Generalized Frobenius numbers: asymptotics and two product families

A Thesis submitted to the faculty of

San Francisco State University

In partial satisfaction of the

requirements for

the Degree

Masters of Arts

in

Mathematics

by

Anika Daisy O'Donnell

San Francisco, California

May 2024

Certification of Approval

I certify that I have read Generalized Frobenius numbers: asymptotics and two product families by Anika Daisy O'Donnell and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirement for the degree Masters of Arts at San Francisco State University.

Matthias Beck, Ph.D
Professor
Thesis Committee Chair

Federico Ardila, Ph.D
Professor

Emily Clader, Ph.D
Associate Professor

Abstract

Given d positive integers $a_1, a_2, \ldots, a_d$ such that $\gcd(a_1, a_2, \ldots, a_d) = 1$, the Frobenius coin-exchange problem asks to find the largest number n that does not have a nonnegative integer solution $(x_1, x_2, \ldots, x_d)$ to the equation $n = a_1x_1 + a_2x_2 + \cdots + a_dx_d$. The generalized Frobenius problem asks to find the largest number n that does not have more than s distinct solutions to the above equation; this is the generalized Frobenius number. We prove that the generalized Frobenius number grows asymptotically like $(s(d-1)!\, a_1a_2 \cdots a_n)^{\frac{1}{d-1}}$. We also find explicit bounds for the generalized Frobenius number in three specific cases.

Acknowledgments

I would like to thank my advisor, Matthias Beck, for introducing the Frobenius problem to me, and for meeting with me nearly every week for two years. His wisdom has guided me to a level of mathematical maturity I could not have imagined when I entered this program. It is with his support that I consider myself a mathematician. I thank Federico Ardila and Emily Clader for their helpful comments on this paper, and for their personal support of my learning, development, and service inside and outside of class. I thank Kim Seashore for the uncountable hours of wisdom that helped me make the most of my Master's program, through teaching, presenting, and living well. I also thank the entire BAMM! community for their financial, professional, academic, and emotional support. I would like to thank every member of the San Francisco State University mathematics department; every class, meeting, event, and friendship has made my time here priceless. I thank my family for encouraging me to continue learning and pursuing math, even when the numbers disappeared. I thank Andrea Padilla for always having an answer to my questions.

This work was supported, in part, by a grant from SFSU BAMM: NSF S-STEM #1930373.

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 McMotivation

Let's say that you're throwing a 35^{th} birthday party for your friend who loves chicken nuggets. While looking at the menu (Figure 1.1), you see three options: a six-piece box, a nine-piece box, and a twenty-piece "Sharebox."

Figure 1.1: The nugget selection at McDonald's in the United Kingdom [21].

You notice that luckily, buying one of each box yields your friend's age in nuggets: $6 + 9 + 20 = 35$. So, you purchase one box of each size as a gift. The next year, you plan to buy a similar gift. While studying the menu, you realize that you can order 36 nuggets in three ways: 4 nine-pieces, 6 six-pieces, or 3 six-pieces and 2 nine-pieces. The following year, when your friend turns 37, you run into a snag. No matter how you combine your options, you find that it is impossible to buy exactly 37 nuggets. What is going on here? What numbers of nuggets are possible to buy? When do you have multiple options? Will there ever be a point when you can always buy your desired number of nuggets?

These questions are all aspects of what is known as the *Frobenius problem*, with study going back to the 1800s. In this paper, given a set A, we obtain results on the size of the *generalized* Frobenius number: the largest number that cannot be constructed in more than s ways by adding elements of A. The above scenario exhibits the case $A = \{6, 9, 20\}$.

1.2 Organization

In Chapter 2, we begin by going deeper into the definitions, history, and classical results of the Frobenius problem. We follow this by restating a generalization introduced by Beck and Kifer, along with some key results of the generalized Frobenius problem. We then, for the sake of completeness, include some preliminary results on quasi-polynomials, Dedekind and Dedekind-Fourier sums, and function asymptotics.

The main result in Chapter 3 is Theorem 3.3. Given an initial set of coprime positive integers $a_1, a_2, \ldots, a_d$, Theorem 3.3 gives an exact characterization of the generalized Frobenius number of A when

$$A = \{a_1 a_2 \cdots a_{d-1},\ a_1 a_2 \cdots a_{d-2} a_d,\ \ldots,\ a_2 a_3 \cdots a_d\},$$

the collection of all products of $d-1$ elements of A. This strengthens a theorem of Beck and Kifer. Our proof has a combinatorial flavor.

The main result in Chapter 4 is Theorem 4.2, an upper bound on the generalized Frobenius number of $A = \{a, b, c\}$ where $a|\mathrm{lcm}(b, c)$. This is a more general form of the $d = 3$ case in Chapter 3. The proof here is more technical, relying on coprimality in multiple ways.

In Chapter 5, the only restriction we place on the set A is for the constituent elements to be pairwise coprime. We first prove Theorem 5.1, giving an asymptotic result on the magnitude of the generalized Frobenius number. We then prove Theorem 5.2, giving a direct bound on the specific case $d = 3$. Finally, we demonstrate a method to find an explicit upper bound on the generalized Frobenius number for a set A of any size. Here, we take advantage of the nature of the restricted partition function with an analytic perspective. Consequently, this is the most technical chapter.

We conclude with future directions for research.

Chapter 2

Preliminaries

We begin with some terminology. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be the natural numbers including zero. We refer to this set as the **nonnegative integers**, and the set $\mathbb{N}\backslash\{0\}$ as the **positive integers**. We call a set $A = \{a_1, a_2, \ldots, a_d\}$ of positive integers **coprime** when $\gcd(a_1, a_2, \ldots, a_d) = 1$. Recall that $\gcd(a_1, a_2, \ldots, a_d) = \gcd(a_1, \gcd(a_2, \ldots, a_d))$. We call a set $A = \{a_1, a_2, \ldots, a_d\}$ of positive integers **pairwise coprime** when $\gcd(a_i, a_j) = 1$ for all $i \neq j$.

2.1 The Frobenius number

Let $A = \{a_1, a_2, \ldots, a_d\}$ be a set of d positive integers. If, for some $n \in \mathbb{N}$, we can find $x = (x_1, x_2, \ldots, x_d) \in \mathbb{N}^d$ that solves

$$n = x_1 a_1 + x_2 a_2 + \cdots + x_d a_d,$$

we call x a **representation** of n using the parts of A. For example, let $a_1 = 6$, $a_2 = 9$, and $a_3 = 20$. Our McMotivation above tells us that $n = 35$ has 1 representation, $n = 36$ has 3 representations, and $n = 37$ has 0 representations.

The **Frobenius problem** (or the **Chicken McNugget problem**[1]) asks which numbers have no representations, and specifically, which is *the greatest*. The following theorem characterizes when such a number exists.

Theorem 2.1. *Suppose that $A = \{a_1, a_2, \ldots, a_d\}$ is a set of positive integers. Then there exists a highest nonrepresentable nonnegative integer if and only if A is coprime.*

Proof. Suppose $\gcd(a_1, a_2, \ldots, a_d) = 1$. By Bézout's lemma, there exists a representation $x \in \mathbb{Z}^d$ for $n = 1$. The entries x_i of x may not be nonnegative. For each entry x_i that is negative, repeatedly add a_1 to x_i until it is positive. We now have a nonnegative integer representation x' for $n = 1 + k_1 a_1$, with k_1 being some positive integer. We may represent any number $n \equiv 1 \mod a_1$ greater than $1 + k_1 a_1$ by increasing k_1. Repeat this process with $2x, 3x, \ldots, a_1 x$ to get representations for $2 + k_2 a_1, 3 + k_3 a_1, \ldots, a_1 + k_{a_1} a_1$, and all numbers greater than them in their respective equivalence classes. Since there are only finitely many numbers less than $i + k_i a_1$ for each i, it follows that there is a unique maximal nonrepresentable element.

Now suppose $\gcd(a_1, a_2, \ldots, a_d) = m \geq 2$. Then only multiples of m are representable, so every number that is not a multiple of m cannot be represented. Therefore, there are

[1] A recent poll [19] shows that 2 out of 3 mathematicians prefer this latter name.

infinitely many nonrepresentable integers, so there is not a highest nonrepresentable integer.

□

We call the greatest nonnegative integer that has no representations the **Frobenius number** $g(A)$. We also write $g(a_1, a_2, \ldots, a_d)$, or just g when A is clear.

2.2 History of the Frobenius problem

Determining the Frobenius number when $d = 2$ is the simplest case, with a folkloric result that inspired the further study for higher d.

Theorem 2.2 (Unknown)**.** *If a and b are coprime positive integers, then $g(a, b) = ab - a - b$.*

The Frobenius problem and the Frobenius number are named after Ferdinand Georg Frobenius. Frobenius popularized the Frobenius problem by occasionally bringing it up in his lectures. Theorem 2.2 is often attributed to Frobenius, but it is unknown who first proved the result. However, it is likely that James Joseph Sylvester knew about Theorem 2.2 when he published the following result.

Theorem 2.3 (Sylvester [29])**.** *If a and b are coprime positive integers, then exactly half of the integers between 1 and $g(a, b) + 1 = ab - a - b + 1$ are representable.*

Not only does Sylvester's result implicitly tell us about Theorem 2.2, it also answers our question above about which numbers are representable at all. The proof given in [10] actually gives us more: for $i \in \{1, 2, \ldots, ab - 1\}$ such that i is not divisible by a or b, there is a correspondence that states that i is representable if and only if $ab - i$ is nonrepresentable. Since there are $(a-1)(b-1) = ab - a - b + 1$ integers between 1 and ab not divisible by a or b, Theorem 2.3 directly follows from the correspondence.

The beauty of Theorems 2.2 and 2.3 inspired research into $g(a, b, c)$ and beyond. Unfortunately, we have come up short. No closed form for $g(a, b, c)$ has been found, and it may not exist at all. Curtis [15] proved that there is no piecewise complex polynomial function that can determine $g(A)$, outside of the result of Theorem 2.2. However, there are many algorithms for determining the Frobenius number in polynomial time; see Chapter 1 of [25]. It is of interest to note that each of these algorithms depend on d being fixed, as Ramírez-Alfonsín [24] showed that varying d as a parameter will no longer admit a polynomial time algorithm.

From a theoretical perspective, we have some identities to rely on to simplify our search. A classic result by Brauer and Shockley says the following.

Theorem 2.4 (Brauer and Shockley [13]). *Let $A = \{a_1, a_2, \ldots, a_d\}$ be coprime. Let r_i be the least nonnegative integer congruent to i mod a_1 that is representable. Then*

$$g(a_1, a_2, \ldots, a_d) = \max_{0 \le i \le a_1 - 1} r_i - a_1.$$

This reduces the problem to searching for a_1 special elements $r_0, r_1, \ldots, r_{a_1-1}$ called the *Apéry numbers.* Apéry [3] introduced this set of “reached values” originally in the context of algebraic geometry.

The next result yields a way to “factor out” the elements of A, in a sense.

Theorem 2.5 (Johnson [18]). *Let* $A = \{a_1, a_2, \ldots, a_d\}$ *be coprime. If* $\gcd(a_2, \ldots, a_d) = m$, *then*

$$g(a_1, a_2, \ldots, a_d) = m\, g\left(a_1, \frac{a_2}{m}, \ldots, \frac{a_d}{m}\right) + a_1(m-1).$$

This greatly simplifies calculations when the elements of A have many common factors. We will remark on a generalization to this theorem in the next section.

Seeing as a simple closed-form formula for $g(A)$ in general remains out of our grasp, much research has been done when different restrictions are imposed on A. For example, Brauer found the Frobenius number of d consecutive elements.

Theorem 2.6 (Brauer [12]). *Let* a *be a positive integer. Then*

$$g(a, a+1, \ldots, a+d-1) = a\left(\left\lfloor \frac{a-2}{d-1} \right\rfloor + 1\right) - 1.$$

Roberts later found a generalization of Brauer's result for general arithmetic progressions.

Theorem 2.7 (Roberts [26]). *Let* a, d, k *be positive integers such that* $\gcd(a, d) = 1$. *Then*

$$g(a, a+k, \ldots, a+dk) = a\left(\left\lfloor \frac{a-2}{d} \right\rfloor + 1\right) + (k-1)(a-1) - 1.$$

Robles-Pérez and Rosales determined the Frobenius numbers of sequences of consecutive triangular numbers $T_n = \binom{n+1}{2}$ and tetrahedral numbers $TH_n = \binom{n+2}{3}$, viewing them as different generalizations of Brauer's result.

Theorem 2.8 (Robles-Pérez and Rosales [27]). *Let* n *be a positive integer. Then*

$$g(T_n, T_{n+1}, T_{n+2}) = \begin{cases} \frac{3n^3+6n^2-3n-10}{4} & \text{if } n = 2k+1, \\ \frac{3n^3+9n^2+6n-4}{4} & \text{if } n = 2k, \end{cases}$$

and

$$g(TH_n, TH_{n+1}, TH_{n+2}, TH_{n+3})$$

$$= \begin{cases} \frac{n-3}{3}TH_{n+1} + nTH_{n+2} + \frac{n}{2}TH_{n+3} - TH_n & \textit{if } n = 6k, \\ (n-1)TH_{n+1} + \frac{n-1}{2}TH_{n+2} + \frac{n-1}{3}TH_{n+3} - TH_n & \textit{if } n = 6k+1, \\ (n-1)TH_{n+1} + \frac{n-2}{3}TH_{n+2} + \frac{n}{2}TH_{n+3} - TH_n & \textit{if } n = 6k+2, \\ \frac{n-3}{3}TH_{n+1} + \frac{n-1}{2}TH_{n+2} + (n+1)TH_{n+3} - TH_n & \textit{if } n = 6k+3, \\ \frac{n+2}{3}TH_{n+2} + \frac{n+2}{2}TH_{n+1} + (n+2)TH_n - TH_{n+3} & \textit{if } n = 6k+4, \\ (n+4)TH_{n+2} + \frac{n+1}{3}TH_{n+1} + \frac{n+1}{2}TH_n - TH_{n+3} & \textit{if } n = 6k+5. \end{cases}$$

The introduction of [27] references many other works that determine the Frobenius number with certain restrictions imposed on A.

2.3 The generalized Frobenius number

As we have seen thus far, the Frobenius problem is incredibly deep. But we remark that it does not concern itself with *multiple* representations of integers. Instead, the Frobenius problem is only interested in when numbers have no representations at all. We take a step towards our desired generalization by first counting the number of representations: the **restricted partition function** $p_A(n)$ counts the number of representations of n that use only the elements of the set A as parts. For example, recalling the values found in our

McMotivation,

$$p_{\{6,9,20\}}(35) = 1, \qquad p_{\{6,9,20\}}(36) = 3, \qquad p_{\{6,9,20\}}(37) = 0.$$

We now can define a generalization of the classical Frobenius number. For $s \in \mathbb{N}$, the **generalized Frobenius number** $g_s^*(A)$ of a coprime set A is the greatest nonnegative integer that has s or fewer[2] representations using the parts of A. In other words,

$$g_s^*(A) = \max\{n \in \mathbb{N} : p_A(n) \leq s\}.$$

As above, we may also write $g_s^*(a_1, a_2, \ldots, a_d)$, or g_s^* when A is clear. Note that $g(A) = g_0^*(A)$. We desire analogues of results about g for g_s^*. One can find analogues to Theorems 2.2, 2.3, 2.4, and 2.5 in [8]. Below we will use their generalization of Theorem 2.5 many times, so we restate it here.

Theorem 2.9 (Beck and Kifer [8]). *Let $A = \{a_1, a_2, \ldots, a_d\}$ be coprime. If*

$$\gcd(a_2, a_3, \ldots, a_d) = m,$$

then

$$g_s^*(a_1, a_2, \ldots, a_d) = m\, g_s^*\left(a_1, \frac{a_2}{m}, \ldots, \frac{a_d}{m}\right) + a_1(m-1).$$

[2]The star in this notation reminds us that other authors have defined the "generalized Frobenius number" to be the greatest number that has *exactly* s representations, following the work of Beck and Robins [9] in 2002. Beck and Kifer [8] in 2010 introduced the g_s^* generalization after realizing that *s or fewer* representations resulted in much more natural results than those following the generalization of Beck and Robins.

2.4 Quasi-polynomials

In order to understand the generalized Frobenius number better, we need to learn more about the restricted partition function. How does it behave? How does it grow? How "strange" is it?

It turns out that for any set A, the restricted partition function $p_A(n)$ is a *quasi-polynomial*. A **quasi-polynomial** $q(n)$ is function on the integers that has the form

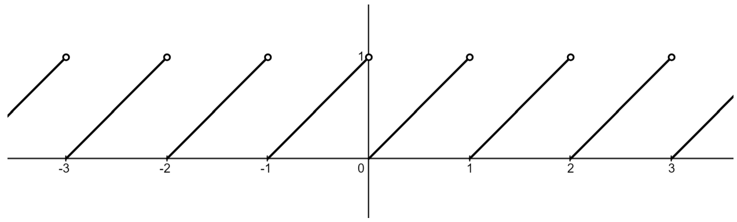
$$q(n) = q_d(n)n^d + q_{d-1}(n)n^{d-1} + \cdots + q_1(n)n + q_0(n),$$

where $q_i(n)$ are periodic functions on the integers satisfying $q_i(n + k_i) = q_i(n)$ for some $k_i \in \mathbb{N}$. Call $P = \operatorname{lcm}(k_0, k_1, \ldots, k_d)$ the period of $q(n)$. An equivalent way of defining a quasi-polynomial is by defining $P - 1$ **constituent polynomials** $p_i(n)$, and setting $q(n) = p_i(n)$ when $n \equiv i \mod P$. For example, we now see that $g(T_n, T_{n+1}, T_{n+2})$ and $g(TH_n, TH_{n+1}, TH_{n+2}, TH_{n+3})$ are quasi-polynomials in n.

In order to investigate the quasi-polynomial behavior of $p_A(n)$, we define the following functions. The **fractional part function** $\{x\}$ is given by $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. The **sawtooth function** $((x))$ is given by

$$((x)) = \begin{cases} \{x\} - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases}$$

Since $\{x\}$ and $((x))$ are themselves periodic (see Figure 2.1), they are trivially quasi-polynomials. They will assist in our construction of more complex quasi-polynomials. We



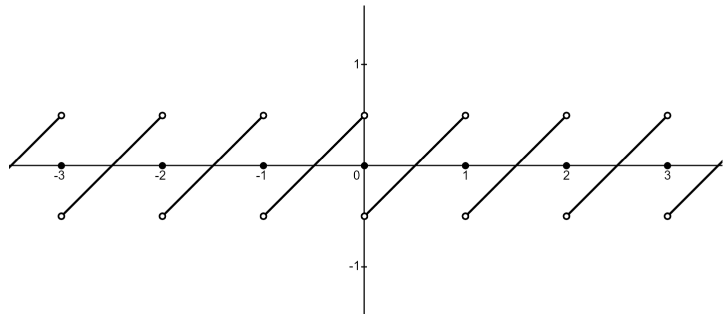


Figure 2.1: The fractional part function $y = \{x\}$ (above) and the sawtooth function $y = ((x))$ (below).

now state a celebrated theorem giving our first example of a concrete formula of a restricted partition function.

Theorem 2.10 (Barlow–Popoviciu formula [5, 22])**.** *If a and b are coprime positive integers, then*

$$p_{\{a,b\}}(n) = \frac{n}{ab} - \left\{ \frac{b^{-1}n}{a} \right\} - \left\{ \frac{a^{-1}n}{b} \right\} + 1,$$

where $b^{-1}b \equiv 1 \mod a$ and $a^{-1}a \equiv 1 \mod b$.

A proof using the machinery of generating functions can be found in Chapter 1 of [10].

It will be convenient for our purposes to decompose a quasi-polynomial $q(n)$ into a "polynomial part" $A(n)$ and a "quasi part" $B(n)$, such that $q(n) = A(n) + B(n)$. However, subtracting any polynomial from $B(n)$ and adding it to $A(n)$ will yield a different decomposition, so we need to work a bit harder to make our decomposition well-defined.

For a quasi-polynomial $q(n)$ with period P, we define the **periodic part** $\mathrm{Quasi}(n)$ of $q(n)$ to be the quasi-polynomial

$$\mathrm{Quasi}(n) = q_d(n)n^d + q_{d-1}(n)n^{d-1} + \cdots + q_1(n)n + q_0(n)$$

that satisfies $\sum_{n=1}^{P} q_i(n) = 0$ for all $0 \leq i \leq d$. We then define the **polynomial part** $\mathrm{Poly}(n)$ of $q(n)$ to be

$$\mathrm{Poly}(n) = q(n) - \mathrm{Quasi}(n).$$

2.5 Asymptotics of a function

Owing to the difficulty of the generalized Frobenius problem, and the quasi-polynomial nature of the restricted partition function, we find it necessary at times to consider the growth rate at a larger scale. The concept of *asymptotics* of a function $f(n)$ is a common one in analytic number theory; we take a step back and estimate the average growth rate as n goes to infinity. In this sense, n^2 grows faster than $\log n$, but grows at roughly the same rate as $n^2 + n$. Two functions $f(n)$ and $g(n)$ are hence considered to be of the same **order** if we have

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

We also denote this as $f \sim g$.

Here we state two theorems having to do with functions being of the same order. We will use these in Chapter 5.

Theorem 2.11 (Entringer [16]). *If $f(x) \to \infty$, $f(x) \sim g(x)$ and $h(x) \sim f^{-1}(x)$ as $x \to \infty$, and if $g^{-1}(x)$ exists, $h(x)$ is monotonic and $\frac{h'(x)}{h(x)} = O(\frac{1}{x})$ for all sufficiently large x, then $h(x) \sim g^{-1}(x)$ and hence $f^{-1}(x) \sim g^{-1}(x)$ as $x \to \infty$.*

Theorem 2.12 (Schur [28]). *Let $A = \{a_1, a_2, \ldots, a_d\}$ be coprime. Then as $n \to \infty$,*

$$p_A(n) \sim \frac{1}{(d-1)!\, a_1 a_2 \cdots a_d} n^{d-1}.$$

In other words, $\mathrm{Poly}_A(n)$ is a degree $(d-1)$ polynomial with leading term shown above.

2.6 Dedekind and Fourier-Dedekind sums

We now look at a class of sums that will determine the quasi-polynomial structure of the restricted partition function for certain sets A. The **Dedekind sum** $s(a, b)$ is given by the following, for coprime positive integers a and b:

$$s(a,b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b}\right)\right) \left(\left(\frac{k}{b}\right)\right).$$

The Dedekind sum has deep connections throughout number theory. The sum was initially found as a consequence of the behavior of Dedekind's eta function $\eta(\tau)$, making a link to the study of modular forms in analytic number theory. Additionally, we will state an elegant reciprocity theorem for the Dedekind sum below; this reciprocity is actually equivalent to the law of quadratic reciprocity.

Proposition 2.13. *Let a and b be coprime positive integers. Then*

$$s(a,b) = s(a \bmod b, b).$$

Proposition 2.14. *Let k be a positive integer. Then*

$$s(1,k) = \frac{(k-1)(k-2)}{12k}.$$

Proposition 2.15 (Dedekind reciprocity)**.** *Let a and b be coprime positive integers. Then*

$$s(a,b) + s(b,a) = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) - \frac{1}{4}.$$

For proofs of the above propositions, and further information on the Dedekind sum, see [4, 10, 23].

Using the machinery of *discrete Fourier analysis*, we now wish to generalize the Dedekind sum. Discrete Fourier analysis allows us to express periodic functions with period b in a new way: as a degree b polynomial in the bth complex root of unity $\xi_b := e^{\frac{2\pi i}{b}}$.

Theorem 2.16. *Let $a(n)$ be any periodic function on $\mathbb{Z}$, with period b. Then we have the following discrete Fourier series expansion of $a(n)$:*

$$a(n) = \sum_{k=0}^{b-1} \hat{a}(k)\xi_b^{nk},$$

where

$$\hat{a}(n) = \frac{1}{b}\sum_{k=0}^{b-1} \frac{a(k)}{\xi_b^{nk}}.$$

For example, [10] shows that the sawtooth function (as a function of $n \in \mathbb{Z}$) is given by the following discrete Fourier series expansion:

$$\left(\left(\frac{n}{b}\right)\right) = \frac{i}{2b}\sum_{k=1}^{b-1}\cot\left(\frac{\pi k}{b}\right)\xi_b^{nk}.$$

We can then immediately use this to restate the classical Dedekind sum:

$$s(a,b) = \frac{1}{4b}\sum_{k=1}^{b-1}\cot\left(\frac{\pi k}{b}\right)\cot\left(\frac{\pi a k}{b}\right).$$

It turns out[3] that finite sums of cotangents of this form may be rewritten as a finite sum of the form

$$\sum_{k=1}^{b-1}\frac{1}{(1-\xi_b^{a_1 k})\cdots(1-\xi_b^{a_d k})}\xi_b^{nk}, \qquad (*)$$

plus some error term we may compute. We will see a derivation of this fact later in Chapter 5. The consequences of this derivation will require the use of Hölder's inequality in order to handle the product structure these sums have. We state a generalized version of the inequality here for convenience.

Theorem 2.17 (Hölder's Generalized Inequality [14])**.** *Let $p, q_1, q_2, \ldots, q_n \in (0,\infty]$ such that*

$$\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n}.$$

Then for all measurable complex-valued functions $f_1, f_2, \ldots, f_n$ we have

$$\left\|\prod_{i=1}^{n} f_i\right\|_p \le \prod_{i=1}^{n}\|f_i\|_{q_i}.$$

[3]Here we will make major simplifications to this fascinating field, as its nuances end up being wiped out by our estimation techniques in later chapters. For more details on the application of discrete Fourier analysis to the Frobenius problem, see Chapter 7 of [10]. For a deep dive into the topic, see Terras' monograph [32].

In particular, we will use the norm

$$\left\| f(\cdot) \right\|_p = \sqrt[p]{\sum_{k=1}^{c-1} |f(k)|^p}$$

for any $p > 0$.

For now, we take the discrete Fourier series seen in $(*)$ as motivation for a generalization of the Dedekind sum $s(a,b)$. We define the **Fourier-Dedekind sum** $s_n(a_1, a_2, \ldots, a_d; b)$ to be the following discrete Fourier series:

$$s_n(a_1, a_2, \ldots, a_d; b) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{1}{(1-\xi_b^{a_1 k})(1-\xi_b^{a_2 k}) \cdots (1-\xi_b^{a_d k})} \xi_b^{nk}.$$

Later we will see how this generalizes the classical Dedekind sum. For now, we remark that this sum easily falls out of the generating function of a restricted partition function (again, see [10] for details). This derivation leads to the following theorem, allowing us precious insight into an entire class of restricted partition functions (and therefore, generalized Frobenius numbers).

Theorem 2.18 (Beck, Diaz, and Robins [6]). *Let $A = \{a_1, a_2, \ldots, a_d\}$ be pairwise coprime. Then the restricted partition function $p_A(n)$ for the set A is a quasi-polynomial with period* $\operatorname{lcm}(a_1, a_2, \ldots, a_d)$, *and the periodic part* $\operatorname{Quasi}(n) := \operatorname{Quasi}_A(n)$ *is given by the sum*

$$s_{-n}(a_2, a_3, \ldots, a_d; a_1) + s_{-n}(a_1, a_3, \ldots, a_d; a_2) + \cdots + s_{-n}(a_1, a_2, \ldots, a_{d-1}; a_d);$$

in particular, $p_A(n)$ is only periodic in its constant term.

An immediate corollary of the above theorem is that $\operatorname{Quasi}_A(n)$ is bounded by some real number $B \geq 0$, since $\operatorname{Quasi}_A(n) = \operatorname{Quasi}(n) = q_0(n)$ takes on a finite number of values. We note that this does not necessarily happen for some term of the form $q_i(n)n^i$ in a generic quasi-polynomial. We now take advantage of the bounded nature of $\text{Quasi}_A(n)$ in the following important lemma.

Lemma 2.19 (Approximation Lemma)**.** *Let* $A = \{a_1, a_2, \ldots, a_d\}$ *be pairwise coprime. If the restricted partition function* $p_A(n)$ *is given by* $\text{Poly}_A(n) + \text{Quasi}_A(n)$ *as above, then the generalized Frobenius number* $g_s^*(A)$ *is bounded above and below by the greatest roots of the polynomials* $\text{Poly}_A(n) - B - s$ *and* $\text{Poly}_A(n) + B - s$*, respectively, where* B *is a bound on the modulus of* $\text{Quasi}_A(n)$*.*

Proof. By definition, the generalized Frobenius number of a set A is the largest nonnegative integer n such that $p_A(n)$ is less than or equal to s. Define N to be the set of solutions in $n \in \mathbb{N}$ to the inequality $p_A(n) \leq s$, so that $g_s^*(A) = \max N$. By Theorem 2.18, we know that for A pairwise coprime $p_A(n)$ is equal to a polynomial $\text{Poly}_A(n)$ plus a periodic function $\text{Quasi}_A(n)$ that satisfies $\text{Quasi}_A(n) = \text{Quasi}_A(n + \text{lcm}(a_1, a_2, \ldots, a_d))$. Since $\text{Quasi}_A(n)$ takes on finite values, there exists some $B \geq 0$ such that $|\text{Quasi}_A(n)| \leq B$ for all $n \in \mathbb{N}$. Therefore, we have the polynomial approximations $\text{Poly}_A(n) - B$ and $\text{Poly}_A(n) + B$ that satisfy

$$\text{Poly}_A(n) - B \leq p_A(n) \leq \text{Poly}_A(n) + B$$

for all $n \in \mathbb{N}$. We define approximations of the set N corresponding to these polynomials:

$$N^+ = \{n \in \mathbb{N} : \text{Poly}_A(n) + B \leq s\} \quad \text{and} \quad N^- = \{n \in \mathbb{N} : \text{Poly}_A(n) - B \leq s\}.$$

We know that N^- will be nonempty since the generalized Frobenius number exists. N^+ may

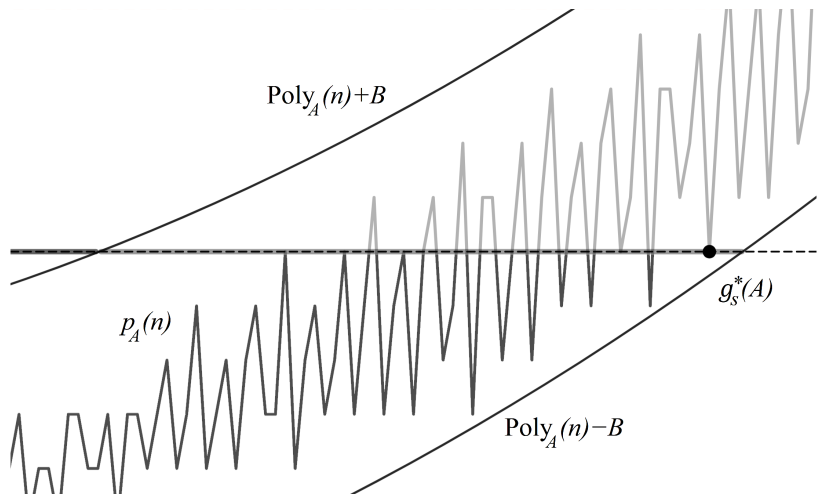


Figure 2.2: The sets N, N^+, and N^- are subsets of the domain of $p_A(n)$. Respectively, we represent them here as the dark gray part of $p_A(n)$, the darker shaded part of the horizontal line $y = s$, and the lighter shaded part of the line $y = s$. In this case, $A = \{6, 9, 20\}$.

be empty, in which case, define $N^+ = \{0\}$. Theorem 2.12 tells us that $\text{Poly}_A(n)$ is increasing, and thus we see that $N^+ \subset N \subset N^-$. Hence,

$$\max N^+ \leq \max N \leq \max N^-.$$

Obtaining $\max N^+$ or $\max N^-$ will give us lower and upper bounds on $g_s^*(A)$ respectively. Since $\text{Poly}_A(n) + B - s$ is continuous on $\mathbb{R}$, $\max N^+$ is precisely the maximum root of $\text{Poly}_A(n) + B - s$.

By a similar argument, $\max N^-$ is the maximum root of $\text{Poly}_A(n) - B - s$. □

We may interpret Lemma 2.19 and its proof graphically; see Figure 2.2. We know that the restricted partition function $p_A(n)$ is a quasi-polynomial, and moreover, the only periodic part of $p_A(n)$ is the constant term. Thus, $y = p_A(n)$ appears graphically as a jagged line, roughly following the path of a polynomial. We note that $p_A(n)$ is only defined on $\mathbb{N}$, but we draw it as a continuous function in Figure 2.2 for ease of viewing.

The generalized Frobenius number $g_s^*(A)$ shows up in Figure 2.2 as (the horizontal coordinate of) the dot to the right, once we plot $y = s$ as the horizontal dotted line. We have darkened the set N on the graph of $p_A(n)$. Thus, $g_s^*(A)$ is the rightmost highlighted value of N. We see in the figure that $N^- \backslash N^+$ contains $g_s^*(A)$.

Chapter 3

Generalized Frobenius numbers of products with many common factors

One prolific author on the Frobenius problem is Amitabha Tripathi, who in [33] proved the following result about a certain family of products.

Theorem 3.1 (Tripathi). *Let $a_1, a_2, \ldots, a_d$ be positive integers with product Π. Let $A_i = \frac{\Pi}{a_i}$ for $1 \leq i \leq d$. Let $\Sigma = A_1 + A_2 + \cdots + A_d$. If $a_1, a_2 \ldots, a_d$ are pairwise coprime, then*

$$g(A_1, A_2, \ldots, A_d) = \Pi(d-1) - \Sigma.$$

For the remainder of this section, we adopt the set of conditions in the first three sentences of the above theorem. Beck and Kifer [8] proved the following for the generalized Frobenius number.

Theorem 3.2 (Beck and Kifer [8]). *Suppose the same conditions as above hold. If we have* *that $a_1, a_2 \ldots, a_d$ are pairwise coprime, then*

$$g_s^*(A_1, A_2, \ldots, A_d) = \Pi(d+t) - \Sigma,$$

where $t \geq 0$ is the unique integer that satisfies

$$\binom{d+t-1}{d-1} \leq s < \binom{d+t}{d-1}.$$

In this chapter, we strengthen this theorem.

Theorem 3.3. *Suppose the same conditions as above hold. If $\gcd(a_1, a_2, \ldots, a_d) = 1$, then*

$$g_s^*(A_1, A_2 \ldots, A_d) = \Pi(d+t) - \Sigma,$$

where $t \geq -1$ is the unique integer that satisfies

$$\binom{d+t-1}{d-1} \leq s < \binom{d+t}{d-1}.$$

We remark that this is quite an extreme case of the behavior of g_s^*. As s varies, the integer t and therefore g_s^* may remain constant for arbitrarily long stretches of increasing s, provided s is sufficiently large.

The proof of Theorem 3.3 involves using Theorem 2.9 repeatedly to "factor out" all of the common factors for each collection of $d-1$ parameters. To investigate this reduced case, we look to the restricted partition function.

Proposition 3.4. *Let A be the multiset consisting of 1 with multiplicity d. Then*

$$p_A(n) = \binom{n+d-1}{d-1}.$$

Proof. We wish to count the number of distinct solutions $(x_1, x_2, \ldots, x_d) \in \mathbb{N}^d$ to

$$n = x_1 + x_2 + \cdots + x_d.$$

This is the combinatorial situation of placing n indistinguishable balls into d distinct boxes. As such, there are $\binom{n+d-1}{d-1}$ ways to do this. □

Proposition 3.5. *Let A be the multiset consisting of 1 with multiplicity d. Then $g_s^*(A) = t$, where $t \geq -1$ is the unique integer that satisfies*

$$\binom{d-1+t}{d-1} \leq s < \binom{d+t}{d-1}.$$

Proof. For n an integer greater than or equal to -1, the restricted partition function $p_A(n) = \binom{n+d-1}{d-1}$ increases as n increases. Therefore, for a given $s \geq 0$ there exists some unique $t \geq -1$ such that

$$\binom{d-1+t}{d-1} \leq s < \binom{d+t}{d-1}.$$

By definition of $g_s^*(A)$, we see t is the maximum value that solves $p_A(t) \leq s$. □

Proof of Theorem 3.3. We use Theorem 2.9 to "factor out" each a_i one by one. We start with a_1:

$$\begin{aligned}
g_s^*(A_1, \ldots, A_d) &= g_s^*\left(\frac{\Pi}{a_1}, \frac{\Pi}{a_2}, \frac{\Pi}{a_3}, \ldots, \frac{\Pi}{a_d}\right) \\
&= a_1\, g_s^*\left(\frac{\Pi}{a_1}, \frac{\Pi}{a_1 a_2}, \frac{\Pi}{a_1 a_3}, \ldots, \frac{\Pi}{a_1 a_d}\right) + \frac{\Pi}{a_1}(a_1 - 1) \\
&= a_1\, g_s^*\left(\frac{\Pi}{a_1}, \frac{\Pi}{a_1 a_2}, \frac{\Pi}{a_1 a_3}, \ldots, \frac{\Pi}{a_1 a_d}\right) + \Pi - A_1.
\end{aligned}$$

We now repeat for a_2:

$$\begin{aligned} g_s^*(A_1,\ldots,A_d) &= a_1\left[a_2\, g_s^*\left(\frac{\Pi}{a_1a_2},\frac{\Pi}{a_1a_2},\frac{\Pi}{a_1a_2a_3},\ldots,\frac{\Pi}{a_1a_2a_d}\right)+\frac{\Pi}{a_1a_2}(a_2-1)\right]+\Pi-A_1 \\ &= a_1a_2\, g_s^*\left(\frac{\Pi}{a_1a_2},\frac{\Pi}{a_1a_2},\frac{\Pi}{a_1a_2a_3},\ldots,\frac{\Pi}{a_1a_2a_d}\right)+\Pi-A_2+\Pi-A_1. \end{aligned}$$

We notice a pattern of each subsequent step multiplying the first term by a_i, then adding Π and subtracting A_i. We continue repeating for each common factor a_i, until we reach a_d:

$$\begin{aligned} &g_s^*(A_1,\ldots,A_d) \\ &\quad= a_1\cdots a_{d-1}\, g_s^*\left(\frac{\Pi}{a_1\cdots a_{d-1}},\frac{\Pi}{a_1\cdots a_{d-1}},\ldots,\frac{\Pi}{a_1\cdots a_{d-1}a_d}\right)+\Pi-A_{d-1}+\cdots+\Pi-A_1 \\ &\quad= a_1\cdots a_{d-1}\left[a_d\, g_s^*\left(\frac{\Pi}{a_1\cdots a_d},\frac{\Pi}{a_1\cdots a_d},\ldots,\frac{\Pi}{a_1\cdots a_d}\right)+\frac{\Pi}{a_1\cdots a_d}(a_d-1)\right] \\ &\qquad+\Pi-A_{d-1}+\cdots+\Pi-A_1 \\ &\quad= \Pi\, g_s^*(1,1,\ldots,1)+\Pi-A_d+\Pi-A_{d-1}+\cdots+\Pi-A_1 \\ &\quad= \Pi\, g_s^*(1,1,\ldots,1)+\Pi d-\Sigma. \end{aligned}$$

We now apply Proposition 3.5. Let $t \geq -1$ be the unique integer that satisfies

$$\binom{d-1+t}{d-1} \leq s < \binom{d+t}{d-1}.$$

Therefore $g_s^*(A_1,\ldots,A_d) = \Pi\, g_s^*(1,1,\ldots,1)+\Pi d-\Sigma = \Pi(d+t)-\Sigma$. □

Chapter 4

Generalized Frobenius number when $a|\mathrm{lcm}(b,c)$

Another paper of Tripathi [34] generalizes Theorem 3.1 in the case when $d = 3$. That is, if $A_1 = a_2a_3, A_2 = a_1a_3$, and $A_3 = a_1a_2$, where a_1, a_2 and a_3 are pairwise coprime, then $A_1|\mathrm{lcm}(A_2, A_3)$. This restriction on A using the least common multiple yields the following remarkable formula for the classical Frobenius number.

Theorem 4.1 (Tripathi)**.** *Let* $A = \{a, b, c\}$*, where* $\gcd(a, b, c) = 1$ *and* $a|\mathrm{lcm}(b, c)$*. Then*

$$g(a, b, c) = \mathrm{lcm}(a, b) + \mathrm{lcm}(a, c) - a - b - c.$$

Using similar methods to the ones we used in Chapter 3, we obtain the following bound on the generalized Frobenius number in this situation.

Theorem 4.2. *Suppose* $\gcd(a,b,c)=1$*, and* $a|\mathrm{lcm}(b,c)$*. Then*

$$g_s^*(a,b,c) \le \left[\frac{1}{4}a^2+\frac{1}{2}a\,\mathrm{lcm}(a,b)+\frac{1}{2}a\,\mathrm{lcm}(a,c)+\frac{1}{4}\,\mathrm{lcm}(a,b)^2+\frac{1}{4}\,\mathrm{lcm}(a,c)^2+\frac{1}{4}bc\,\mathrm{lcm}(a,b)\right.$$
$$\left.+\frac{1}{4}bc\,\mathrm{lcm}(a,c)+2abcs\right]^{\frac{1}{2}}+bc+\frac{1}{2}\,\mathrm{lcm}(a,b)+\frac{1}{2}\,\mathrm{lcm}(a,c)-\frac{3}{2}a-b-c,$$

and a similar lower bound holds.

By fixing a, b, and c, we obtain the following corollary describing the asymptotic behavior of g_s^*.

Corollary 4.3. *Suppose* $\gcd(a,b,c)=1$*, and* $a|\mathrm{lcm}(b,c)$*. Then*

$$g_s^*(a,b,c) \sim \sqrt{2abcs}.$$

We prove Theorem 4.2 by reframing the condition of $a|\mathrm{lcm}(b,c)$ in a different way. Taking inspiration from Chapter 3, we find that we can rewrite $A=\{a,b,c\}$ as a set of products instead.

Lemma 4.4 (Tripathi [34])**.** *Suppose* a,b,c *are positive integers, with* $\gcd(a,b,c)=1$*. Let* $\beta=\gcd(a,b)$ *and* $\gamma=\gcd(a,c)$*. If* $a|\mathrm{lcm}(b,c)$*, then* $a=\beta\gamma$*. Moreover,* $\gcd(\beta,\gamma)=1$*.*

Therefore, we may reduce finding $g_s^*(a,b,c)$ when $a|\mathrm{lcm}(b,c)$ and $\gcd(a,b,c)=1$ to finding $g_s^*(\beta\gamma,\beta m,\gamma n)$ with $a=\beta\gamma$, $\frac{b}{\beta}=m$, and $\frac{c}{\gamma}=n$. So, if we are able to determine the generalized Frobenius number $g_s^*(1,m,n)$ then the proof of Theorem 4.2 follows a similar argument to the proof of Theorem 3.3 in Chapter 3 by using Theorem 2.9 twice. We now claim a bound on this reduced generalized Frobenius number.

Proposition 4.5. *We have*

$$g_s^*(1,a,b) \leq ab\left\lfloor \frac{\sqrt{1+2a+2b+a^2+b^2+a^2b+ab^2+8abs}-1-a-b}{2ab}\right\rfloor + ab - 1.$$

Assuming this claim, we now prove Theorem 4.2.

Proof of Theorem 4.2. Let $a|\operatorname{lcm}(b,c)$. Set $\beta = \gcd(a,b)$ and $\gamma = \gcd(a,c)$. Using Proposition 4.5, Lemma 2.5, and Lemma 4.4, we obtain

$$\begin{aligned}
g_s^*(a,b,c) &= \beta\, g_s^*\left(\gamma, \frac{b}{\beta}, c\right) + c(\beta-1) \\
&= \beta\left(\gamma\, g_s^*\left(1, \frac{b}{\beta}, \frac{c}{\gamma}\right) + \frac{b}{\beta}(\gamma-1)\right) + c(\beta-1) \\
&= a\, g_s^*\left(1, \frac{b}{\beta}, \frac{c}{\gamma}\right) + b\gamma - b + c\beta - c \\
&\leq a\left(\frac{bc}{a}\left\lfloor \frac{\sqrt{1+\frac{2b}{\beta}+\frac{2c}{\gamma}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}+\frac{b^2c}{\beta a}+\frac{bc^2}{a\gamma}+\frac{8bcs}{a}}-1-\frac{b}{\beta}-\frac{c}{\gamma}}{\frac{2bc}{a}}\right\rfloor + \frac{bc}{a} - 1\right) \\
&\quad + b\gamma - b + c\beta - c \\
&= bc\left\lfloor \frac{a\sqrt{1+\frac{2b}{\beta}+\frac{2c}{\gamma}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}+\frac{b^2c}{\beta a}+\frac{bc^2}{a\gamma}+\frac{8bcs}{a}}-a-b\gamma-c\beta}{2bc}\right\rfloor + bc - a \\
&\quad + b\gamma - b + c\beta - c \\
&= bc\left\lfloor \frac{\sqrt{a^2+2ab\gamma+2ac\beta+b^2\gamma^2+c^2\beta^2+b^2c\gamma+bc^2\beta+8abcs}-a-b\gamma-c\beta}{2bc}\right\rfloor \\
&\quad + bc - a + b\gamma - b + c\beta - c.
\end{aligned}$$

We finish by using the identities $b\gamma = b\frac{a}{\beta} = \frac{ab}{\gcd(a,b)} = \operatorname{lcm}(a,b)$ and $c\beta = \operatorname{lcm}(a,c)$ to obtain the estimation

$$\begin{aligned}
g_s^*(a,b,c) \le bc \Bigg\lfloor &\frac{1}{2bc}(a^2 + 2a\,\mathrm{lcm}(a,b) + 2a\,\mathrm{lcm}(a,c) + \mathrm{lcm}(a,b)^2 + \mathrm{lcm}(a,c)^2 + bc\,\mathrm{lcm}(a,b) \\
&+bc\,\mathrm{lcm}(a,c) + 8abcs)^{\frac{1}{2}} + \frac{1}{2bc}\left(-a - \mathrm{lcm}(a,b) - \mathrm{lcm}(a,c)\right)\Bigg\rfloor + bc + \mathrm{lcm}(a,b) \\
&+ \mathrm{lcm}(a,c) - a - b - c \\
\le\ & \frac{1}{2}(a^2 + 2a\,\mathrm{lcm}(a,b) + 2a\,\mathrm{lcm}(a,c) + \mathrm{lcm}(a,b)^2 + \mathrm{lcm}(a,c)^2 + bc\,\mathrm{lcm}(a,b) \\
&+ bc\,\mathrm{lcm}(a,c) + 8abcs)^{\frac{1}{2}} + \frac{1}{2}\left(-a - \mathrm{lcm}(a,b) - \mathrm{lcm}(a,c)\right) + bc + \mathrm{lcm}(a,b) \\
&+ \mathrm{lcm}(a,c) - a - b - c \\
=\ & \frac{1}{2}(a^2 + 2a\,\mathrm{lcm}(a,b) + 2a\,\mathrm{lcm}(a,c) + \mathrm{lcm}(a,b)^2 + \mathrm{lcm}(a,c)^2 + bc\,\mathrm{lcm}(a,b) \\
&+ bc\,\mathrm{lcm}(a,c) + 8abcs)^{\frac{1}{2}} + bc + \frac{1}{2}\,\mathrm{lcm}(a,b) + \frac{1}{2}\,\mathrm{lcm}(a,c) - \frac{3}{2}a - b - c \\
\le\ & \Bigg[\frac{1}{4}a^2 + \frac{1}{2}a\,\mathrm{lcm}(a,b) + \frac{1}{2}a\,\mathrm{lcm}(a,c) + \frac{1}{4}\,\mathrm{lcm}(a,b)^2 + \frac{1}{4}\,\mathrm{lcm}(a,c)^2 + \frac{1}{4}bc\,\mathrm{lcm}(a,b) \\
&+ \frac{1}{4}bc\,\mathrm{lcm}(a,c) + 2abcs\Bigg]^{\frac{1}{2}} + bc + \frac{1}{2}\,\mathrm{lcm}(a,b) + \frac{1}{2}\,\mathrm{lcm}(a,c) - \frac{3}{2}a - b - c. \quad \square
\end{aligned}$$

Now let us prove the claim of Proposition 4.5. The arguments used here will act as a warmup for those seen in Chapter 5. We begin with a direct combinatorial proof.

Proposition 4.6. *Let A be a set that does not contain 1. Then*

$$p_{A\cup\{1\}}(n) = \sum_{i=0}^{n} p_A(i).$$

Proof. Any representation of an integer between 0 and n using only the elements of A can be made into a representation of n by adding an appropriate number of 1's. Any representation of n using the elements of $A\cup\{1\}$ can be made into a representation of some integer between 0 and n using only the elements of A by removing the 1's. $\square$

Proposition 4.7. *Let a and b be coprime positive integers. Then*

$$\frac{n(n+1)}{2ab}+\frac{\lfloor\frac{n}{a}\rfloor+\lfloor\frac{n}{b}\rfloor}{2}-\frac{a}{8}-\frac{b}{8}+\frac{1}{4}\le p_{\{1,a,b\}}(n).$$

Proof. We first use Proposition 4.6 to reduce the number of parts we must work with:

$$p_{\{1,a,b\}}(n)=\sum_{i=0}^{n}p_{\{a,b\}}(i).$$

We then use Theorem 2.10 to determine $p_{\{a,b\}}(i)$ explicitly. Recall that we define b^{-1} to be the inverse of $b \mod a$; that is, $b^{-1}b\equiv 1 \mod a$. We thus have

$$\begin{aligned}\sum_{i=0}^{n}p_{\{a,b\}}(i)&=\sum_{i=0}^{n}\left[\frac{i}{ab}-\left\{\frac{b^{-1}i}{a}\right\}-\left\{\frac{a^{-1}i}{b}\right\}+1\right]\\&=\sum_{i=0}^{n}\left[\frac{i}{ab}-\left(\left\{\frac{b^{-1}i}{a}\right\}-\frac{1}{2}\right)-\left(\left\{\frac{a^{-1}i}{b}\right\}-\frac{1}{2}\right)\right]\\&=\sum_{i=0}^{n}\frac{i}{ab}-\sum_{i=0}^{n}\left(\left(\frac{b^{-1}i}{a}\right)\right)+\sum_{\substack{0\le i\le n\\ b|a^{-1}i}}\frac{1}{2}-\sum_{i=0}^{n}\left(\left(\frac{a^{-1}i}{b}\right)\right)+\sum_{\substack{0\le i\le n\\ a|b^{-1}i}}\frac{1}{2}.\end{aligned}$$

Since a and b are coprime, $a|b^{-1}i$ implies $a|b^{-1}bi$, and hence $a|i$. We also note that the sawtooth function is periodic, with period a, and that the values attained in each period are exactly the multiples of $\frac{1}{a}$ between -0.5 and 0.5. Therefore, since the multiples of $\frac{1}{a}$ are symmetric for any $a-1$ consecutive terms of the sequence $\left(\left(\frac{b^{-1}i}{a}\right)\right)$, their sum vanishes. Noting that $a\left\lfloor\frac{n}{a}\right\rfloor$ is the largest multiple of a less than or equal to n (since $\frac{n}{a}\ge\left\lfloor\frac{n}{a}\right\rfloor$), we obtain

$$\begin{aligned}
p_{\{1,a,b\}}(n) &= \sum_{i=0}^{n} \frac{i}{ab} - \sum_{i=0}^{n} \left(\left(\frac{b^{-1}i}{a}\right)\right) - \sum_{i=0}^{n} \left(\left(\frac{a^{-1}i}{b}\right)\right) + \sum_{\substack{0\le i\le n \\ b|a^{-1}i}} \frac{1}{2} + \sum_{\substack{0\le i\le n \\ a|b^{-1}i}} \frac{1}{2} \\
&= \sum_{i=0}^{n} \frac{i}{ab} - \sum_{i=0}^{a\left\lfloor \frac{n}{a}\right\rfloor -1} \left(\left(\frac{b^{-1}i}{a}\right)\right) - \sum_{i=a\left\lfloor \frac{n}{a}\right\rfloor}^{n} \left(\left(\frac{b^{-1}i}{a}\right)\right) - \sum_{i=0}^{b\left\lfloor \frac{n}{b}\right\rfloor -1} \left(\left(\frac{a^{-1}i}{b}\right)\right) \\
&\quad - \sum_{i=b\left\lfloor \frac{n}{b}\right\rfloor}^{n} \left(\left(\frac{a^{-1}i}{b}\right)\right) + \sum_{\substack{0\le i\le n \\ b|i}} \frac{1}{2} + \sum_{\substack{0\le i\le n \\ a|i}} \frac{1}{2} \\
&= \sum_{i=0}^{n} \frac{i}{ab} - \sum_{i=a\left\lfloor \frac{n}{a}\right\rfloor}^{n} \left(\left(\frac{b^{-1}i}{a}\right)\right) - \sum_{i=b\left\lfloor \frac{n}{b}\right\rfloor}^{n} \left(\left(\frac{a^{-1}i}{b}\right)\right) + \sum_{0\le i\le n/a} \frac{1}{2} + \sum_{0\le i\le n/b} \frac{1}{2} \\
&= \frac{n(n+1)}{2ab} + \frac{\left\lfloor \frac{n}{a}\right\rfloor + \left\lfloor \frac{n}{b}\right\rfloor}{2} - \sum_{i=a\left\lfloor \frac{n}{a}\right\rfloor}^{n} \left(\left(\frac{b^{-1}i}{a}\right)\right) - \sum_{i=b\left\lfloor \frac{n}{b}\right\rfloor}^{n} \left(\left(\frac{a^{-1}i}{b}\right)\right).
\end{aligned}$$

Let us focus on the sums shown in the preceding line. We know that

$$\sum_{i=a\left\lfloor \frac{n}{a}\right\rfloor}^{n} \left(\left(\frac{b^{-1}i}{a}\right)\right) = \sum_{i=0}^{n-a\left\lfloor \frac{n}{a}\right\rfloor} \left(\left(\frac{b^{-1}i}{a}\right)\right)$$

is summing at most $a-1$ terms. The effect of multiplying the fractions $\frac{i}{a}$ by b^{-1} can be approximated by a random permutation of the ath fractions. Taking the sawtooth function of these yields terms of the form

$$\frac{1}{2} - \frac{i}{a} = \frac{a-2i}{2a}.$$

The worst case scenario, in terms of the absolute value of this sum, would be when we sum up all of the positive terms in the period, then stop before we add any of the negative terms. In other words, this would be when $b^{-1} = 1$, and $n = \left\lfloor \frac{a-1}{2}\right\rfloor$. We make these heuristics formal with the estimation

$$\begin{aligned}
\left|\sum_{i=0}^{n-a\lfloor\frac{n}{a}\rfloor}\left(\left(\frac{b^{-1}i}{a}\right)\right)\right| &\leq \sum_{i=0}^{\lfloor\frac{a-1}{2}\rfloor}\left(\left(\frac{i}{a}\right)\right)\\
&= 0+\frac{a-2}{2a}+\frac{a-4}{2a}+\cdots+\frac{a-2\lfloor\frac{a-1}{2}\rfloor}{2a}\\
&= \frac{1}{2a}\left(a-2+a-4+\cdots+a-2\left\lfloor\frac{a-1}{2}\right\rfloor\right)\\
&= \frac{1}{2a}\left(\sum_{i=1}^{\lfloor\frac{a-1}{2}\rfloor}a-\sum_{i=1}^{\lfloor\frac{a-1}{2}\rfloor}2i\right)\\
&= \frac{1}{2a}\left(a\left\lfloor\frac{a-1}{2}\right\rfloor-2\left(\frac{\lfloor\frac{a-1}{2}\rfloor\left(\lfloor\frac{a-1}{2}\rfloor+1\right)}{2}\right)\right)\\
&= \frac{1}{2a}\left((a-1)\left\lfloor\frac{a-1}{2}\right\rfloor-\left\lfloor\frac{a-1}{2}\right\rfloor^2\right).
\end{aligned}$$

This bound is sharp. However, straying from sharpness we obtain

$$\begin{aligned}
\left|\sum_{i=0}^{n-a\lfloor\frac{n}{a}\rfloor}\left(\left(\frac{b^{-1}i}{a}\right)\right)\right| &\leq \frac{1}{2a}\left((a-1)\left\lfloor\frac{a-1}{2}\right\rfloor-\left\lfloor\frac{a-1}{2}\right\rfloor^2\right)\\
&\leq \frac{1}{2a}\left((a-1)\frac{a-1}{2}-\left(\frac{a-1}{2}\right)^2\right)\\
&= \frac{(a-1)^2}{8a}\\
&\leq \frac{a-1}{8}.
\end{aligned}$$

Returning to $p_{\{1,a,b\}}(n)$, we finally obtain

$$\frac{n(n+1)}{2ab}+\frac{1}{2}\left\lfloor\frac{n}{a}\right\rfloor+\frac{1}{2}\left\lfloor\frac{n}{b}\right\rfloor-\frac{a-1}{8}-\frac{b-1}{8}\leq p_{\{1,a,b\}}(n)$$

and

$$\frac{n(n+1)}{2ab}+\frac{1}{2}\left\lfloor\frac{n}{a}\right\rfloor+\frac{1}{2}\left\lfloor\frac{n}{b}\right\rfloor+\frac{a-1}{8}+\frac{b-1}{8}\leq p_{\{1,a,b\}}(n).$$

□

Proof of Proposition 4.5. By Lemma 2.19 and the lower bound in Proposition 4.7,

$$g_s^*(1,a,b) \leq \max N^- = \max\left\{n \in \mathbb{N} : \frac{n(n+1)}{2ab} + \frac{1}{2}\left\lfloor\frac{n}{a}\right\rfloor + \frac{1}{2}\left\lfloor\frac{n}{b}\right\rfloor - \frac{a-1}{8} - \frac{b-1}{8} \leq s\right\}.$$

We attempt solving the condition for n in terms of s. Note that the inequality below comes from the fact that for any real number x and positive integer m, we have $\frac{x}{m} \geq \left\lfloor\frac{x}{m}\right\rfloor$, which implies $x \geq m\left\lfloor\frac{x}{m}\right\rfloor$. Hence,

$$\begin{aligned}
0 &\geq \frac{n^2}{ab} + \frac{n}{ab} + \left\lfloor\frac{n}{a}\right\rfloor + \left\lfloor\frac{n}{b}\right\rfloor - \frac{a-1}{4} - \frac{b-1}{4} - 2s \\
&\geq ab\left\lfloor\frac{n}{ab}\right\rfloor^2 + \left\lfloor\frac{n}{ab}\right\rfloor + b\left\lfloor\frac{n}{ab}\right\rfloor + a\left\lfloor\frac{n}{ab}\right\rfloor - \frac{a-1}{4} - \frac{b-1}{4} - 2s \\
&= ab\left\lfloor\frac{n}{ab}\right\rfloor^2 + (1+a+b)\left\lfloor\frac{n}{ab}\right\rfloor - \frac{a-1}{4} - \frac{b-1}{4} - 2s.
\end{aligned}$$

Geometrically, this inequality describes the solutions between the two roots of a quadratic function in $\left\lfloor\frac{n}{ab}\right\rfloor$. Therefore,

$$\begin{aligned}
\left\lfloor\frac{n}{ab}\right\rfloor &\leq \frac{-(1+a+b) + \sqrt{(1+a+b)^2 - 4(ab)(-\frac{a-1}{4} - \frac{b-1}{4} - 2s)}}{2ab} \\
&= \frac{-(1+a+b) + \sqrt{1+2a+2b+a^2+b^2+a^2b+ab^2+8abs}}{2ab}.
\end{aligned}$$

Now rewrite n as $abq+r$, with $0 \leq r \leq ab-1$. We see that in order to maximize n, we have to maximize $\lfloor\frac{abq+r}{ab}\rfloor$. Therefore we set $r = ab-1$ and now maximize $q = \left\lfloor\frac{n}{ab}\right\rfloor \in \mathbb{N}$,

$$\begin{aligned}
g_s^*(1,a,b) \leq \max&\left\{n \in \mathbb{N} : \frac{n(n+1)}{2ab} + \frac{1}{2}\left\lfloor\frac{n}{a}\right\rfloor + \frac{1}{2}\left\lfloor\frac{n}{b}\right\rfloor - \frac{a-1}{8} - \frac{b-1}{8} \leq s\right\} \\
\max&\left\{abq+ab-1 \in \mathbb{N} : \right. \\
&\left. q \leq \frac{-(1+a+b) + \sqrt{1+2a+2b+a^2+b^2+a^2b+ab^2+8abs}}{2ab}\right\}.
\end{aligned}$$

Some simplification yields

$$g_s^*(1,a,b) \leq ab \left\lfloor \frac{-(1+a+b)+\sqrt{1+2a+2b+a^2+b^2+a^2b+ab^2+8abs}}{2ab} \right\rfloor + ab - 1. \square$$

Proof of Corollary 4.3. We may get a lower bound on $g_s^*(a,b,c)$ when $a|\mathrm{lcm}(b,c)$ by following a similar argument to the one above. Namely, we find a lower bound on $g_s^*(1,a,b)$ by way of the lower bound on $p_{1,a,b}(n)$. The argument follows exactly the same from there, only this time with a different expression in terms of a and b under the square root than $1+2a+2b+a^2+b^2+a^2b+ab^2$. However, the $8abcs$ remains untouched, yielding our desired asymptotic result once we bring the $\frac{1}{2}$ inside the square root. $\square$

Chapter 5

Generalized Frobenius numbers of pairwise coprime elements

In the previous two chapters, we required the elements of $A = \{a_1, a_2, \ldots, a_d\}$ to each be products of some common factors. In this chapter, we forgo this requirement. Instead, we impose a simplifying assumption for the remainder of the chapter: that $\gcd(a_i, a_j) = 1$ for $i \neq j$. In this setting, we now wish to find a bound on the generalized Frobenius number by way of the geometry of polynomials.

5.1 Asymptotics of g_s^* in terms of s

As seen in the last two chapters, as well as the wider literature, the generalized Frobenius number is difficult to determine. For this reason, we investigate the growth of g_s^* as $s \to \infty$.

Aliev, Fukshansky, and Henk [1] used the geometry of lattices to obtain the following bounds on $g_s^*(A)$, when A is coprime:

$$g_s^*(A) \geq ((s+1)(d-1)!\, a_1 a_2 \cdots a_d)^{\frac{1}{d-1}} - a_1 - a_2 - \cdots - a_d,$$

$$g_s^*(A) \leq (s(d-1)!\, a_1 a_2 \cdots a_d)^{\frac{1}{d-1}} + g_0^*(A).$$

Comparable results using similar geometrical arguments are given in [2, 17]. Note that both the upper and lower bounds have the same growth rate as $s \to \infty$:

$$(s(d-1)!\, a_1 a_2 \cdots a_d)^{\frac{1}{d-1}}.$$

Thus an immediate corollary to their work is that for a fixed A, we have

$$g_s^*(A) \sim (s(d-1)!\, a_1 a_2 \cdots a_d)^{\frac{1}{d-1}}.$$

Kevin Woods [35] corroborated this asymptotic result using recurrences on $p_A(n)$ and generating functions. Here we reprove this asymptotic result using solely the growth of the restricted partition function.

Theorem 5.1. *Let $A = \{a_1, a_2 \ldots, a_d\}$ be pairwise coprime. Then as $s \to \infty$,*

$$g_s^*(A) \sim (s(d-1)!\, a_1 a_2 \cdots a_d)^{\frac{1}{d-1}} .$$

Proof. Let $N_s^+ = N^+, N_s = N$, and $N_s^- = N^-$ be as in the proof of Lemma 2.19, for any desired s. In order to show

$$g_s^*(a_1, a_2, \ldots, a_d) \sim (s(d-1)!\, a_1 a_2 \cdots a_d)^{\frac{1}{d-1}} ,$$

we must show that the following limit is 1:

$$\lim_{s\to\infty} \frac{g_s^*(a_1,a_2,\ldots,a_d)}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}} = \lim_{s\to\infty} \frac{\max N_s}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}}.$$

We show that the limit on the right is equal to 1 by replacing N_s with N_s^+ and N_s^-, so that

$$\lim_{s\to\infty} \frac{\max N_s^+}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}} \le \lim_{s\to\infty} \frac{\max N_s}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}} \le \lim_{s\to\infty} \frac{\max N_s^-}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}}.$$

If we can show the leftmost and rightmost limits are both equal to 1, we are done.

First consider N_s^+. Since $\mathrm{Poly}_A(n) + B$ is a polynomial that is eventually increasing, there exists a point M such that for all $n \ge M$, $\mathrm{Poly}_A(n) + B$ is an increasing and therefore invertible function. Define $t(n)$ to be the inverse of $\mathrm{Poly}_A(n) + B$ for $n \ge M$. That is,

$$t(\mathrm{Poly}_A(n) + B) = n = \mathrm{Poly}_A(t(n)) + B$$

for all $n \ge M$. Namely, for $s \ge M$, we see that $n = t(s)$ is the greatest root of $\mathrm{Poly}_A(n)+B-s$. Therefore

$$\lim_{s\to\infty} \frac{\max N_s^+}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}} = \lim_{s\to\infty} \frac{t(s)}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}}.$$

We claim that $t(n)$ has the same order as the inverse of $\frac{1}{(d-1)!a_1a_2\cdots a_d}n^{d-1}$. Assuming the claim, the inverse of $\frac{1}{(d-1)!a_1a_2\cdots a_d}n^{d-1}$ at s is $(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}$, and thus we conclude that

$$\lim_{s\to\infty} \frac{t(s)}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}} = \lim_{s\to\infty} \frac{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}}$$

and therefore

$$\lim_{s\to\infty} \frac{\max N_s^+}{(s(d-1)!\,a_1a_2\cdots a_d)^{\frac{1}{d-1}}} = 1$$

as desired.

We now prove the claim. Let $f(n) = \frac{1}{(d-1)!a_1a_2\cdots a_d}n^{d-1}$, and $g(n) = \text{Poly}_A(n) + B$. Let $\tilde{g}$ be g restricted to $n \geq M$. Hence $g^{-1} \sim \tilde{g}^{-1}$. We now show that $f^{-1} \sim \tilde{g}^{-1}$. First, we know $f(n) \to \infty$. From Theorem 2.12, $f(n) \sim p_A(n)$, and furthermore $p_A(n) \sim \tilde{g}(n)$ by

$$\lim_{n\to\infty} \frac{p_A(n)}{\text{Poly}_A(n)} = \lim_{n\to\infty} \frac{\text{Poly}_A(n)}{\text{Poly}_A(n)} + \frac{\text{Quasi}_A(n)}{\text{Poly}_A(n)} = \lim_{n\to\infty} 1 + \frac{B}{\text{Poly}_A(n)} = 1$$

and

$$\lim_{n\to\infty} \frac{\tilde{g}(n)}{\text{Poly}_A(n)} = \lim_{n\to\infty} \frac{\text{Poly}_A(n) + B}{\text{Poly}_A(n)} = 1.$$

Let $h(n) = (n(d-1)!\, a_1a_2\cdots a_d)^{\frac{1}{d-1}}$. By construction $h(n) \sim f^{-1}(n)$. Since $\tilde{g}(n)$ is increasing, $\tilde{g}^{-1}(n)$ exists. Additionally $h(n)$ is monotonic. We lastly have

$$\frac{h'(n)}{h(n)} = \frac{\frac{1}{d-1}\left((d-1)!\, a_1a_2\cdots a_d\right)^{\frac{1}{d-1}} n^{\left(\frac{1}{d-1}-1\right)}}{\left((d-1)!\, a_1a_2\cdots a_d\right)^{\frac{1}{d-1}} n^{\frac{1}{d-1}}} = \frac{1}{d-1}n^{-1} = O\left(\frac{1}{n}\right).$$

By Theorem 2.11, we are done. □

5.2 Explicit bounds when $A = \{a, b, c\}$

In Section 5.3, we will demonstrate a method to obtain bounds on $g_s^*(A)$ that are asymptotically sharp. In this section, we tackle the case $A = \{a, b, c\}$ as a warmup.

Theorem 5.2. *If $A = \{a, b, c\}$ is pairwise coprime, then*

$$g_s^*(A) \leq \left\lfloor \sqrt{2abcs + \frac{(a+b+c)^2 - 2abc(a+b+c)}{12}} - \frac{1}{2}(a+b+c) \right\rfloor,$$

and

$$g_s^*(A) \geq \left\lfloor \sqrt{2abcs + \frac{(a+b+c)^2 + 2abc(a+b+c) - 4(ab+ac+bc)}{12}} - \frac{1}{2}(a+b+c) \right\rfloor.$$

Corollary 5.3. *If $A = \{a, b, c\}$ is pairwise coprime, then*

$$g_s^*(A) \sim \sqrt{2abcs}.$$

We sketch the proof as follows: first we use Lemma 2.19 to approximate $g_s^*(a, b, c)$ as a root of a polynomial. To do this, we find a suitable bound on $\mathrm{Quasi}_{\{a,b,c\}}(n)$. We then use the quadratic formula to solve our polynomial that we constructed, yielding our desired bound.

We begin by bounding $\mathrm{Quasi}_{\{a,b,c\}}(n)$. Recall that for $a, b,$ and c pairwise coprime, Theorem 2.18 tells us

$$\mathrm{Quasi}_{\{a,b,c\}}(n) = s_{-n}(a, b; c) + s_{-n}(a, c; b) + s_{-n}(b, c; a).$$

Proposition 5.4. *For all $n \in \mathbb{Z}$ and all pairwise coprime $a, b, c \in \mathbb{N}$, we have*

$$|s_n(a, b; c)| \leq s_0(c - 1, 1; c).$$

Proof. By the definition of the Fourier-Dedekind sum and the triangle inequality,

$$|s_n(a, b; c)| = \left| \frac{1}{c} \sum_{k=1}^{c-1} \frac{1}{(1 - \xi_c^{ak})(1 - \xi_c^{bk})} \xi_c^{kn} \right| \leq \frac{1}{c} \sum_{k=1}^{c-1} \left| \frac{1}{(1 - \xi_c^{ak})(1 - \xi_c^{bk})} \right|.$$

Define $f(k) = \frac{1}{1-\xi_c^k}$. Then

$$\frac{1}{c} \sum_{k=1}^{c-1} \left| \frac{1}{(1 - \xi_c^{ak})(1 - \xi_c^{bk})} \right| = \frac{1}{c} \sum_{k=1}^{c-1} |f(ak)f(bk)|.$$

We now apply the Cauchy–Schwarz inequality (Theorem 2.17 with $p = 1$ and $q_1 = q_2 = 2$) to obtain

$$\frac{1}{c}\left\|f(a(\cdot))f(b(\cdot))\right\|_1 \leq \frac{1}{c}\left\|f(a(\cdot))\right\|_2\left\|f(b(\cdot))\right\|_2 = \frac{1}{c}\sqrt{\sum_{k=1}^{c-1}|f(ak)|^2}\sqrt{\sum_{k=1}^{c-1}|f(bk)|^2}.$$

We note that a and b are coprime to c. Thus, if $f(ak_1) = f(ak_2)$, then $ak_1 \equiv ak_2 \mod c$, and thus $k_1 \equiv k_2 \mod c$. Hence, the set of values $\{f(1), f(2), \ldots, f(c-1)\}$ is equivalent to the set of values $\{f(a), f(2a), \ldots, f((c-1)a)\}$. We call this a **complete residue system**. Thus we have the identity

$$\frac{1}{c}\sum_{k=1}^{c-1}|f(ak)|^2 = \frac{1}{c}\sum_{k=1}^{c-1}|f(bk)|^2 = \frac{1}{c}\sum_{k=1}^{c-1}|f(k)|^2.$$

Therefore

$$\frac{1}{c}\sqrt{\sum_{k=1}^{c-1}|f(ak)|^2}\sqrt{\sum_{k=1}^{c-1}|f(bk)|^2} = \frac{1}{c}\sum_{k=1}^{c-1}|f(k)|^2 = \frac{1}{c}\sum_{k=1}^{c-1}\overline{f(k)}f(k).$$

Multiplying each term of the last sum by ξ_c^{0i} yields

$$\begin{aligned}\frac{1}{c}\sum_{k=1}^{c-1}\overline{\frac{1}{1-\xi_c^k}}\frac{1}{1-\xi_c^k}\xi_c^{0k} &= \frac{1}{c}\sum_{k=1}^{c-1}\frac{1}{1-\overline{\xi_c^k}}\frac{1}{1-\xi_c^k}\xi_c^{0k} \\ &= \frac{1}{c}\sum_{k=1}^{c-1}\frac{1}{\left(1-\xi_c^{(c-1)k}\right)\left(1-\xi_c^k\right)}\xi_c^{0k} \\ &= s_0(c-1,1;c). \quad \square\end{aligned}$$

Taking advantage of the classical Dedekind sum, we can now determine $s_0(c-1,1;c)$.

Proposition 5.5. *If a, b are coprime positive integers, then $s_0(b-1,1;b) = \frac{b}{12} - \frac{1}{12b}$.*

Proof. Euler's formula gives us the following:

$$\cot\left(\frac{\pi x}{y}\right) = i\,\frac{\xi_y^{x/2}+\xi_y^{-x/2}}{\xi_y^{x/2}-\xi_y^{-x/2}} = -i\,\frac{1+\xi_y^x}{1-\xi_y^x}.$$

Furthermore,

$$\frac{1}{2}\left(\frac{1+x}{1-x}\right) = \frac{1}{1-x} - \frac{1}{2}.$$

Putting these together, we see that

$$\frac{1}{1-\xi_y^x} - \frac{1}{2} = \frac{i}{2}\cot\left(\frac{\pi x}{y}\right).$$

Additionally, we see

$$\left(\frac{1}{1-\xi_b^{ak}}-\frac{1}{2}\right)\left(\frac{1}{1-\xi_b^{k}}-\frac{1}{2}\right) = \left(\frac{1}{1-\xi^{ak}}\right)\left(\frac{1}{1-\xi^{k}}\right) - \frac{1}{2}\left(\frac{1}{1-\xi^{ak}}\right) - \frac{1}{2}\left(\frac{1}{1-\xi^{k}}\right) + \frac{1}{4}.$$

Using the above two identities, we change $s_0(a,1;b)$ from a complex sum to a real sum:

$$\begin{aligned}
s_0(a,1;b) &= \frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{(1-\xi_b^{ak})(1-\xi_b^{k})} \\
&= \frac{1}{b}\sum_{k=1}^{b-1}\left(\frac{1}{1-\xi_b^{ak}}-\frac{1}{2}\right)\left(\frac{1}{1-\xi_b^{k}}-\frac{1}{2}\right) + \frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{2}\left(\frac{1}{1-\xi^{ak}}\right) \\
&\quad + \frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{2}\left(\frac{1}{1-\xi^{k}}\right) - \frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{4} \\
&= \frac{1}{b}\left(\frac{i}{2}\right)^2\sum_{k=1}^{b-1}\cot\left(\frac{\pi ak}{b}\right)\cot\left(\frac{\pi k}{b}\right) + \frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{2}\left(\frac{1}{1-\xi^{ak}}\right) \\
&\quad + \frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{2}\left(\frac{1}{1-\xi^{k}}\right) - \frac{b-1}{4b}.
\end{aligned}$$

We now use the fact that $\frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{2}\left(\frac{1}{1-\xi^{ak}}\right)$ and $\frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{2}\left(\frac{1}{1-\xi^{k}}\right)$ reduce to $\frac{1}{4}-\frac{1}{4b}$, courtesy of [10]. The authors use restricted partition function machinery to determine this identity on the way to proving Theorem 2.10. We also know the first sum is actually the classical Dedekind sum, so we now obtain

$$s_0(a,1;b) = -s(a,b) + \frac{b-1}{4b}.$$

We now have a relationship between the Fourier-Dedekind and classical Dedekind sums. In order to simplify further, we first apply Proposition 2.15, then Proposition 2.13, and finally Proposition 2.14:

$$\begin{aligned} s_0(b-1,1;b) &= -s(b-1,b) + \frac{b-1}{4b} \\ &= s(b,b-1) - \frac{1}{12}\left(\frac{b-1}{b} + \frac{b}{b-1} + \frac{1}{b(b-1)}\right) + \frac{1}{4} + \frac{b-1}{4b} \\ &= s(1,b-1) - \frac{1}{12}\left(\frac{b-1}{b} + \frac{b}{b-1} + \frac{1}{b(b-1)}\right) + \frac{1}{4} + \frac{b-1}{4b} \\ &= \frac{(b-2)(b-3)}{12(b-1)} - \frac{1}{12}\left(\frac{b-1}{b} + \frac{b}{b-1} + \frac{1}{b(b-1)}\right) + \frac{1}{4} + \frac{b-1}{4b} \\ &= \frac{b}{12} - \frac{1}{12b}. \quad \square \end{aligned}$$

We now have the tools to bound $\mathrm{Quasi}_{\{a,b,c\}}(n)$, so we proceed with the proof of Theorem 5.2.

Proof of Theorem 5.2. We begin with the following estimation:

$$\begin{aligned} p_{\{a,b,c\}}(n) &= \mathrm{Poly}_{\{a,b,c\}}(n) + \mathrm{Quasi}_{\{a,b,c\}}(n) \\ &= \mathrm{Poly}_{\{a,b,c\}}(n) + s_{-n}(a,b;c) + s_{-n}(a,c;b) + s_{-n}(b,c;a) \\ &\le \mathrm{Poly}_{\{a,b,c\}}(n) + s_0(c-1,1;c) + s_0(b-1,1;b) + s_0(a-1,1;a). \end{aligned}$$

Now using Proposition 5.5, we obtain

$$\begin{aligned} p_{\{a,b,c\}}(n) &\leq \frac{n^2}{2abc} + \frac{n}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) \\ &\quad + \frac{a}{12} - \frac{1}{12a} + \frac{b}{12} - \frac{1}{12b} + \frac{c}{12} - \frac{1}{12c} \\ &= \frac{n^2}{2abc} + \frac{n}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{(a+b+c)(abc+a+b+c)}{12abc}. \end{aligned}$$

Moving directly forward as before,

$$\begin{aligned} g_s^*(a,b,c) &= \max\left\{n \in \mathbb{N} : p_{\{a,b,c\}}(n) - s \leq 0\right\} \\ &\leq \max\left\{n \in \mathbb{N} : \frac{n^2}{2abc} + \frac{n}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right)\right. \\ &\qquad\qquad \left. + \frac{(a+b+c)(abc+a+b+c)}{12abc} - s \leq 0\right\} \\ &= \max\left\{n \in \mathbb{N} : \frac{n^2}{2} + \frac{n}{2}(a+b+c) + \frac{(a+b+c)(abc+a+b+c)}{12} - abcs \leq 0\right\}. \end{aligned}$$

We once again take advantage of the geometry of our estimation of the restricted partition function, using the quadratic formula to determine that n is at most the greater root of the quadratic function. Thus, the last line above is equal to

$$\max\left\{n \in \mathbb{N} : n \leq -\frac{1}{2}(a+b+c) \right. \\ \left. + \sqrt{\frac{1}{4}(a+b+c)^2 - 4 \cdot \frac{1}{2}\left(\frac{(a+b+c)(abc+a+b+c)}{12} - abcs\right)}\right\}.$$

After some simplification, we obtain

$$\begin{aligned} g_s^*(a,b,c) &\leq \max\left\{n \in \mathbb{N} : n \leq -\frac{1}{2}(a+b+c) + \sqrt{2abcs + \frac{(a+b+c)^2 - 2abc(a+b+c)}{12}}\right\} \\ &= \left\lfloor -\frac{1}{2}(a+b+c) + \sqrt{2abcs + \frac{(a+b+c)^2 - 2abc(a+b+c)}{12}} \right\rfloor. \end{aligned}$$

Similarly, we can bound $g_s^*(a,b,c)$ from below by using

$$p_{\{a,b,c\}}(n) \geq \frac{n^2}{2abc} + \frac{n}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{(a+b+c)^2 - abc(a+b+c) + 2(ab+ac+bc)}{12abc},$$

which yields

$$g_s^*(a,b,c) \geq \left\lfloor -\frac{1}{2}(a+b+c) + \sqrt{2abcs + \frac{(a+b+c)^2 + 2abc(a+b+c) - 4(ab+ac+bc)}{12}} \right\rfloor.$$

□

5.3 Explicit upper bounds when $A = \{a_1, a_2, \ldots, a_d\}$

Emboldened by our success in the previous section, we attempt to generalize each step to give a similar bound on $g_s^*(A)$, when $A = \{a_1, a_2, \ldots, a_d\}$ is pairwise coprime. The biggest obstacle is the use of the quadratic formula. Since there is no way to precisely determine the largest root of $\mathrm{Poly}_A(n) \pm B$ in general, we need to use a different method. We turn to the *geometry of polynomials* to help us. There is a long history of finding the maximum modulus of the roots of polynomial equations. Lagrange and Cauchy were the first to find bounds using only the coefficients of a polynomial, and estimations have improved since (see chapter VII of Marden [20], and chapter VI of Yap [36]). A classical result of Lagrange will suffice for us.

Theorem 5.6 (Lagrange)**.** *Let $f(z) = c_d z^d + c_{d-1} z^{d-1} + \cdots + c_1 z + c_0$ be a polynomial. Let R be the set*

$$R = \left\{ \left| \frac{c_k}{c_d} \right|^{\frac{1}{d-k}} : c_k \text{ is a coefficient of } f \right\}.$$

If r_1, r_2 are the two greatest (not necessarily distinct) elements of R, then any root r of f satisfies $|r| \leq r_1 + r_2$.

Therefore, once we determine our approximation of the restricted partition function of A, we may calculate an upper bound on $g_s^*(A)$ which perfectly matches the asymptotics for $g_s^*(A)$ found in the last section.

5.3.1 Four lemmas

First, some lemmas.

Lemma 5.7 (Cotangent Expansion). *Let $A = \{a_1, a_2, \ldots, a_d\}$ pairwise coprime. Then for every k,*

$$\frac{1}{\left(1-\xi_{a_d}^{a_1 k}\right)\cdots\left(1-\xi_{a_d}^{a_{d-1} k}\right)}$$
$$= \left(\frac{i}{2}\right)^{d-1} \cot\left(\frac{\pi a_1 k}{a_d}\right)\cdots\cot\left(\frac{\pi a_{d-1} k}{a_d}\right) - \sum_{j=0}^{d-2}\left[\left(-\frac{1}{2}\right)^{d-j} \sum_{\substack{S\subset[d-1],\\ |S|=j}} \prod_{i\in S} \frac{1}{1-\xi_{a_d}^{a_i k}}\right].$$

In particular, there is a product of $d-1$ factors on the left hand side, and only products of $d-2$ or less factors on the right hand side (modulo the term made up of cotangents).

Proof. Let $f_i(k) = \frac{1}{1-\xi_b^{a_i k}}$. We now investigate

$$\left(f_1 - \frac{1}{2}\right)\left(f_2 - \frac{1}{2}\right)\cdots\left(f_{d-1} - \frac{1}{2}\right) = \sum_{j=0}^{d-1}\left[\left(-\frac{1}{2}\right)^{d-j} \sum_{\substack{S\subset[d-1],\\ |S|=j}} \prod_{i\in S} f_i\right],$$

which may be rewritten as

$$f_1 f_2 \cdots f_{d-1} = \left(f_1 - \frac{1}{2}\right)\left(f_2 - \frac{1}{2}\right)\cdots\left(f_{d-1} - \frac{1}{2}\right) - \sum_{j=0}^{d-2}\left[\left(-\frac{1}{2}\right)^{d-j} \sum_{\substack{S\subset[d-1],\\ |S|=j}} \prod_{i\in S} f_i\right].$$

Recall that $f_i(k) - \frac{1}{2} = \frac{1}{1-\xi_b^{a_i k}} - \frac{1}{2} = \frac{1}{2}\frac{1+\xi_b^{a_i k}}{1-\xi_b^{a_i k}} = \cot\left(\frac{\pi a_i k}{b}\right)$. Therefore

$$\frac{1}{\left(1-\xi_{a_d}^{a_1 k}\right)\cdots\left(1-\xi_{a_d}^{a_{d-1} k}\right)}$$
$$= \left(\frac{i}{2}\right)^{d-1} \cot\left(\frac{\pi a_1 k}{a_d}\right)\cdots\cot\left(\frac{\pi a_d k}{a_d}\right) - \sum_{j=0}^{d-2}\left[\left(-\frac{1}{2}\right)^{d-j} \sum_{\substack{S\subset[d-1],\\ |S|=j}} \prod_{i\in S} \frac{1}{1-\xi_{a_d}^{a_i k}}\right]. \quad \square$$

Lemma 5.8 (Cosecant Bound)**.** *If $b \geq 2$, then*

$$\sum_{k=1}^{b-1}\left|\frac{1}{1-\xi_b^k}\right| = \frac{1}{2}\sum_{k=1}^{b-1}\csc\left(\frac{\pi k}{b}\right) < \frac{1}{2\pi}b^2.$$

Proof. Finding the modulus of $1-\xi_b^k$ is equivalent to finding the distance between 1 and the kth power of the first bth root of unity. So,

$$\sum_{k=1}^{b-1}\left|\frac{1}{1-\xi_b^k}\right| = \sum_{k=1}^{b-1}\frac{1}{\sqrt{(1-\cos(\frac{2\pi k}{b}))^2 + (\sin(\frac{2\pi k}{b}))^2}} = \sum_{k=1}^{b-1}\frac{1}{\sqrt{2(1-\cos(\frac{2\pi k}{b}))}}.$$

Using the double angle formula, we get

$$\sum_{k=1}^{b-1}\frac{1}{\sqrt{2(1-\cos(\frac{2\pi k}{b}))}} = \sum_{k=1}^{b-1}\frac{1}{\sqrt{2(2\sin^2(\frac{\pi k}{b}))}} = \sum_{k=1}^{b-1}\frac{1}{2\sin(\frac{\pi k}{b})} = \frac{1}{2}\sum_{k=1}^{b-1}\csc\left(\frac{\pi k}{b}\right).$$

By the work of Blagouchine and Moreau [11],

$$\sum_{k=1}^{b-1}\csc\left(\frac{\pi k}{b}\right) < \frac{2b}{\pi}\left(\log\frac{2b}{\pi} + \gamma\right) - \frac{\pi}{36b} + \frac{7\pi^3}{21600b^3},$$

where $\gamma = 0.577...$ is the Euler-Mascheroni constant. When $b \geq 2$, we may use the cruder upper bound $\frac{1}{2\pi}b^2$, by estimating $\log b < \frac{1}{2}b$. $\square$

Lemma 5.9 (Fourier Alignment). *Let f be a periodic function with period b. Let $a_1, \ldots, a_n$ be integers with* $\gcd(a_i, b) = 1$. *Then*

$$\sum_{k=1}^{b-1} |f(a_1 k) f(a_2 k) \cdots f(a_n k)| \leq \sum_{k=1}^{b-1} |f(k)|^n.$$

Proof. We interpret $\sum_{k=1}^{b-1} |f(a_1 k) f(a_2 k) \cdots f(a_n k)|$ as a norm, much like in the proof of Proposition 5.5. We apply Theorem 2.17 with $p = 1$ and $q_1 = q_2 = \cdots = q_n = n$ to obtain

$$\|f(a_1(\cdot)) f(a_2(\cdot)) \cdots f(a_n(\cdot))\|_1 \leq \|f(a_1(\cdot))\|_n \|f(a_2(\cdot))\|_n \cdots \|f(a_n(\cdot))\|_n .$$

Since a_k and b are coprime for all k, we have a complete residue system over the set $\{1, 2, \ldots, b-1\}$, and thus the identity

$$\sum_{k=1}^{b-1} |f(a_k(\cdot))|^p = \sum_{k=1}^{b-1} |f(1(\cdot))|^p$$

holds for any p. Hence $\|f(a_k(\cdot))\|_n = \|f(\cdot)\|_n$ for all k. Therefore

$$\|f(a_1(\cdot))\|_n \|f(a_2(\cdot))\|_n \cdots \|f(a_n(\cdot))\|_n = \|f(\cdot)\|_n^n = \sum_{k=1}^{b-1} |f(k)|^n. \qquad \square$$

Lemma 5.10 (Cotangent Bound). *Let $d \geq 2, b \geq 2$. Then*

$$\sum_{k=1}^{b-1} \left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1} < \frac{1}{4\pi^{d-1}} b^d + \frac{3b}{4}.$$

Proof. As the absolute value of the cotangent function (and any of its positive powers) is concave on the interval $(0, \pi)$, we may bound it above by the three lines given in Figure 5.1. We begin by splitting up the interval $0 < k < b$ into quarters, and the sum into 4 smaller

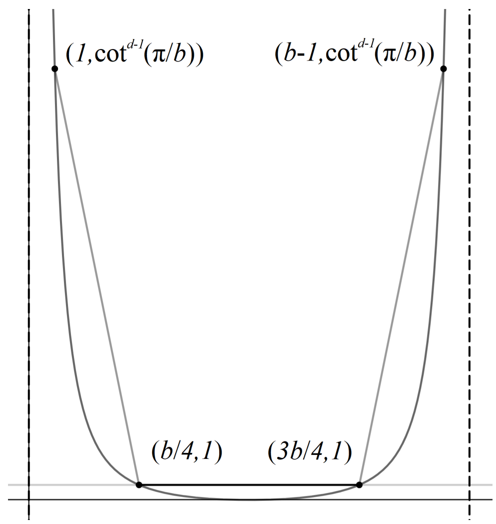


Figure 5.1: The function $\left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1}$, plotted from 0 to b. We estimate the finite sum $\sum_{k=1}^{b-1}\left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1}$ by using the lines connecting the four labeled points.

sums. By the symmetry of $\left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1}$, we can just consider the first two sums doubled:

$$\sum_{k=1}^{b-1}\left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1} \leq 2\left(\sum_{k=1}^{\lfloor \frac{b}{4}\rfloor}\left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1} + \sum_{k=\lfloor \frac{b}{4}\rfloor}^{\lfloor \frac{b}{2}\rfloor}\left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1}\right).$$

We bound above the function values in the first sum by the line from $\left(1, \cot^{d-1}\left(\frac{\pi}{b}\right)\right)$ to $\left(\frac{b}{4}, 1\right)$. The region under this line is a trapezoid with width $\lfloor \frac{b}{4}\rfloor$, and heights $\cot^{d-1}\left(\frac{\pi}{b}\right)$ and 1. Since we are doubling this quantity, we can instead view the region as a rectangle with the same width, but a height of $\cot^{d-1}\left(\frac{\pi}{b}\right)+1$. Hence,

$$2\left(\sum_{k=1}^{\lfloor \frac{b}{4}\rfloor}\left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1}\right) \leq \sum_{k=1}^{\lfloor \frac{b}{4}\rfloor}\left(\cot^{d-1}\left(\frac{\pi}{b}\right)+1\right).$$

We bound above the function values in the second sum by 1; we write

$$2\left(\sum_{k=\lfloor \frac{b}{4}\rfloor}^{\lfloor \frac{b}{2}\rfloor} \left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1}\right) \leq 2\sum_{k=\lfloor \frac{b}{4}\rfloor}^{\lfloor \frac{b}{2}\rfloor} 1.$$

Since neither of these estimations depend on k, we multiply them by the width of one quarter of the period. So,

$$\sum_{k=1}^{b-1}\left|\cot\left(\frac{\pi k}{b}\right)\right|^{d-1} \leq \left(\cot^{d-1}\left(\frac{\pi}{b}\right)+1\right)\left(\frac{b}{4}\right)+(2)\left(\frac{b}{4}\right) = \frac{b}{4}\cot^{d-1}\left(\frac{\pi}{b}\right)+\frac{3b}{4}.$$

Recalling the Taylor expansion of cotangent around the origin, we have that $\cot(x) < \frac{1}{x}$ for $x \in (0, \pi/2]$. Hence $\cot^{d-1}\left(\frac{\pi}{b}\right) < \left(\frac{b}{\pi}\right)^{d-1}$ for $b \geq 2$. Therefore,

$$\frac{b}{4}\cot^{d-1}\left(\frac{\pi}{b}\right)+\frac{3b}{4} < \frac{1}{4\pi^{d-1}}b^d+\frac{3b}{4}.$$ □

5.3.2 Determining an upper bound for $g_s^*(A)$

Theorem 5.11. *Let $A = \{a_1, a_2, \ldots, a_d\}$ be pairwise coprime. Then there exists an upper bound of $g_s^*(A)$ that is constructable. Moreover, when s is sufficiently large, the upper bound will be have the form $(s(d-1)!\, a_1 a_2 \cdots a_d)^{\frac{1}{d-1}} + c$, where c is a constructable number.*

Proof. We begin by bounding $\text{Quasi}_A(n)$. Since the elements of A are pairwise coprime,

$$\text{Quasi}_A(n) = s_{-n}(a_1, a_2, \ldots, a_{d-1}; a_d) + s_{-n}(a_1, a_2, \ldots, a_{d-2}, a_d; a_{d-1}) + \cdots + s_{-n}(a_2, a_3, \ldots, a_d; a_1).$$

Without loss of generality, we bound $s_{-n}(a_1, a_2, \ldots, a_{d-1}; a_d)$. Let $\xi = \xi_{a_d}$. Then

$$\begin{aligned} |s_{-n}(a_1, a_2, \ldots, a_{d-1}; a_d)| &= \left|\frac{1}{a_d}\sum_{k=1}^{a_d-1}\frac{1}{(1-\xi^{a_1 k})(1-\xi^{a_2 k})\cdots(1-\xi^{a_{d-1}k})}\xi^{-kn}\right| \\ &= \frac{1}{a_d}\sum_{k=1}^{a_d-1}\left|\frac{1}{(1-\xi^{a_1 k})(1-\xi^{a_2 k})\cdots(1-\xi^{a_{d-1}k})}\right|. \end{aligned}$$

We now repeatedly use the cotangent expansion lemma on any remaining term that has more factors than one. We now have an expression of the form

$$\frac{1}{a_d}\sum_{k=1}^{a_d-1}\left|\frac{B_1}{1-\xi^{a_1k}}+\frac{B_2}{1-\xi^{a_2k}}+\cdots+\frac{B_{d-1}}{1-\xi^{a_{d-1}k}}+C+\sum_{\substack{S\subset[d-1],\\|S|=d-1}}D_S\, i^{|S|}\prod_{j\in S}\cot\left(\frac{\pi a_j k}{a_d}\right)\right.$$
$$\left.+\sum_{\substack{S\subset[d-1],\\|S|=d-2}}D_S\, i^{|S|}\prod_{j\in S}\cot\left(\frac{\pi a_j k}{a_d}\right)+\cdots+\sum_{\substack{S\subset[d-1],\\|S|=2}}D_S\, i^{|S|}\prod_{j\in S}\cot\left(\frac{\pi a_j k}{a_d}\right)\right|,$$

where B_ℓ, C, and D_S are real constants for all $\ell \in [d-1]$ and $S \subset [d-1]$. Applying the triangle inequality hence yields

$$\sum_{\ell}|B_\ell|\left(\frac{1}{a_d}\sum_{k=1}^{a_d-1}\left|\frac{1}{1-\xi^{a_\ell k}}\right|\right)+|C|\left(\frac{1}{a_d}\sum_{k=1}^{a_d-1}|1|\right)+\sum_{S\subset[d-1]}|D_S|\left(\frac{1}{a_d}\sum_{k=1}^{a_d-1}\prod_{j\in S}\left|\cot\left(\frac{\pi a_j k}{a_d}\right)\right|\right).$$

Using the fact that the inner sum of the first term covers the complete residue system, we may replace a_ℓ by 1. Using the cosecant bound lemma, we have

$$\sum_{k=1}^{a_d-1}\left|\frac{1}{1-\xi^k}\right|\le\frac{1}{2\pi}a_d^2.$$

Using the Fourier alignment lemma, we see that

$$\sum_{k=1}^{a_d-1}\prod_{j\in S}\left|\cot\left(\frac{\pi a_j k}{a_d}\right)\right|\le\sum_{k=1}^{a_d-1}\left|\cot\left(\frac{\pi k}{a_d}\right)\right|^{|S|}.$$

for any $S\subset[d-1]$. Furthermore, by the cotangent bound lemma, we have

$$\sum_{k=1}^{a_d-1}\left|\cot\left(\frac{\pi k}{a_d}\right)\right|^{|S|}<\frac{1}{4\pi^{|S|}}a_d^{|S|}+\frac{3a_d}{4}.$$

All of these bounds thus yield

$$|s_{-n}(a_1, a_2, \ldots, a_{d-1}; a_d)| < \sum_{\ell} |B_\ell| \left(\frac{1}{a_d} \left(\frac{1}{2\pi} a_d^2 \right) \right) + \frac{|C|(a_d - 1)}{a_d} + \sum_{S \subset [d-1]} |D_S| \left(\frac{1}{a_d} \left(\frac{1}{4\pi^{|S|}} a_d^{|S|} + \frac{3a_d}{4} \right) \right).$$

We have now bounded the Fourier-Dedekind sum by a specific polynomial in a_d of degree at most $d-2$. Therefore, there exist d polynomials b_i in a_i with real coefficients of degree at most $d-2$ such that $|\mathrm{Quasi}_A(n)| < b_1 + b_2 + \cdots + b_d$. Recall that since A is fixed, $b_1 + b_2 + \cdots + b_d$ acts as a constant depending on A. Call this bound b_A.

We now turn our attention to $\mathrm{Poly}_A(n)$. Beck, Gessel, and Komatsu in [7] detail an explicit method using the Bernoulli numbers to calculate $\mathrm{Poly}_A(n)$ by way of generating functions. Thus, we may denote $\mathrm{Poly}_A(n)$ as $p_{d-1}n^{d-1} + p_{d-2}n^{d-2} + \cdots + p_0$, with $p_k \in \mathbb{R}$.

We conclude by using Lagrange's bound (Theorem 5.6) on the polynomial

$$p_{d-1}n^{d-1} + p_{d-2}n^{d-2} + \cdots + p_0 - b_A - s.$$

We know that $p_{d-1} = \frac{1}{(d-1)!a_1a_2\cdots a_d}$ from Theorem 2.12, so we now construct the set

$$R = \{1, |p_{d-2}(d-1)!\, a_1a_2\cdots a_d|^{\frac{1}{3}}, |p_{d-3}(d-1)!\, a_1a_2\cdots a_d|^{\frac{1}{4}}, \ldots, |p_1(d-1)!\, a_1a_2\cdots a_d|^{\frac{1}{d-2}}, |(p_0 - b_A - s)(d-1)!\, a_1a_2\cdots a_d|^{\frac{1}{d-1}}\}.$$

We obtain our bound on $g_s^*(A)$ by choosing the two largest elements from R. In the case that s is sufficiently large, the element $|(p_0 - b_A - s)(d-1)!\, a_1a_2\cdots a_d|^{\frac{1}{d-1}}$ will be one of our choices. This bound matches our asymptotic result from above; namely, when our second choice from R is M, we have

$$((s - b_A - p_0)(d-1)!\, a_1 a_2 \cdots a_d)^{\frac{1}{d-1}} + M \sim (s(d-1)!\, a_1 a_2 \cdots a_d)^{\frac{1}{d-1}}. \qquad \square$$

Chapter 6

Future Work

Most of the results of this paper are consequences of the approximation lemma from Chapter 2; that is, from approximating $p_A(n)$ by $\mathrm{Poly}_A(n)$. How good is this approximation? That is, how big is $\mathrm{Quasi}_A(n)$? Are there cases where the Fourier-Dedekind sums are large compared to $\mathrm{Poly}_A(n)$? We remark that this direction of research requires a fair amount of discrete Fourier analysis in order to refine our bounds.

Additionally, our approximation lemma requires A to be pairwise coprime. This is quite a strong condition. There is literature on the structure of the restricted partition function when A is just coprime, some coming directly from Sylvester [30, 31]. Does this more general case admit approximations as nice as the pairwise coprime case?

The results of Chapters 3 and 4 follow easily from the powerful Theorem 2.9. In the wide world of classical Frobenius numbers, what other results are there when A is a collection of products? Do they also have direct generalizations like the ones in this paper?

Is there a way to generalize the work in Chapter 4 to $d > 3$? There is a great amount of symmetry in the lemmas of Chapter 4. Is there more structure here to take advantage of?

The bound for $g_s^*(A)$ discussed in Chapter 5 has several crude estimations. How can we improve on these? Additionally, are there other methods in the geometry of polynomials we can use to make this bound sharper and/or more explicit?

Bibliography

[1] Iskander Aliev, Lenny Fukshansky, and Martin Henk. “Generalized Frobenius numbers: Bounds and average behavior”. In: *Acta Arithmetica* 155 (2011).

[2] Iskander Aliev, Martin Henk, and Eva Linke. “Integer Points in Knapsack Polytopes and s-covering Radius”. In: *Electronic Journal of Combinatorics* 20.2 (2012).

[3] Roger Apéry. “Sur les branches superlinéaires des courbes algébriques”. In: *C. R. Acad. Sci., Paris* 222 (1946), pp. 1198–1200.

[4] Tom Apostol. *Modular Functions and Dirichlet Series in Number Theory*. Graduate Texts in Mathematics. Springer New York, 2012.

[5] Peter Barlow. *An Elementary Investigation of the Theory of Numbers*. J. Johnson, 1811.

[6] Matthias Beck, Ricardo Diaz, and Sinai Robins. “The Frobenius problem, rational polytopes, and Fourier-Dedekind Sums”. In: *Journal of Number Theory* 96 (2002).

[7] Matthias Beck, Ira Gessel, and Takao Komatsu. "The polynomial part of a restricted partition function related to the Frobenius problem". In: *Electronic Journal of Combinatorics* 8.1 (2001).

[8] Matthias Beck and Curtis Kifer. "An extreme family of generalized Frobenius numbers". In: *Integers* 11.5 (2011), pp. 639–645.

[9] Matthias Beck and Sinai Robins. "An extension of the Frobenius coin-exchange problem". In: *New York Number Theory Seminar 2003* (2004), pp. 17–23.

[10] Matthias Beck and Sinai Robins. *Computing the continuous discretely. Integer-point enumeration in polyhedra*. Undergraduate Texts in Mathematics. New York, NY: Springer, 2007.

[11] Iaroslav Blagouchine and Eric Moreau. *On a finite sum of cosecants appearing in various problems*. Preprint. 2023. arXiv: `2312.16657 [math.NT]`.

[12] Alfred Brauer. "On a problem of partitions". In: *Am. J. Math.* 64 (1942), pp. 299–312.

[13] Alfred Brauer and James Shockley. "On a problem of Frobenius". In: *J. Reine Angew. Math.* 211 (1962), pp. 215–220.

[14] Wing-Sum Cheung. "Generalizations of Hölder's inequality". In: *International Journal of Mathematics and Mathematical Sciences* 26 (2001).

[15] Frank Curtis. "On Formulas for the Frobenius Number of a Numerical Semigroup". In: *Mathematica Scandinavica* 67.2 (1990), pp. 190–192.

[16] Roger Entringer. "Functions and Inverses of Asymptotic Functions". In: *The American Mathematical Monthly* 74.9 (1967), pp. 1095–1097.

[17] Lenny Fukshansky and Achill Schürmann. "Bounds on generalized Frobenius numbers". In: *European Journal of Combinatorics* 32.3 (Apr. 2011), pp. 361–368.

[18] Selmer Johnson. "A linear Diophantine problem". In: *Can. J. Math.* 12 (1960), pp. 390–398.

[19] Thomas Lam. *100 Questions: A Mathematical Conventions Survey*. [Online; accessed March 16, 2024]. 2024. URL: `https://cims.nyu.edu/~tjl8195/survey/results.html`.

[20] Morris Marden. *Geometry of Polynomials*. Vol. 3. American Mathematical Society, 1949.

[21] McDonald's. *Chicken McNuggets® & Selects® & Veggie Dippers*. [Online; accessed September 5, 2023]. 2017-2023. URL: `https://www.mcdonalds.com/gb/en-gb/menu/chicken-mcnuggets-and-selects.html`.

[22] Tiberiu Popoviciu. "Asupra unei probleme de partitie a numerelor". In: *Acad. Republicii Populare Romane, Filiala Cluj, Studii si cercetari stiintifice* 4 (1953), pp. 7–58.

[23] Hans Rademacher and Emil Grosswald. *Dedekind Sums*. Vol. 16. Mathematical Association of America, 1972.

[24] Jorge Ramírez-Alfonsín. "Complexity of the Frobenius problem". In: *Combinatorica* 16 (1996), pp. 143–147.

[25] Jorge Ramírez-Alfonsín. *The Diophantine Frobenius Problem.* Oxford Lecture Series in Mathematics and Its Applications. OUP Oxford, 2005.

[26] Joe Roberts. "Note on linear forms". In: *Proc. Am. Math. Soc.* 7 (1956), pp. 465–469.

[27] Aureliano Robles-Pérez and José Carlos Rosales. "The Frobenius number for sequences of triangular and tetrahedral numbers". In: *Journal of Number Theory* 186 (May 2018), pp. 473–492.

[28] Issai Schur. *Zur additiven Zahlentheorie.* Sitzungsberichte der Preussischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse. 1926.

[29] James Sylvester. *Mathematical Questions and Solutions, from the "Educational Times": With Many Papers and Solutions in Addition to Those Published in the "Educational Times," Problem 7382.* Vol. 41. Hodgson, 1884.

[30] James Sylvester. "On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order". In: *Amer. J. Math* 5.1 (1882).

[31] James Sylvester. "On the partition of numbers". In: *Quarterly J. Pure Appl. Math* 1 (1857).

[32] Audrey Terras. *Fourier Analysis on Finite Groups and Applications.* London Mathematical Society Student Texts. Cambridge University Press, 1999.

[33] Amitabha Tripathi. “On a linear Diophantine problem of Frobenius”. In: *Integers* 6 (2006).

[34] Amitabha Tripathi. “On a special case of the Frobenius problem”. In: *J. Integer Seq.* 20.7 (2017).

[35] Kevin Woods. “The generalized Frobenius problem via restricted partition functions”. In: *Electronic Journal of Combinatorics* 30.3 (2023).

[36] Chee Yap. *Fundamental Problems of Algorithmic Algebra*. Oxford University Press, 2000.