Classification of Ehrhart quasi-polynomials of half-integral polygons

A thesis presented to the faculty of
San Francisco State University
In partial fulfilment of
The Requirements for
The Degree

Master of Arts
In
Mathematics

by

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San Francisco, California

August 2010

CERTIFICATION OF APPROVAL

I certify that I have read *Classification of Ehrhart quasi-polynomials of half-integral polygons* by Andrew John Herrmann and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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In the 1960s, Eugene Ehrhart developed Ehrhart theory to enumerate lattice points in convex polytopes. An important tool in Ehrhart theory is the Ehrhart quasi-polynomial, which encodes information about continuous and discrete area, lattice boundary points, and lattice interior points. Here, we will give an introduction to Ehrhart theory and outline some of the methods used to characterize polytopes based on their corresponding Ehrhart quasi-polynomials. We will discuss work done recently, and then expand on this work to classify all half-integral polygons by the coefficients of their corresponding Ehrhart quasi-polynomials.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

ACKNOWLEDGMENTS

I would like to thank Serkan Hosten and Federico Ardila for their support and time during my defense. I would especially like to thank Matthias Beck—without his continued support, encouragement, and criticism, this work would not have been possible. I also wish to acknowledge Christopher O'Neill for his many helpful conversations. Finally, I thank Shauna Crawford for her love, time and support, for allowing me to talk through many proofs, and for all the ice cream runs.

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Chapter 1

Introduction

In 1899, Georg Pick discovered a useful relation between the area, interior lattice points, and boundary lattice points of an integral polygon.

Theorem 1.1. *(Pick's Theorem)[5] Let P be an integral polygon with i interior lattice points and b boundary lattice points. Then the area a of P is given by*

$$a = i + \frac{b}{2} - 1. \tag{1.1}$$

One can translate this to the language of Ehrhart theory:

Theorem 1.2. *(Pick's Theorem (rewritten)) Let P be an integral polygon. Let $L_P(t)$ be the number of lattice points in the t^{th} dilate of P. Then*

$$L_P(t) = at^2 + \frac{b}{2}t + 1, \tag{1.2}$$

where $a \geq \frac{b}{2} - 1$.

Pick's Theorem classifies all possible integral polygons by their area, interior, and boundary lattice-point counts—all possible Ehrhart polynomials of integral polygons are given by (1.2), with the constraint $a \geq \frac{b}{2} - 1$. Presently, there is no statement analogous to Theorem 1.2 for rational polygons or for higher dimensional polytopes; the Ehrhart (quasi)-polynomials of these objects are not classified in any way.

Here, we focus on a subset of that problem: rational polygons with denominator two. Rather than examining each constituent of the Ehrhart quasi-polynomials of these polygons, we focus on the odd constituents—dilating any half-integral polygon by a multiple of two results in an integral polygon, which are classified by Pick's Theorem. We use interior and boundary lattice points in our classification of half-integral polygons. This results in Theorem 3.7 and Theorem 3.8 for half-integral polygons without interior lattice points, and Theorem 3.14 and Theorem 3.12 for half-integral polygons with interior lattice points. For half-integral polygons without interior lattice points, there is a one-to-one correspondence between half-integral polygons we can construct with Theorem 3.8 and half-integral polygons that satisfy Theorem 3.7. This is not the case for half-integral polygons with interior lattice points. Conjecture 3.1 states that Theorem 3.14 is lacking, and the bound it presents regarding area and boundary lattice points is not a tight bound. We end the thesis with justification for Conjecture 3.1, and give several possible approaches to proving it.

Chapter 2

Background

2.1 Some preliminaries

Definition 2.1. A subset P of $\mathbb{R}^n$ is *convex* if for each $x, y \in P$, we have

$$\{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\} \subseteq P.$$

Definition 2.2. A *convex polytope* is the set

$$P = \left\{ \lambda_1 v_1 + \cdots + \lambda_k v_k : \ \lambda_j \geq 0, \quad \sum_{j=1}^{k} \lambda_j = 1, \right\} \tag{2.1}$$

for some $v_1, \ldots, v_k \in \mathbb{R}^n$.

In other words, a polytope is the convex hull of a finite number of points in $\mathbb{R}^n$, and we denote P in (2.1) as $\text{conv}(v_1, \ldots, v_k)$.

A *closed polygonal chain* is a non-intersecting path in $\mathbb{R}^2$ consisting of a finite number of line segments. Any closed polygonal chain separates $\mathbb{R}^2$ into two subspaces—one with finite area, and one with infinite area.

Definition 2.3. Let B be a polygonal chain. Let B' be the subspace defined by B with finite area. Then

$$P = B \cup B'$$

is a polygon.

This alternate definition of a dimension-two polytope is desirable as it does not require convexity.

Definition 2.4. Let $a_1, \ldots, a_n, c \in \mathbb{R}$, with at one non-zero. Then

$$H = \{v_1, \ldots, v_n \in \mathbb{R}^n : a_1 v_1 + \cdots + a_n v_n = c\}$$

is a *hyperplane* in $\mathbb{R}^n$.

A hyperplane divides $\mathbb{R}^n$ into two halves. For a convex polytope $P \in \mathbb{R}^n$, we say $v \in P$ is a vertex of P if for some hyperplane H containing v, $P \backslash v$ is contained in exactly one of the half-spaces determined by H and $P \backslash v \cap H = \emptyset$. If P has vertices with integer (rational) coordinates, we say P is an integral (rational) polytope. If a polytope P contains at least n linearly independent vectors, P is n-dimensional and we say P is a *n-polytope*.

Triangles and tetrahedra are examples of *simplices*. An n-dimensional simplex is a n-polytope with $n+1$ vertices.

We define the interior P° of P (as defined in 2.1) to be

$$\left\{\lambda_1 v_1 + \cdots + \lambda_k v_k : \lambda_j > 0, \qquad \sum_{j=1}^{k} \lambda_j = 1\right\}.$$

We say the *boundary* of P is $P \backslash P^\circ$.

We call $\mathbb{Z}^n \subset \mathbb{R}^n$ the integer lattice. The set $P \cap \mathbb{Z}^n$ refers to the integer lattice points (or simply lattice points) contained in the polytope P. When we refer to interior lattice points of P, we mean the set $P^\circ \cap \mathbb{Z}^n$; likewise, the set $\partial P \cap \mathbb{Z}^n$ contains the boundary lattice points of P. With this in mind, we define the discrete volume of P to be the number of elements in $P \cap \mathbb{Z}^n$.

We define the t^{th} dilate of P to be the set

$$\{tx : x \in P\}.$$

Call this set tP. For any $P \subset \mathbb{R}^d$, we can count lattice points in tP. We denote this lattice-point count of a polytope P by $L_P(t)$. With these ideas introduced, we begin our story.

2.2 Ehrhart Theory

In the early 1960s, the French math teacher Eugene Ehrhart noticed that the lattice-point counts of certain polygons, together with their dilates, formed patterns. Ehrhart hoped to describe $L_P(t)$ for generic polytopes. To explain his success, we first require a definition:

Definition 2.5. A function $g(t) = c_n(t)t^n + c_{n-1}(t)t^{n-1} + \cdots + c_1(t)t + c_0(t)$ is a *quasi-polynomial* if each $c_j(t)$ is a periodic function with integral period. The period of g is given by

$$p = \operatorname{lcm}\left(\{\operatorname{period}(c_k(t))\}_{k=0}^{n}\right).$$

One can think of a quasi-polynomial as a set of polynomials. Each polynomial in a quasi-polynomial is called a *constituent*. Since the coefficient of each t^j depends only on the value of t mod p,

$$g(t) = \begin{cases} c_n(0)t^n + \cdots + c_1(0)t + c_0(0) & \text{if} \quad t \equiv 0 \mod p, \\ c_n(1)t^n + \cdots + c_1(1)t + c_0(1) & \text{if} \quad t \equiv 1 \mod p, \\ \vdots & \vdots \\ c_n(p-1)t^n + \cdots + c_1(p-1)t + c_0(p-1) & \text{if} \quad t \equiv (p-1) \mod p \end{cases}$$

Theorem 2.1. *(Ehrhart's Theorem)[2] Let P be a convex, rational, n-polytope in $\mathbb{R}^n$. Then $L_P(t)$ is a quasi-polynomial, where each constituent has degree n. If P is a convex, integral polytope, $L_P(t)$ is a polynomial of degree n.*

We call $L_P(t)$ the *Ehrhart quasi-polynomial* of P. If two polytopes have the same Ehrhart quasi-polynomial, we say they are *Ehrhart equivalent*. We can construct an infinite series using $L_P(t)$:

$$\mathrm{Ehr}_P(z) = 1 + \sum_{t \geq 1} L_P(t) z^t.$$

This is the *Ehrhart series* of P.

Theorem 2.1 tells us—at least in the integral case—that $L_P(t)$ is among a well understood class of functions. Even when our polytope is rational, $L_P(t)$ can be thought of as a set of polynomials. Also convenient is that there is a relationship (discovered by Ehrhart) between $L_P(t)$ (which is concerned with the discrete volume of dilates of P) and the traditional measure of volume.

Theorem 2.2. *[2] Let P be a rational n-polytope in $\mathbb{R}^n$, with corresponding Ehrhart quasi-polynomial $L_P(t) = c_n(t)t^n + \cdots + c_1(t)t + c_0(t)$. Then* $\mathrm{vol}(P) = c_n(t)$.

This theorem tells us that $c_n(t)$ is a constant, rather than a periodic function. In general, determining the meaning behind, or possible values of, coefficients of a polytope's Ehrhart quasi-polynomial is a non-trivial task, but an attractive one. Since we are concerned with enumerating lattice points contained in polytopes, our view of polytopes is highly combinatorial. As such, one might hope the coefficients of $L_P(t)$ measure some attribute of P. While each coefficient (or coefficient function) of $L_P(t)$ may not count an attribute of P, we are able to characterize these coefficient functions by the possible values they may attain. The following theorem describes bounds on the coefficient functions of an Ehrhart quasi-polynomial of a polytope by placing a bound on the coefficients in its Ehrhart series. As with Theorem 2.1, a definition is required first.

Definition 2.6. Let P be a rational polytope with vertex set $\{v_1, \ldots, v_k\}$. Define the denominator of a vertex v_j to be the least common multiple of the denominator of each of its coordinates. Define the denominator d of P to be the least common multiple of the denominator of each vertex.

Theorem 2.3. *(Stanley's nonnegativity theorem)[7] Let P be a rational n-polytope denominator d. Then*

$$Ehr_P(z) = 1 + \sum_{t \geq 1} L_P(t) z^t = \frac{f(z)}{(1 - z^d)^{n+1}},$$

where f is a polynomial with nonnegative integer coefficients.

Since $L_P(t)$ determines $\mathrm{Ehr}_P(t)$, Theorem 2.3 helps describe the values attained by the coefficients of $L_P(t)$. One might hope the coefficients of $L_P(t)$ are likewise nonnegative; unfortunately, this is incorrect. A quick counterexample comes with the triangle T defined by $\mathrm{conv}((\frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}), (\frac{1}{2}, 1))$. This triangle has Ehrhart quasi-polynomial

$$L_T(t) = \begin{cases} \frac{1}{8}t^2 - \frac{1}{8} & \text{if } t \text{ is odd} \\ \frac{1}{8}t^2 + \frac{3}{4}t + 1 & \text{if } t \text{ is even} \end{cases} \tag{2.2}$$

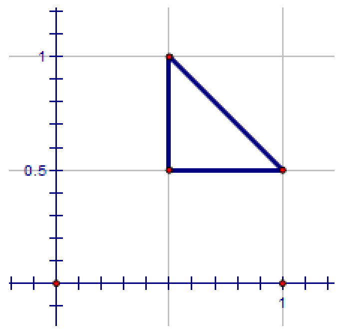

Figure 2.1: A polygon whose Ehrhart quasi-polynomial is given by 2.2

Ehrhart tells us we can also determine one additional cofficient of $L_P(t)$.

Proposition 2.4. *[2] Let P be a rational polytope. Then $L_P(0) = 1$.*

Thus if our polytope has an Ehrhart quasi-polynomial with period p, then the constant term of $L_P(pt)$ is 1. For integral polytopes, the constant term of the Ehrhart polynomial is identically 1. For an integral polytope, two coefficients of the Ehrhart polynomial are now known.

Proposition 2.4 illustrates that even though $L_P(t)$ is constructed to only be meaningful for positive integers, evaluating at 0 may provide additional information about our polytope. Theorem 2.5 extends this idea by evaluating the function at negative integers:

Theorem 2.5. *(Ehrhart–Macdonald reciprocity theorem)[1] Let P be a rational n-polytope. Then*

$$L_P(-t) = (-1)^n L_{P^\circ}(t).$$

The word "reciprocity" above refers to evaluations of $L_P(t)$ at additive inverses of t. With Theorem 2.5 in mind, we can determine the Ehrhart quasi-polynomial of the boundary of P.

Corollary 2.6. *Let P be a rational n-polytope. Then*

$$L_{\partial P}(t) = L_P(t) + (-1)^{n-1} L_P(-t).$$

Note that if P is an integral polygon with $L_P(t) = at^2 + \frac{b}{2}t + 1$, then

$$L_{\partial P}(t) = L_P(t) - L_P(-t) = bt.$$

For an integral polygon P, the linear coefficient of $L_P(t)$ is equal to half the boundary lattice-point count of the polygon. Corollary 2.6 is a simple example of the inclusion/exclusion principle.

Theorem 2.7. *(Inclusion/exclusion principle) If $A_1, \ldots, A_n$ are finite sets, then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i,j\,:\,1\le i<j\le n} |A_i \cap A_j| + \sum_{i,j,k\,:\,1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

This principle is essential to enumerative combinatorics. We have seen already that it allows us to determine the boundary lattice-point count of a polytope P from

$L_P(t)$. It also allows us to determine the lattice-point count of a rational polytope by dividing the polytope into smaller rational polytopes. This second idea will be expanded upon in Chapter 3.

Previously, we hoped to assign meaning or values to the coefficients of $L_P(t)$ for a general polytope P. If P is an integral polygon, $L_P(t)$ is well understood. The quadratic term of $L_P(t)$ is the area of P (Theorem 2.2), the linear term counts half the boundary lattice points of P (Corollary 2.6), and the constant term is 1 (Proposition 2.4). Unfortunately, this victory does not immediately extend to rational polygons or higher-dimensional polytopes P.

There has been much work recently examining periods of Ehrhart quasi-polynomials. Before we end this section and leave rational n-polytopes behind, we will briefly mention some of these results. Recall the denominator d of a polytope is the least common multiple of the denominators of the vertices. Further, if $L_P(t)$ has period q, then $q|d$[2].

By Definition 2.5, each coefficient function of a quasi-polynomial is periodic. For some polytope P, let p_j be the period of $c_j(t)$. The sequence of integers $(p_n, \ldots, p_0)$ is called the period squence of P.

Theorem 2.8. *[4] Given positive integers s and t, there exists a polygon P with period sequence $(1, s, t)$.*

For polygons in $\mathbb{R}^2$, we see any period sequence is possible within the restriction on $c_2(t)$ provided by Theorem 2.2. As a final thought, consider the following example, provided by Stanley:

Example 2.1. [7] Let P be the pyramid with vertices $(0,0,0), (1,0,0), (0,1,0), (1,1,0)$, and $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$. Then $L_P(t) = \binom{n+3}{3}$

Here we have an example of a rational polytope with denominator two whose Ehrhart quasi-polynomial is a polynomial. In those instances where the period of the quasi-polynomial of a polytope is less than the denominator of a polytope, we have *period collapse*. This phenomenon is mentioned here because later, as we classify polygons with denominator two, it is important to realize that their Ehrhart quasi-polynomials may be polynomials.

2.3 Pick and Scott

In the previous section, we began with theorems about general polytopes and discovered more information is available if we restrict ourselves to integral polygons. Specifically, for integral polygons, we can say much about the coefficients of Ehrhart polynomials, especially in regard to their interior and boundary lattice-point counts. Recall that Pick's Theorem classifies integral polygons by their area and interior and boundary lattice-point counts. Note that unlike some statements regarding polytopes, Pick's Theorem does not require our polygons be convex; the relationship between area, interior lattice-points and boundary lattice points holds for all integral polygon. Pick's Theorem is useful when working with integral polygons: given two of (a, i, b), the third can be determined by (1.2). Thus, not all triples (a, i, b) are possible.

If Ehrhart theory had been developed first, Pick's Theorem would have been no surprise. For some integral polygon P, consider $L_P(t) = at^2 + \frac{b}{2} + 1$. Recall that a and b are the respective area and boundary lattice-point counts of P. Thus the number of lattice points in P is

$$L_P(1) = a + \frac{b}{2} + 1 = i + b.$$

Some algebra yields Pick's Theorem; thus, it arises as a special case in Ehrhart theory. Pick's Theorem generalizes to higher dimensions, though more information about the polytope is required. Rather than knowing the interior and boundary lattice-point counts, as is sufficient in two dimensions, determining the volume of an integral n-polytope P requires knowing n different lattice-point counts of dilates of P. This is a result of Theorem 2.2 and linear algebra.

While Pick's Theorem places constraints on polygons once two of (a, i, b) are fixed, it does not give any bounds on possible values for i and b. A priori i and b can vary wildly and independently. It turns out, this intuition is true but with a simple restriction. Since all vertices are integral, and since there are no two-sided polygons, we must have at least three boundary lattice points. With this, we also see that any polygon must have area at least $\frac{1}{2}$ (in fact, Pick's Theorem tells us the area of any integral polygon is some multiple of $\frac{1}{2}$). For general polygons, the above observations and Pick's theorem show that i and b do vary with little limitations—for any value of i, we can find some polygon P with b boundary lattice points, where $b \geq 3$. We can see how to construct such a polygon in Figure 2.2.

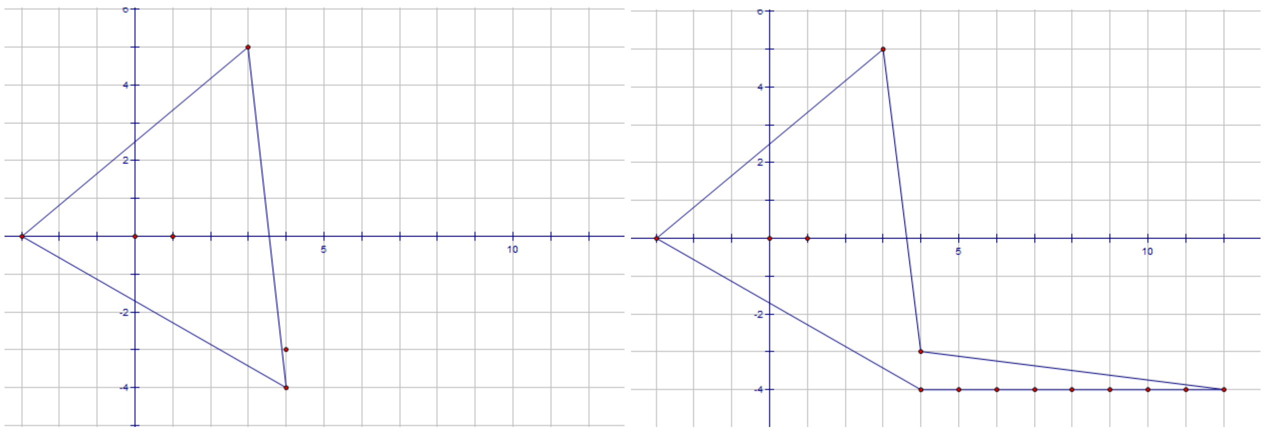

Figure 2.2: One can create an integral polygon with arbitrary interior and boundary lattice-point counts.

Recall that when we originally defined polytopes, we were concerned with convex subsets of $\mathbb{R}^n$. Requiring convexity does place constraints on the possible interior and boundary lattice point combinations of polygons. In 1976, P.R. Scott found these bounds for integral polygons.

Theorem 2.9. *(Scott's Inequality)[6] Let P be a convex integral polygon, with i interior and b boundary lattice points. Then the integer pair (i, b) can take on the following values*

1. $(0, b)$, $\qquad b \geq 3$
2. $(b, 2i + 6)$, $\qquad b \leq 2i + 6$

3. $(1, 9)$.

Scott's Inequality describes all convex integral polygons by their interior and boundary lattice points. Combining Theorem 2.9 with Pick's Theorem, we see the following are equivalent

1. $b \leq 2i + 6$

2. $a \leq 2i + 2$

3. $b \leq a + 4$

Scott's inequality has not been extended to rational polygons or higher-dimensional polytopes. This is a common theme in math—the leap from two- to three-dimensional objects is generally far more difficult than that from n- to $(n + 1)$-dimensional objects.

The goal of this work is to classify demoninator-two (or *half-integral*) polygons based on their area, interior lattice-point and boundary lattice-point counts. We will include all half-integral polygons in this classification and not restrict ourselves to convex polygons.

Chapter 3

Classification of half-integral polygons

3.1 Triangulations

Definition 3.1. Let P be an n-polytope. A *triangulation* S of P is a finite set of n-dimensional simplices such that

1. if $S = \{T_1, \ldots, T_k\}$, then $P = \bigcup_{j=1}^{k} T_j$;

2. any two simplices in T intersect in a lower dimensional simplex, or do not intersect.

Every polytope admits some triangulation [1]. While triangulating polytopes often raises numerous questions, here we will focus on the questions it helps to answer.

Definition 3.2. Let

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} : \{a, b, c, d, m, n\} \in \mathbb{Z} \text{ and } ad - bc = \pm 1$$

Then M is called an *affine unimodular transformation.*

Transformations of this form are helpful when applied to triangulations of polytopes. When acting on a polytope, a unimodular transformation preserves the interior and boundary lattice-point counts, as well as the area and denominator.

Definition 3.3. Let P and Q be polytopes, with $\{T_j\}_{j=1}^{k}$ a triangulation of P. Let M_j be a unimodular transformation of T_j. If

$$\{M_j(T_j)\}_{j=1}^{k}$$

is a triangulation of Q, then we say P and Q are *unimodularly equivalent.*

If two polytopes are unimodularly equivalent, they have the same Ehrhart quasi-polynomial [3]. If two *polygons* have the same Ehrhart quasi-polynomial, they are unimodularly equivalent [3]. Unimodular equivalence allows us to generate infinitely many polygons with identical area, interior lattice-point count, boundary lattice-point count, and denominator. Since these qualities are preserved under unimodular transformations, these transformations are a powerful tool when working with rational polytopes.

Lemma 3.1. *Let P be a polygon with denominator p. Then the area of P is a multiple of $\frac{1}{2p^2}$.*

Proof. Let P be a polygon with denominator p. Then pP is an integral polygon. When we dilate by p, the area of P is multiplied by p^2. By Pick's Theorem, the area of pP is a multiple of $\frac{1}{2}$. Therefore, the area of P is a multiple of $\frac{1}{2p^2}$. □

Lemma 3.2. *Let P be a polygon with denominator p. Then P can be triangulated into triangles of area $\frac{1}{2p^2}$.*

Proof. Every polygon admits some triangulation, and so it is sufficient to show Lemma 3.2 is true for an arbitrary triangle T with denominator p. By Lemma 3.1, the area of T is a multiple of $\frac{1}{2p^2}$. Therefore, we can induct on the area of pT. If pT has area $\frac{1}{2}$, then T has area $\frac{1}{2p^2}$ and the triangulation of T is trivial.

Now assume Lemma 3.2 holds for all triangles with denominator p and area less than $\frac{k}{2p^2}$, where $k > 1$. Let T have area $\frac{k}{2p^2}$. Then pT has area $\frac{k}{2}$. Since $k > 1$, Pick's theorem tells us pT contains at least four lattice points. Choose one lattice point q that is not a vertex of pT. We can now choose the three vertices of pT and q as vertices for triangles in a triangulation of pT. By induction, each of these triangles can be broken up into a union of triangles, where the area of each is a multiple of $\frac{1}{2}$.

Since T is a triangle with area $\frac{k}{2p^2}$ and since pT can be triangulated into triangles with area $\frac{1}{2}$, then T be triangulated into triangles with area $\frac{1}{2p^2}$. □

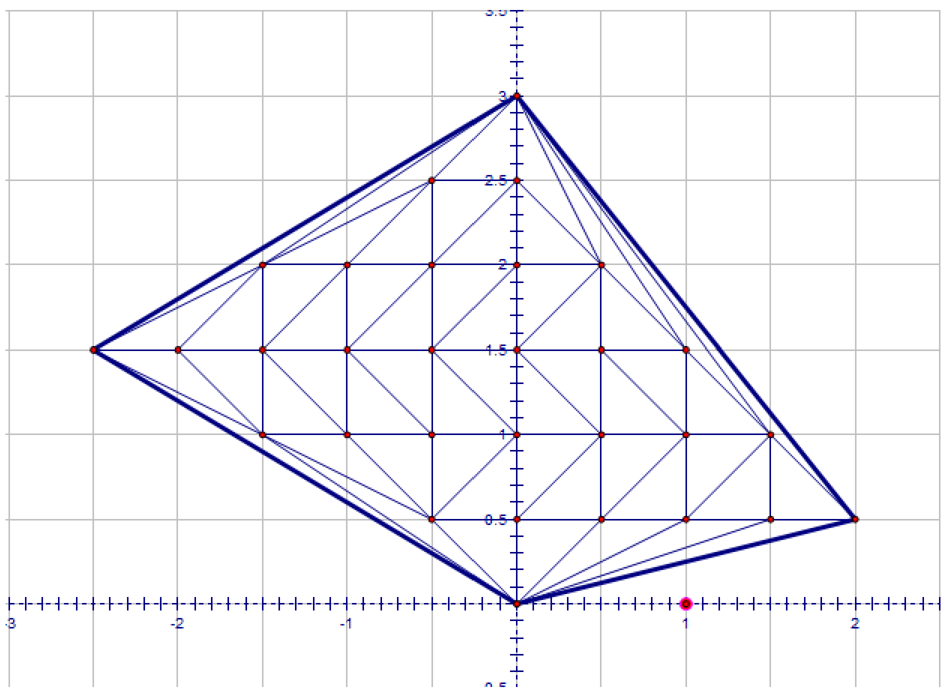

Figure 3.1: One possible triangulation of a polygon with denominator two. Note each triangle has area $\frac{1}{8}$.

3.2 Minimal Triangles

Lemma 3.2 states that for any half-integral polygon, there exists some triangulation S such that each triangle in S has area $\frac{1}{8}$. Call any half-integral triangle with area $\frac{1}{8}$ *minimal*. These triangles allow us to describe a half-integral polygon by breaking it up into smaller components. Instead of computing the Ehrhart quasi-polynomial for the entire polygon P, we first triangulate into minimal triangles. We then compute the Ehrhart quasi-polynomial for each triangle, and use the inclusion/exclusion principle to determine the quasi-polynomial of our original polygon. In practice, these operations may take far more effort than determining $L_P(t)$ using other methods. However, this approach will allow us to describe all half-integral polygons by their respective Ehrhart quasi-polynomials.

In our half-integral case, we have (at most) two constituents: one applies when t is even, the other when t is odd. For some half-integral polygon P, denote the odd constituent by $O_P(t)$.

Lemma 3.3. *Let P be a half-integral polygon, where $O_P(t) = at^2 + \frac{b}{2}t + c$. Then a is the area of P and b is the number of boundary lattice points in P.*

Proof. Let P be as above, with corresponding $O_P(t)$. By Theorem 2.2, a is the area of P. By Theorem 2.5,

$$O_P(t) - O_P(-t) = 2b,$$

since $t \equiv -t \mod 2$. □

Note this lemma is not true for general period p. For a polygon P, let $L_P^{(k)}(t)$ be the k^{th} constituent of $L_P(t)$. Then

$$L_P^{(k)}(-t) = L_P^{(p-k)}(t).$$

The key in Lemma 3.3 is that $-1 \equiv 1 \mod 2$.

Lemma 3.4. *Let T be a half-integral triangle with area $\frac{1}{8}$. Then T has one or zero lattice points. If T has one lattice point, $O_T(t) = \frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}$. If T has zero lattice* *points,* $O_T(t) = \frac{1}{8}t^2 - \frac{1}{8}$.

Proof. Let T be a triangle with area $\frac{1}{8}$. Suppose T contains two or more lattice points. Two lattice points differ in either their x- or y-coordinate by at least one. This implies the area of T is at least $\frac{1}{4}$, a contradiction. Suppose then that T contains exactly one lattice point. This point must be on the boundary—since T has area $\frac{1}{8}$, $2T$ has area $\frac{1}{2}$. By Pick's Theorem, $2T$ has no interior lattice points. Since we know the leading coefficient of $L_T(t)$ is the area of T, and since Lemma 3.3 tells us that the linear coefficient is half the boundary lattice points,

$$O_T(t) = \frac{1}{8}t^2 + \frac{1}{2}t + c, \tag{3.1}$$

where c is some constant. Evaluating 3.1 at $t = 1$ gives

$$1 = \frac{1}{8} + \frac{1}{2} + c$$

and so $c = \frac{3}{8}$. If T contains no lattice points, then again by Lemma 3.3 and our knowledge about the leading coefficient, we have

$$O_T(t) = \frac{1}{8}t^2 + c.$$

Evaluating at $t = 1$ yields

$$0 = \frac{1}{8} + c,$$

and so $c = -\frac{1}{8}$. □

Let us name the two types of half-integral minimal triangles given by Lemma 3.4.

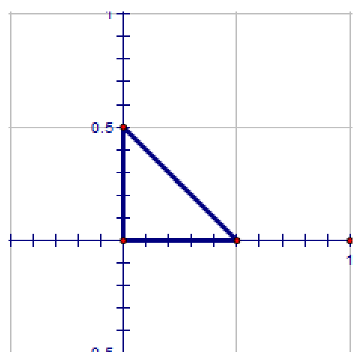

Figure 3.2: An example of a non-empty minimal triangle.

Definition 3.4. Let T be a half-integral minimal triangle. We say T is an *empty* minimal triangle if it contains no lattice points. We say T is a *non-empty* minimal triangle if it contains exactly one lattice point.

Earlier, we said that for some polygon P, we could determine $L_P(t)$ by first triangulating and then using the inclusion/exclusion principle. If T and T' are distinct minimal half-integral triangles, we would like to determine $O_{T\cup T'}(t)$.

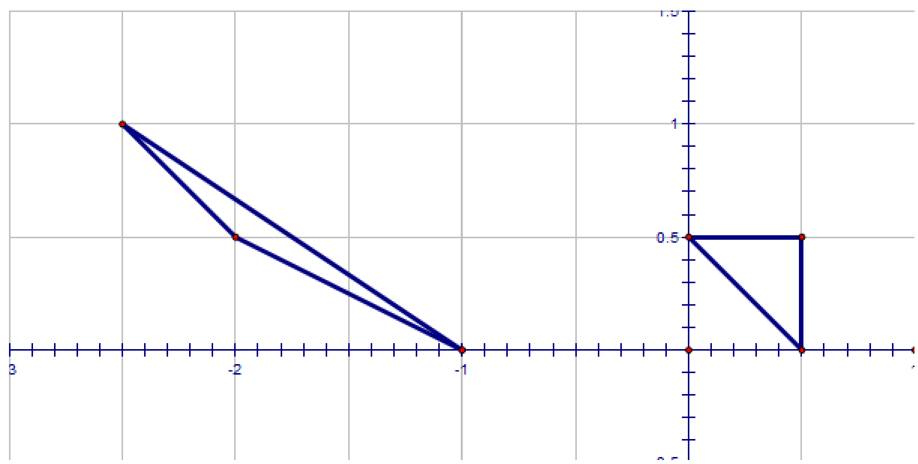

Figure 3.3: Two examples of empty minimal triangles.

Lemma 3.5. *Let P be a half-integral polygon, and let S be a triangulation of P. Let T and T' be two distinct minimal half-integral triangles in S.*

1. *If $T \cap T'$ is a line segment containing no lattice points, or is a non-integral point, then $O_{T \cup T'}(t) = O_T(t) + O_{T'}(t)$.*

2. *Suppose $T \cap T'$ contains some lattice point q. Let $S' = \{T_j \in S : q \in T_j\}$, and suppose S' contains k triangles. We have two cases:*

   (a) If q is an interior lattice point of P, then $O_S(t) = \left(\frac{3}{8}t^2 + \frac{5}{8}\right) + (k - 3)\left(\frac{1}{8}t^2 - \frac{1}{8}\right)$.

   (b) If q is a boundary lattice point of P, then $O_S(t) = \left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + (k - 1)\left(\frac{1}{8}t^2 - \frac{1}{8}\right)$.

Proof. Lemma 3.5 follows entirely from the inclusion/exclusion principle.

1. If $T \cap T'$ is a line segment containing no lattice points, or is a non-integral point, then no odd dilate of $T \cap T'$ will contain a lattice point. Therefore, $O_{T \cap T'}(t) = 0$, and by the inclusion/exclusion principle, $O_{T \cup T'}(t) = O_T(t) + O_{T'}(t)$.

2. Suppose $T \cap T'$ contains some lattice point q. Let $S' = \{T_j \in S : q \in T_j\}$, and suppose S' contains k triangles. Also suppose q is an interior lattice point of P. Then by the inclusion/exclusion principle and Lemma 3.4,

$$O_{S'}(t) = k\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) - k\left(\frac{1}{2}t + \frac{1}{2}\right) - \binom{k}{2} + k - \sum_{j=3}^{k}\binom{k}{j}(-1)^j. \tag{3.2}$$

   Here, we count the odd constituents of k non-empty minimal triangles, subtract the odd constituent of k line segments they intersect at, and add and subtract the lattice point q, as it was counted multiple times. Simplifying (3.2), and using $\sum_{j=0}^{k}\binom{k}{j}(-1)^j = 0$, we have

$$O_{S'}(t) = \left(\frac{3}{8}t^2 + \frac{5}{8}\right) + (k-3)\left(\frac{1}{8}t^2 - \frac{1}{8}\right).$$

   If q is a boundary lattice point of P, then by the inclusion/exclusion principle and Lemma 3.4,

$$O_{S'}(t) = k\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) - (k-1)\left(\frac{1}{2}t + \frac{1}{2}\right) - \binom{k}{2} + (k-1) - \sum_{j=3}^{k}\binom{k}{j}(-1)^j. \tag{3.3}$$

In (3.3) we subtract off only the odd constituent of $k-1$ line segments. Simplifying (3.3) yields

$$O_{S'}(t) = \left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + (k-1)\left(\frac{1}{8}t^2 - \frac{1}{8}\right).$$

□

Lemma 3.5 gives us a dictionary between half-integral polygons and the odd constituents of their Ehrhart quasi-polynomial. As a result, for a polygon P with i interior lattice points and b boundary lattice points, we have

$$O_P(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right) \tag{3.4}$$

for some integer k.

3.3 Minimal Segments

Let P be a half-integral polygon, and triangulate P. Call e a *minimal segment* of P if e is a half-integral line segment in P and e contains exactly two half-integral points. Let E be the set of minimal segments in P. Call minimal segments in ∂P *boundary* minimal segments. If a boundary minimal segment contains a lattice point, call it a *non-empty* boundary minimal segment. Otherwise, call it an *empty* boundary minimal segment.

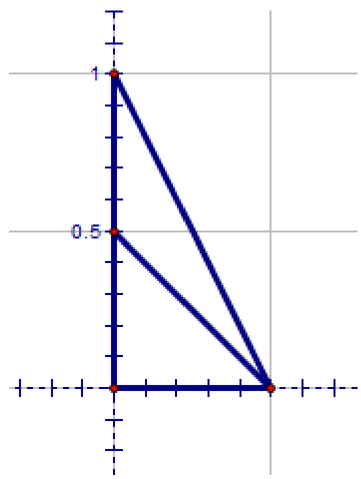

Figure 3.4: A triangle has five minimal segments, and four boundary minimal segments.

Since all half-integral polygons can be triangulated into two types of minimal triangles, we can construct any possible half-integral polygon by choosing a starting minimal triangle and adding additional triangles, one at a time, to boundary minimal segments of the polygon.

Lemma 3.6. *Let P be a half-integral polygon with N lattice points. Let T be a minimal triangle such that $T \cap P$ is exactly one boundary minimal segment of P. Let P have s empty boundary minimal segments.*

1. *If* $|(T \cup P) \cap \mathbb{Z}^2| = |P \cap \mathbb{Z}^2| + 1$, $T \cap P$ *does not contain a lattice point. Further,* $P \cup T$ *has* $s - 1$ *empty boundary minimal segments.*

2. *If* $|(T \cup P) \cap \mathbb{Z}^2| = |P \cap \mathbb{Z}^2|$ *and* $T \cup P$ *has one more minimal segment than* P*, then* $T \cup P$ *has* $s + 1$ *empty boundary minimal segments.*

3. *Let* Q *be a half-integral polygon with one interior lattice point and no boundary lattice points such that if* S_Q *is a triangulation of* Q*, each* $T \in S$ *contains a lattice point. If* Q *has area* $\frac{a_Q}{8}$*, and* $P \cap Q$ *is an empty minimal segment, then* $P \cup Q$ *has* $s + a_Q - 2$ *empty boundary minimal segments.*

Proof. 1. T contains exactly one lattice point. Since $T \cup P$ has one more lattice point than P, then $T \cap P$ cannot contain a lattice point. Further, only one minimal segment of T is an empty boundary minimal segment. That minimal segment is $T \cap P$, which is on the interior of $T \cup P$. Therefore, $T \cup P$ has $s - 1$ empty boundary minimal segment.

2. We have two cases. First, suppose T contains a lattice point, and P contains s boundary minimal segment. Since $|(T \cup P) \cap \mathbb{Z}^2| = |P \cap \mathbb{Z}^2|$, $T \cap P$ must contain a lattice point. Thus adding T to P changes $T \cap P$ to an interior minimal segment. Of the other two minimal segments of T, one of them does not contain a lattice point. Therefore, $P \cup T$ contains $s + 1$ empty boundary minimal segment.

   Suppose T does not contain a lattice point. Then $T \cap P$ is an empty boundary minimal segment of P. Adding T to P changes $T \cap P$ to an interior minimal segment. The two remaining minimal segment of T are now boundary minimal segment of $T \cup P$. Since T contained no lattice point, $T \cup P$ has $s+1$ empty boundary minimal segment.

3. Since Q has area $\frac{a_Q}{8}$, and since any triangle in a triangulation of Q contains a lattice point, Q has a_Q empty boundary minimal segments. Since $P \cap Q$ is a minimal segment, and since $P \cap Q$ is on the interior of $P \cup Q$, then $P \cup Q$ has $s + a_Q - 2$ minimal segments.

□

Call Q in Lemma 3.6 (3) a *basic polygon* While Lemma 3.6 is by no means a surprising result, it gives us a useful rule for constructing half-integral polygons. A second interpretation of Lemma 3.6 is the following: if P is a polygon where every boundary minimal segment contains a lattice point, then one cannot increase the boundary lattice-point count of P by adding exactly one non-empty triangle. Rather, one must first increase the number of empty boundary minimal segment (by adding a triangle that does not increase the lattice-point count of P).

3.4 Half-integral Polygons without Interior Lattice Points

The ideas in the previous section developed tools needed to classify half-integral polygons containing no interior lattice points. Half-integral polygons with interior lattice points are more complicated, and are dealt with separately.

Theorem 3.7. *Let P be a half-integral polygon containing no interior lattice points. Then*

$$O_P(t) = b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right) \tag{3.5}$$

where b is the number of boundary lattice points in P, $k \geq b-2$ and $\frac{k+b}{8}$ is the area of P.

Proof. Let P be a half-integral polygon with no interior lattice points. By Lemma 3.5, $O_P(t)$ must be of the form (3.5). Thus we need only show the bound on k in Theorem 3.7 holds. First note that if a half-integral polygon has one or two boundary lattice points, Theorem 3.7 holds trivially. By Lemma 3.2, any half-integral polygon P can be triangulated into minimal triangles. Conversely, one can construct any half-integral polygon by beginning with a single minimal triangle T' and adding additional minimal triangles to T' at the boundary minimal segment of T' (and subsequently to the boundary minimal segment of the additional triangles, and so forth).

Let T' be a non-empty minimal triangle. Since T' has one empty boundary minimal segment e, we can add a non-empty triangle T'' at e. Thus $T' \cup T''$ has two boundary lattice points. By Lemma 3.6, if we wish to add additional lattice points to our polygon $T' \cup T''$, we must first increase the number of empty boundary minimal segments. Again by Lemma 3.6, adding one empty boundary minimal segment is equivalent to adding one triangle that does not increase the lattice-point count of the polygon. By Lemma 3.5, each triangle that does not alter the lattice-point count of a polygon contributes $\frac{1}{8}t^2 - \frac{1}{8}$ to the odd constituent of the Ehrhart quasi-polynomiarel of the polygon.

Therefore, when constructing any polygon P with $b \geq 3$ boundary lattice points and no interior lattice points, we need at least $b-1$ empty boundary minimal segments. Since a non-empty triangle has one empty boundary minimal segment, we need a total of $b-2$ empty boundary minimal segments to construct P. Each of these boundary minimal segments appears by adding one triangle that does not alter the lattice-point count of P. Thus for any half-integral polygon P with no interior lattice points,

$$O_P(t) = b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right),$$

where $k \geq b-2$. □

Theorem 3.7 describes all possible odd constituents of Ehrhart quasi-polynomials of half-integral polygons. Theorem 3.8 states the converse.

Theorem 3.8. *Consider the polynomial*

$$f(t) = b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right), \tag{3.6}$$

where $k \geq b-2$, and $b, k \in \mathbb{Z}_{\geq}0$. Then there exists a half-integral convex polygon P such that $O_P(t) = f(t)$.

Proof. Let $P = \operatorname{conv}\big((0,0), (\frac{1}{2}, \frac{1}{2}), (b-1, 0), (-\frac{k-b-1}{2}, \frac{1}{2})\big)$. This trapezoid has area $\frac{b+k}{8}$, b boundary lattice points and no interior lattice points. By Theorem 2.2 and Lemma 3.3,

$$O_P(t) = \frac{b+k}{8}t^2 + \frac{b}{2}t + c.$$

Since $O_P(1) = b$,

$$b = \frac{b+k}{8} + \frac{b}{2} + c,$$

and $c = \frac{3b-k}{8}$. Thus

$$O_P(t) = b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right).$$

□

3.5 Half-integral Polygons with Interior Lattice Points

Note in the proof of Theorem 3.8, the example given is a convex polygon. However, Theorem 3.7 does not assume convexity when describing the odd constituents of Ehrhart quasi-polynomials of half-integral polygons. Thus, all half-integral polygons without interior lattice points satisfy Theorem 3.7, regardless of convexity. Further, every quasi-polynomial described by Theorem 3.8 can be realized by a half-integral *convex* polygon with no interior lattice points. We would like to repeat this success for half-integral polygons with interior lattice points.

Lemma 3.9. *Let P be a half-integral polygon with interior lattice points. Then P must have area at least $\frac{3}{8}$.*

Proof. Let P be some half-integral polygon with at least one interior lattice point and area $\frac{n}{8}$. By Lemma 3.4, P cannot have area $\frac{1}{8}$. If P had area $\frac{1}{4}$, $2P$ would have area 1. By Pick's Theorem, this cannot happen, as each integral polygon must have at least three boundary lattice points. Figure 3.5 shows one example of a half-integral polygon with area $\frac{3}{8}$ and one interior lattice point. □

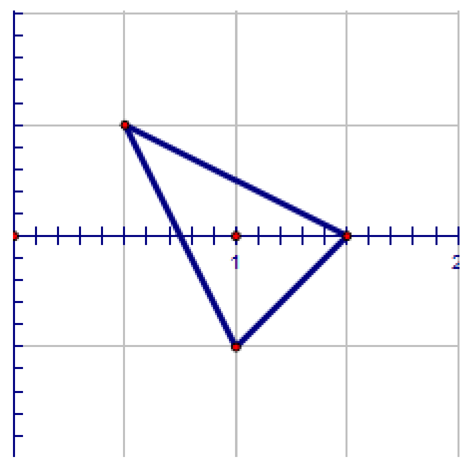

Figure 3.5: A polygon with one interior lattice point and area $\frac{3}{8}$, the minimum possible for a half-integral polygon.

In Lemma 3.9, the minimum area for a half-integral polygon with one interior lattice point is $\frac{3}{8}$. Lemma 3.10 shows us how this generalizes to i interior lattice points.

Lemma 3.10. *Let P be a half-integral polygon with i interior lattice points. Then P has area at least $\frac{3i}{8}$.*

Proof. Suppose P has one interior lattice point. By Lemma 3.9, P has area at least $\frac{3}{8}$.

Suppose for all half-integral polygons with $k < i$ interior lattice points, the area of the polygon is at least $\frac{3k}{8}$. Let P be a half-integral polygon with N interior lattice points. Triangulate P into minimal triangles. Choose a triangle T such that at least one minimal segment of T is on the boundary of P, and such that $P \backslash T$ is connected. If T has exactly one boundary minimal segment, remove that minimal segment from P. If T has two boundary minimal segments, remove these minimal segments and the vertex the minimal segments share. In either case, the new polygon P' has area $\frac{1}{8}$ less than that of P.

Continue this process until one of our removed triangles has an interior lattice point as a vertex. Now remove all triangles that share this vertex; there will be at least three, as seen in the proof of Lemma 3.9. If the remaining polygon is connected, it has $i - 1$ interior lattice points and area at least $\frac{3(i-1)}{8}$. If not, we have several half-integral polygons with a total of $i - 1$ interior lattice points between them. By the induction hypothesis, the total area of these polygons is at least $\frac{3(i-1)}{8}$. Note we decreased the area of P by at least $\frac{3}{8}$, and removed precisely one interior lattice point. Thus P has area at least $\frac{3i}{8}$. □

Lemma 3.11 shows that given any i, there exists a polygon with i interior lattice

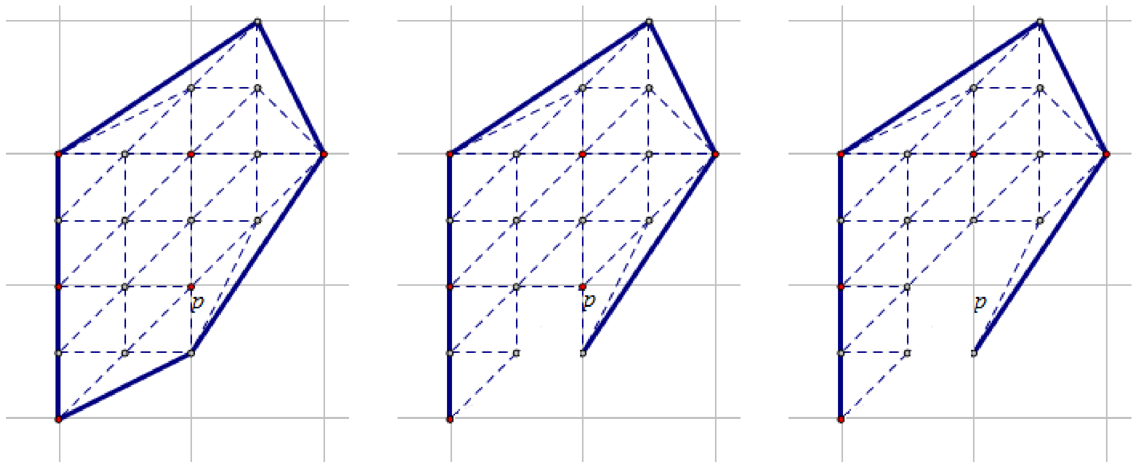

Figure 3.6: Proof of Lemma 3.9.

points and area $\frac{3i}{8}$.

Lemma 3.11. *Let* $\binom{x}{y} \in \mathbb{R}^2$. *Let* M *be the affine unimodular transformation given by*

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.7}$$

and let T *be the triangle with vertices* $\left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right)$. *Then*

$$P = \bigcup_{k=0}^{i-1} M^k(T) \tag{3.8}$$

is a polygon with i *interior lattice points,* 0 *boundary lattice points and area* $\frac{3i}{8}$, *with*

$$O_P(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right).$$

Proof. We first note that for any polygon Q, $M^k(Q)$ is a polygon with the same interior and boundary lattice-point counts and area. Further, the denominator of each vertex of Q is fixed under M. Thus, we need to show P given by (3.8) is connected and for all $j \neq k$, $\text{area}\big(M^j(T) \cap M^k(T)\big) = 0$.

For $i = 1$, we have $P = T$. See Figure 3.7 for $i = 2$.

Assume Lemma 3.11 is true for all $j < i$. In particular,

$$P' = \bigcup_{k=0}^{i-2} M^k(T) \tag{3.9}$$

is connected, has $i-1$ interior lattice points and has area $\frac{3(i-1)}{8}$. Since M is unimodular, $M(P')$ is connected, has $i-1$ lattice points and has area $\frac{3(i-1)}{8}$. The minimal segment e with endpoints $\left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, 0\right)$ is contained in $T \cap M(P')$. If $e \neq T \cap M(P')$, then by the unimodularity of M, $T \subset M(P')$. Note $M\binom{x}{y} = \binom{3x-y+1}{-2x+y-1}$. Thus for any $x \geq 0, y \leq 0$, $(1,0) \cdot M\binom{x}{y} \geq x$ and $(0,1) \cdot M\binom{x}{y} \leq y$. This implies that $(0, \frac{1}{2}) \notin M(P')$; otherwise, some triangle $M^j(T)$ equals T, which cannot happen. Therefore, $e = T \cap M(P')$. Therefore, the polygon $T \cup M(P')$ has area $\frac{3i}{8}$, i interior lattice points and no boundary lattice points.

Note that for M as above, $M^j(T) \cap M^k(T) \neq \emptyset$ if and only if j and k are consecutive. Further, since T has no boundary lattice points, $M^j(T) \cap M^{j+1}(T)$ has no lattice points in its intersection; thus,

$$O_{M^j(T) \cap M^{j+1}(T)}(t) = 0.$$

Since $M^j(T) \cap M^k(T)$, where $j \neq k$, contains no boundary lattice points,

$$O_P(t) = \sum_{j=0}^{i-1} M^j(T) = iO_T(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right).$$

□

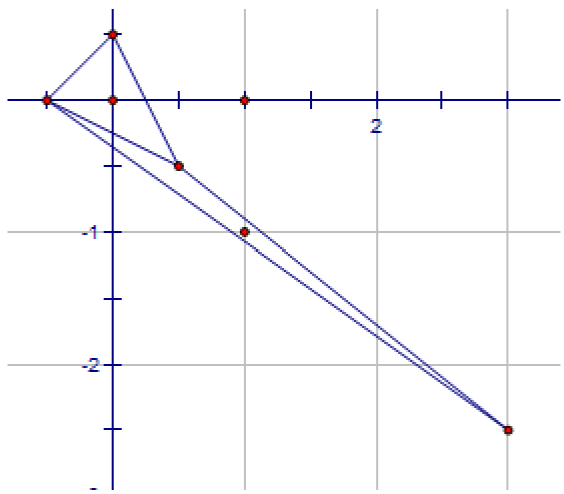

Figure 3.7: A half-integral polygon with two interior lattice points.

Lemma 3.11 allows us to construct a half-integral polygon with i interior lattice points and area $\frac{3}{8}i$. We can use this lemma to generate half-integral polygons with various interior and boundary lattice-point counts.

Theorem 3.12. *Consider the polynomial*

$$f(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right) \tag{3.10}$$

where $k \geq b-3$, and $i,k,b \in \mathbb{Z}_{\geq 0}$. Then there exists a half-integral polygon P such that $O_P(t) = f(t)$, with i interior lattice points and b boundary lattice points.

Proof. Let $f(t)$ be as in (3.10). By Lemma 3.11, the polygon

$$P' = \bigcup_{k=0}^{i-1} M^k(T)$$

has i interior lattice points, 0 boundary lattice points and area $\frac{3i}{8}$ (where T is the triangle with vertices $\left(\frac{1}{2} - \frac{1}{2}\right), \left(-\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right)$ and M is given by (3.7)). By Lemma 3.11,

$$O_{P'}(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right).$$

Let

$$\begin{aligned}
T_1 &= \operatorname{conv}\left(\left(\frac{1}{2}, -\frac{1}{2}\right), (0,1), \left(0, \frac{1}{2}\right)\right) \\
T_2 &= \operatorname{conv}\left(\left(0, \frac{1}{2}\right), \left(-\frac{1}{2}, 0\right), (-1,0)\right) \\
T_3 &= M^{i+1}\left(\operatorname{conv}\left(\left(-\frac{1}{2}, 0\right), (0,0), \left(\frac{1}{2}, -\frac{1}{2}\right)\right)\right).
\end{aligned}$$

For $b = 1,2,3$, let $P^{(b)} = P' \cup T_1, P' \cup T_1 \cup T_2, P' \cup T_1 \cup T_2 \cup T_3$, respectively. The intersection of T_i with P' contains no lattice points, and so

$$O_{P^{(b)}}(t) = O_{P'}(t) + b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right).$$

For $b > 3$, let

$$P^{(b)} = (\mathrm{conv}(T_1, (0, b-2))) \cup P^{(3)}.$$

The resulting polygon $P^{(b)}$ has i interior lattice points and b bounday lattice points. Further, we added $2(b-3)$ nonempty minimal triangles to $P^{(3)}$. When the intersection of any two of these triangles contains a boundary lattice point, the odd constituent of the Ehrhart quasi-polynomial of that intersection is $\frac{1}{2}t + \frac{1}{2}$. There are $b-3$ places where two non-empty minimal triangles intersect such that the intersection contains a boundary lattice point, and so

$$O_{P^{(b)}}(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + (b-3)\left(\frac{1}{8}t^2 - \frac{1}{8}\right).$$

Let $T_4 = \mathrm{conv}\left((-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, 0), (0, \frac{1}{2})\right)$. Then

$$O_{P^{(b)} \cup T_4}(t) = O_{P^{(b)}}(t) + \frac{1}{8}t^2 - \frac{1}{8} = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + (b-2)\left(\frac{1}{8}t^2 - \frac{1}{8}\right)$$

since $P^{(b)} \cap T_4$ is a line segment with no boundary lattice points. For $k > b-2$, let

$$P = \left(\mathrm{conv}\left(-\frac{k-b-3}{2}, \frac{1}{2}\right), T_4\right) \cup P^{(b)}.$$

By Lemma 3.5, the polygon $Q = \mathrm{conv}\left(\left(-\frac{k-b-3}{2}, \frac{1}{2}\right), T_4\right)$ has Ehrhart quasi-polynomial

$$O_Q(t) = k\left(\frac{1}{8}t^2 - \frac{1}{8}\right),$$

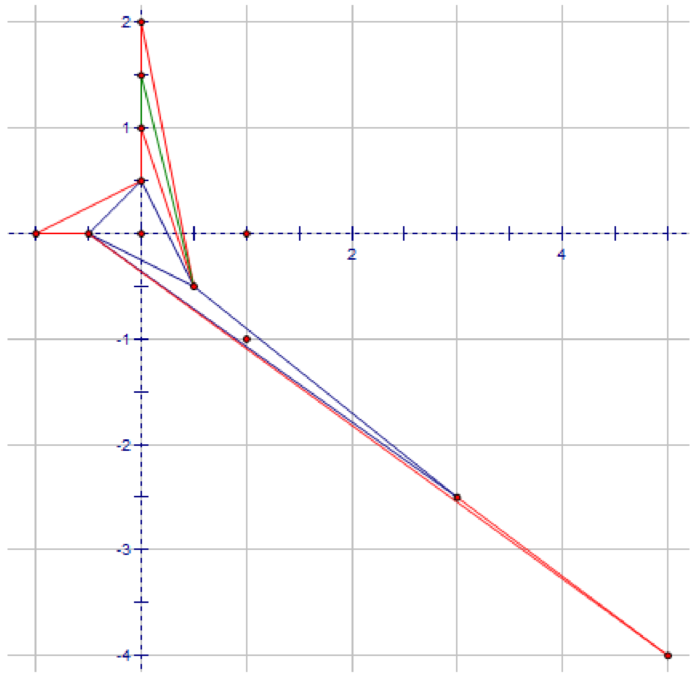

Figure 3.8: An half-integral polygon with two interior lattice points, four boundary lattice points and area $\frac{11}{8}$.

since it consists of k empty triangles. Let $P = Q \cup P^{(b)}$. Then

$$O_P(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right).$$

$\square$

In Theorem 3.7, we showed that given a half-integral polygon P with no interior lattice points, $O_P(t)$ was given by (3.6), where $k \geq b - 2$. Further, given $f(t)$ as in (3.6), one could construct a half-integral polygon P containing no interior lattice points such that $f(t) = O_P(t)$. If P has interior lattice points, however, this is no

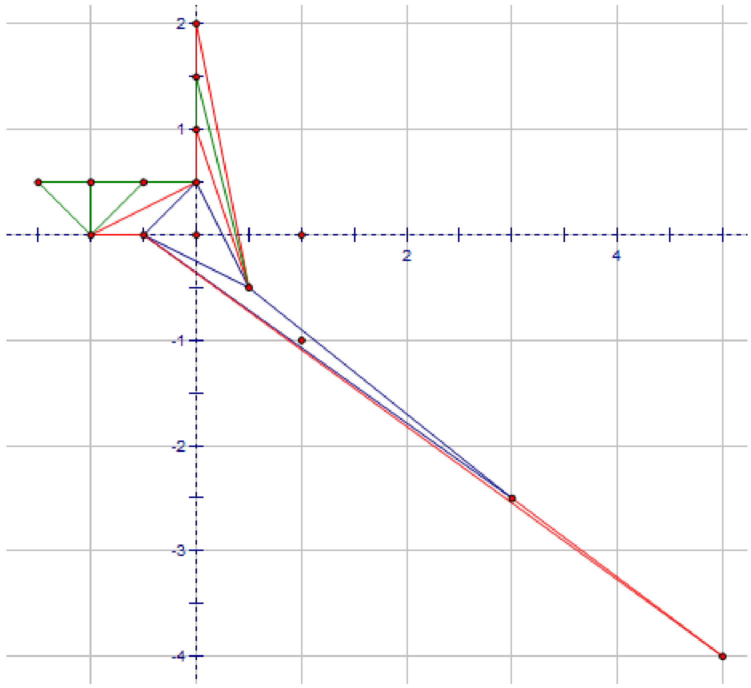

Figure 3.9: The final step in Theorem 3.12.

longer the case. Given $f(t)$ as in (3.10), we can construct a half-integral polygon P such that $O_P(t) = f(t)$. However, there may exist a polygon whose Ehrhart quasi-polynomial is not given by (3.10).

Lemma 3.13. *Let P be a half-integral polygon with interior lattice points. Let S be a triangulation of P into minimal triangles, and let q be an interior lattice point of P. Let $S' = \{T \in S : q \in T\}$. Then S' contains $k \geq 3$ elements, and $Q = \bigcup_{S'} T$ has k boundary empty minimal segments.*

Proof. Let P, S, q, S' and Q be as stated. By Lemma 3.9, S' has $k \geq 3$ elements. Since q is an interior lattice point, and since every triangle T containing q is in S',

q is an interior lattice point of Q. Further, for any triangle $T \in S'$, two minimal segments of T are interior minimal segments. Thus, Q has k boundary empty minimal segments. □

Definition 3.5. Let P be a half-integral polygon with boundary lattice points, and let S be a triangulation of P. Suppose e is an empty minimal segment in P. Call e a *sub-boundary empty minimal segment* if there exists a boundary lattice point $q \in P$ such that $\mathrm{conv}(e, q)$ is a minimal triangle.

For each boundary lattice point in P, there must be at least one sub-boundary empty minimal segment, since in any triangulation of P, each boundary lattice point is contained in at least one non-empty minimal triangle.

Theorem 3.14. *Let P be a half-integral polygon with i interior lattice points and b boundary lattice points. Then*

$$O_P(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2} + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}t^2\right) \tag{3.11}$$

where $k + i + 2 \geq b$.

Proof. Let P be a half-integral polygon, and let S be a triangulation of P. Suppose P has b boundary lattice points and $i \geq 2$ interior lattice points. By Lemma 3.5,

$$O_P(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2} + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}t^2\right),$$

and so we need only show that the bound $k + i + 2 \geq b$ holds.

For each interior lattice point q_j in P, let $P_j = \{T \in S : q_j \in P_j\}$. Let $\frac{a_j}{8}$ be the area of P_j. Then by Lemma 3.5,

$$O_{P_j}(t) = \left(\frac{3}{8}t^2 + \frac{5}{8}\right) + (a_j - 3)\left(\frac{1}{8}t^2 - \frac{1}{8}\right).$$

Further, P_j contains a_j empty minimal segments. Let $Q = \bigcup_{j=1}^{i} P_j$. By unimodularity, we can assume without loss of generality that Q is connected. We have

$$O_Q(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + \left(\sum_{j=1}^{i}(a_j - 3)\right)\left(\frac{1}{8}t^2 - \frac{1}{8}\right),$$

since $P_i \cap P_j$ is a line segment containing no lattice points.

Except for at most two, each basic polygon P_j shares at least two of its empty minimal segments with another basic polygon. Thus Q contains at most $i + 2 + \sum_{j=1}^{i}(a_j - 3)$ sub-boundary empty minimal segments. By Lemma 3.6, each triangle in S' adds one empty minimal segment to Q. Then S' contains at most $|S'|$ empty minimal segments, and

$$O_{Q \cup S'}(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + \left(\sum_{j=1}^{i}(a_j - 3) + |S'|\right)\left(\frac{1}{8}t^2 - \frac{1}{8}\right).$$

For each boundary lattice point b_j in P, let $R_j = \{T \in S : b_j \in T\}$. Let $\frac{c_j}{8}$ be the area of R_j. By Lemma 3.5,

$$O_{R_j}(t) = \frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8} + (c_j - 1)\left(\frac{1}{8}t^2 - \frac{1}{8}\right).$$

R_j is unimodulary equivalent to the union of one non-empty minimal triangle and $c_j - 1$ empty minimal triangles. By Lemma 3.6, these $c_j - 1$ empty minimal triangles add at most $c_j - 1$ sub-boundary empty minimal segments to $Q \cup S'$. Thus, P has at most $i + 2 + \sum_{j=1}^{i}(a_j - 3) + \sum_{j=1}^{b}(c_j - 1) + |S'|$ sub-boundary empty minimal segments, and

$$O_P(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2} + \frac{3}{8}\right) + \left(\sum_{j=1}^{i}(a_j - 3) + \sum_{j=1}^{b}(c_j - 1) + |S'|\right)\left(\frac{1}{8}t^2 - \frac{1}{8}\right).$$

Let $k = \sum_{j=1}^{i}(a_j - 3) + \sum_{j=1}^{b}(c_j - 1) + |S'|$. Let E be the set of sub-boundary empty minimal segments in P. Then $k + i + 2 \geq |E| \geq b$. □

3.6 A Final Conjecture

For half-integral polygons with interior lattice points, we are not able to construct all the half-integral polygons with Ehrhart quasi-polynomials that are given by Theorem 3.14. We believe the bound given by Theorem 3.14 is not tight.

Conjecture 3.1. *Let P be a polygon with i interior lattice points and b boundary* *lattice points. Then*

$$O_P(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right),$$

where $k \geq b - 3$.

Conjecture 3.1 tightens the bound on k, resulting in a one-to-one correspondence between half-integral polygons that can be constructed and half-integral polygons that are possible using Theorem 3.12. The remainder of this document will be dedicated to providing the motivation behind Conjecture 3.1.

Lemma 3.15. *Let* P *be a half-integral polygon with two interior lattice points and area* $\frac{5}{4}$ *such that*

$$(T \cup M(T)) \subset P \tag{3.12}$$

where T *is the triangle with vertices* $(-\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})$ *and* M *is given by (3.7). Then* P *has at most three boundary lattice points.*

Proof. Suppose $T \cup M(T) \subset P$ and P has area $\frac{5}{4}$. Here, $T \cup M(T)$ has four empty boundary minimal segments. Suppose P has four boundary lattice points and area $\frac{5}{4}$. Then we can construct P by adding a minimal triangle to each empty boundary minimal segments of $T \cup M(T)$, where each of these triangles adds exactly one boundary lattice point to $T \cup M(T)$.

Assume the minimal segment $\operatorname{conv}\big((0, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\big)$ is contained in a non-empty minimal triangle T_1. Then T_1 has vertex set $\big\{(0, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (a_1, -2a_1 + 1)\big\}$, where

a_1 is an integer. If a_1 is positive, however, P is self-intersecting (Figure 3.10). Thus

$$a_1 \leq 0. \tag{3.13}$$

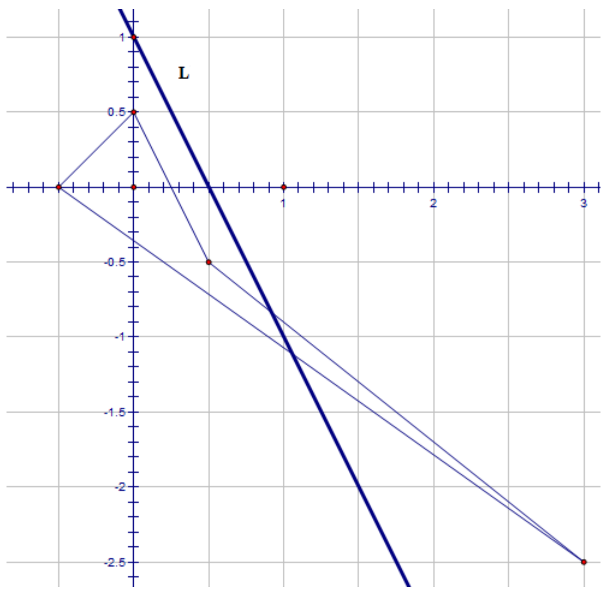

Figure 3.10: Line L contains possible vertices for a half-integral triangle that contains $\mathrm{conv}\big((0, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\big)$.

If the minimal segment $\mathrm{conv}\big((0, \frac{1}{2}), (-\frac{1}{2}, 0)\big)$ is contained in a non-empty minimal triangle T_2, T_2 has vertex set $\big\{(0, \frac{1}{2}), (-\frac{1}{2}, 0), (a_2, a_2 + 1)\big\}$, where a_2 is an integer. If $a_2 \geq 0$, then P is self-intersecting or shares a lattice point with T_1. If $T_2 \cap T_1$ contains a lattice point, then P has at most three boundary lattice points (due to the bound on the area of P) (Figure 3.11). Thus,

$$a_2 \leq -1. \tag{3.14}$$

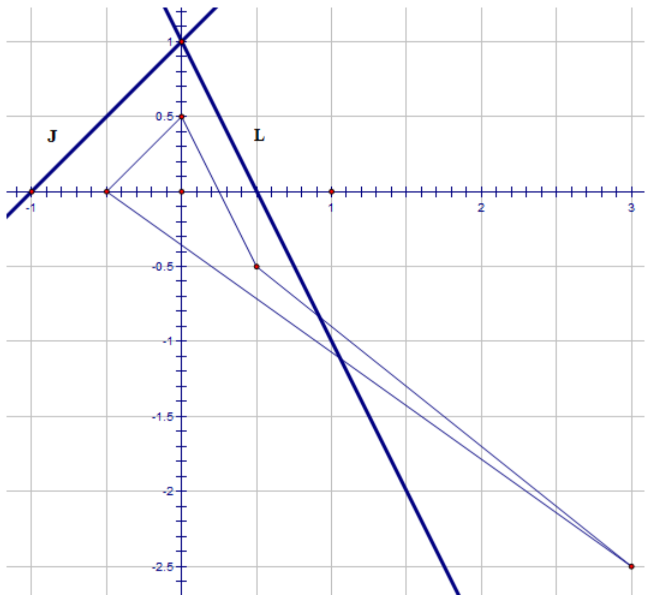

Figure 3.11: Line J contains possible vertices for a half-integral triangle that contains $\text{conv}\big((0, \frac{1}{2}), (-\frac{1}{2}, 0)\big)$.

If the minimal segment $\text{conv}\big(M\left((-\frac{1}{2}, 0), (\frac{1}{2}, -\frac{1}{2})\right)\big)$ is contained in a non-empty minimal triangle T_3, T_3 has vertex set $\big\{(-\frac{1}{2}, 0), (3, -\frac{5}{2}), (a_3, -\frac{5}{7}a_3 - \frac{3}{7})\big\}$, where a_3 and $-\frac{5}{7}a_3 - \frac{3}{7}$ are integers. If $a_3 \leq -2$, P is self-intersecting (Figure 3.12). Thus,

$$a_3 \geq 5. \tag{3.15}$$

Finally if the last minimal segment $\text{conv}\big(M\left((0,\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\right)\big)$ is contained in a non-empty minimal triangle T_4, T_4 has vertex set $\left\{(\frac{1}{2},-\frac{1}{2}),(3,-\frac{5}{2}),(a_4,-\frac{4}{5}a_4)\right\}$. a_4 must be a multiple of five; however, $a_4 \leq 0$ will cause P to be self-intersecting; if $a_4 \geq 5$, P will be self-intersecting or T_4 will intersect T_3 at a line segment containing a boundary lattice point (Figure 3.13). In either case, P has three boundary lattice points. If we assume $\text{conv}\big((0,\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\big)$ is not contained in a non-empty triangle, then $T \cup M(T)$ has only three empty mimimal segments. By Lemma 3.6, only three triangles that add boundary lattice points to $T \cup M(T)$ may be added to $T \cup M(T)$ (by our bound on the area of P), and so P cannot have more than three boundary lattice points.

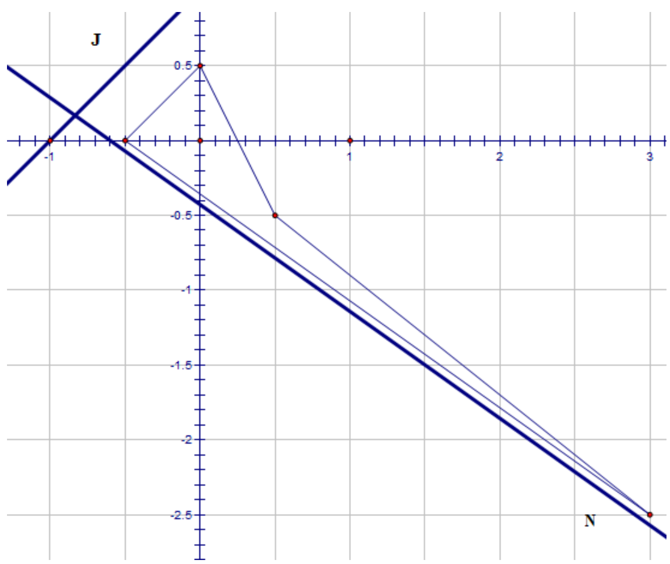

Figure 3.12: Line N contains possible vertices for a half-integral triangle that contains $\text{conv}\big(M\left((-\frac{1}{2},0),(\frac{1}{2},-\frac{1}{2})\right)\big)$.

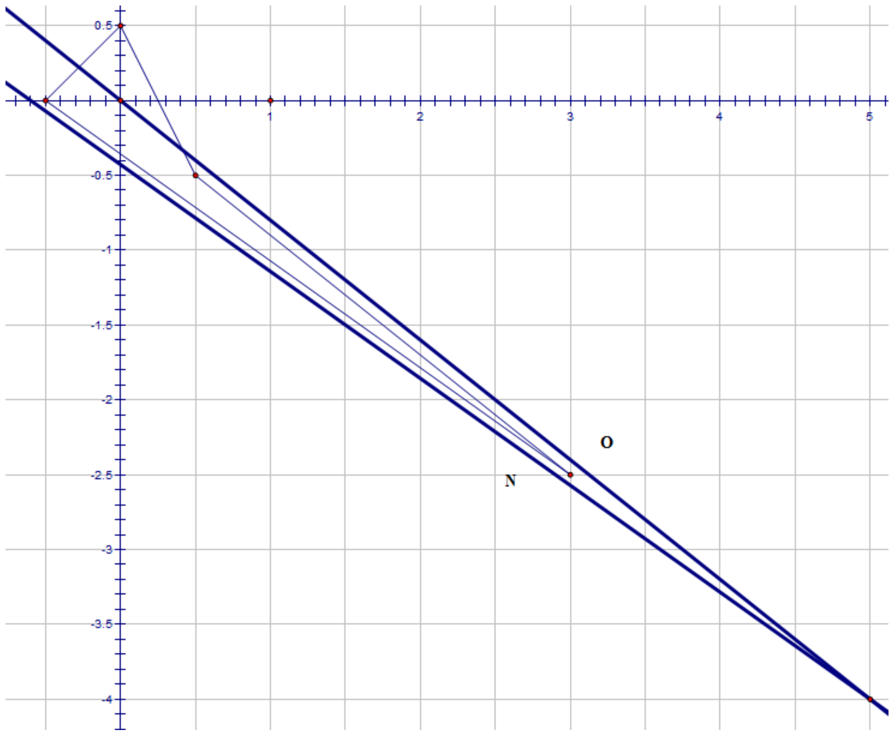

Figure 3.13: Line O contains possible vertices for a half-integral triangle that contains $\text{conv}\big(M\left((0,\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\right)\big)$. Note no vertices may be integral.

□

We can expand the argument from Lemma 3.15 to characterize a larger class of half-integral polygons.

Lemma 3.16. *Let P be a half-integral polygon with i interior lattice points and area $\frac{3}{8}i+\frac{1}{2}$ such that*

$$\left(\bigcup_{k=0}^{i-1} M^k(T)\right) \subset P, \tag{3.16}$$

where T is the triangle with vertices $(-\frac{1}{2},0),(0,\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})$ and M is given by (3.7). Then P has at most three boundary lattice points.

Proof. From Lemma 3.3, if P contains four boundary lattice points, any triangulation of P must contain at least four non-empty minimal triangles, each containing a boundary lattice point. Further, no two of these triangles may intersect at a line segment containing a lattice point.

Suppose $i=1$, $P \supset T$, and P has area $\frac{7}{8}$. By Lemma 3.6, since T has only three empty boundary minimal segments, P can have at most three boundary lattice points.

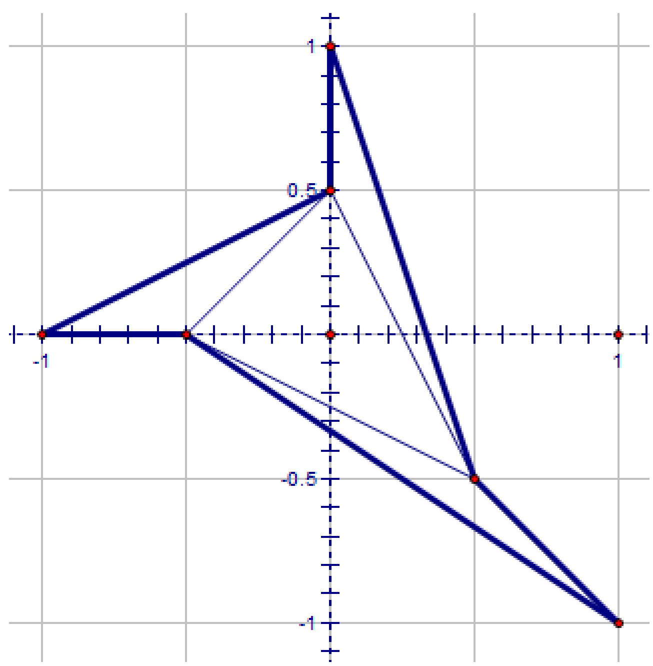

Figure 3.14: A polygon with area $\frac{3}{4}$ and three boundary lattice points. Any polygon with four boundary lattice points has area at least 1.

Suppose $T \cup M(T) \subset P$, P has area $\frac{5}{4}$ and P has two interior lattice points. Then by Lemma 3.15, P has at most three boundary lattice points.

Suppose $i > 1$, and suppose Lemma 3.16 holds for all $2 < j < i$. Also suppose that for all $j < i$, only the following can be sub-boundary empty minimal segments (for $i = 2$ this was trivially true):

$$E_1 = \text{conv}\left(\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)\right) \qquad E_2 = \text{conv}\left(\left(0, \frac{1}{2}\right), \left(-\frac{1}{2}, 0\right)\right)$$

$$E_3 = \text{conv}\left(M^{j-1}\left(\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, -\frac{1}{2}\right)\right)\right) \qquad E_4 = \text{conv}\left(M^{j-1}\left(\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)\right)\right).$$

Suppose P has area $i\frac{3}{8} + \frac{1}{2}$, and

$$\bigcup_{k=0}^{i-1} M^k(T) \subset P.$$

Note

$$P \supset Q = \bigcup_{k=1}^{i-1} M^k(T) = M\left(\bigcup_{k=0}^{i-2} M^k(T)\right).$$

By the induction hypothesis and unimodularity of M, if $P' \supset Q$ and P' has area $\frac{3}{8}(i-1) + \frac{1}{2}$, P' can have at most three boundary lattice points. Further, if the non-empty minimal triangles that contain these lattice points each share an minimal segment with Q, then these minimal segments are

$$E_5 = \text{conv}\left(M\left((\frac{1}{2},-\frac{1}{2}),(0,\frac{1}{2})\right)\right) \qquad E_6 = \text{conv}\left(M\left((0,\frac{1}{2}),(-\frac{1}{2},0)\right)\right)$$
$$E_7 = \text{conv}\left(M^{i-1}\left((-\frac{1}{2},0),(\frac{1}{2},-\frac{1}{2})\right)\right) \qquad E_8 = \text{conv}\left(M^{i-1}\left((\frac{1}{2},-\frac{1}{2}),(0,\frac{1}{2})\right)\right).$$

For E_5 to be a sub-boundary empty minimal segment, it must be contained in $\text{conv}\big((\frac{1}{2},\frac{1}{2}),(3,-\frac{5}{2}),(a_5,-\frac{4}{5}a_5)\big)$, where $a_5 \leq 0$ (otherwise, P' would be self-intersecting). For E_6 to be a sub-boundary empty minimal segment, it must be contained in $\text{conv}\big((-\frac{1}{2},0),(\frac{1}{2},-\frac{1}{2}),(a_6,-\frac{1}{2}a_6)\big)$.

Note $Q \cup T \subset P$. Thus, non-empty minimal triangles can only contain minimal segments E_1, E_2, E_7, E_8. E_6 can no longer be a sub-boundary empty minimal segment, and no other empty minimal segment can be a sub-boundary empty minimal segment, less P become self-intersecting; this follows from our induction hypothesis, since $Q \subset P$. We need only show that of these four minimal segments, no more than three may be sub-boundary empty minimal segments.

Recall from Lemma 3.15 that if E_1 and E_2 were contained in minimal triangles that add boundary lattice points to $Q \cup T$, and we arrived at inequalities (3.13) and (3.14), respectively. Also, $E_1 \subset T_1$ and $E_2 \subset T_2$, where T_1, T_2 are given by Lemma 3.15. Since these minimal segments appear again in $Q \cup T$, the same inequalities and set containments hold. Thus, suppose E_7 is a sub-boundary empty minimal segment. Then it is contained in $M^{i-2}(T_3)$ (where T_3 is given by Lemma 3.15). Two of the vertices of $M^{i-2}(T_3)$ are vertices of E_7. The third vertex q is on the line $M^{i-2}(a_3, -\frac{5}{7}a_3 - \frac{3}{7})$. By the induction hypothesis, in P',

$$(1,0)M^{i-2}\left(a_3, -\frac{5}{7}a_3 - \frac{3}{7}\right) \geq (1,0)M^{i-2}\left(3, -\frac{5}{2}\right);$$

otherwise, $M^{i-2}(T_3) \cap M(T_2)$ was non-empty, or they share an integer lattice point. This implies there are no integer points on the line $M^{i-2}(a_3, -\frac{5}{7}a_3 - \frac{3}{7})$ between $a_3 = 0$ and where $M^{i-2}(a_3, -\frac{5}{7}a_3 - \frac{3}{7})$ intersects the line given by $(a_6, -\frac{1}{2}a_6)$. Since the line given by $(a_6, -\frac{1}{2}a_6)$ has a more negative slope than the line given by $(a_2, a_2 + 1)$ (which contains a vertex of T_2), then for $M^{i-2}(T_3) \in P$,

$$(1,0)M^{i-2}\left(a_3, -\frac{5}{7}a_3 - \frac{3}{7}\right) \geq (1,0)M^{i-2}\left(3, -\frac{5}{2}\right).$$

Now we have that E_1, E_2, and E_7 are each contained in distinct non-empty minimal triangles. Further, $M^{i-2}(T_3) \in P$,

$$(1,0)M^{i-2}\left(a_3, -\frac{5}{7}a_3 - \frac{3}{7}\right) \geq (1,0)M^{i-2}\left(3, -\frac{5}{2}\right).$$

By the induction hypothesis again, when this constraint on the x-coordinate of q appeared for $q \in P'$, E_8 was not a sub-boundary empty minimal segment. Since $E_7, E_8 \subset P$, then when E_1, E_2, E_7 are each sub-boundary empty minimal segments, E_8 cannot be. Thus, P contains at most three sub-boundary non-empty minimal segments. By Lemma 3.6, for any half-integral polygon P with area $i\frac{3}{8}+\frac{1}{2}$ such that (3.16) holds, P has at most three boundary lattice points. □

Lemma 3.16 has a second interpretation. Let

$$P = \bigcup_{k=0}^{i-1} M^k(T)$$

for some i. Then Lemma 3.16 tells us we can add non-empty minimal triangles to P at exactly four minimal segments. Further, we can add triangles to no more than three of these minimal segments simultaneously.

This second interpretation finds a use in Lemma 3.17.

Lemma 3.17. *Let P be a half-integral polygon with i interior lattice points and b boundary lattice points such that (3.16) holds. Then*

$$O_P(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right), \tag{3.17}$$

where $k \geq b - 3$.

Proof. Suppose P is a half-integral polygon with i interior lattice points, and b boundary lattice points such that (3.16) holds. As in the proof of Theorem 3.7, since we can triangulate P, we can construct P by adding half-integral triangles to

$$Q = \bigcup_{k=0}^{i-1} M^k(T).$$

Q has $i + 2$ empty boundary minimal segments. Lemma 3.16 states that only three of these are sub-boundary empty minimal segments. By Lemma 3.6, for each additional boundary lattice point in P, we must add a sub-boundary empty minimal segment. By Lemma 3.6, this is accomplished by adding a triangle that does not add a boundary lattice point to Q.

By Lemma 3.5,

$$O_P(t) = i\left(\frac{3}{8}t^2 + \frac{5}{8}\right) + b\left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}\right) + k\left(\frac{1}{8}t^2 - \frac{1}{8}\right),$$

where $k \geq b - 3$. □

For the polygon in Lemma 3.17, the bound $k \geq b - 3$ holds. It appears that no matter how one arranges basic polygons and minimal triangles, one can only decrease the number of sub-boundary empty minimal segments. This observation inspires Conjecture 3.1.

Conjecture 3.2. *Let P be a half-integral polygon with i interior lattice points. Then P is unimodularly equivalent to a half-integral polygon containing*

$$\bigcup_{j=0}^{i-1} M^j(T).$$

Conjecture 3.3. *Let P and Q be two unimodularly equivalent half-integral polygons. Let T be a minimal triangle such that $P \cap T$ is a minimal segment. Then there exists* *some affine unimodular transformation M such that $M(T) \cap Q$ is a minimal segment.*

Presently, there is no clear way to proceed in proving Conjectures 3.2 or 3.3, but both imply Conjecture 3.1. Every polygon containing $\bigcup_{j=0}^{i-1} M^j(T)$ is described by Lemma 3.17, and so Conjecture 3.1 follows directly from Conjecture 3.2. With Conjecture 3.3, let P be a half-integral polygon that is unimodularly equivalent to Q, where $\bigcup_{j=0}^{i-1} M^j(T) \subset Q$. If there does not exist a minimal triangle T such that $T \cup Q$ has one more lattice boundary point than Q, then there does not exist T' such that $T' \cup P$ has one more lattice boundary point than P. Thus since Q satisfies Lemma 3.17, P satisfies Conjecture 3.1.

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