

UNLABELLING SIGNED GRAPH COLORINGS AND ACYCLIC ORIENTATIONS

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ABSTRACT. Several well-known graph invariants have a geometric interpretation via hyperplane arrangements. In this paper we review geometric methods for computing the chromatic polynomial and the number of acyclic orientations of a given graph. We then provide a compatible interpretation of Hanlon's work on unlabeled graph coloring and acyclic orientations. By extending these tools we give counting formulas for the number of labeled and unlabeled k -colorings of a signed graph, and its number of labeled and unlabeled acyclic orientations.

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0.1. **Introduction.** A **graph** g consists of a set of vertices $V(g)$ and a set of unordered pairs of those vertices called edges $E(g)$. The cardinality of $V(g)$ is called the **order** of g . We define the notation $[n]$ to mean the set $\{1, 2, \dots, n\}$, so that “the order of g is n ” and “ $V(g) = [n]$ ” are equivalent statements. All of our graphs will be finite.

Example 1. Let the graph g be as in Figure 1 with vertex set $V(g) = [5]$ and edge set $E(g) = \{(1, 2), (2, 3), (2, 4), (3, 5), (4, 5)\}$.

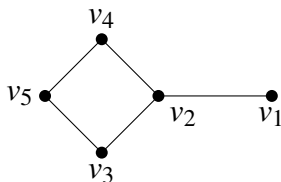


FIGURE 1. A graph on 5 vertices.

Although graphs have many interesting properties and applications (see, e.g., [7]), we are primarily concerned with questions of coloring. A graph coloring is a map from the vertex set to some set of labels; for our purposes the set of labels will be the non-negative integers, which we will refer to as **colors**.

A historical note on the synesthetic use of colors to refer to numbers—the classical context for graph vertex labeling is the coloring of maps subject to the restriction that territories sharing a border should be assigned different colors. This is the source of the famous Four Color Theorem [1], which states colloquially that any map of territories may be colored respecting the coloring rule using just four colors (try it out!).

The connection to maps of territories may be severed by generating a graph that encodes the relevant rules for coloring the map. We generate a vertex set by identifying a vertex with each territory and generate an edge set by including edge (i, j) if territories i and j share a border. Such vertices are called **adjacent**. Under this scheme a **proper** coloring of a graph is a labeling of the vertex set such that adjacent vertices are assigned distinct colors.

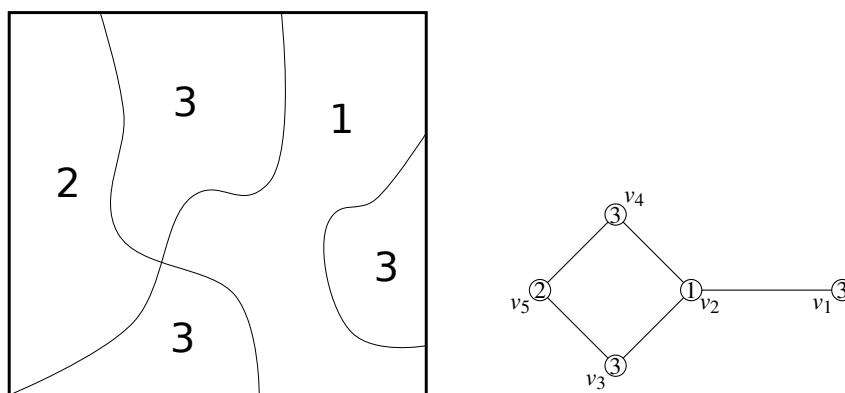


FIGURE 2. Interpreting a map coloring as a graph coloring.

Natural questions that arise include:

- (1) how many colors are required to properly color a given map or graph, and
- (2) given a finite collection of colors, how many distinct colorings exist?

In the vocabulary of graph theory, the first question is asking for the **chromatic number** of the graph. The Four Color Theorem says that for graphs arising from maps of territories (planar graphs), the chromatic number is at most 4. The second question is most commonly answered by defining a function called the **chromatic polynomial** (chroma is Greek for color) whose input is a number k of colors and whose output is the number of possible proper *colorings* using k colors.

The proper context for this paper is to extend the second question to read “up to some natural symmetry of the graph”.

For a graph g of order n , an **automorphism** is a permutation in S_n such that $\pi \cdot E(g) = E(g)$, where the action of π is given by

$$\pi \cdot E(g) := \{(\pi(i), \pi(j)) : (i, j) \in E(g)\}.$$

Example 2. Let g be as in Figure 3, with $E(g) = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$. The image of edge $e_1 = (1, 2)$ under the action of automorphism $a = (24)$ is $a \cdot e_1 = (1, 4)$. The middle graph is the result of acting on $E(g)$ with the permutation (24) , while the graph on the right is the result of acting by permutation (123) . By inspection, (24) is an automorphism, while (123) is not.

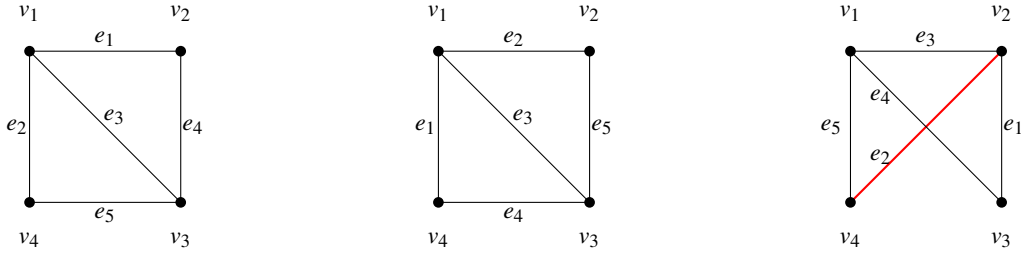


FIGURE 3. A graph automorphism example and a non-example.

Because the vertices of a graph g inherit an ordering from $[n]$, there is a natural interpretation of a graph coloring as a vector in \mathbb{R}^n . Informally, we list the color assigned to each vertex in order, with commas between, as with the coloring $(3, 1, 3, 3, 2)$ in Figure 0.1. This leads to a definition of the action of a permutation $\pi \in S^n$ on a coloring $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ by

$$\pi \cdot \sigma := (\sigma_{\pi(1)}, \sigma_{\pi(2)}, \dots, \sigma_{\pi(n)}).$$

The action extends to all of \mathbb{R}^n .

In Section 1 we develop Lemma 1.1, our main tool for counting colorings of graphs, and later, signed graphs. In Section 2 we define the geometry of graph coloring and show an application of our counting lemma, resulting in Theorem 3.1, a formula for the chromatic polynomial of a graph. As a corollary we get a new formula for the number of acyclic orientations of a graph using the reciprocity result of Stanley. We characterize the orbits of \mathbb{R}^n under a given permutation in Section 3.3, leading to Theorem 3.5, a formula for the unlabeled chromatic polynomial of a graph. Section 3.4 introduces the geometry of unlabeled acyclic orientations, and we present a formula for the number of unlabeled acyclic orientations of a graph (Corollary 3.10). The latter part of the paper repeats these results in the context of signed graphs, with Theorem 4.1 and Corollary 4.3 giving the signed chromatic polynomial and number of acyclic orientations, respectively, of a signed graph. Unlabeled versions are stated in Theorems 4.4 and 4.5.

1. HYPERPLANE ARRANGEMENTS

For $a \in \mathbb{R}^n \setminus 0$ and $b \in \mathbb{R}$, the solution set to the equation $ax = b$ is an affine subspace called a **hyperplane**. For the purposes of this paper, $b = 0$ and therefore a hyperplane is a linear subspace of codimension 1. A collection \mathcal{A} of hyperplanes is called an **arrangement**. An arrangement with nonempty intersection (such as an arrangement consisting of linear hyperplanes) is **central**. A maximal connected subset of $\mathbb{R}^n \setminus \mathcal{A}$ is a **region** of the arrangement.

1.1. **The Intersection Lattice.** To each subcollection of \mathcal{A} we identify its intersection and call this a **flat** of the arrangement. It is possible for more than one subcollection to give rise to the same flat, as in the case of the arrangement of hyperplanes in \mathbb{R}^3

$$\mathcal{A} = \{\{x \in \mathbb{R}^3 : x_1 = x_2\}, \{x \in \mathbb{R}^3 : x_1 = x_3\}, \{x \in \mathbb{R}^3 : x_2 = x_3\}\}.$$

Here we note that any collection of two hyperplanes is identified with the same flat of the arrangement since their intersection will be the set $\{x \in \mathbb{R}^3 : x_1 = x_2 = x_3\}$.

Flats of any arrangement have a partial order \succeq by reverse containment, e.g.,

$$\{x \in \mathbb{R}^3 : x_1 = x_2 = x_3\} \succeq \{x \in \mathbb{R}^3 : x_1 = x_2\}$$

since the first set is contained by the second.

A partially ordered finite set (finite poset) P is a **lattice** if for all elements $p, q \in P$, there exist a unique minimum element of P greater than and a unique maximum element of P less than both p and q . Lattices have unique minimal/maximal elements $\hat{0}$ and $\hat{1}$, respectively.

We include as a flat of \mathcal{A} the “empty” intersection corresponding to \mathbb{R}^n . The intersection poset $L(\mathcal{A})$ of a central arrangement is a lattice whose elements are the flats of the arrangement [11].

Lattices arising from arrangements come equipped with a **rank** function that maps each element to a natural number such that $\text{rank } \hat{0} = 0$ and whenever p immediately succeeds q (there exist no elements between p and q), $\text{rank } p = \text{rank } q + 1$. Since $\hat{0}$ in $L(\mathcal{A})$ is \mathbb{R}^n , we observe that a flat of rank k has dimension $n - k$.

Example 3. We give examples of A_n , the arrangement consisting of all hyperplanes of the form $\{x \in \mathbb{R}^n : x_i = x_j\}$ for $1 \leq i < j \leq n$; see Figure 4.

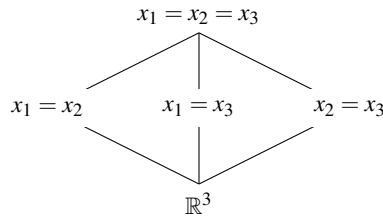


FIGURE 4. $L(A_3)$.

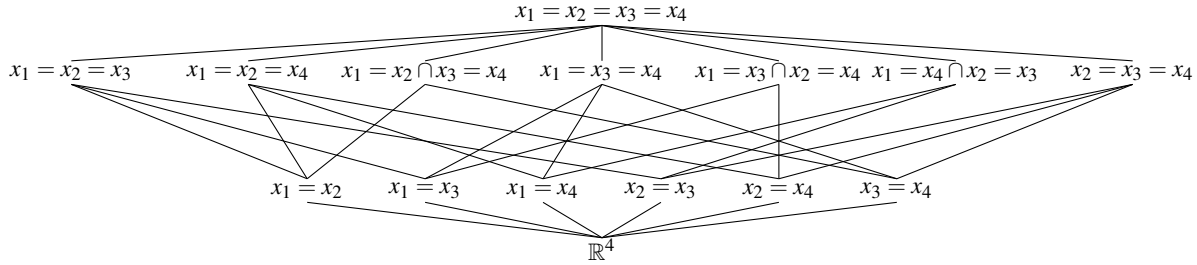


FIGURE 5. $L(A_4)$.

For a poset P equipped with a rank function, we define the **Whitney number** w_i^P to be the number of elements of P with rank i . From Figures 4 and 5, $w_1^{L(A_3)} = 3$ and $w_2^{L(A_4)} = 7$.

1.2. A Handy Partitioning. The **induced arrangement** on a flat F of the arrangement \mathcal{A} is the collection

$$\mathcal{A}^F := \{h \cap F : h \in \mathcal{A}, h \cap F \notin \{\emptyset, F\}\}.$$

The arrangement \mathcal{A}^F can be thought of as the hyperplanes of \mathcal{A} as viewed from *within* F .

There is a nice partitioning of \mathbb{R}^n via its flats: each point in \mathbb{R}^n lies in a region of the induced arrangement \mathcal{A}^F for exactly one $F \in L(\mathcal{A})$.

Example 4. Consider B_2 , the arrangement consisting of the coordinate hyperplanes in \mathbb{R}^2 . Elements of the lattice of flats of B_2 (Figure 6) are represented geometrically in Figure 7.

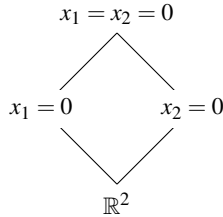


FIGURE 6. $L(B_2)$.

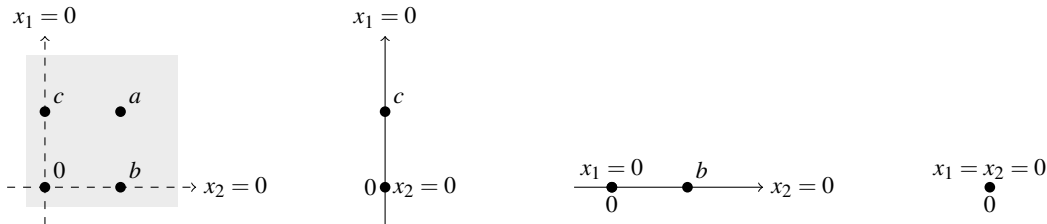


FIGURE 7. Partitioning by flats of B_2 .

In the depiction of B_2 in Figure 7, the point a is an element of $\mathbb{R}_2 \setminus B_2$, and thus is in a region (the positive orthant in this case) of the flat \mathbb{R}^2 . The point c is not in $\mathbb{R}_2 \setminus B_2$ since it lies on the

hyperplane $x_1 = 0$ which is in the collection B_2 . However, if we restrict to the flat $x_1 = 0$ and look at the induced arrangement on the flat, we see the second picture from the left. In this context, c is in a region of the induced arrangement consisting of the origin (the image of $x_2 = 0$ is the point 0). A similar argument places the point b in a region of the flat $x_2 = 0$. The point 0 is its own flat and thus is trivially in a region.

We can partition any subset of \mathbb{R}^n according to the scheme described above. For an arrangement \mathcal{A} and a subset S of \mathbb{R}^n ,

$$S = \bigcup_{F \in L(\mathcal{A})} S \cap F \setminus \mathcal{A}^F.$$

If S is a finite set,

$$|S| = \sum_{F \in L(\mathcal{A})} |S \cap F \setminus \mathcal{A}^F|.$$

Example 5. Let S be the points in the cube $[0, 3]^2$ with coordinates in \mathbb{Z} as illustrated in Figure 8. We count the number of integer points in the cube by summing on the flats of the arrangement $\mathcal{A} = \{\{x_1 = 0\}, \{x_2 = 0\}, \{x_1 = x_2\}\}$.

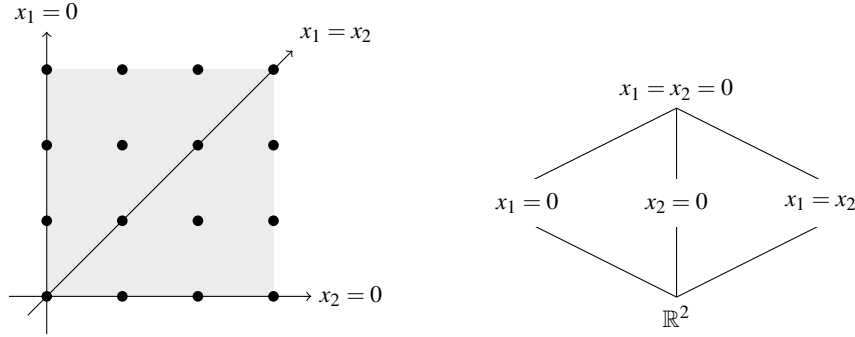


FIGURE 8. Flats of \mathcal{A} partitioning integer points.

The number of integer points in the cube that are not on any hyperplane in \mathcal{A} is $|S \cap \mathbb{R}^2 \setminus \mathcal{A}| = 6$. Summing bottom-to-top through $L(\mathcal{A})$,

$$\sum_{F \in L(\mathcal{A})} |S \cap \mathbb{R}^2 \setminus \mathcal{A}^F| = 6 + 3 + 3 + 3 + 1 = 16.$$

Let \mathcal{B} be a sub-arrangement of \mathcal{A} and $P(\mathcal{B})$ be the poset $\{F \in L(\mathcal{A}) : F \not\subseteq h \in \mathcal{B}\}$ with partial order inherited from $L(\mathcal{A})$. We will define the rank of $P(\mathcal{B})$ to be the largest rank (in $L(\mathcal{A})$) of any of its members. Then $P(\mathcal{B})$ is the poset of flats of $L(\mathcal{A})$ not contained in any of the hyperplanes of \mathcal{B} and

$$S \setminus \mathcal{B} = \bigcup_{F \in P(\mathcal{B})} S \cap F \setminus \mathcal{A}^F.$$

Lemma 1.1. If S is a finite set,

$$|S \setminus \mathcal{B}| = \sum_{F \in P(\mathcal{B})} |S \cap F \setminus \mathcal{A}^F|.$$

Example 6. Let S and \mathcal{A} be as in Example 5 and \mathcal{B} be $\{x_1 = 0\}$. The integer points in $[0, 3]^2$ that avoid \mathcal{B} may be counted by summing over the flats of $L(\mathcal{A})$ not contained in \mathcal{B} ; these are precisely the flats of $P(\mathcal{B})$.

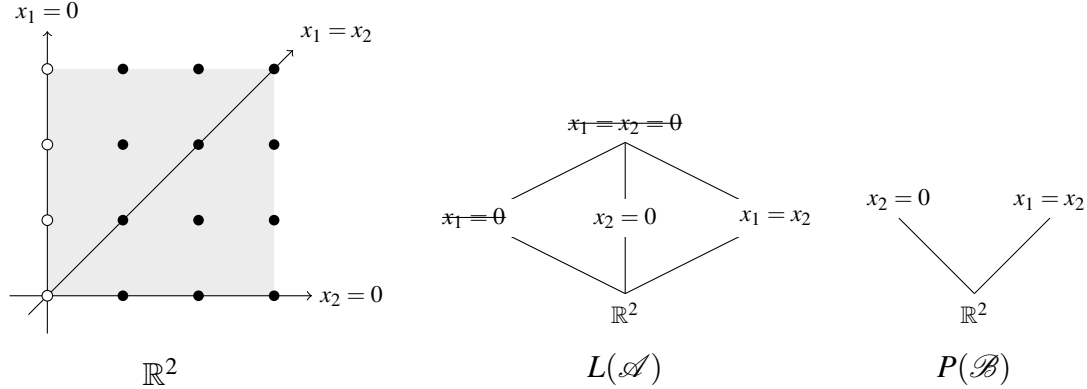


FIGURE 9. Counting integer points avoiding a hyperplane

The sum $\sum_{F \in P(\mathcal{B})} |S \cap \mathbb{R}^2 \setminus \mathcal{A}^F|$ is $6 + 3 + 3 = 12$.

2. GRAPHS

2.1. Chromatic Polynomials. The fact that the counting function for the number of proper colorings of a graph in k colors is polynomial in k is not *a priori* obvious. One should expect it to be an increasing function, since if one can properly color a graph using colors from $[k]$ the same coloring uses colors from $[k + 1]$. We address here one method for computing this function which shows it to be a sum of polynomials and hence polynomial.

We interpret colorings of a graph on n vertices as points in \mathbb{R}^n and count them by flats of a distinguished arrangement. The fundamental ingredient used is Lemma 1.1 with the summands on the right-hand side shown to be polynomials depending only on the dimension of the flat and the number of allowed colors.

The natural geometry of graph coloring is as follows:

- (1) The **braid arrangement** A_n in dimension n consists of all hyperplanes of the form $h_{ij} := \{x \in \mathbb{R}^n : x_i = x_j\}$ for $1 \leq i < j \leq n$.
- (2) The **graphic arrangement** \mathcal{B}_g associated to a given graph g is the subcollection of the braid arrangement that includes the hyperplane h_{ij} if and only if (i, j) is an edge of g .
- (3) A coloring of a graph on n vertices, considered as a map $\sigma : [n] \rightarrow \mathbb{Z}_{\geq 0}$, is a vector in $\mathbb{Z}_{\geq 0}^n$.
- (4) The condition of being a proper coloring is equivalent to being a vector in $\mathbb{Z}_{\geq 0}^n \setminus \mathcal{B}_g$.

This is an example of the **inside out polytope** construction presented in [2]; we are interested in counting lattice points *inside* a polytope (in this case, a copy of the unit cube that has been

scaled to k times its normal size) but *outside* a collection of fixed hyperplanes (in this case, the graphic arrangement \mathcal{B}_g).

To elaborate on (4), just as edges in graphs encode the properness conditions of maps of territories, the graphic arrangement encodes properness geometrically as a union of linear subspaces to be avoided. The question of how many proper colorings there are of g using k colors (a k -**coloring**) is thus reinterpreted as how many points in $[k]^n \cap \mathbb{Z}_{\geq 0}^n$ do not lie on any hyperplane of \mathcal{B}_g .

To use our scheme of partitioning by flats we need to know which flats contain the points we want to count and how many points there are on these flats. Regarding the first point, we answer simply that for points $\mathbf{x} = (x_1, x_2, \dots, x_n)$ on any flat contained by a hyperplane of \mathcal{B}_g , there exist indices i and j with $(i, j) \in E(g)$ such that $x_i = x_j$, so that \mathbf{x} is not a proper coloring of g . Therefore proper colorings necessarily lie on flats in the poset $P(\mathcal{B}_g)$ (hereafter denoted $P(g)$) with partial ordering descending from $L(A_n)$.

To answer the second question, we observe that a flat of dimension i arises as the intersection of $n - i$ hyperplanes, each equating the value of one coordinate to that of another. Thus even though all points in \mathbb{R}^n have n coordinates, those on a flat of $P(g)$ of dimension i are defined by only i distinct values.

If we want to count k -colorings lying in regions of a flat of dimension i , there are $\binom{k}{i}$ ways to choose the values and $i!$ ways to order them. Using the factorial definition of binomial coefficients and simplifying, we get

$$\begin{aligned} \binom{k}{i} \cdot i! &= \frac{k!}{i!(k-i)!} \cdot i! \\ &= k(k-1) \cdots (k-i+1) \\ &=: \langle k \rangle_i, \end{aligned}$$

a polynomial! Again, note that this polynomial depends only on k and the dimension $\dim F$ of the flat F . It follows from Lemma 1.1 with S the set of (not necessarily proper) k -colorings of the graph g and $\chi_g(k)$ the number of proper k -colorings of g ,

$$\chi_g(k) = \sum_{F \in P(g)} \langle k \rangle_{\dim F}.$$

Summing by dimension, we obtain:

Theorem 2.1. For a graph g of order n , the chromatic polynomial $\chi_g(k)$ is given by

$$\chi_g(k) = \sum_{i=0}^n w_i^{P(g)} \langle k \rangle_{n-i}.$$

Corollary 2.2. For a graph g of order n , the chromatic number is $n - \text{rank}(P(g))$.

Proof. Each falling factorial term $\langle k \rangle_{n-i}$ of $\chi_g(k)$ is zero for values less than $n - i$. Thus the smallest k for which $\chi_g(k)$ is not zero is when the value of i is maximal such that $w_i^{P(g)}$ is non-zero, i.e., for $i = \text{rank}(P(g))$. \square

Example 7. Let the C_4 be a 4-cycle as in Figure 10.

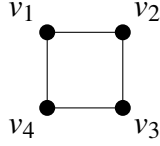


FIGURE 10. C_4 .

We construct $P(C_4)$ from $L(A_4)$ by removing the elements that are contained by hyperplanes of the graphic arrangement.

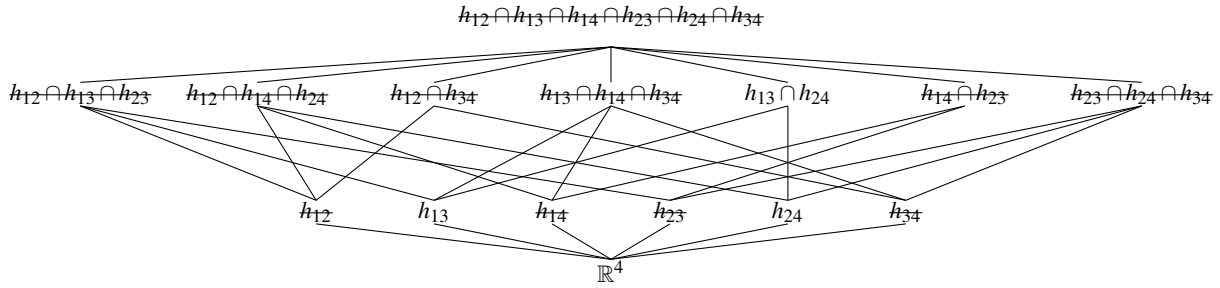


FIGURE 11. Constructing $P(C_4)$.

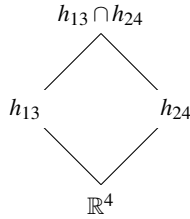


FIGURE 12. $P(C_4)$.

We compute $\chi_{C_4}(k)$ using Theorem 2.1:

$$\begin{aligned} \chi_{C_4}(k) &= \langle k \rangle_4 + 2\langle k \rangle_3 + \langle k \rangle_2 \\ &= (k^4 - 6k^3 + 11k^2 - 6k) + 2(k^3 - 3k^2 + 2k) + (k^2 - k) \\ &= k^4 - 4k^3 + 6k^2 - 3k. \end{aligned}$$

There are no 1-dimensional flats containing proper colorings of C_4 , but the graph may be 2-colored by noting the defining relations of the flat given by $x_1 = x_3$ and $x_2 = x_4$. Thus $(1, 2, 1, 2)$ is a proper 2-coloring of C_4 , and the chromatic number of C_4 is 2.

2.2. Acyclic Orientations. An edge of a graph may be directed by choosing an ordering of its endpoints. Unlike with edges (i, j) of graphs, which are unordered, directed edges \overrightarrow{ij} and \overrightarrow{ji} are distinct, with the first implying $v_i > v_j$ and the second implying $v_j > v_i$. This may be represented graphically (by convention of [10]) by drawing an arrowhead on the edge directed from the greater vertex to the lesser, so that the arrowhead can be read as an inequality.

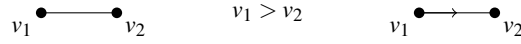


FIGURE 13. Edge orientation convention.

The assignment of a direction to each edge of a graph is an **orientation** of the graph. If orderings are chosen for each pair of adjacent vertices such that no contradictions arise (orderings $v_i < v_j < \dots < v_k < v_i$), the result is an **acyclic orientation** of the graph— an orientation containing no directed cycle; see Figure 14: the first orientation implies the ordering $v_1 < v_2 < v_3 < v_4 < v_1$, a contradiction, while the second orientation implies $v_4 < v_1 < v_2 < v_3$, and hence is acyclic.

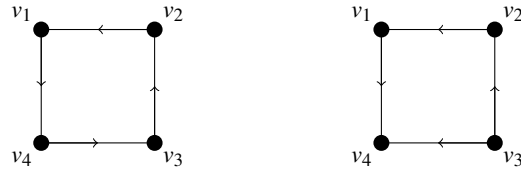


FIGURE 14. Orientations of C_4 .

Let $\Delta(g)$ be the set of acyclic orientations of g .

Theorem 2.3 (Stanley [10]). For a graph g of order n ,

$$|\Delta(g)| = (-1)^n \chi_g(-1).$$

Notice that $\langle -1 \rangle_j = (-1)(-1-1)\cdots(-1-j+1) = (-1)^j j!$, so that we obtain:

Corollary 2.4. For a graph g of order n ,

$$|\Delta(g)| = \sum_{i=0}^n (-1)^i w_i^{P(g)} (n-i)!.$$

Example 8. Consider the graph C_4 from Example 15. There are $2^4 = 16$ possible orientations on C_4 , all but 2 of which are acyclic, so that $|\Delta(C_4)|$ should be 14. Applying Corollary 2.4,

$$|\Delta(C_4)| = 4! - 2 \cdot 3! + 2! = 14,$$

as expected.

2.3. Unlabeled Graphs. Presented by Hanlon in [6], the notion of coloring an unlabeled graph (the orbit of a graph under its automorphism group) follows from the work of Robinson [9] among others, employing the cycle index counting methods of Pólya [8]. Equivalent though distinct from these viewpoints, we consider unlabeled *colorings* of labeled graphs.

An **unlabeled coloring** of a graph g is the orbit of a coloring of g under the action of g 's automorphism group. The counting function for the number of unlabeled proper k -colorings of a graph g is called the unlabeled chromatic polynomial of g and is denoted $\hat{\chi}_g(k)$.

Example 9. The orbit of the coloring $\sigma = (1, 2, 3, 2)$ of the graph g in Example 2 under its automorphism group $A(g) = \{\text{id}, (1, 3), (2, 4), (1, 3)(2, 4)\}$ is $\{(1, 2, 3, 2), (3, 2, 1, 2)\}$; this set is an unlabeled coloring of g .

Burnside's Lemma (see, e.g., [4]) gives us the following expression for the number of orbits of a set S under the action of a finite group A :

$$|\text{orbits of } S \text{ under } A| = \frac{1}{|A|} \sum_{a \in A} |\text{fix}(S, a)|$$

where $\text{fix}(S, a) := \{s \in S : a \cdot s = s\}$.

Given a graph automorphism, the set of points $\mathbf{x} \in \mathbb{R}^n$ fixed by its action has the following definition. Consistent with our earlier notation, let e_i be the i 'th standard basis vector of \mathbb{R}^n and

$$\mathbf{x} = (x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i e_i.$$

The point \mathbf{x} is fixed by a precisely when for each $i \in [n]$, $e_i \cdot a\mathbf{x} = x_i$ (with the usual inner-product.) Thus we can describe $\text{fix}(a)$ as the intersection of the sets

$$h_i^a := \{x \in \mathbb{R}^n : x_i = e_i \cdot a\mathbf{x}\} = h_{i, a^{-1}(i)},$$

so that

$$\text{fix}(\mathbb{R}^n, a) = \bigcap_{i \in [n]} h_i^a =: \hat{a}.$$

Example 10. Let $a = (3, 1, 2)(4, 5)$. For $\mathbf{x} \in \mathbb{R}^5$, $a\mathbf{x} = (x_2, x_3, x_1, x_5, x_4)$ so that

$$\begin{aligned} h_1^a &= \{x \in \mathbb{R}^5 : x_1 = x_2\} = h_{12} & h_4^a &= \{x \in \mathbb{R}^5 : x_4 = x_5\} = h_{45} \\ h_2^a &= \{x \in \mathbb{R}^5 : x_2 = x_3\} = h_{23} & h_5^a &= \{x \in \mathbb{R}^5 : x_5 = x_4\} = h_{45} \\ h_3^a &= \{x \in \mathbb{R}^5 : x_3 = x_1\} = h_{13} & \hat{a} &= h_{12} \cap h_{13} \cap h_{23} \cap h_{45}. \end{aligned}$$

In general, \hat{a} is a flat of the braid arrangement with dimension $\dim(\hat{a})$. To count the number of proper k -colorings fixed by a we can sum by the flats of $P(g)$ that are greater than \hat{a} . We point out the similarity to [6, Thm 2.1].

We define $Q(g, a) := \{q \in P(g) : q \succeq \hat{a}\}$.

Proper k -colorings of a graph g in a region of a flat of $Q(g, a)$ of dimension i have coordinates defined by precisely i distinct values. There are $\binom{k}{i}$ ways to choose them and $i!$ ways to order

them. Recalling that a flat of rank i in $Q(g, a)$ has dimension $\dim(a) - i$, such a flat contains in its regions $\langle k \rangle_{\dim(a)-i}$ proper k -colorings of g .

Theorem 2.5. For a graph g with automorphism group $A(g)$,

$$\hat{\chi}_g(k) = \frac{1}{|A(g)|} \sum_{a \in A(g)} \sum_{i=0}^n w_i^{Q(g,a)} \langle k \rangle_{\dim(\hat{a})-i}.$$

Example 11. For the graph C_4 (see Example 15),

$$A(C_4) = \{\text{id}, (13), (24), (13)(24), (12)(34), (14)(23), (1234), (1432)\}.$$

The automorphisms for which $Q(C_4, a)$ is not the empty poset are id , (13) , (24) and $(13)(24)$.

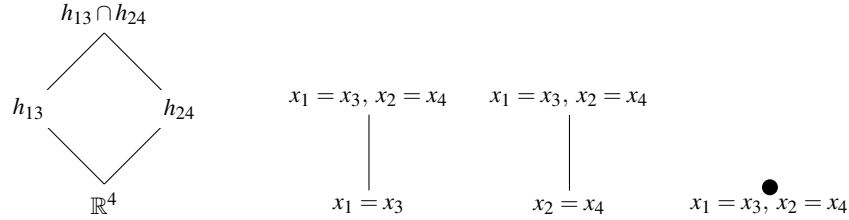


FIGURE 15. Posets containing proper colorings fixed by automorphisms.

Summing over $A(C_4)$,

$$\begin{aligned} \hat{\chi}_{C_4}(k) &= \frac{1}{8} \left[\left(\sum_{i=0}^4 w_i^{Q(C_4, \text{id})} \langle k \rangle_{4-i} \right) + \left(\sum_{i=0}^4 w_i^{Q(C_4, (13))} \langle k \rangle_{3-i} \right) \right. \\ &\quad \left. + \left(\sum_{i=0}^4 w_i^{Q(C_4, (24))} \langle k \rangle_{3-i} \right) + \left(\sum_{i=0}^4 w_i^{Q(C_4, (13)(24))} \langle k \rangle_{2-i} \right) \right] \\ &= \frac{1}{8} \left[\left(\langle k \rangle_4 + 2\langle k \rangle_3 + \langle k \rangle_2 \right) + \left(\langle k \rangle_3 + \langle k \rangle_2 \right) + \left(\langle k \rangle_3 + \langle k \rangle_2 \right) + \left(\langle k \rangle_2 \right) \right] \\ &= \frac{1}{8} \left[\langle k \rangle_4 + 4\langle k \rangle_3 + 4\langle k \rangle_2 \right] \\ &= \frac{1}{8} \left[k^4 - 2k^3 + 3k^2 - 2k \right]. \end{aligned}$$

2.4. Unlabeled Acyclic Orientations. We can craft a counting formula for the number of acyclic orientations of a graph up to the action of the graph's automorphism group similar to the unlabeled chromatic polynomial. An **unlabeled acyclic orientation** [9] is an orbit of an acyclic orientation of the underlying graph g under the action of $A(g)$.

There is a natural association of an acyclic orientation of a graph with a particular region of the graphic arrangement. The hyperplane associated to the edge (i, j) of the graph breaks up the ambient space into two open half-spaces defined by $x_i > x_j$ and $x_i < x_j$. Thus directing an edge selects a half-space, and a graph orientation can be associated with the intersection of this

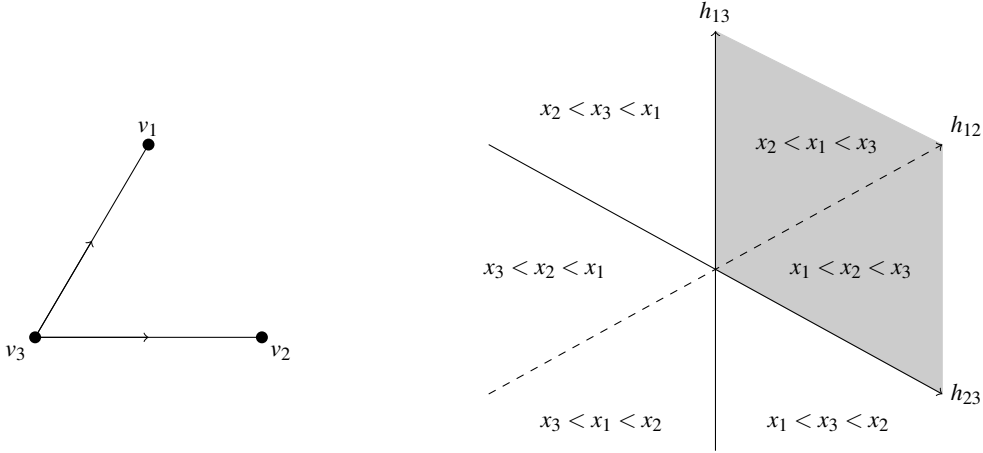


FIGURE 16. The region of \mathcal{B}_g associated with orientation of the graph on the left.

collection of half-spaces. This intersection will be non-empty if and only if the orientation is acyclic [5].

The depiction of the arrangement A_3 in Figure 16 shows \mathbb{R}^3 as viewed along the main diagonal $(1, 1, 1)$.

For the graph in Figure 16, the directed edge $\vec{31}$ in orientation o selects the half-space to the right of h_{13} and the directed edge $\vec{32}$ selects the half-space above h_{23} . Their intersection is the shaded region $r(o)$ of the graphic arrangement.

Lemma 2.6. Let a graph g have automorphism group $A(g)$. Then for each automorphism $a \in A(g)$ and orientation $o \in \Delta(g)$, we have $o \in \text{fix}(\Delta(g), a)$ if and only if $\hat{a} \cap r(o) \neq \emptyset$.

Proof. Let $a \in \text{fix}(\Delta(g), a)$. For $y \in r(o)$, $\langle a \rangle y$ is a subset of $r(o)$, and thus the barycenter \bar{y} of $\langle a \rangle y$ is in $r(o)$ and is fixed by a . A point in \mathbb{R}^n is fixed by a if and only if it is contained by \hat{a} , thus $r(o) \cap \hat{a}$ is non empty since it contains \bar{y} .

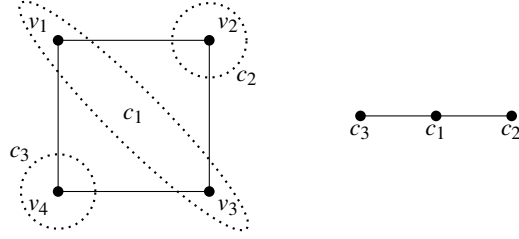
Conversely, let $y \in r(o) \cap \hat{a}$. An automorphism a is continuous and so preserves connectedness of sets, so that $a \cdot r(o)$ is connected. Thus for all y' in $a \cdot r(o)$ there exists a path from y to y' in $\mathbb{R}^n \setminus \mathcal{B}_g$. Since $r(o)$ is defined to be a maximally connected subset of $\mathbb{R}^n \setminus \mathcal{B}_g$, y' is in $r(o)$. Let instead y' be in $r(o)$. Then there exists a path from y to y' in $\mathbb{R}^n \setminus \mathcal{B}_g$, so that y' is in $a \cdot r(o)$ (again because $a \cdot r(o)$ is maximally connected). \square

For a permutation π with cycles c_1, \dots, c_m (ordered lexicographically), we define the quotient graph g/π as follows:

$$V(g/\pi) = [m], \quad E(g/\pi) = \{(ij) : \exists (uv) \in E(g) \text{ with } u \in c_i, v \in c_j\}.$$

Example 12. For the graph C_4 and the permutation $(1, 3)(2)(4) \in S^4$, we construct $C_4/(1, 3)$ by identifying v_1 and v_3 (Figure 17). Note that the order of g/π is the number of cycles of π .

Lemma 2.7. For a graph g with automorphism a , $|\text{fix}(\Delta(g), a)| = |\Delta(g/a)|$.

FIGURE 17. Constructing the graph C_4/π for $\pi = (1, 3)$.

Proof. By Lemma 2.6, each element $o \in \text{fix}(\Delta(g), a)$ appears as a region of $\hat{a} \setminus \mathcal{B}_g = \hat{a} \setminus \mathcal{B}_g^{\hat{a}}$. The arrangement $\mathcal{B}_g^{\hat{a}}$ is again a graphic arrangement (with respect to \hat{a}), and corresponds to the graph g/a . By Greene's observation [5], each region of $\hat{a} \setminus \mathcal{B}_g^{\hat{a}}$ corresponds to a unique acyclic orientation of g/a . \square

Theorem 2.8. For a graph g with automorphism group $A(g)$,

$$|\hat{\Delta}(g)| = \frac{1}{|A(g)|} \sum_{a \in A(g)} (-1)^{\dim(\hat{a})} \chi_{g/a}(-1).$$

Proof. By Burnside's lemma,

$$|\hat{\Delta}(g)| = \frac{1}{|A(g)|} \sum_{a \in A(g)} |\text{fix}(\Delta(g), a)|.$$

Using Lemma 2.7,

$$|\hat{\Delta}(g)| = \frac{1}{|A(g)|} \sum_{a \in A(g)} |\Delta(g/a)|.$$

We conclude by Theorem 2.3, noting that the order of g/a is the dimension of \hat{a} . \square

This reproduces [6, Cor. 4.1].

Lemma 2.9. For a graph g with automorphism a , as posets, $P(g/a) \cong Q(g, a)$.

Proof. We assume that \hat{a} is in $P(g)$. By [11], the poset consisting of the flats of $L(A_n)$ that are above \hat{a} is isomorphic to $L(A_n^{\hat{a}})$. Since $G \in L(A_n^{\hat{a}})$ is contained by $\mathcal{B}_g^{\hat{a}}$ if and only if it is contained by \mathcal{B}_g , G is in $P(g/a)$ if and only if it is in $Q(g, a)$. \square

Corollary 2.10. For a graph g of order n and automorphism group $A(g)$,

$$|\hat{\Delta}(g)| = \frac{1}{|A(g)|} \sum_{a \in A(g)} \sum_{i=0}^n (-1)^i w_i^{Q(g,a)} (\dim(\hat{a}) - i)!.$$

Proof. Applying Theorem 2.1 to Theorem 3.5, and substituting $Q(g, a)$ for $P(g/a)$,

$$\begin{aligned} |\hat{\Delta}(g)| &= \frac{1}{|A(g)|} \sum_{a \in A(g)} (-1)^{\dim(\hat{a})} \sum_{i=0}^n w_i^{Q(g,a)} \langle -1 \rangle_{\dim(\hat{a})-i} \\ &= \frac{1}{|A(g)|} \sum_{a \in A(g)} \sum_{i=0}^n (-1)^{\dim(\hat{a})} w_i^{Q(g,a)} (-1)^{\dim(\hat{a})-i} (\dim(\hat{a}) - i)! \\ &= \frac{1}{|A(g)|} \sum_{a \in A(g)} \sum_{i=0}^n (-1)^i w_i^{Q(g,a)} (\dim(\hat{a}) - i)!. \end{aligned}$$

□

Example 13. Consider the graph C_4 as in Example 11. Summing Corollary 2.10 over the elements of $A(C_4)$ for which $Q(C_4, a)$ is non-empty,

$$\begin{aligned} |\hat{\Delta}(C_4)| &= \frac{1}{8} \left[\left(\sum_{i=0}^4 (-1)^i w_i^{Q(C_4, \text{id})} (4-i)! \right) + \left(\sum_{i=0}^4 (-1)^i w_i^{Q(C_4, (1,3))} (3-i)! \right) \right. \\ &\quad \left. + \left(\sum_{i=0}^4 (-1)^i w_i^{Q(C_4, (2,4))} (3-i)! \right) + \left(\sum_{i=0}^4 (-1)^i w_i^{Q(C_4, (1,3)(2,4))} (2-i)! \right) \right] \\ &= \frac{1}{8} \left[(4! - 2 \cdot 3! + 2!) + (3! - 2!) + (3! - 2!) + (2!) \right] \\ &= \frac{1}{8} [14 + 4 + 4 + 2] \\ &= 3. \end{aligned}$$

The three unlabeled orientations of C_4 are represented in Figure 18. A vertex with all edges directed outward is called a **source**, and a vertex with all edges directed inward is a **sink**. Note that no permutation in $A(C_4)$ can map one orientation to another, since the first has one sink and one source adjacent, while the second has them non-adjacent. The third unlabeled orientation has two sinks and two sources. The sizes of the orbits of these orientations are 8, 4, and 2, respectively.

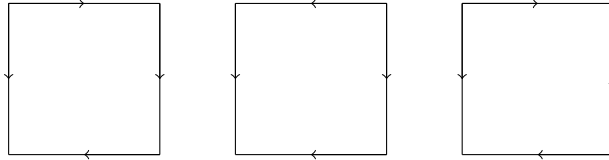


FIGURE 18. The three unlabeled orientations of C_4 .

3. SIGNED GRAPHS

A **signed graph** Γ is an ordinary graph g equipped with a sign $\varepsilon(e) \in \{\pm\}$ on each edge $e \in E(\Gamma)$. We allow up to two edges (of different sign) between any two vertices, so that each

edge is associated with its endpoints and its sign. Proper colorings of a signed graph reflect this added structure: a signed graph coloring is an assignment of integers (rather than positive integers) to the vertex set, with proper colorings $\sigma \in \mathbb{Z}^n$ satisfying $\sigma_i \neq \varepsilon(e)\sigma_j$ for any edge $e = (i, j, \varepsilon(e)) \in E(\Gamma)$. Note that a signed graph with $\varepsilon(e) = +$ for all edges e is essentially an ordinary graph, and the proper coloring rule for signed graphs is consistent with that of ordinary graphs.

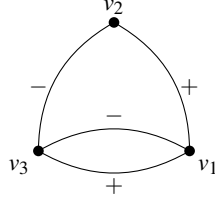


FIGURE 19. A signed graph.

Example 14. Let the signed graph Γ be as in Figure 19. The properness condition associated with the edge $(1, 2, +)$ is $\sigma_1 \neq \sigma_2$, and for the edge $(1, 3, -)$ it is $\sigma_1 \neq -\sigma_3$.

We develop a geometric construction for signed graph coloring parallel to the one for ordinary graphs using the method of inside out polytopes. To define the geometry of signed graphs, we need a hyperplane arrangement that will encode the properness of colorings. The Coxeter arrangement BC_n consists of the the following hyperplanes:

$$h_{i,j}^+ := \{x \in \mathbb{R}^n : x_i = x_j, 1 \leq i < j \leq n\}$$

$$h_{i,j}^- := \{x \in \mathbb{R}^n : x_i = -x_j, 1 \leq i, j \leq n\}.$$

The hyperplanes $h_{i,j}^\pm$ with $i \neq j$ are precisely what are needed to interpret proper signed graph colorings as integer points in \mathbb{R}^n avoiding the **signed graphic arrangement** \mathcal{B}_Γ , where

$$\mathcal{B}_\Gamma := \bigcup_{e \in E(\Gamma)} h_{i,j}^{\varepsilon(e)}.$$

We may take advantage of the unused (by ordinary edges) hyperplanes $h_{i,i}^-$ by the following scheme— a **loop** is an edge whose endpoints are the same vertex. In the context of ordinary graph coloring, a graph containing a loop at vertex i would have no proper colorings since the (degenerate) hyperplane corresponding to this edge is $\{x \in \mathbb{R}^n : x_i = x_i\} = \mathbb{R}^n$, so that no integer points would avoid the graphic arrangement. In the context of signed graph coloring, if the sign of loop e at vertex i is positive, the hyperplane $h_{i,i}^+$ is the degenerate hyperplane \mathbb{R}^n , and we have the same condition. However, if $\varepsilon(e) = -$, then we have the following relation to be avoided by proper colorings σ : $\sigma_i = -\sigma_i$, which can be restated as $\sigma_i = 0$. Thus subarrangements of BC_n may be thought of as encoding the properness of colorings of a signed graph possibly including negative loops. The signed graph for which BC_n is the signed graphic arrangement is the complete signed graph, depicted in Figure 20 with $n = 3$.

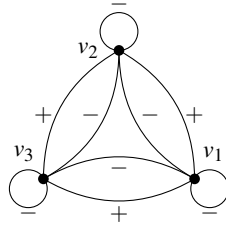


FIGURE 20. The complete signed graph on three vertices.

Much of the construction from the case of ordinary graphs is the same for signed graphs. There is a lattice $L(BC_n)$ of intersections of the hyperplanes of the arrangement BC_n , and we define the poset $P(\Gamma)$ by $P(\Gamma) := \{F \in L(BC_n) : F \not\subseteq h \in \mathcal{B}_\Gamma\}$. When drawing these posets and labeling their elements, we will use the $\cap h_{i,j}^\pm$ notation and let $h_{i,\dots,k} := h_{i,i}^- \cap \dots \cap h_{k,k}^-$.

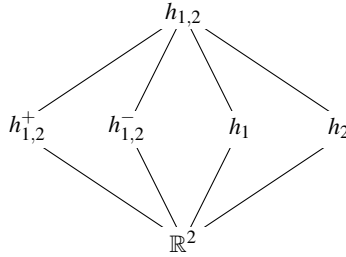


FIGURE 21. The lattice of flats of BC_2 .

3.1. Signed Chromatic Polynomials. Since a coloring of a signed graph uses integers rather than positive integers, we modify our notion of a k -coloring to mean a coloring with values taken from $[-k, k] \cap \mathbb{Z}$. As such, we will construct a counting function for a given signed graph Γ whose input is a *magnitude* k , and whose output is the number of proper colorings of Γ using colors from $[-k, k] \cap \mathbb{Z}$. We call this function the **signed chromatic polynomial** $\chi_\Gamma(k)$. This is a departure from the work of Zaslavsky in [12], which defines the signed chromatic polynomial $\chi'_\Gamma(k)$ only for odd arguments, and treats separately the case where the color 0 is not used. To compare the two perspectives, observe that $\chi_\Gamma(k) = \chi'_\Gamma(2k + 1)$.

Similarly to before, we derive a counting function for the number of integer points avoiding the induced arrangement on a flat F of $L(BC_n)$. For a flat F of dimension i , an integer point in $F \setminus BC_n^F \cap \mathbb{Z}^n \cap [-k, k]$ has i distinct (nonzero) magnitudes among its coordinates. There are $\binom{k}{i}$ ways to choose them, $i!$ ways to order them, and 2^i ways to sign them. Thus

$$|F \cap \mathbb{Z}^n \cap [-k, k] \setminus BC_n^F| = 2^i \binom{k}{i}.$$

To count the proper k -colorings of a signed graph Γ of order n , we again employ Lemma 1.1 with S the set of (not necessarily proper) k -colorings of Γ , \mathcal{A} the arrangement BC_n , and \mathcal{B} the signed graphic arrangement \mathcal{B}_Γ .

Theorem 3.1. For a signed graph Γ of order n , the signed chromatic polynomial is given by

$$\chi_\Gamma(k) = \sum_{i=0}^n w_i^{P(\Gamma)} 2^{n-i} \langle k \rangle_{n-i}.$$

This result may be compared with [12, Lemma 2.1], which can be restated as

$$\chi_\Gamma(k) = \sum_{i=0}^n a_i(\Gamma) 2^{n-i} \langle k \rangle_{n-i},$$

with $a_i(\Gamma)$ the number of symmetry classes of exact proper signed i -colorings of Γ .

Example 15. Let the graph Γ be as in Figure 22.

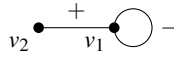


FIGURE 22. Signed graph Γ .

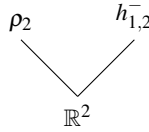


FIGURE 23. $P(\Gamma)$.

We compute $\chi_\Gamma(k)$ using Theorem 3.1:

$$\chi_\Gamma(k) = 2^2 \langle k \rangle_2 + 2 \cdot 2^1 \langle k \rangle_1 = 4k^2.$$

The four 1-colorings of Γ are $(1, 0)$, $(-1, 0)$, $(1, -1)$, and $(-1, 1)$.

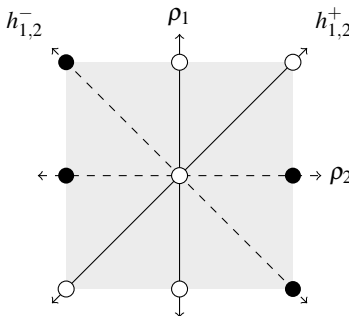


FIGURE 24. The signed graphic arrangement \mathcal{B}_Γ and proper 1-colorings of Γ .

3.2. Acyclic Orientations of Signed Graphs. Signed graphs Γ have a notion of orientation similar to that of ordinary graphs, except that, rather than directed edges, signed graphs have directed **incidences** (e, v) , where $e \in E(\Gamma)$ and $v \in V(\Gamma)$ is an endpoint of e . We represent a directed incidence graphically by an arrowhead on the edge pointing either into or out of the endpoint. We denote algebraically the direction at the incidence by $\tau(e, v) = \pm$, with the convention that $\tau(e, v) = +$ is associated with an arrowhead directed out of vertex v along edge e .¹ The orientation thus respects the structure of the underlying ordinary graph (including loops and multiple edges). An orientation of edge $e = (u, v, \varepsilon(e)) \in E(\Gamma)$ respects the added signed graph structure by conforming to the relation

$$\tau(e, u) = -\varepsilon(e)\tau(e, v).$$

For $\varepsilon(e) = +$, the directions on the incidences will be coherent, and for a signed graph with all positive edges, an orientation will be consistent with an orientation of an ordinary graph.

The notion of an acyclic orientation of a signed graph is defined in [12] as an orientation of Γ such that every subgraph has either a sink or a source vertex.

Theorem 3.2. (Zaslavsky, [12]). For signed graph Γ with set of acyclic orientations $\Delta(\Gamma)$ and $|V(\Gamma)| = n$,

$$|\Delta(\Gamma)| = (-1)^n \chi_{\Gamma}(-1).$$

Corollary 3.3. For signed graph Γ with set of acyclic orientations $\Delta(\Gamma)$ and $|V(\Gamma)| = n$,

$$|\Delta(\Gamma)| = \sum_{i=0}^n (-1)^i 2^{n-i} w_i^{P(\Gamma)} (n-i)!.$$

3.3. Unlabeled Signed Graphs. We call an edge and its endpoints **incident**, and **switching** a signed graph at a vertex changes the sign of all edges incident to the vertex.

The Hyperoctahedral group is the group of symmetries of the n -cube, and its elements are the compositions of reflections about hyperplanes of type $h_{i,j}^+$ or h_i for $1 \leq i < j \leq n$ [3]. We call these group elements **signed permutations**.

Let signed permutation b be defined by the product of some reflections. Then the action of b on signed graph Γ is to permute the vertex set by transposition (ij) for the reflection $h_{i,j}^+$ and switch at vertex i for the reflection h_i . We denote a product of switching at vertices i, j , and k with the shorthand notation $h_{i,j,k}$. A **signed graph automorphism** of Γ is a signed permutation that fixes $E(\Gamma)$.

3.4. Unlabeled Signed Chromatic Polynomials. The counting function for the number of unlabeled proper k -colorings of a signed graph Γ is called the unlabeled chromatic polynomial of Γ and is denoted $\hat{\chi}_{\Gamma}(k)$. Proper signed graph colorings are precisely those contained in $P(\Gamma)$, and colorings fixed by a given signed permutation also have a description in terms of $L(BC_n)$. Following the definition from unlabeled ordinary graphs, a point is fixed by automorphism a if and only if it is contained by the flat $\hat{a} \in L(BC_n)$. Thus we have the following theorem.

¹This convention is the opposite of that adopted in [13] in order to maintain consistency within the paper.

Theorem 3.4. For a given signed graph Γ of order n with automorphism group A ,

$$\hat{\chi}_\Gamma(k) = \frac{1}{|A|} \sum_{a \in A} \sum_{i=0}^n 2^{\dim(a)-i} w_i^{Q(\Gamma,a)} \langle k \rangle_{\dim(a)-i},$$

where $Q(\Gamma, a) := \{p \in P(\Gamma) : p \succeq \hat{a}\}$.

Example 16. Let the signed graph Γ be as in Figure 19. We compute $\hat{\chi}_\Gamma(k)$ as follows.

The automorphism group of Γ is $A = \{\text{id}, h_{1,3} \cdot h_{1,3}^+, h_2 \cdot h_{1,3}^+, h_{1,2,3}\}$. The images of A in $L(BC_n)$ are \mathbb{R}^3 , $h_{1,3}^-$, $h_2 \cap h_{1,3}^+$, and $h_{1,2,3}$. The associated posets $Q(\Gamma, a)$ are empty with the exception of $Q(\Gamma, \text{id})$ (Figure 25).

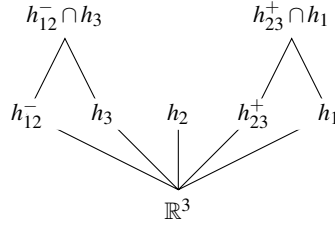


FIGURE 25. $P(\Gamma) = Q(\Gamma, \text{id})$ for Γ as in Figure 19.

We compute $\hat{\chi}_\Gamma(k)$ using Theorem 3.4.

$$\begin{aligned} \hat{\chi}_\Gamma(k) &= \frac{1}{4} [2^3 \langle k \rangle_3 + 5 \cdot 2^2 \langle k \rangle_2 + 2 \cdot 2^1 \langle k \rangle_1] \\ &= 2k^3 - k^2. \end{aligned}$$

The four 1-colorings $(0, 1, 1)$ and $(0, -1, -1)$ from $h_{2,3}^+ \cap h_1$ and $(1, -1, 0)$ and $(-1, 1, 0)$ from $h_{1,2}^- \cap h_3$ are an orbit under $A(\Gamma)$, and represent the only unlabeled 1-coloring of Γ , since $\hat{\chi}_\Gamma(1) = 1$.

3.5. Unlabeled Acyclic Orientations of Signed Graphs. It is proved in [13] that regions of the signed graphic arrangement of Γ correspond to acyclic orientations of Γ . The statements of Lemmas 2.6, 2.7, and 2.9 generalize naturally to the signed case.

Theorem 3.5. For a signed graph Γ with automorphism group $A(\Gamma)$,

$$|\hat{\Delta}(\Gamma)| = \frac{1}{|A(\Gamma)|} \sum_{a \in A(\Gamma)} \sum_{i=0}^n (-1)^i 2^{\dim(\hat{a})-i} w_i^{Q(\Gamma,a)} (\dim(\hat{a}) - i)!.$$

Proof. By Burnside's lemma,

$$|\hat{\Delta}(\Gamma)| = \frac{1}{|A(\Gamma)|} \sum_{a \in A(\Gamma)} |\text{fix}(\Delta(\Gamma), a)|.$$

Lemma 2.6 applied to signed graphic arrangements says that the region $r(o)$ associated to a signed acyclic orientation $o \in \Delta(\Gamma)$ has non-empty intersection with \hat{a} if and only if $o \in \text{fix}(\Delta(\Gamma), a)$. Thus we can count $\text{fix}(\Delta(\Gamma), a)$ by counting the regions of the induced arrangement $\mathcal{B}_\Gamma^{\hat{a}}$ on \hat{a} .

Restating the proof of Lemma 2.7 for the signed case, $\mathcal{B}_\Gamma^{\hat{a}}$ is a signed graphic arrangement in the context of \hat{a} , and the chromatic polynomial of its underlying signed graph (which we will call Γ/a) can be computed via Theorem 3.1 using the poset $Q(\Gamma, a)$, which by Lemma 2.9 is isomorphic to $P(\Gamma/a)$. \square

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