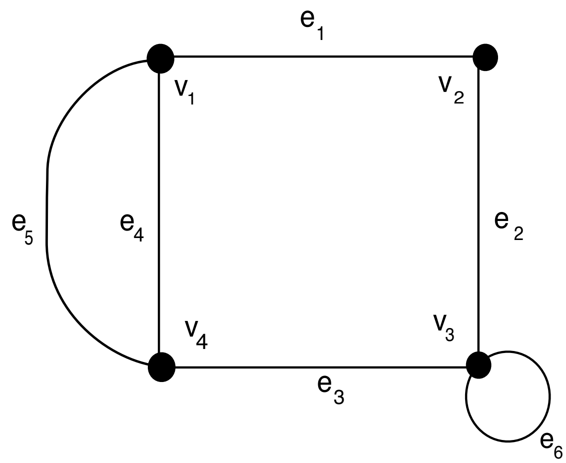

Figure 2.1: A graph with six edges on four vertices.

note that this varies slightly from the definitions of oriented and directed graphs in [5]. If G is a digraph and $e = uv \in E$ is directed from u to v, we write $e = \overrightarrow{uv}$. Intuitively, we think of an orientation as assigning a direction from $t(e)$ (the **tail** of e) to $h(e)$ (the **head** of e).

Definition 2.2. A **(real-valued) flow** ϕ on an oriented graph G is a map $\phi : E \to \mathbb{R}$ such that

$$\sum_{h(e)=v} \phi(e) = \sum_{t(e)=v} \phi(e)$$

for every $v \in V$. A **nowhere-zero (nwz) flow** ϕ is a flow with the added condition that $\phi(e) \neq 0$ for all $e \in E$.

Notice that if ϕ is a flow on $G = (V, E)$ and there is a bridge $e \in E$, then $\phi(e) = 0$. So when considering nowhere-zero flows on any graph G we will tacitly assume that G is bridgeless. We also note that in this paper we only consider real and integer valued flows. In particular we will not discuss modular flows which are flows taking values in a finite group. Modular flows are interesting in their own right as they provide a notion of duality to graph colorings. They are studied, for example, in [5, Chapter 6].

A vertex v in a digraph G is called a **source** (respectively, a **sink**) if v is the tail (respectively, the head) of all edges incident to v.

Let $C = v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_n = v_0$ be a cycle in an oriented graph G. We call C a **circuit** if $t(e_i) = h(e_{i+1})$ for $i = 1, 2, \dots, n$ or if $h(e_i) = t(e_{i+1})$ for $i = 1, 2, \dots, n$. So, intuitively, a circuit is an oriented cycle whose orientation 'goes around' the cycle. Two edges of a cycle $e_1 = uv, e_2 = xy$ are called **coherently oriented** if the cycle C' obtained by contracting the edges between v and x and those between y and u is a circuit. We may write C as a disjoint union $C = C_+ \coprod C_-$ where $e_i \in C_+$ if e_1 and e_i are coherently oriented and $e_i \in C_-$ otherwise. We note that if C is a circuit then $C_- = \emptyset$.

Definition 2.3. A **(real-valued) tension** τ on a graph G is a map $\tau : E \to \mathbb{R}$ such that for every cycle C of G

$$\sum_{e \in C_+} \tau(e) = \sum_{e \in C_-} \tau(e).$$

A **nowhere-zero (nwz) tension** τ is a tension with added condition that $\tau(e) \neq 0$ for all $e \in E$.

It is clear that if τ is a tension on G and if $e \in E$ is a loop, then $\tau(e) = 0$. So when considering nowhere-zero tensions on a graph we will tacitly assume that G is loopless. We also note that every flow ϕ may be thought of as a vector $\mathbf{x} = (x_1, \ldots, x_{|E|}) \in \mathbb{R}^E$ where $x_i = \phi(e_i)$ and that a nwz-flow has $x_i \neq 0$ for all i. The situation is similar for tensions and nowhere-zero tensions.

We briefly note a connection between this corner of graph theory and the mysterious (at least to the author) world of electrical circuits. The flow conditions in definition 2.2 are also known in graph theory as the **cycle space** equations. Electrical engineers call them Kirchhoff's Current Law. The tension conditions in definition 2.3 are known as the **cut space** equations to graph theorists and as Kirchhoff's Voltage Law to electrical engineers. The former may be restated as "at any point, the amount of current flowing in to the point must equal the amount of current flowing out". We may think of the latter as "the directed sum of the electrical potential differences around a cycle must be zero". In the next section we justify the above terminology by showing that the cut space and cycle space of a graph are indeed subspaces of the edge space.

2.2 Flows and Tensions

Let $G = (V, E, \epsilon)$ be a connected digraph with fixed orientation ϵ. In this section we study flows and tensions on directed graphs and the subspaces they inhabit in the edge space, $\mathbb{R}^E$.

Let $T = T(G, \epsilon) \subset \mathbb{R}^E$ be the set of all tensions on a digraph G with orientation ϵ and let $F = F(G, \epsilon)$ be the set of all flows on G. Our first goal is to prove the following:

Theorem 2.1. *The sets F and T are subspaces of $\mathbb{R}^E$ and $F = T^{\perp}$.*

One way to view the connection between flows and tensions is through a certain linear transformation on the edge space:

Definition 2.4. The **boundary operator** $\partial : \mathbb{R}^E \to \mathbb{R}^V$ is given by $\mathbf{e}_i \mapsto \mathbf{v}_k - \mathbf{v}_j$ where $e_i = \overrightarrow{v_j v_k}$.

If we denote by M the $(|V| \times |E|)$-matrix corresponding to ∂ with respect to the standard bases of $\mathbb{R}^E$ and $\mathbb{R}^V$, then M is the (vertex-edge) **incidence matrix** of the digraph G with entries

$$M_{ij} = \begin{cases} 1 & \text{if } v_i = h(e_j), \\ -1 & \text{if } v_i = t(e_j), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.2. *The kernel of the boundary map is the set of all flows. That is,*

$$\ker \partial = F.$$

Proof. Let G be a graph with orientation ϵ with $|E| = n$. Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^E$ and fix a vertex v_i. Then $\mathbf{x} \in \ker \partial$ if and only if

$$\begin{aligned}(\partial \mathbf{x})_i &= (M\mathbf{x})_i \\ &= \sum_{v_i = h(e_j)} x_j - \sum_{v_i = t(e_j)} x_j \\ &= 0\end{aligned}$$

for each i. Thus $\mathbf{x}$ is a flow if and only if $\mathbf{x} \in \ker \partial$. □

The dimension of F is $|E|-|V|+\kappa(G)$ where $\kappa(G)$ is the number of connected components of G. In particular, if G is connected we have $\dim F = |E| - |V| + 1$. One can prove this for a connected graph by considering the fundamental cycles of an arbitrary spanning tree of G and showing that they span the cycle space (see, e.g., Theorem 1.9.5 in [5]).

It is a fundamental result from linear algebra that if $S : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation represented by the matrix M (with respect to some fixed bases for $\mathbb{R}^m$ and $\mathbb{R}^n$), then $\ker(M) = \operatorname{im}(M^t)^\perp$ where M^t is the transpose of M. So Theorem 2.1 will follow from Lemma 2.2 and the following lemma:

Lemma 2.3. *The image of the transpose of the boundary map is exactly the set of all tensions:*

$$T = \mathrm{im}\partial^t.$$

Proof. For a connected graph G we have $\dim(\mathrm{im}\partial^t) = |V|-1$ since $\dim(\ker \partial) = |E|-|V|-1$. For each vertex $v \in V$ define the **basic tension**, τ_v, associated with v to be

$$\tau_v(e) = \begin{cases} 1 & \text{if } v = h(e), \\ -1 & \text{if } v = t(e), \\ 0 & \text{otherwise.} \end{cases}$$

(Notice that τ_v corresponds to the row of the incidence matrix M associated with the vertex v.) Now for a fixed vertex w, the set $\{\tau_v : v \in V \setminus w\}$ forms a basis for $\mathrm{im}\partial^t$. Thus $\mathrm{im}\partial^t$ is a subspace of T and a dimension count gives the opposite inclusion. □

To completely understand the linear algebra of our situation we would like to switch the roles of flows and tensions in the above discussion. More precisely, we would like to find a new function $\gamma : \mathbb{R}^E \to \mathbb{R}^n$ (for some natural number n) such that $\ker \gamma = T$ and $\mathrm{im}\gamma^t = F$. We can produce this function (in matrix form) directly from the graph G as follows.

Suppose there are c cycles in G. For every cycle C in G, choose an edge $e \in C$ arbitrarily. Let v_C be the vector in $\mathbb{R}^E$ such that

$$(v_C)_i = \begin{cases} 1 & \text{if } e_i \in C \text{ and if } e \text{ and } e_i \text{ are coherently oriented,} \\ -1 & \text{if } e_i \in C \text{ and if } e \text{ and } e_i \text{ are not coherently oriented,} \\ 0 & \text{if } e_i \notin C. \end{cases}$$

Let N be the $(c \times e)$-matrix obtained by using the v_C's as rows.

Proposition 2.4. *The kernel of N is the tension space T of G.*

Proof. Let G be a graph with orientation ϵ with $|E| = n$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^E$ and fix a cycle C and an edge $e \in C$. Then we may write C as the disjoint union of C_+ and C_- as in section 1, where e_i is in C_+ if it is coherently oriented with the distinguished edge e. Then $\mathbf{x} \in \ker N$ if and only if

$$\begin{aligned}(N\mathbf{x})_i &= v_C \cdot \mathbf{x} \\ &= \sum_{e \in C_+} \tau(e) - \sum_{e \in C_-} \tau(e) \\ &= 0\end{aligned}$$

for each i. Thus $\mathbf{x}$ is a tension if and only if $\mathbf{x} \in \ker N$. $\square$

Chapter 3

Enumeration of Nowhere-Zero k-Tensions

Let $G = (V, E)$ be a graph with a fixed orientation ϵ. Let

$$[-1,1]^E := \{\mathbf{x} \in \mathbb{R}^E \mid x_i \in [-1,1] \text{ for each } i\}.$$

An **integral tension** on G is a tension $\tau : E \to \mathbb{Z}$. We denote the set of integral tensions by $T_{\mathbb{Z}} = T_{\mathbb{Z}}(G, \epsilon)$. It is clear from Theorem 2.1 that

$$T_{\mathbb{Z}} = \mathrm{im}\partial^t \cap \mathbb{Z}^E,$$

where $\mathbb{Z}^E$ is the integer lattice in $\mathbb{R}^E$.

A **k-tension** ($k \in \mathbb{N}$) is an integral tension τ such that $|\tau(e)| < k$ for every e in E. We denote the set of all k-tensions by $T_k = T_k(G, \epsilon)$ and note that

$$T_k = T_{\mathbb{Z}} \cap (k-1)[-1,1]^E,$$

where $(k-1)[-1,1]^E$ is the $(k-1)^{st}$ dilate of the cube $[-1,1]^E$.

Let us denote by $T_{nwz,k}$ the set of all nowhere-zero k-tensions. Then

$$T_{nwz,k} = T_k \setminus \bigcup \mathcal{B},$$

where $\mathcal{B}$ is the boolean hyperplane arrangement

$$\mathcal{B} = \{B_1, B_2, \ldots, B_{|E|}\},$$

where $B_i = \{\mathbf{x} \in \mathbb{R}^E \ : \ x_i = 0\}$.

Define the counting functions $\tau(k) := |T_k|$ and $\tau_{nwz}(k) := |T_{nwz,k}|$ for $k \in \mathbb{N}$. We will use Ehrhart theory (resp. inside-out polytope theory) to show that $\tau(k)$ (resp. $\tau_{nwz}(k)$) is a polynomial depending on k and that evaluating these polynomials at negative integers yields data about the original graph.

In sections 3.1 and 3.2, we give the necessary background in Ehrhart theory (following [1]) and inside-out polytopes. Then in section 3.3 we apply the theory to enumerate nowhere-zero k-tensions. A similar approach for enumerating tensions was taken in [4]. The theory of inside-out polytopes was used in [3] to enumerate flows.

3.1 Ehrhart Theory

In this section we recall various definitions from the wonderful world of polytopes and state an equally wonderful theorem of Ehrhart. For details and generalizations see [22] for polytopes and [1] for Ehrhart Theory.

A **(convex) polytope** is the convex hull of finitely many points in $\mathbb{R}^d$ for some $d \geq 0$. That is, given a finite point set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^d$, the polytope $\mathcal{P}$ is given by

$$\mathcal{P} = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \;\mid\; \lambda_i \geq 0 \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

We write $\mathcal{P} = \mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n\}$ and call this the **vertex description** of $\mathcal{P}$. Equivalently, a polytope is the bounded intersection of finitely many half spaces. More precisely, if $\mathcal{P} = \mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n\}$ then there exists a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ such that

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}. \tag{3.1}$$

Each inequality $\sum_{j=1}^{d} a_{ij} x_j \leq b_i$ in $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ describes a halfspace in $\mathbb{R}^d$. When $\sum_{j=1}^{d} a_{ij} x_j = b_i$ we obtain a hyperplane in $\mathbb{R}^d$. So we call (3.1) the **hyperplane description** of $\mathcal{P}$. It is a theorem (see, for example, [BR], [Z]) that from the vertex description of a polytope one may obtain the hyperplane description, and *vice versa*.

Let $\mathcal{P}$ be a polytope. The **dimension** of $\mathcal{P}$ in $\mathbb{R}^d$ is the dimension of its affine span. If $\mathcal{P}$ is d-dimensional, then we write $\dim\mathcal{P} = d$ and call $\mathcal{P}$ a d-polytope. A linear inequality of the form $\mathbf{c}\mathbf{x} \leq c_0$ is called **valid** for $\mathcal{P}$ if it is satisfied for all $\mathbf{x} \in \mathcal{P}$. A **face** F of $\mathcal{P}$ is any set of the form

$$\mathcal{P} \cap \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{c}\mathbf{x} \leq c_0\}$$

where $\mathbf{c}\mathbf{x} \leq c_0$ is valid. One checks that the empty set is a face of every polytope and that every polytope is a face of itself.

An **integral lattice point** in $\mathbb{R}^d$ is a vector with integer coordinates. We call $\mathcal{P}$ **integral** (resp. **rational**) if it is the convex hull of vertices in $\mathbb{Z}^d$ (resp. $\mathbb{Q}^d$).

A **polytopal complex** $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{R}^d$ such that

1. the empty polytope is in $\mathcal{C}$,
2. if $\mathcal{P} \in \mathcal{C}$, then every face of $\mathcal{P}$ is also in $\mathcal{C}$,
3. the intersection $\mathcal{P} \cap \mathcal{Q}$ of two polytopes in $\mathcal{P}, \mathcal{Q} \in \mathcal{C}$ is a face of both $\mathcal{P}$ and $\mathcal{Q}$.

We call a polytope $\mathcal{P} \in \mathcal{C}$ a **region** of $\mathcal{C}$ if $\mathcal{P} \not\subset \mathcal{Q}$ for any $\mathcal{Q} \in \mathcal{C}'$. Polytopal complexes are the main players in the following section and in chapter 4.

A **simplex** is a d-dimensional polytope with $d+1$ vertices. We note that every face of a simplex is a simplex. A **simplicial complex** is a polytopal complex whose regions are all simplices.

For each positive integer k we define the k^{th} **dilate** of a polytope $\mathcal{P}$ to be

$$k\mathcal{P} = \{k\mathbf{x} \mid \mathbf{x} \in \mathcal{P}\}.$$

Ehrhart Theory is concerned with counting lattice points in dilates of polytopes. Given a d-polytope $\mathcal{P}$ we define two counting functions:

$$\mathcal{L}_{\mathcal{P}}(k) = \#\{\mathbf{x} \in k\mathcal{P} \cap \mathbb{Z}^d\}$$

and

$$\mathcal{L}_{\mathcal{P}^\circ}(k) = \#\{\mathbf{x} \in k\mathcal{P}^\circ \cap \mathbb{Z}^d\},$$

where $k \in \mathbb{Z}_{>0}$. Then we have the following theorems:

Theorem 3.1. *([6]) If $\mathcal{P}$ is a convex integral d-polytope, then $\mathcal{L}_{\mathcal{P}}(k)$ is a polynomial in k of degree d.*

Theorem 3.2. *([7, 14]) If $\mathcal{P}$ is a convex integral d-polytope, then $\mathcal{L}_{\mathcal{P}}(-k) = (-1)^d \mathcal{L}_{\mathcal{P}^\circ}(k)$.*

Both theorems have an analogue for rational polytopes but we will only need to consider the integral case. The second theorem is known as Ehrhart–Macdonald reciprocity.

The **Ehrhart series** of a polytope $\mathcal{P}$, denoted $\mathrm{Ehr}_P(z)$, is the generating function of $\mathcal{L}_{\mathcal{P}}(k)$:

$$\mathrm{Ehr}_P(z) = 1 + \sum_{k\geq 1} \mathcal{L}_{\mathcal{P}}(k) z^k.$$

Since $\mathcal{L}_p(k)$ is a polynomial, we can write $\mathrm{Ehr}_P(z)$ as a rational function

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_0 + h_1 z + \cdots + h_d z^d}{(1-z)^{d+1}}.$$

We call the vector $(h_0, h_1, \ldots, h_d)$ consisting of coefficients of the numerator of the Ehrhart series the **$h^\star$-vector** of P.

Theorem 3.3. *([17]) Suppose $\mathcal{P}$ is an integral d-polytope with Ehrhart series*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_0 + h_1 z + \cdots + h_d z^d}{(1-z)^{d+1}}.$$

Then $h_i \geq 0$ for all $i = 1, 2 \ldots, d$.

An integral polytope $\mathcal{P}$ is said to be **reflexive** if

$$\mathcal{P}^\circ \cap \mathbb{Z}^d = \{\mathbf{0}\} \text{ and for all } k \in \mathbb{Z}_{>0},\ (k+1)\mathcal{P}^\circ \cap \mathbb{Z}^d = k\mathcal{P} \cap \mathbb{Z}^d,$$

where

$$\mathcal{P}^\circ := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} < \mathbf{b}\} \tag{3.2}$$

is the **interior** of $\mathcal{P}$. The set $\partial\mathcal{P} := \mathcal{P} \setminus \mathcal{P}^\circ$ is called the **boundary** of $\mathcal{P}$. We call $\mathcal{P}$ **centrally symmetric** if $-\mathbf{x} \in \mathcal{P}$ for every $\mathbf{x} \in \mathcal{P}$.

Theorem 3.4. *([10]) Suppose $\mathcal{P}$ is an integral d-polytope that contains the origin in its interior and that has Ehrhart series*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_0 + h_1 z + \cdots + h_d z^d}{(1-z)^{d+1}}.$$

Then $\mathcal{P}$ is reflexive if and only if $h^\star$ is palindromic, i.e., $h_k = h_{d-k}$ for $0 \leq k \leq d/2$.

We may now rephrase our introductory comments preceding this section in the language of polytopes. For a finite graph $G = (V, E)$ with orientation ϵ, the set of k-tensions T_k are integer points in the polytope $\mathcal{P} = \text{im}\partial^t \cap [-1, 1]^E$. In section 3.3 we will show that this polytope is integral and thus (re)prove that the counting function $\tau(k)$ (= the number of k-tensions on G) is a polynomial depending only on k. We will also see that this polytope is reflexive as well as centrally symmetric.

But in order to show that $T_{nwz,k}$ is a polynomial depending only on k we will need a more sophisticated version of Ehrhart theory which we introduce presently.

3.2 Inside-Out Polytopes

In what follows we outline the concepts and results from [2] which the interested reader may mine for details.

An arrangement of hyperplanes $\mathcal{H}$ is said to be **integral** if it is given by equations with integral coefficients. Given a polytope $\mathcal{P}$, an arrangement $\mathcal{H}$ is **transverse** to $\mathcal{P}$ if any hyperplane that intersects $\mathcal{P}$ also intersects the interior of $\mathcal{P}$.

Let $\mathcal{P}$ be a closed, convex polytope and let $\mathcal{H}$ be an arrangement of hyperplanes that is transverse to $\mathcal{P}$. The pair $(\mathcal{P}, \mathcal{H})$ is called an **inside-out polytope**. A **region** of $(\mathcal{P}, \mathcal{H})$ is one of the components of $\mathcal{P} \setminus \bigcup \mathcal{H}$ or the closure of such a component. Furthermore, the regions of $(\mathcal{P}, \mathcal{H})$ (i.e., the maximal dimensional polytopes in this complex) all have the same dimension, the dimension of $\mathcal{P}$. Thus every inside-out polytope $(\mathcal{P}, \mathcal{H})$ is a polytopal complex whose faces consist of faces of the regions of $(\mathcal{P}, \mathcal{H})$. The set of **vertices** of $(\mathcal{P}, \mathcal{H})$ is the union of the sets of vertices of the regions. We call $(\mathcal{P}, \mathcal{H})$ an **integral inside-out polytope** if all of its vertices are integral.

For simplicity, we restrict our focus to integral polytopes and hyperplane arrangements though the theory goes through (with appropriate modifications) for rational polytopes and arrangements.

The **multiplicity** of $\mathbf{x} \in \mathbb{R}^d$ with respect to $\mathcal{H}$ is

$$m_{\mathcal{H}}(\mathbf{x}) := \text{the number of closed regions of } \mathcal{H} \text{ that contain } \mathbf{x}.$$

The **multiplicity** of $\mathbf{x}$ with respect to $(\mathcal{P}, \mathcal{H})$ is

$$m_{\mathcal{P},\mathcal{H}}(\mathbf{x}) := \begin{cases} \text{the number of closed regions of } \mathcal{H} \text{ that contain } \mathbf{x} & \text{if } \mathbf{x} \in \mathcal{P}, \\ 0 & \text{if } \mathbf{x} \notin \mathcal{P}. \end{cases}$$

For $\mathbf{x} \in \mathcal{P}$ this definition agrees with the previous definition if we assume transversality.

Let the **closed** and **open Ehrhart polynomials** of $(\mathcal{P}, \mathcal{H})$ be defined by

$$E_{\mathcal{P},\mathcal{H}}(k) := \sum_{\mathbf{x} \in k^{-1}\mathbb{Z}^d} m_{\mathcal{P},\mathcal{H}}(\mathbf{x})$$

and

$$E^{\circ}_{\mathcal{P}^\circ,\mathcal{H}}(k) := \#(k^{-1}\mathbb{Z}^d \cap [\mathcal{P} \setminus \bigcup \mathcal{H}])$$

for $k \in \mathbb{Z}_{>0}$. Then we have the following generalization of Ehrhart's theorem for integral polytopes:

Theorem 3.5. *([2]) If $(\mathcal{P},\mathcal{H})$ is a closed, full dimensional integral inside-out polytope in $\mathbb{R}^d$ such that $\mathcal{H}$ does not contain the degenerate hyperplane, then $E_{\mathcal{P},\mathcal{H}}(k)$ and $E^{\circ}_{\mathcal{P}^\circ,\mathcal{H}}(k)$ are both polynomials in k with leading term $c_d t^d$ where $c_d = \mathrm{Vol}(P)$ and constant term $E_{\mathcal{P},\mathcal{H}}(0)$ equal to the numer of region of $(\mathcal{P},\mathcal{H})$. Furthermore,*

$$E^{\circ}_{\mathcal{P}^\circ,\mathcal{H}}(k) = (-1)^d E_{\mathcal{P},\mathcal{H}}(-k).$$

We may now apply the theorems in the previous two sections to obtain our desired results regarding tensions.

3.3 Enumeration of k- and nowhere-zero Tensions

Let G be a digraph with orientation ϵ. Let us first prove that the counting function $\tau(k) := |T_k|$ is a polynomial depending only on k. To see this recall that

$$T_k = T_{\mathbb{Z}} \cap (k-1)[-1,1]^E$$

where $T_{\mathbb{Z}} = T \cap \mathbb{Z}^E$. Our first goal is to show that $T \cap [-1,1]^E$ is an integral polytope. To this end recall that a matrix M is **totally unimodular** if every minor of M is either $0, 1$, or -1. Totally unimodular matrices and integral polytopes are intimately related (see [16, Thm.19.3] for the following characterization and many more):

Theorem 3.6. *Let A be a matrix with entries $0, +1, -1$. Then the following are equivalent:*

1. *A is totally unimodular;*
2. *for each integral vector $\mathbf{b}$ the polyhedron $\{\mathbf{x} | A\mathbf{x} \leq \mathbf{b}\}$ has only integral vertices.*

The fact that the incidence matrix M of a directed graph G is totally unimodular follows from the following proposition:

Proposition 3.7. *[16, page 274] A $\{-1,0,1\}$-matrix M that has in each column exactly one $+1$ and one -1 is totally unimodular.*

Proof. We proceed by induction. The case for 1×1 submatrices is trivial. Let N be any square submatrix of M. If N contains a column with at most one nonzero entry, then expansion of the determinant of N by this column has determinant 0 or ± 1 by induction. Otherwise, each column of N contains both a $+1$ and a -1. In this case the determinant of N is 0. □

Thus the incidence matrix M of an oriented graph G is totally unimodular because it has exactly one 1 and one -1 in each column. It follows that M^t is totally unimodular since any minor of M^t is a minor of M. Let I be the $(|V| \times |V|)$ identity matrix. Let A be the matrix

$$A := \begin{pmatrix} I \\ -I \end{pmatrix}.$$

Then A is totally unimodular ([16, page 274],). Finally, the tension polytope $\mathcal{T}$ is given by

$$\mathcal{T} = \left\{ A\mathbf{x} \leq \mathbf{1} \mid \mathbf{x} \in \mathrm{im} M^t \right\}$$

where $\mathbf{1} := (1, 1, \ldots, 1)$. Since adding a row or column with one nonzero entry that is either ± 1 preserve total unimodularity ([16, page 280]), the matrix

$$\begin{pmatrix} M^t \\ I \\ -I \end{pmatrix}$$

is totally unimodular. Thus $\mathcal{T}$ is an integral polytope.

Proposition 3.8. *Let $\mathcal{P}$ be a reflexive d-polytope and let S be a linear subspace of $\mathbb{R}^d$. If $\mathcal{Q} := \mathcal{P} \cap S$ is an integral polytope, then $\mathcal{Q}$ is reflexive.*

Proof. Since S is a subspace, we have $\mathcal{Q}^\circ \cap \mathbb{Z}^d = \mathcal{P}^\circ \cap S \cap \mathbb{Z}^d = \{\mathbf{0}\}$. If $\mathbf{x} \in (t+1)\mathcal{Q}^\circ \cap \mathbb{Z}^d \setminus t\mathcal{Q} \cap \mathbb{Z}^d$ for some $t \in \mathbb{Z}_{>0}$, then $\mathbf{x}$ is also in $t\mathcal{P} \setminus (t+1)\mathcal{P}$. This contradicts the fact that $\mathcal{P}$ is reflexive. Thus $\mathcal{Q}$ is reflexive. $\square$

Applying Theorems 3.1, 3.2, 3.3, and 3.4, as well as Proposition 3.8, we have:

Theorem 3.9. *The counting function $\tau(k) := |T_k|$ for $k \in \mathbb{Z}_{>0}$ is a polynomial in k of degree $|V| - 1$ satisfying*

$$\tau(k) = \mathcal{L}_{\mathcal{T}^\circ}(k) = (-1)^{|V|-1}\mathcal{L}_{\mathcal{T}}(-k).$$

Furthermore, the $h^\star$-vector of $\mathcal{T}$ has nonnegative entries and is palindromic.

Example 3.1. A **tree** $T = T(V, E)$ is a connected graph with no cycles. Let T be any tree with d edges. Given any orientation on T we note that every function $\tau : E \to \mathbb{Z}$ is a tension as the condition in 2.3 is vacuous. Thus the tension polytope $\mathcal{T}$ is $[-1, 1]^d$ (see Figure 3.1 for the case $d = 3$). So the tension polynomial of T is $\tau(k) = (2k + 1)^d$.

Example 3.2. Denote the graph consisting of one cycle with three edges by C_3. Suppose C_3 is oriented cyclically. Then the tension polytope is

$$\mathcal{T}_{C_3} = \text{conv}\{(1, 0, -1), (1, -1, 0), (0, 1, -1), (0, -1, 1), (-1, 1, 0), (-1, 0, 1)\}$$

(see Figure 3.2). This polytope is obtained by taking the convex hull of all permutations of coordinates of the vector $(1, 0, -1)$. A polytope obtained in this way is called a **permutahedron**.

We are now in a position to prove that $\tau_{nwz}(k) := |T_{nwz,k}|$ for $k \in \mathbb{Z}_{>0}$ is a polynomial. We have seen that $\mathcal{T} := T \cap [-1, 1]^E$ is an integral polytope. For a tension to be nowhere zero, it must not lie on the **Boolean hyperplane arrangement** $\mathcal{B}$ consisting of all coordinate hyperplanes $\mathcal{H}_i := \{\mathbf{x} \in \mathbb{R}^E \mid x_i = 0\}$ for $i = 1, 2, \ldots, |E|$. This is clearly an integral arrangement. We now have

Theorem 3.10. *The inside-out polytope $(\mathcal{T}, \mathcal{B})$ is a closed, integral, inside-out polytope of dimension $|V| - 1$.*

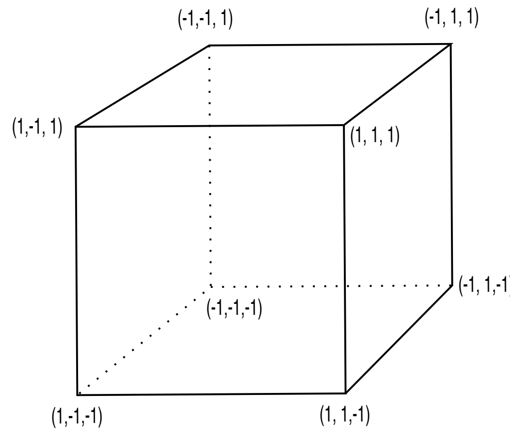

Figure 3.1: The tension polytope of any tree on three vertices.

Proof. The inside-out polytope $(\mathcal{T}, \mathcal{B})$ is closed since the tension polytope $\mathcal{T}$ is a closed convex body. Its dimension is the dimension of the tension space. Viewing$(\mathcal{T}, \mathcal{B})$ as a polytopal complex then the vertices of $(\mathcal{T}, \mathcal{B})$ consist of the vertices of each polytope in this complex. The regions of $(\mathcal{T}, \mathcal{B})$ are of the form $\mathcal{P}_{\mathcal{O}} := \mathcal{T} \cap \mathcal{O}$ where $\mathcal{O}$ is the topological closure of an orthant in $\mathbb{R}^E$ corresponding to some acyclic orientation of G. Let $\mathbf{e}_i$ denote the ith standard unit vector in $\mathbb{R}^E$. Let $B_{\mathcal{O}}$ be the $(|E| \times |E|)$-matrix with rows $\mathbf{e}_i$ or $-\mathbf{e}_i$ for each $i = 1, 2, \dots, |E|$ where we take $\mathbf{e}_i$ if $x_i \geq 0$ for all $x \in \mathcal{O}$ and $-\mathbf{e}_i$ otherwise. The inequalities involved in the hyperplane description of such a polytope are

1. the equations $M^t\mathbf{x} = \mathbf{0}$ defining the tension space,
2. the hyperplane description $A\mathbf{x} \leq \mathbf{1}$ defining the cube $[-1, 1]^E$, and
3. the hyperplane description of the orthant $\mathcal{O}$ given by $B_{\mathcal{O}}\mathbf{x} \leq \mathbf{0}$.

Then the matrix equation describing $\mathcal{P}_{\mathcal{O}}$ is

$$\begin{pmatrix} M \\ B_{\mathcal{O}} \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}.$$

Since adding a row consisting of exactly one nonzero entry that is either 1 or -1 to a totally unimodular matrix yields a totally unimodular matrix, we have that the matrix on the left is totally unimodular. Thus $\mathcal{P}_{\mathcal{O}}$ is an integral polytope. $\square$

By applying Theorem 3.5 to Theorem 3.10 we obtain the following:

Theorem 3.11. *The closed and open Ehrhart polynomials of $(\mathcal{T}, \mathcal{B})$ satisfy*

$$(-1)^d E_{\mathcal{T},\mathcal{B}}(-k) = E^{\circ}_{\mathcal{T}^{\circ},\mathcal{B}}(k) = \tau_{nwz}(k).$$

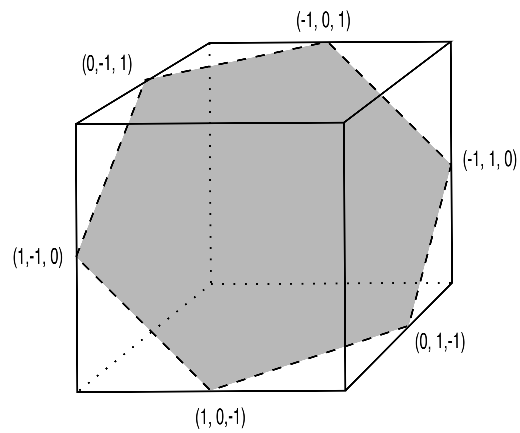

Figure 3.2: The tension polytope of C_3.

Now we know that $\tau_{nwz}(k)$ is a polynomial that, when evaluated at $k \in \mathbb{Z}_{>0}$, gives the number of nowhere-zero k-tensions of the graph G. We now give an interpretation of $\tau_{nwz}(k)$ when evaluated at negative integers.

Given a digraph G with initial orientation ϵ and a nowhere-zero tension τ, we obtain a new orientation ϵ_τ by declaring

$$\epsilon_\tau(e) := \begin{cases} \epsilon(e) & \text{if } \tau(e) > 0, \\ -\epsilon(e) & \text{if } \tau(e) < 0. \end{cases}$$

(Note that each reorientation depends on the initial orientation ϵ.) We say that a tension τ and an orientation ϵ' are **compatible** (with respect to the initial orientation ϵ) if $\epsilon_\tau = \epsilon'$. For example, a tension τ is compatible with the initial orientation ϵ precisely when $\tau_i > 0$ for all i. In general, each orthant $\mathcal{O}$ of $\mathbb{R}^E$ corresponds to a sign vector $sgn(\mathcal{O}) \in \{-1, 1\}^E$ and each sign vector corresponds to a reorientation of G obtained by switching the orientation on the edge e_i if the ith component of $sgn(\mathcal{O})$ is -1. Thus a nowhere-zero tension τ is compatible with an orientation ϵ' if and only if τ lies in the orthant corresponding to ϵ'.

An orientation is **acyclic** if it has no circuits (i.e., no coherently oriented cycles).

Theorem 3.12. *The number of pairs (ϵ, τ) consisting of an acyclic orientation of a graph G and a compatible nowhere-zero k-tension τ equals $(-1)^{|V|-1}\tau_{nwz}(-k)$. In particular, $(-1)^{|V|-1}\tau_{nwz}(-1)$ is the number of acyclic orientations of G.*

Proof. From Theorem 3.11 we have

$$E_{\mathcal{T},\mathcal{B}}(k) = (-1)^d \tau_{nwz}(-k).$$

The function $E_{\mathcal{T},\mathcal{B}}(k)$ counts the number of pairs (τ, R) where τ is a k-tension, R is a closed region of $\mathcal{B}$, and $\tau \in R$. But a closed region of $\mathcal{B}$ is the topological closure of some orthant in $\mathbb{R}^E$ which corresponds to some orientation ϵ of G. It was shown in [12] that the tension space T of a graph G intersects exactly those orthants of $\mathbb{R}^E$ that correspond to acyclic orientations of G. So τ is compatible with ϵ if and only if $\tau \in R$. The final assertion follows from the evaluation of $E(0)$ in Theorem 3.5. $\square$

Chapter 4

From Inside-Out Polytopes to Simplicial Complexes

In [18], Steingrimsson constructed, given a simple graph G, a monomial ideal K in a certain face ring A with the property that the monomials in K are in one-to-one correspondence with the proper colorings of G.

In the previous chapter we saw that the number of k-tensions on a finite digraph G is counted by the polynomial $\tau(k)$. It was shown in [11] that the number of k-flows is also counted by a polynomial which we denote by $\phi(k)$.

Our goal in this chapter is to realize the nowhere-zero flow and nowhere-zero tension polynomials as the Hilbert polynomials of certain ideals of a polynomial ring. The common theme is that each of these polynomials is the open inside-out polynomial of an integral inside-out polytope with vertices on the $\{-1, 1\}$-cube. In the next section, we introduce the needed terminology from polyhedral geometry and commutative algebra. In Section 4.2, we prove a general statement regarding certain polytopal complexes of which the three aforementioned complexes are examples. Finally, we apply our theory to our examples and state some directions for further research.

The following is joint work with Felix Breuer (Freie Universität Berlin).

4.1 Background

4.1.1 Polyhedral Geometry

We briefly present some needed concepts from polyhedral geometry. For a thorough treatment see [22] or [8].

Let $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d$. The **support** of $\mathbf{u}$, denoted $\mathrm{supp}(\mathbf{u})$, is defined by

$$\mathrm{supp}(\mathbf{u}) := \{i \in [d] \mid u_i \neq 0\}.$$

The **1-norm** of $\mathbf{u}$ is given by $\|\mathbf{u}\|_1 := \sum_{i=1}^{d} |u_i|$ and the **infinity norm** of $\mathbf{u}$ is given by $\|\mathbf{u}\|_\infty := \max\{|u_1|, \ldots, |u_d|\}$. We denote by $\mathrm{Mx}(\mathbf{u})$ the set

$$\mathrm{Mx}(\mathbf{u}) := \{i \in [d] \mid u_i = \|\mathbf{u}\|_\infty\}.$$

Given a nonempty set $S = \{\mathbf{s}_1, \dots, \mathbf{s}_n\} \subset \mathbb{R}^d$, the **cone over** S, denoted $\operatorname{cone}_{\mathbb{R}}(S)$, is the set

$$\operatorname{cone}_{\mathbb{R}}(S) := \{\lambda_1 \mathbf{s}_1 + \cdots + \lambda_n \mathbf{s}_n \mid \lambda_i \in \mathbb{R}_{\geq 0} \text{ for all } i\}.$$

Similarly, we define

$$\operatorname{cone}_{\mathbb{Z}}(S) := \{\lambda_1 \mathbf{s}_1 + \cdots + \lambda_n \mathbf{s}_n \mid \lambda_i \in \mathbb{Z}_{\geq 0} \text{ for all } i\}.$$

A **polyhedron** $P \subseteq \mathbb{R}^d$ is any set of the form

$$P = \operatorname{conv}(V) + \operatorname{cone}_{\mathbb{R}}(S)$$

where $V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, $S = \{\mathbf{s}_1, \dots, \mathbf{s}_n\} \subset \mathbb{R}^d$ and the sum is the **Minkowski sum** which, for any two sets $A, B \subset \mathbb{R}^d$, is given by

$$A + B := \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A,\ \mathbf{b} \in B\}.$$

We note that if a polyhedron is bounded, then it is a polytope.

A **polyhedral complex** $\mathcal{C}$ is a finite collection of polyhedra in $\mathbb{R}^d$ such that

1. the empty polyhedron is in $\mathcal{C}$,
2. if $P \in \mathcal{C}$ and F is a face of P, then $F \in \mathcal{C}$,
3. the intersection $P \cap Q$ of two polyhedra $P, Q \in \mathcal{C}$ is a face of P as well as a face of Q.

We call a polyhedral complex a **polytopal complex** if all the polyhedra in $\mathcal{C}$ are bounded.

Let $\mathcal{C}$ be a polytopal complex with vertices $\operatorname{vert}(\mathcal{C}) = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and let $F \in \mathcal{C}$ be a face of $\mathcal{C}$. We define the **support of** F to be

$$\operatorname{supp}(F) := \{i \in [n] \ \mid \ \mathbf{a}_i \text{ is a vertex of } F\}.$$

4.1.2 Combinatorial Commutative Algebra

We now collect the needed ideas from commutative algebra. We follow [15] and [19]. Let $\Bbbk$ be a field. Let $\Bbbk[\mathbf{x}] := \Bbbk[x_1, \dots, x_n]$ be the polynomial ring over $\Bbbk$ with n indeterminates. Let $\Bbbk[\mathbf{z}^{\pm 1}] := \Bbbk[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$ be the Laurent polynomial ring over $\Bbbk$ with d indeterminates. We write $\mathbf{x}^{\mathbf{u}}$ for the monomial $x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n} \in \Bbbk[\mathbf{x}]$ and, similarly, $\mathbf{z}^{\mathbf{s}}$ for the monomials in $\Bbbk[\mathbf{z}^{\pm 1}]$. We identify the monomials in $\Bbbk[\mathbf{x}]$ with vectors in $\mathbb{N}^n$ via the natural correspondence $\mathbf{x}^{\mathbf{u}} \leftrightarrow (u_1, \dots, u_n)$. Similarly, the monomials in $\Bbbk[\mathbf{z}^{\pm 1}]$ are identified with vectors in $\mathbb{Z}^d$.

As a vector space over $\Bbbk$, the polynomial ring $\Bbbk[\mathbf{x}]$ is a direct sum

$$\Bbbk[\mathbf{x}] = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_{\mathbf{a}},$$

where $S_{\mathbf{a}} := \Bbbk[\mathbf{x}^{\mathbf{a}}]$ is the vector subspace of $\Bbbk[\mathbf{x}]$ spanned by the monomial $\mathbf{x}^{\mathbf{a}}$. We call $S_{\mathbf{a}}$ a **graded component** of $\Bbbk[\mathbf{x}]$. Thus $\Bbbk[\mathbf{x}]$ is a $\mathbb{N}^n$**-graded** $\Bbbk$**-algebra** since the product $S_{\mathbf{a}} \cdot S_{\mathbf{b}} = S_{\mathbf{a} \cdot \mathbf{b}}$.

A **term order** $\prec$ on $\mathbb{N}^n$ is a total order such that

1. the zero vector is the unique minimal element, and
2. $\mathbf{a} \prec \mathbf{b}$ implies $\mathbf{a} + \mathbf{c} \prec \mathbf{b} + \mathbf{c}$.

One term order of particular importance to us is the **graded reverse lexicographic** term order, $\prec_{revlex}$, which is defined by $\mathbf{x}^{\mathbf{a}} \prec_{revlex} \mathbf{x}^{\mathbf{b}}$ if $\deg(\mathbf{x}^{\mathbf{a}}) \leq \deg(\mathbf{x}^{\mathbf{b}})$ or if $\deg(\mathbf{x}^{\mathbf{a}}) = \deg(\mathbf{x}^{\mathbf{b}})$ and the rightmost nonzero entry of $\mathbf{b} - \mathbf{a}$ is negative.

Given a term order $\prec$, every non-zero polynomial $f \in \Bbbk[\mathbf{x}]$ has a unique **initial monomial**, denoted $in_{\prec}(f)$, whose exponent vector is greatest with respect to the term order. If I is an ideal in $\Bbbk[\mathbf{x}]$, then the **initial ideal** of I with respect to $\prec$ is

$$in_{\prec}(I) := \langle in_{\prec}(f) \mid f \in I \rangle.$$

An ideal I of $\Bbbk[\mathbf{x}]$ is a **monomial ideal** if it is generated by finitely many monomials, i.e., if

$$I = \langle \mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2}, \ldots, \mathbf{x}^{\mathbf{u}_r} \rangle.$$

A monomial ideal is called **square-free** if, for each generator $\mathbf{x}^{\mathbf{u}}$, the exponent vector $\mathbf{u}$ is a $\{0, 1\}$-vector. Square-free ideals in $\Bbbk[\mathbf{x}]$ are especially useful in combinatorial commutative algebra because they correspond to simplicial complexes (see, e.g., Theorem 1.7 in [15]).

Given a monomial ideal I of $\Bbbk[\mathbf{x}]$, let $R = \Bbbk[\mathbf{x}]/I$. Then the **Hilbert function** of R is the map that sends each graded component of R to its dimension as a $\Bbbk$-vector space, that is, $R_{\mathbf{a}} \mapsto \dim_{\Bbbk}(R_{\mathbf{a}})$. The **Hilbert polynomial** of R is the polynomial in $\mathbb{Z}[\mathbf{x}]$ that agrees with the Hilbert function for almost all $\mathbf{a} \in \mathbb{Z}^n$ (such a polynomial exists by Hilbert's Theorem [13, Theorem X.6.2]). The **Hilbert series** of R is the series

$$\begin{aligned} H(R, \mathbf{x}) &= \sum_{\mathbf{a} \in \mathbb{N}^n} \dim_{\Bbbk}(R_{\mathbf{a}}) \cdot \mathbf{x}^{\mathbf{a}} \\ &= \text{sum of all monomials not in } I. \end{aligned}$$

If the Hilbert series of R is written as a rational function

$$H(R, \mathbf{x}) = \frac{\mathcal{K}(R, \mathbf{x})}{(1 - x_1) \cdots (1 - x_n)},$$

then its numerator $\mathcal{K}(R, \mathbf{x})$ is the **$\mathcal{K}$-polynomial** of R.

There is a connection between these algebraic notions and the world of polyhedral geometry. Suppose Δ is a simplicial complex on $[n]$. Then the **f-vector** of Δ is

$$f_{\Delta} := (f_0, f_1, \ldots, f_{n-1})$$

where f_i is the number of i-dimensional faces of Δ. The **h-vector** of Δ, $h_{\Delta} := (h_0, \ldots, h_d)$, is a transformation of the f-vector given by

$$h_k = \sum_{i=1}^{n} (-1)^{k-i} \binom{n-i}{k-i} f_{i-1}.$$

We associate to Δ the **Stanley-Reisner ideal** I_Δ in the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ defined by

$$I_\Delta := \langle x_{i_1} x_{i_2} \cdots x_{i_r} \mid i_1 < i_2 < \cdots < i_r,\ \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta \rangle.$$

The h-vector of Δ is related to the $\mathcal{K}$-polynomial of I_Δ via

$$\mathcal{K}(\Bbbk[\mathbf{x}]/I_\Delta, t) = h_0 + h_1 t + \cdots + h_d t^d,$$

where we work with the course grading of $\Bbbk[\mathbf{x}]$ which is defined by setting $x_i = t$ for each $i \in [n]$.

Finally, let us fix a set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of n vectors in $\mathbb{Z}^d$. Let $\pi : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{z}^{\pm 1}]$ be the ring homomorphism defined by $x_i \mapsto \mathbf{z}^{\mathbf{a}_i}$. The kernel of π is called the **toric ideal** of $\mathcal{A}$ and will be denoted by $I_\mathcal{A}$. It is shown in [19] that $I_\mathcal{A}$ is spanned as a $\Bbbk$-vector space by the set of binomials

$$\left\{ \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \mid \sum_{i=1}^n u_i \mathbf{a}_i = \sum_{i=1}^n v_i \mathbf{a}_i \right\}.$$

Notice that this ideal is generated by binomials and that in the quotient ring $\Bbbk[\mathbf{x}]/I_\mathcal{A}$ two monomials are identified if they are sent to the same monomial in $\Bbbk[\mathbf{z}^{\pm 1}]$ under π.

4.2 The Main Result

We define a partial order $\preceq$ on $\mathbb{R}^d$ by $\mathbf{v} \preceq \mathbf{w}$ if

1. $|v_i| \le |w_i|$ for all $i \in [d]$ and
2. there is no $j \in [d]$ such that v_j and w_j are nonzero and $v_j = -w_j$.

Our main theorem is

Theorem 4.1. *Let $\mathcal{C}$ be a d-dimensional polytopal complex and $\mathcal{C}' \subseteq \mathcal{C}$ a subcomplex. Suppose that for every face F of $\mathcal{C}$ we have*

1. *the vertices of F are $\{-1, 0, 1\}$-vectors,*
2. *F lies in one orthant,*
3. *F lies on one face of $\partial[-1, 1]^d$, and*
4. *if $\mathbf{z}$ lies in the interior of $\operatorname{cone}_\mathbb{R}(\operatorname{vert}(F)) \cap \mathbb{Z}^d$ and $\mathbf{y} \in \operatorname{vert}(F)$ with $\mathbf{y} \prec \mathbf{z}$, then $\mathbf{z} - \mathbf{y} \in \operatorname{cone}_\mathbb{R}(\operatorname{vert}(F))$.*

Then there is a Hilbert function $f : \mathbb{N} \to \mathbb{N}$ with

$$f(k) = \#\{\mathbf{z} \in \mathbb{Z}^d \cap \operatorname{cone}_\mathbb{R}(\mathcal{C}) \setminus \operatorname{cone}_\mathbb{R}(\mathcal{C}') \mid \|\mathbf{z}\|_\infty = k\}$$

and f is a polynomial.

We will devote the rest of this section to proving this theorem. Before doing so let us say why this theorem is useful in the case of enumerating tensions on a directed graph G. Consider the subdivision, $\mathcal{C}$, of the boundary of the tension polytope, $\partial\mathcal{T}$, obtained by intersecting $\partial\mathcal{T}$ with the Boolean arrangement $\mathcal{B}$. Then $\mathcal{C}$ is a polytopal complex that contains $\partial\mathcal{T}\cap\mathcal{B}$ as a subcomplex. We will show in Section 4.3 that a certain triangulation of $\mathcal{C}$ satisfies the conditions in Theorem 4.1 and so the theorem implies that every nowhere-zero tension polynomial is the Hilbert polynomial of some graded algebra.

We now set up the notations needed for our proof.

Let $\mathcal{A} := \mathrm{vert}(\mathcal{C}) = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ be the set of vertices of $\mathcal{C}$ ordered in any fixed way. Let $\Bbbk$ be any field and consider the polynomial ring $\Bbbk[\mathbf{x}] := \Bbbk[x_1 \dots, x_n]$ where we have one indeterminate for each vertex of $\mathcal{C}$. The ordering of the vertices induces a linear ordering of the variables in a natural way. Let $\prec_{revlex}$ be the reverse lexicographic term order on $\Bbbk[\mathbf{x}]$ with respect to the given linear ordering of the variables. We may view $\prec_{revlex}$ as a well ordering of $\mathbb{N}^n$ by declaring

$$\mathbf{u} \prec_{revlex} \mathbf{v} \Leftrightarrow \mathbf{x}^{\mathbf{u}} \prec_{revlex} \mathbf{x}^{\mathbf{v}}.$$

Let A denote the $(d\times n)$-matrix whose i^{th} column is $\mathbf{a}_i$ and let $\pi : \Bbbk[\mathbf{x}] \to \mathbb{Z}^d$ be the ring homomorphism given by $\pi(\mathbf{x}^{\mathbf{u}}) = A\mathbf{u}$. For a given $\mathbf{z} \in \mathbb{Z}^d$, we call a vector $\mathbf{u} \in \mathbb{N}^n$ a **valid representation** of $\mathbf{z}$ if $A\mathbf{u} = \mathbf{z}$ and $\mathrm{supp}(\mathbf{u}) \subseteq \mathrm{supp}(F)$ for some face $F \in \mathcal{C}$. We say that $\mathbf{u}$ is **valid** if it is a valid representation of some $\mathbf{z} \in \mathbb{Z}^d$.

Lemma 4.2. *Let $\mathcal{C}$ be a polytopal complex satisfying the conditions in Theorem 4.1 and let F be a face of $\mathcal{C}$. Let $\mathbf{z} \in \mathbb{Z}^d$ lie in the interior of $\mathrm{cone}_{\mathbb{R}}(\mathrm{vert}(F))$. Then for any $\mathbf{a}_j \in \mathrm{vert}(F)$ such that $\mathbf{a}_j \prec \mathbf{z}$ there is a valid representation $\mathbf{u} \in \mathbb{N}^n$ of $\mathbf{z}$ such that $u_j \neq 0$.*

Proof. Let $\mathbf{z}' := \mathbf{z} - \mathbf{a}_j$. Then $\mathbf{z}' \in \mathbb{Z}^d \cap \mathrm{cone}_{\mathbb{R}}(F)$ by condition (4) in Theorem 4.1. If $\mathbf{z}'$ is in the interior of $\mathrm{cone}_{\mathbb{R}}(F')$ for some $F' \subseteq F$, then $\mathbf{z}'$ has a valid representation $\mathbf{u}'$ with $\mathrm{supp}(\mathbf{u}) \subseteq \mathrm{supp}(F')$. It follows that $\mathbf{u} = \mathbf{u}' + \mathbf{e}_j$ is a valid representation of $\mathbf{z}$ (where $\mathbf{e}_j$ is the j^{th} standard unit vector in $\mathbb{R}^n$). If $\mathbf{z}'$ does not lie in the interior of any such $\mathrm{cone}_{\mathbb{R}}(F')$, then $\mathbf{z}' = \mathbf{0}$ in which case $\mathbf{z} = \mathbf{a}_j$. So $\mathbf{e}_j$ is a valid representation of $\mathbf{z}$. $\square$

Our next goal is to pass to a quotient ring R of $\Bbbk[\mathbf{x}]$ with the property that a monomial $\mathbf{x}^{\mathbf{v}} \in R$ if and only if $\mathbf{v}$ is the $\prec_{revlex}$-maximal valid representation of some $\mathbf{z}$. To this end let I be the ideal in $\Bbbk[\mathbf{x}]$ defined by

$$I := \langle \mathbf{x}^{\mathbf{u}} \mid \mathrm{supp}(\mathbf{u}) \not\subset \mathrm{supp}(F) \text{ for any face } F \text{ of } \mathcal{C} \rangle$$

and let

$$I' := \langle \mathbf{x}^{\mathbf{u}} \mid \ \mathbf{u} \text{ is valid and there is a valid } \mathbf{v} \text{ with } A\mathbf{u} = A\mathbf{v} \text{ and } \mathbf{v} \prec_{revlex} \mathbf{u} \ \rangle.$$

We note that I is a generalization of a Stanley-Reisner ideal of a simplicial complex to the polytopal complex $\mathcal{C}$ and that the monomials in $\Bbbk[\mathbf{x}]/I$ are of the form $\mathbf{x}^{\mathbf{u}}$ where $\mathbf{u}$ is valid. Also, the definition of I' makes sense because there are only finitely many valid representations of a given $\mathbf{z} \in \mathbb{Z}^d$ (we prove this in Lemma 4.4).

Lemma 4.3. *The ideal $I + I'$ is square-free.*

Proof. Let $\mathbf{x}^{\mathbf{u}} \in I + I'$ where $\mathbf{u} = (u_1, \ldots, u_n)$ and suppose $u_i \geq 2$ for some $i \in [n]$. Our goal is to show that there is a $\mathbf{u}' = (u'_1, \ldots, u'_n) \in \mathbb{N}^n$ such that $\mathbf{u} \prec_{revlex} \mathbf{u}'$, $u'_i = u_i - 1$, and $\mathbf{x}^{\mathbf{u}'} \in I + I'$. First suppose $\mathbf{x}^{\mathbf{u}} \in I$. Define $\mathbf{u}'$ by setting $u'_i = 1$ if $i \in \mathrm{supp}(\mathbf{u})$ and $u'_i = 0$ otherwise. Then $\mathbf{u} \prec_{revlex} \mathbf{u}'$ since $u'_i \leq u_i$ for all $i \in [n]$ and $\mathbf{x}^{\mathbf{u}'} \in I$ since $\mathrm{supp}(\mathbf{u}') = \mathrm{supp}(\mathbf{u})$.

So we may assume without loss of generality that $\mathbf{x}^{\mathbf{u}}$ is a nonzero monomial in $(I + I')/I$. But then $\mathbf{u}$ is a valid representation of some $\mathbf{z}$ and there exists a $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{N}^n$ such that $\mathbf{v}$ is a valid representation of $\mathbf{z}$ and $\mathbf{u} \prec_{revlex} \mathbf{v}$. If $j \in \mathrm{supp}(\mathbf{v}) \cap \mathrm{supp}(\mathbf{u})$, then $A(\mathbf{u} - \mathbf{e}_j) = A(\mathbf{v} - \mathbf{e}_j)$. Furthermore, $\mathbf{u} - \mathbf{e}_j, \mathbf{v} - \mathbf{e}_j$ are both valid.

Thus we may suppose that $\mathrm{supp}(\mathbf{u}) \cap \mathrm{supp}(\mathbf{v}) = \emptyset$. We may also assume that, for some face $F \in \mathcal{C}$, we have $\mathbf{z} \in \mathrm{cone}_{\mathbb{R}}(\mathrm{vert}(F))$ and $\mathrm{supp}(\mathbf{u}), \mathrm{supp}(\mathbf{v}) \subseteq \mathrm{vert}(F)$. Since $u_i \geq 2$ we have $(A\mathbf{u})_l \geq 2$ for all $l \in \mathrm{supp}(\mathbf{a}_i)$. So for any $\{-1, 0, 1\}$-vector $\mathbf{z}'$ we have $\mathbf{z}' \prec A(\mathbf{u} - \mathbf{e}_i)$ if and only if $\mathbf{z}' \prec A\mathbf{u}$, where $\mathbf{e}_i$ is the i^{th} standard unit vector in $\mathbb{R}^n$. So we set $\mathbf{u}' = \mathbf{u} - \mathbf{e}_i$ which is clearly valid.

Now we must find a valid representation $\mathbf{v}'$ of $A(\mathbf{u} - \mathbf{e}_i)$ such that $\mathbf{v}' \prec_{revlex} \mathbf{u}'$. Let $k \in [n]$ be the greatest such that $v_k \geq 1$ and $u_j = 0$ for all $j \leq k$. (Such a k exists since $\mathbf{u} \prec_{revlex} \mathbf{v}$ and $\mathrm{supp}(\mathbf{u}) \cap \mathrm{supp}(\mathbf{v}) = \emptyset$.) Then $\mathbf{a}_k \prec A\mathbf{u}$ and so $\mathbf{a}_k \prec A\mathbf{u}'$. Since $A\mathbf{u}' = A(\mathbf{u} - \mathbf{e}_i) \in \mathrm{cone}_{\mathbb{R}}(\mathrm{vert}(F))$ we can find a valid representation $\mathbf{v}'$ of $A\mathbf{u}'$ such that $v'_k > 0$ by Lemma 4.2. But this means $\mathbf{v}' \prec_{revlex} \mathbf{u}'$ as desired. $\square$

We set $R := \Bbbk[\mathbf{x}]/(I + I')$ and define the ideal J of R by

$$J := \langle \mathbf{x}^{\mathbf{u}} \mid \mathrm{supp}(\mathbf{u}) \not\subset \mathrm{supp}(F) \text{ for any face } F \text{ of } \mathcal{C}' \rangle.$$

So J is the Stanley–Reisner ideal of $\mathcal{C}'$.

Let $\tilde{\pi} : R \to \mathbb{Z}^d$ be the ring homomorphism induced by π (recall that $\pi : \Bbbk[\mathbf{x}] \to \mathbb{Z}^d$ was defined by $\pi(\mathbf{x}^{\mathbf{u}}) = A\mathbf{u}$).

Lemma 4.4. *The restriction of the map $\tilde{\pi}$ to the ideal J gives a bijection between the monomials of degree k in J and the vectors $\mathbf{z} \in \mathbb{Z}^d \cap \mathrm{cone}_{\mathbb{R}}(\mathcal{C}) \setminus \mathrm{cone}_{\mathbb{R}}(\mathcal{C}')$ with $\|\mathbf{z}\|_\infty = k$.*

Proof. Let $\mathbf{x}^{\mathbf{u}} \in J$ be any nonzero monomial. We start by showing

$$A\mathbf{u} \in \mathrm{cone}_{\mathbb{R}}(\mathcal{C}) \setminus \mathrm{cone}_{\mathbb{R}}(\mathcal{C}') \cap \mathbb{Z}^d.$$

As $A\mathbf{u}$ is an integral combination of integer vectors, it is itself integral. As $\mathbf{x}^{\mathbf{u}}$ is non-zero, $\mathbf{u}$ is valid and hence there is a face F such that

$$A\mathbf{u} \in \mathrm{cone}_{\mathbb{R}}(\mathrm{vert}(F)) \subset \mathrm{cone}_{\mathbb{R}}(\mathrm{vert}(\mathcal{C})).$$

Now assume that $A\mathbf{u} \in \mathrm{cone}_{\mathbb{R}}(F_0)$ for some $F_0 \in \mathcal{C}'$. Then

$$A\mathbf{u} \in \mathrm{cone}_{\mathbb{R}}(F_0 \cap F)$$

and $F_0 \cap F$ is a face of F. Therefore $\mathrm{supp}(\mathbf{u}) \subset \mathrm{vert}(F_0 \cap F)$. But $F_0 \cap F$ is also a face of $\mathcal{C}'$ and so $\mathbf{x}^{\mathbf{u}} \notin J$, which is a contradiction. So

$$A\mathbf{u} \in \mathrm{cone}_{\mathbb{R}}(\mathcal{C}) \setminus \mathrm{cone}_{\mathbb{R}}(\mathcal{C}') \cap \mathbb{Z}^d$$

for any non-zero monomial $\mathbf{x}^{\mathbf{u}} \in J$.

Next we show that $\tilde{\pi}$ is degree preserving. Since $\mathbf{u}$ is valid we have that those $\mathbf{a}_i$ with $i \in \operatorname{supp}(\mathbf{u})$ all lie in one face of $[-1,1]^d$. Therefore, there is a $\sigma \in \{-1,1\}$ and a $j_0 \in [d]$ such that $(\mathbf{a}_i)_{j_0} = \sigma$ for all $i \in \operatorname{supp}(\mathbf{u})$. Now

$$\begin{aligned}\|A\mathbf{u}\|_\infty &= \max_j \left|\sum u_i(\mathbf{a}_i)_j\right| \\ &\geq \left|\sum u_i(\mathbf{a}_i)_{j_0}\right| \\ &= \left|\sum u_i\sigma\right| \\ &= |\sigma| \cdot \left|\sum u_i\right| \\ &= \sum u_i\end{aligned}$$

where the summations are over all $i \in \operatorname{supp}(\mathbf{u})$. Conversely, since $(\mathbf{a}_i)_j \in \{-1,0,+1\}$,

$$\begin{aligned}\sum u_i &= \left|\sum u_i\right| \\ &\geq \left|\sum u_i(\mathbf{a}_i)_j\right|\end{aligned}$$

for all $j \in [d]$. Hence $\|A\mathbf{u}\|_\infty = \sum u_i = \deg(\mathbf{x}^{\mathbf{u}})$ where $\deg(\mathbf{x}^{\mathbf{u}})$ is the total degree. So $\tilde{\pi}$ is degree preserving.

To see that $\tilde{\pi}$ is injective first notice that since $\tilde{\pi}$ is degree preserving, there are only finitely many valid representations of a given $\mathbf{z} \in \mathbb{Z}^d$. As $\prec_{revlex}$ is a well order, any finite set has a unique minimal element. All non-minimal elements are, by definition, in I' and hence zero in R.

Finally, we show that $\tilde{\pi}$ is surjective. First we note that if a face F satisfies the four conditions in Theorem 4.1, then $\operatorname{cone}_{\mathbb{R}}(\operatorname{vert}(F)) \cap \mathbb{Z}^d = \operatorname{cone}_{\mathbb{Z}}(\operatorname{vert}(F))$. Let

$$\mathbf{z} \in \operatorname{cone}_{\mathbb{R}}(\mathcal{C}) \setminus \operatorname{cone}_{\mathbb{R}}(\mathcal{C}') \cap \mathbb{Z}^d.$$

Then there exists a face $F \in \mathcal{C}$ such that $\mathbf{z} \in \operatorname{cone}_{\mathbb{R}}(\operatorname{vert}(F))$ and hence in $\operatorname{cone}_{\mathbb{Z}}(\operatorname{vert}(F))$. This shows that there are valid representations of $\mathbf{z}$. As $\prec_{revlex}$ is a linear order on $\mathbb{N}^n$, there is a representation $\mathbf{u}$ that is maximal with respect to $\prec_{revlex}$. Hence $\mathbf{x}^{\mathbf{u}}$ is nonzero in R. Since $\mathbf{z} \notin \operatorname{cone}_{\mathbb{R}}(\mathcal{C}')$ we have

$$\operatorname{supp}(\mathbf{u}) \not\subset \operatorname{supp}(F)$$

for any face $F \in \mathcal{C}'$. So $\mathbf{x}^{\mathbf{u}} \in J$ and $Au = \mathbf{z}$. So $\tilde{\pi}$ is surjective. This completes the proof of the lemma. □

The final step in the proof Theorem 4.1 is to prove:

Lemma 4.5. *The Hilbert function*

$$f(k) = \#\{\mathbf{z} \in \mathbb{Z}^d \cap \operatorname{cone}_{\mathbb{R}}(\mathcal{C}) \setminus \operatorname{cone}_{\mathbb{R}}(\mathcal{C}')\} \mid \|\mathbf{z}\|_\infty = k\}$$

is a polynomial.

Proof. For any cone over a face $F \in \mathcal{C}$ satisfying the four conditions in Theorem 4.1 the Hilbert function of F is a polynomial (see for example [13, Theorem X.6.2]). The Hilbert function $f(k)$ is obtained by adding the Hilbert polynomials of the faces of $\mathcal{C}$ and subtracting the Hilbert polynomials of those faces in $\mathcal{C}'$. Thus $f(k)$ is itself a polynomial. $\square$

Thus Theorem 4.1 is proved.

4.3 Applications of the Main Theorem

In this section we apply the main theorem to the inside-out tension and flow polytopes to obtain simplicial complexes whose Hilbert polynomials are the tension and flow polynomials, respectively.

4.3.1 The Tension Complex

Let $G = (V, E, \epsilon)$ be an oriented loopless graph with $|E| = d$ and let T, $\mathcal{T}$ and $(\mathcal{T}, \mathcal{B})$ be the corresponding tension space, tension polytope and inside-out tension polytope, respectively (see Section 3.3). The Boolean arrangement $\mathcal{B}$ subdivides $\partial\mathcal{T}$ (viewed as a polytopal complex) into a new polytopal complex, $\mathcal{P}$, that contains $\mathcal{P}' := \mathcal{B} \cap \partial\mathcal{T}$ as a subcomplex. Since $\mathrm{vert}(\mathcal{T}) \subset \{-1, 0, 1\}^d$ and the regions of $\mathcal{B}$ are simply the orthants of $\mathbb{R}^d$, it is clear that $\mathcal{P}$ satisfies the first two conditions of Theorem 4.1. In what follows we find a unimodular triangulation $\mathcal{C}$ of $\mathcal{P}$ (and hence of $\mathcal{P}'$) such that $\mathcal{C}$ satisfies conditions (3) and (4) of Theorem 4.1 and the Hilbert functions of $\mathcal{C}$ and $\mathcal{C}' := \mathcal{C} \cap \mathcal{B}$ coincide with that of $\mathcal{P}$ and $\mathcal{P}'$, respectively.

Let $\mathcal{O}$ be the topological closure of some fixed orthant in $\mathbb{R}^d$ such that $T \cap \mathcal{O}^\circ \neq \emptyset$. Let $\mathcal{A} = \mathcal{A}_{\mathcal{O}} := \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ be the vertices of $\mathcal{T}$ that lie in $\mathcal{O}$ ordered so that

$$\|\mathbf{a}_i\|_\infty \geq \|\mathbf{a}_{i+1}\|_\infty$$

for all $i \in [n-1]$. We have seen that $\mathcal{A}$ is a collection of $\{-1, 0, 1\}$-vectors and so

$$\mathrm{Mx}(\mathbf{a}_i) = \mathrm{supp}(\mathbf{a}_i).$$

Let A denote the matrix whose i^{th} column is $\mathbf{a}_i$. Given a tension $\tau \in T \cap \mathcal{O}$ we call a vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ a **representation** of τ if $A\mathbf{u} = \tau$. A representation $\mathbf{u}$ of τ is **valid** if

$$\bigcap_{i \in \mathrm{supp}(\mathbf{u})} \mathrm{supp}(\mathbf{a}_i) \neq \emptyset.$$

Lemma 4.6. *Let $\tau \in T \cap \mathcal{O}$ for some fixed orthant $\mathcal{O}$. Let $\mathbf{a}_i \in \mathcal{A}$ such that*

$$\mathrm{Mx}(\tau) \subseteq \mathrm{supp}(\mathbf{a}_i) \subseteq \mathrm{supp}(\tau).$$

Then there is a valid representation $\mathbf{u} \in \mathbb{N}^n$ of τ such that $u_i \neq 0$.

Proof. We proceed by induction on $\|\tau\|_\infty$. If $\|\tau\|_\infty$=1 and $\mathbf{a}_i \in \mathcal{A}$ such that

$$\mathrm{Mx}(\tau) \subseteq \mathrm{supp}(\mathbf{a}_i) \subseteq \mathrm{supp}(\tau),$$

then $\tau \in \mathcal{A}$ and so $\mathrm{Mx}(\tau) = \mathrm{supp}(\tau)$. It follows that $\tau = \mathbf{a}_i$ which is a valid representation.

Now suppose $\|\tau\|_\infty = k$. Since $\mathrm{Mx}(\tau) \subseteq \mathrm{supp}(\mathbf{a}_i) \subseteq \mathrm{supp}(\tau)$, we have that $\mathrm{Mx}(\tau) \subseteq \mathrm{Mx}(\tau - \mathbf{a}_i)$ and if $j \in \mathrm{Mx}(\tau - \mathbf{a}_i)$ then $(\tau - \mathbf{a}_i)_j = k - 1$. By the induction hypothesis we have a valid representation $\mathbf{u} \in \mathbb{N}^n$ of $\tau - \mathbf{a}_i$ with $\tau - \mathbf{a}_i = \sum_{j=1}^{n} u_j \mathbf{a}_j$. So $\tau = \sum_{j=1}^{n} u_j \mathbf{a}_j + \mathbf{a}_i$ corresponds to a valid representation $\mathbf{u}'$ of τ with $u'_i \neq 0$. The lemma follows by induction. $\square$

Theorem 4.7. *Let $\tau \in T \cap \mathcal{O}$ be a tension with $\|\tau\|_\infty = k$. If $\mathbf{u} \in \mathbb{N}^n$ is a valid representation of τ, then $\|\mathbf{u}\|_1 = k$.*

Proof. Let $\mathbf{u}$ be a valid representation of the tension $\tau \in T \cap \mathcal{O}$. Let $j \in \cap_{i \in \mathrm{supp}(\mathbf{u})} \mathrm{supp}(\mathbf{a}_i)$. Then there is a $\sigma \in \{-1, 1\}$ such that $\sigma = (\mathbf{a}_r)_j$ for all $r \in \mathrm{supp}(\mathbf{u})$. Thus

$$\begin{aligned} |\tau_j| &= \left| \sum_{i=1}^{n} u_i (a_i)_j \right| \\ &= |\sigma| \left| \sum_{i=1}^{n} u_i |(a_i)_j| \right| \\ &= \sum_{i=1}^{n} u_i. \end{aligned}$$

Since $j \in \bigcap_{i \in \mathrm{supp}(\mathbf{u})} \mathrm{supp}(\mathbf{a}_i)$, we have $\tau_j \in \mathrm{Mx}(\tau)$. But this implies $\sum_{i=1}^{n} u_i = k$. $\square$

Recall that $\prec_{revlex}$ denotes the reverse lexicographic term order on $\mathbb{N}^n$.

Corollary 4.8. *Every tension τ has a unique $\prec_{revlex}$-maximal valid representation.*

Let $\Bbbk$ be any field and let $\Bbbk[\mathbf{x}]$ be the polynomial ring over $\Bbbk$ with $n = |\mathcal{A}|$ indeterminates. Let $\Bbbk[\mathbf{z}^{\pm 1}]$ be the Laurent polynomial ring with one indeterminate for each edge in our graph. Let π be the map $\pi : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{z}^{\pm 1}]$ defined by $x_i \mapsto \mathbf{z}^{\mathbf{a}_i}$. If $\mathbf{x}^{\mathbf{u}}$ is some monomial in $\Bbbk[\mathbf{x}]$, then $\pi(\mathbf{x}^{\mathbf{u}}) = \mathbf{z}^{\tau}$ where $\tau = \sum_{i=1}^{n} u_i \mathbf{a}_i$, i.e., π is a map that sends monomials corresponding to a representation of a tension to the monomial corresponding to the tension itself.

Let G be the ideal of $\Bbbk[\mathbf{x}]$ given by

$$G := \left\langle x_{i_1} x_{i_2} \cdots x_{i_r} \;\middle|\; \bigcap_{j=1}^{r} \mathrm{supp}(\mathbf{a}_{i_j}) = \emptyset \right\rangle.$$

The monomials in the quotient ring $\Bbbk[\mathbf{x}]/G$ correspond to valid representations. Geometrically, we have produced a subdivision of the polytopal complex $\mathcal{P} = \partial\mathcal{T} \cap \mathcal{O}$ such that each face of the subdivision lies on some face of $[-1, 1]^d$, i.e., this subdivision satisfies condition (3) in Theorem 4.1. Furthermore, each face of this subdivision satisfies condition (4) by Lemma 4.6.

Repeating this process for each orthant $\mathcal{O}$ of $\mathbb{R}^d$ we obtain a polyhedral subdivision $\mathcal{C}$ of $\partial\mathcal{T}$ that satisfies the conditions of Theorem 4.1. Consider the subcomplex $\mathcal{C}' := \mathcal{C} \cap \mathcal{B}$

where $\mathcal{B}$ is the Boolean arrangement. We note that a tension $\tau \in \mathcal{C} \setminus \mathcal{C}'$ if and only if it τ is a nowhere-zero tension. Applying Theorem 4.1 to $\mathcal{C}$ and $\mathcal{C}'$ yields a Hilbert polynomial $f(k)$ which counts those nowhere-zero tensions τ with $\max(\tau) = k$ for $k \in \mathbb{Z}_{>0}$. So $f(k) = \tau_{nwz}(k+1) - \tau_{nwz}(k)$. On the level of generating functions we have

$$\begin{aligned}
\sum_{k\geq 1} f(k)x^k &= \sum_{k\geq 1} \tau_{nwz}(k+1)x^k - \sum_{k\geq 1} \tau_{nwz}(k)x^k && (4.1)\\
&= \sum_{k\geq 2} \tau_{nwz}(k)x^{k-1} - \sum_{k\geq 1} \tau_{nwz}(k)x^k && (4.2)\\
&= x^{-1}\sum_{k\geq 2} \tau_{nwz}(k)x^k - \sum_{k\geq 1} \tau_{nwz}(k)x^k && (4.3)\\
&= x^{-1}\sum_{k\geq 1} \tau_{nwz}(k)x^k - \sum_{k\geq 1} \tau_{nwz}(k)x^k && (4.4)\\
&= \frac{1-x}{x}\sum_{k\geq 1} \tau_{nwz}(k)x^k, && (4.5)
\end{aligned}$$

where equality (4.4) holds since $\tau_{nwz}(1) = 0$ (the only 1-tension is the tension that assigns to each edge the value 0).

Since $\sum_{k\geq 1} f(k)x^k$ is the Hilbert series of a graded ring R, we may write

$$\sum_{k\geq 1} f(k)x^k = \frac{\mathcal{K}(R,x)}{(1-x)^d},$$

where $\mathcal{K}(R,x)$ is a polynomial of degree less than d. Thus

$$\sum_{k\geq 1} \tau_{nwz}(k)x^k = \frac{x\mathcal{K}(R,x)}{(1-x)^{d+1}}$$

and so we have shown

Theorem 4.9. *There is a graded ring whose Hilbert polynomial is the nowhere-zero tension polynomial.*

4.3.2 The Flow Complex

Let $G = (V, E, \epsilon)$ be an oriented bridgeless graph with $|E| = d$ and let F, $\mathcal{F}$ and $(\mathcal{F}, \mathcal{B})$ be the corresponding flow space, flow polytope and inside-out flow polytope, respectively (see [3]). The goal of this subsection is to realize the nowhere-zero flow polynomial, $\phi_{nwz}(k)$, which counts the number of nowhere-zero k-flows on G, as the Hilbert polynomial of some graded ring. The approach we use is similar to that used in proving the same result for the nowhere-zero tension polynomial in the previous subsection.

The Boolean arrangement $\mathcal{B}$ subdivides $\partial\mathcal{F}$ (viewed as a polytopal complex) into a new polytopal complex, $\mathcal{P}$, that contains $\mathcal{P}' := \mathcal{B} \cap \partial\mathcal{F}$ as a subcomplex. Since $\mathrm{vert}(\mathcal{F}) \subset \{-1, 0, 1\}^d$ and the regions of $\mathcal{B}$ are simply the orthants of $\mathbb{R}^d$, it is clear that $\mathcal{P}$ satisfies the first two conditions of Theorem 4.1. As in the tension case, we find a polytopal subdivision $\mathcal{C}$ of $\mathcal{P}$ (and hence of $\mathcal{P}'$) such that $\mathcal{C}$ satisfies conditions (3) and (4) of Theorem 4.1 and the Hilbert functions of $\mathcal{C}$ and $\mathcal{C}' := \mathcal{C} \cap \mathcal{B}$ coincide with that of $\mathcal{P}$ and $\mathcal{P}'$, respectively.

Let $\mathcal{O}$ be the topological closure of some fixed orthant in $\mathbb{R}^d$ such that $F \cap \mathcal{O}^\circ \neq \emptyset$. Let $\mathcal{A} = \mathcal{A}_\mathcal{O} := \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ be the vertices of $\mathcal{T}$ that lie in $\mathcal{O}$ ordered so that

$$\|\mathbf{a}_i\|_\infty \geq \|\mathbf{a}_{i+1}\|_\infty$$

for all $i \in [n-1]$. By [2, Theorem 3.1] $\mathcal{A}$ is a collection of $\{-1, 0, 1\}$-vectors and so

$$\mathrm{Mx}(\mathbf{a}_i) = \mathrm{supp}(\mathbf{a}_i).$$

Let A denote the matrix whose i^{th} column is $\mathbf{a}_i$. Given a flow $\phi \in F \cap \mathcal{O}$ we call a vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ a **representation** of ϕ if $A\mathbf{u} = \phi$. A representation $\mathbf{u}$ of ϕ is **valid** if

$$\bigcap_{i \in \mathrm{supp}(\mathbf{u})} \mathrm{supp}(\mathbf{a}_i) \neq \emptyset.$$

Lemma 4.10. *Let $\phi \in F \cap \mathcal{O}$ for some fixed orthant $\mathcal{O}$. Let $\mathbf{a}_i \in \mathcal{A}$ such that*

$$\mathrm{Mx}(\phi) \subseteq \mathrm{supp}(\mathbf{a}_i) \subseteq \mathrm{supp}(\phi).$$

Then there is a valid representation $\mathbf{u} \in \mathbb{N}^n$ of ϕ such that $u_i \neq 0$.

Proof. We proceed by induction on $\|\phi\|_\infty$. If $\|\phi\|_\infty$=1 and $\mathbf{a}_i \in \mathcal{A}$ such that

$$\mathrm{Mx}(\phi) \subseteq \mathrm{supp}(\mathbf{a}_i) \subseteq \mathrm{supp}(\phi),$$

then $\phi \in \mathcal{A}$ and so $\mathrm{Mx}(\phi) = \mathrm{supp}(\phi)$. It follows that $\phi = \mathbf{a}_i$ which is a valid representation.

Now suppose $\|\phi\|_\infty = k$. Since $\mathrm{Mx}(\phi) \subseteq \mathrm{supp}(\mathbf{a}_i) \subseteq \mathrm{supp}(\phi)$, we have that $\mathrm{Mx}(\phi) \subseteq \mathrm{Mx}(\phi - \mathbf{a}_i)$ and if $j \in \mathrm{Mx}(\phi - \mathbf{a}_i)$ then $(\phi - \mathbf{a}_i)_j = k-1$. By the induction hypothesis we have a valid representation $\mathbf{u} \in \mathbb{N}^n$ of $\phi - \mathbf{a}_i$ with $\phi - \mathbf{a}_i = \sum_{j=1}^n u_j \mathbf{a}_j$. So $\phi = \sum_{j=1}^n u_j \mathbf{a}_j + \mathbf{a}_i$ corresponds to a valid representation $\mathbf{u}'$ of ϕ with $u_i' \neq 0$. The lemma follows by induction. $\square$

Theorem 4.11. *Let $\phi \in F \cap \mathcal{O}$ be a flow with $\|\phi\|_\infty = k$. If $\mathbf{u} \in \mathbb{N}^n$ is a valid representation of ϕ, then $\|\mathbf{u}\|_1 = k$.*

Proof. Let $\mathbf{u}$ be a valid representation of the flow $\phi \in F \cap \mathcal{O}$. Let $j \in \cap_{i \in \mathrm{supp}(\mathbf{u})} \mathrm{supp}(\mathbf{a}_i)$. Then there is a $\sigma \in \{-1, 1\}$ such that $\sigma = (\mathbf{a}_r)_j$ for all $r \in \mathrm{supp}(\mathbf{u})$. Thus

$$\begin{aligned} |\phi_j| &= \left| \sum_{i=1}^n u_i (a_i)_j \right| \\ &= |\sigma| \left| \sum_{i=1}^n u_i |(a_i)_j| \right| \\ &= \sum_{i=1}^n u_i. \end{aligned}$$

Since $j \in \bigcap_{i \in \mathrm{supp}(\mathbf{u})} \mathrm{supp}(\mathbf{a}_i)$, we have $\phi_j \in \mathrm{Mx}(\tau)$. But this implies $\sum_{i=1}^n u_i = k$. $\square$

Recall that $\prec_{revlex}$ denotes the reverse lexicographic term order on $\mathbb{N}^n$.

Corollary 4.12. *Every flow ϕ has a unique $\prec_{revlex}$-maximal valid representation.*

Let $\Bbbk$ be any field and let $\Bbbk[\mathbf{x}]$ be the polynomial ring over $\Bbbk$ with $n = |\mathcal{A}|$ indeterminates. Let $\Bbbk[\mathbf{z}^{\pm 1}]$ be the Laurent polynomial ring with one indeterminate for each edge in our graph. Let π be the map $\pi : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{z}^{\pm 1}]$ defined by $x_i \mapsto \mathbf{z}^{\mathbf{a}_i}$. If $\mathbf{x}^{\mathbf{u}}$ is some monomial in $\Bbbk[\mathbf{x}]$, then $\pi(\mathbf{x}^{\mathbf{u}}) = \mathbf{z}^{\phi}$ where $\phi = \sum_{i=1}^{n} u_i \mathbf{a}_i$, i.e., π is the map that sends a monomial corresponding to a representation of a flow to a monomial corresponding to the flow itself.

Let G be the ideal of $\Bbbk[\mathbf{x}]$ given by

$$G := \left\langle x_{i_1} x_{i_2} \cdots x_{i_r} \;\middle|\; \bigcap_{j=1}^{r} \operatorname{supp}(\mathbf{a}_{i_j}) = \emptyset \right\rangle .$$

The monomials in the quotient ring $\Bbbk[\mathbf{x}]/G$ correspond to valid representations. Geometrically, we have produced a subdivision of the polytopal complex $\mathcal{P} = \partial\mathcal{F} \cap \mathcal{O}$ such that each face of the subdivision lies on some face of $[-1, 1]^d$, i.e., this subdivision satisfies condition (3) in Theorem 4.1. Furthermore, each face of this subdivision satisfies condition (4) by Lemma 4.10.

Repeating this process for each orthant $\mathcal{O}$ of $\mathbb{R}^d$ we obtain a polyhedral subdivision $\mathcal{C}$ of $\partial\mathcal{F}$ that satisfies the conditions of Theorem 4.1. Consider the subcomplex $\mathcal{C}' := \mathcal{C} \cap \mathcal{B}$ where $\mathcal{B}$ is the Boolean arrangement. We note that a flow $\phi \in \mathcal{C} \setminus \mathcal{C}'$ if and only if it ϕ is a nowhere-zero flow. Applying Theorem 4.1 to $\mathcal{C}$ and $\mathcal{C}'$ yields a Hilbert polynomial $f(k)$ which counts the number of nowhere-zero flows ϕ with $\max(\phi) = k$ for $k \in \mathbb{Z}_{>0}$. So $f(k) = \phi_{nwz}(k+1) - \phi_{nwz}(k)$. On the level of generating functions we have

$$\begin{aligned}
\sum_{k\geq 1} f(k)x^k &= \sum_{k\geq 1} \phi_{nwz}(k+1)x^k - \sum_{k\geq 1} \phi_{nwz}(k)x^k && (4.6)\\
&= \sum_{k\geq 2} \phi_{nwz}(k)x^{k-1} - \sum_{k\geq 1} \phi_{nwz}(k)x^k && (4.7)\\
&= x^{-1}\sum_{k\geq 2} \phi_{nwz}(k)x^{k} - \sum_{k\geq 1} \phi_{nwz}(k)x^k && (4.8)\\
&= x^{-1}\sum_{k\geq 1} \phi_{nwz}(k)x^{k} - \sum_{k\geq 1} \phi_{nwz}(k)x^k && (4.9)\\
&= \frac{1-x}{x}\sum_{k\geq 1} \phi_{nwz}(k)x^{k}, && (4.10)
\end{aligned}$$

where equality 4.9 holds since $\phi_{nwz}(1) = 0$ (the only 1-flow is the flow that assigns to each edge the value 0).

Since $\sum_{k\geq 1} f(k)x^k$ is the Hilbert series of a graded ring R, we may write

$$\sum_{k\geq 1} f(k)x^k = \frac{\mathcal{K}(R, x)}{(1-x)^d},$$

where $\mathcal{K}(R,x)$ is a polynomial of degree less than d. Thus

$$\sum_{k\geq 1} \phi_{nwz}(k)x^k = \frac{x\mathcal{K}(R,x)}{(1-x)^{d+1}}.$$

So we have shown

Theorem 4.13. *There is a graded ring whose Hilbert polynomial is the nowhere-zero flow polynomial.*

4.3.3 The Signed Coloring Complex

We will now show that Theorem 4.1 is a generalization of Steingrimsson's result in [18] concerning the chromatic polynomial of a graph. To do so we introduce signed graphs and signed graph colorings. Signed graph colorings generalize the notion of graph colorings discussed in Chapter 2. The paper [2] introduced two related inside-out polytopes whose open Ehrhart polynomials count the number of signed graph k-colorings (denoted $\chi^{\star}(k)$)and nowhere-zero signed-graph k-colorings (denoted $\chi^{\star}_{nwz}(k)$), respectively. We apply Theorem 4.1 to a subdivision of this inside-out polytope to show that there is a graded ring R whose Hilbert polynomial is $\chi^{\star}_{nwz}(k)$. We begin with the requisite definitions.

For any positive integer n let $[n] := \{1,\ldots,n\}$ and

$$[-n,n]_{\mathbb{Z}} := \{-n,-(n-1),\ldots,0,\ldots,n-1,n\}.$$

A **signed graph** $S = (G,\sigma)$ consists of a graph $G = (V,E)$ that, in addition to edges with two endpoints (as is the case for ordinary graphs), may have **half edges** (edges with only one endpoint), and **loose edges** (edges with no endpoints), and a **signature** σ that labels each edge with two endpoints with a sign, $+$ or $-$. The **order** of G is the number of vertices of G. A k**-coloring** (see [21]) of a signed graph with vertex set $V = [n]$ is a function

$$\mathbf{c} : V \to [-k,k]_{\mathbb{Z}}.$$

We call **c proper** if, whenever there is an edge ij with sign ϵ, then $\mathbf{c}(i) \neq \epsilon\mathbf{c}(j)$. A signed graph coloring $\mathbf{c}$ is called **zero-free** if $\mathbf{c}(i) \neq 0$ for all $i \in V$.

Geometrically, a vector $\mathbf{c} = (c_1,\ldots,c_d) \in [-k,k]^d_{\mathbb{Z}}$ (where $d = |V|$) is a proper k-coloring if

$$\mathbf{c} \in [-k,k]_{\mathbb{Z}} \setminus \bigcup \mathcal{H}(G)$$

where

$$\begin{aligned}\mathcal{H}(G) :=&\{h^{\epsilon}_{ij} \mid G \text{ has an edge } ij \text{ with sign } \epsilon\}\\ &\cup\{c_i = 0 \mid G \text{ has a halfedge at vertex } v_i\}\\ &\cup\{0 = 0 \mid G \text{ has a loose edge }\}\end{aligned}$$

and h^{ϵ}_{ij} is the hyperplane $x_i = \epsilon x_j$. The hyperplane $\{0 = 0 \mid G$ has a loose edge $\}$ is the **degenerate hyperplane**, $\mathbb{R}^d$. A vector is a zero-free proper k-coloring if

$$\mathbf{c} \in [-k,k] \setminus \bigcup \mathcal{H}',$$

where $\mathcal{H}' := \mathcal{H}(G) \cup \mathcal{B}$ and $\mathcal{B}$ is the Boolean hyperplane arrangement.

We now consider the inside-out polytope $([-1,1]^d, \mathcal{H}')$ and show that its boundary complex $\mathcal{P} = (\partial[-1,1]^d, \mathcal{H}')$ satisfies the conditions in Theorem 4.1. In [2] it is shown that $([-1,1]^d, \mathcal{H}')$ has integral vertices and so $\mathcal{P}$ satisfies the first condition of Theorem 4.1. Condition (2) is satisfied since each face of $\mathcal{P}$ lies in some region of the Boolean arrangement.

Let $\mathcal{O}$ be the topological closure of some fixed orthant in $\mathbb{R}^d$. Let $\mathcal{A} = \mathcal{A}_{\mathcal{O}} := \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ be the vertices of $\mathcal{P}$ that lie in $\mathcal{O}$ ordered so that

$$\|\mathbf{a}_i\|_\infty \geq \|\mathbf{a}_{i+1}\|_\infty$$

for all $i \in [n-1]$. By [21, Theorem 2.2] $\mathcal{A}$ is a collection of $\{-1,0,1\}$-vectors and so

$$\mathrm{Mx}(\mathbf{a}_i) = \mathrm{supp}(\mathbf{a}_i)$$

for each $i \in [n]$. Let A denote the matrix whose i^{th} column is $\mathbf{a}_i$. Given a coloring $\mathbf{c} \in \mathcal{O}$ we call a vector $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{N}^n$ a **representation** of $\mathbf{c}$ if $A\mathbf{u} = \mathbf{c}$. A representation $\mathbf{u}$ of $\mathbf{c}$ is **valid** if

$$\bigcap_{i \in \mathrm{supp}(\mathbf{u})} \mathrm{supp}(\mathbf{a}_i) \neq \emptyset.$$

Lemma 4.14. *Let $\mathbf{c} \in \mathcal{O}$ for some fixed orthant $\mathcal{O}$. Let $\mathbf{a}_i \in \mathcal{A}$ such that*

$$\mathrm{Mx}(\mathbf{c}) \subseteq \mathrm{supp}(\mathbf{a}_i) \subseteq \mathrm{supp}(\mathbf{c}).$$

Then there is a valid representation $\mathbf{u} \in \mathbb{N}^n$ of $\mathbf{c}$ such that $u_i \neq 0$.

Proof. We proceed by induction on $\|\mathbf{c}\|_\infty$. If $\|\mathbf{c}\|_\infty$=1 and $\mathbf{a}_i \in \mathcal{A}$ such that

$$\mathrm{Mx}(\mathbf{c}) \subseteq \mathrm{supp}(\mathbf{a}_i) \subseteq \mathrm{supp}(\mathbf{c}),$$

then $\mathbf{c} \in \mathcal{A}$ and so $\mathrm{Mx}(\mathbf{c}) = \mathrm{supp}(\mathbf{c})$. It follows that $\mathbf{c} = \mathbf{a}_i$ which is a valid representation.

Now suppose $\|\mathbf{c}\|_\infty = k$. Since $\mathrm{Mx}(\mathbf{c}) \subseteq \mathrm{supp}(\mathbf{a}_i) \subseteq \mathrm{supp}(\mathbf{c})$, we have that $\mathrm{Mx}(\mathbf{c}) \subseteq \mathrm{Mx}(\mathbf{c}-\mathbf{a}_i)$ and if $j \in \mathrm{Mx}(\mathbf{c}-\mathbf{a}_i)$ then $(\mathbf{c}-\mathbf{a}_i)_j = k-1$. By the induction hypothesis we have a valid representation $\mathbf{u} \in \mathbb{N}^n$ of $\mathbf{c}-\mathbf{a}_i$ with $\mathbf{c}-\mathbf{a}_i = \sum_{j=1}^n u_j \mathbf{a}_j$. So $\mathbf{c} = \sum_{j=1}^n u_j \mathbf{a}_j + \mathbf{a}_i$ corresponds to a valid representation $\mathbf{u}'$ of $\mathbf{c}$ with $u'_i \neq 0$. The lemma follows by induction. □

Theorem 4.15. *Let $\mathbf{c} \in \mathcal{O}$ be a coloring with $\|\mathbf{c}\|_\infty = k$. If $\mathbf{u} \in \mathbb{N}^n$ is a valid representation of $\mathbf{c}$, then $\|\mathbf{u}\|_1 = k$.*

Proof. Let $\mathbf{u}$ be a valid representation of the coloring $\mathbf{c} \in \mathcal{O}$. Let $j \in \cap_{i \in \mathrm{supp}(\mathbf{u})} \mathrm{supp}(\mathbf{a}_i)$. Then there is a $\sigma \in \{-1,1\}$ such that $\sigma = (\mathbf{a}_r)_j$ for all $r \in \mathrm{supp}(\mathbf{u})$. Thus

$$\begin{aligned}
|\mathbf{c}_j| &= \left| \sum_{i=1}^n u_i (a_i)_j \right| \\
&= |\sigma| \left| \sum_{i=1}^n u_i |(a_i)_j| \right| \\
&= \sum_{i=1}^n u_i.
\end{aligned}$$

Since $j \in \bigcap_{i \in \mathrm{supp}(\mathbf{u})} \mathrm{supp}(\mathbf{a}_i)$, we have $\mathbf{c}_j \in \mathrm{Mx}(\tau)$. But this implies $\sum_{i=1}^{n} u_i = k$. $\square$

Corollary 4.16. *Every signed graph coloring* $\mathbf{c}$ *has a unique* $\prec_{revlex}$*-maximal valid representation.*

Let $\Bbbk$ be any field and let $\Bbbk[\mathbf{x}]$ be the polynomial ring over $\Bbbk$ with $n = |\mathcal{A}|$ indeterminates. Let $\Bbbk[\mathbf{z}^{\pm 1}]$ be the Laurent polynomial ring with one indeterminate for each vertex in our graph. Let π be the map $\pi : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{z}^{\pm 1}]$ defined by $x_i \mapsto \mathbf{z}^{\mathbf{a}_i}$. If $\mathbf{x}^{\mathbf{u}}$ is some monomial in $\Bbbk[\mathbf{x}]$, then $\pi(\mathbf{x}^{\mathbf{u}}) = \mathbf{z}^{\mathbf{c}}$ where $\mathbf{c} = \sum_{i=1}^{n} u_i \mathbf{a}_i$, i.e., π is the map that sends a monomial corresponding to a representation of a coloring to a monomial corresponding to the coloring itself.

Let G be the ideal of $\Bbbk[\mathbf{x}]$ given by

$$G := \left\langle x_{i_1} x_{i_2} \cdots x_{i_r} \;\middle|\; \bigcap_{j=1}^{r} \mathrm{supp}(\mathbf{a}_{i_j}) = \emptyset \right\rangle.$$

The monomials in the quotient ring $\Bbbk[\mathbf{x}]/G$ correspond to valid representations. Geometrically, we have produced a subdivision of the polytopal complex $\mathcal{P}$ such that each face of the subdivision lies on some face of $[-1,1]^d$, i.e., this subdivision satisfies condition (3) in Theorem 4.1. Furthermore, each face of this subdivision satisfies condition (4) by Lemma 4.10.

Repeating this process for each orthant $\mathcal{O}$ of $\mathbb{R}^d$ we obtain a polyhedral subdivision $\mathcal{C}$ of $\mathcal{P}$ that satisfies the conditions of Theorem 4.1. Consider the subcomplex $\mathcal{C}' := \mathcal{C} \cap \mathcal{H}'$. We note that a coloring $\mathbf{c} \in \mathcal{C} \setminus \mathcal{C}'$ if and only if it $\mathbf{c}$ is a zero-free proper coloring. Applying Theorem 4.1 to $\mathcal{C}$ and $\mathcal{C}'$ yields a Hilbert polynomial $f(k)$ which counts the number of zero-free colorings $\mathbf{c}$ with $\max(\mathbf{c}) = k$ for $k \in \mathbb{Z}_{>0}$. So $f(k) = \chi^{\star}_{nwz}(k) - \chi^{\star}_{nwz}(k-1)$. On the level of generating functions we have

$$\begin{aligned}
\sum_{k \geq 1} f(k) x^k &= \sum_{k \geq 1} \chi^{\star}_{nwz}(k) x^k - \sum_{k \geq 1} \chi^{\star}_{nwz}(k-1) x^k && (4.11)\\
&= \sum_{k \geq 1} \chi^{\star}_{nwz}(k) x^k - \sum_{k \geq 0} \chi^{\star}_{nwz}(k) x^k + 1 && (4.12)\\
&= \sum_{k \geq 1} \chi^{\star}_{nwz}(k) x^k - x \sum_{k \geq 0} \chi^{\star}_{nwz}(k) x^k && (4.13)\\
&= \sum_{k \geq 1} \chi^{\star}_{nwz}(k) x^k - x \sum_{k \geq 1} \chi^{\star}_{nwz}(k) x^k && (4.14)\\
&= (1-x) \sum_{k \geq 1} \chi^{\star}_{nwz}(k) x^k, && (4.15)
\end{aligned}$$

where equality 4.14 holds since $\chi^{\star}_{nwz}(0) = 0$ (the only 0-coloring is the coloring that assigns to each vertex the value 0).

Since $\sum_{k \geq 1} f(k) x^k$ is the Hilbert series of a graded ring R, we may write

$$\sum_{k \geq 1} f(k) x^k = \frac{\mathcal{K}(R, x)}{(1-x)^d},$$

where $\mathcal{K}(R, x)$ is a polynomial of degree less than d. Thus

$$\sum_{k \geq 1} \chi^{\star}_{nwz}(k) x^k = \frac{\mathcal{K}(R, x)}{(1-x)^{d+1}}.$$

So we have shown

Theorem 4.17. *There is a graded ring whose Hilbert polynomial is the zero-free signed graph coloring polynomial.*

In particular this theorem implies Steingrimsson's result ([18, Corollary 10]) that the chromatic polynomial, $\chi_G(k)$, of an ordinary (i.e., unsigned) graph G is a Hilbert polynomial. To see this consider a signed graph G with no half edges nor loose edges and whose edges all have positive sign. In this case a signed graph k-coloring of G may be interpreted as an ordinary coloring of the vertices of G using the $2k$ colors $\pm[k]$ where the signs of the edges are ignored. So $\chi^{\star}_{nwz}(k) = \chi(2k)$. Combining this with Theorem 4.17 yields the desire result.

We note that a result similar to Theorem 4.17 is attainable for the polynomial $\chi^{\star}(k)$ that counts the (not necessarily zero-free) colorings of a signed graph G.

Chapter 5

Concluding Remarks

In this chapter we propose some directions for further study and pay our dues.

5.1 Tension Polytopes

In Section 3.2, the two examples of tension polytopes were both examples of *zonotopes*. It would be interesting to know if all tension polytopes are zonotopes. This question is quite natural because we obtain tension polytopes by intersecting a cube with a subspace.

Furthermore, in Example 3.2 we saw that the tension polytope for C_3, the cycle on three vertices, was in fact a permutahedron. It is not the case that the tension polytope of any cycle is a permutahedron. Indeed, this fails for C_4 since its tension polytope $\mathcal{T}$ is

$$\begin{aligned}\mathrm{conv}\{&(-1,-1,1,1),(-1,1,-1,1),(-1,1,1,-1),(1,-1,-1,1),(1,-1,1,-1),(1,1,-1,-1),\\&(-1,1,0,0),(0,-1,1,0),(-1,0,1,0),(0,1,-1,0),(1,-1,0,0),(1,0,-1,0)\\&(0,-1,0,1),(0,0,1,-1),(-1,0,0,1),(0,1,0,-1),(0,0,-1,1),(1,0,0,-1)\}.\end{aligned}$$

We notice that the polytopes

$$\begin{aligned}&\mathrm{conv}\{(-1,-1,1,1),(-1,1,-1,1),(-1,1,1,-1),(1,-1,-1,1),(1,-1,1,-1),(1,1,-1,-1)\},\\&\mathrm{conv}\{(-1,1,0,0),(0,-1,1,0),(-1,0,1,0),(0,1,-1,0),(1,-1,0,0),(1,0,-1,0)\},\\&\mathrm{conv}\{(0,-1,0,1),(0,0,1,-1),(-1,0,0,1),(0,1,0,-1),(0,0,-1,1),(1,0,0,-1)\}\end{aligned}$$

are all permutahedra. We ask if there is a classification of tension polytopes of cycles in terms of permutahedra and to what extent such a classification can be extended to arbitrary tension polytopes.

5.2 Complexes arising from Inside-Out Polytopes

In our proof of Theorem 4.1 we provide unimodular triangulations for a class of inside-out polytopes via the algebraic machinery of toric ideals. In the tension, flow, and signed graph coloring cases it would be interesting to understand these triangulations in terms of the underlying graphs.

We may also ask whether any simpicial complex arising from a polytopal complex satisfying the conditions of Theorem 4.1 are constructible, Cohen-Macaulay, shellable, or permit convex ear decompositions. Answering these questions for the tension, flow, or signed coloring case would yield bounds on the coefficients of the respective $\mathcal{K}$-polynomials.

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