Partition Analysis and Ehrhart Theory

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by
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CERTIFICATION OF APPROVAL

I certify that I have read *Partition Analysis and Ehrhart Theory* by Dorothy L. Moorefield and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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ABSTRACT

In the early 1900s, Major Percy A. MacMahon developed the Ω Operator as a tool for enumerating partitions via their corresponding diophantine relations. In this paper, we will give an introduction to MacMahon's techniques provided in his now classic *Combinatory Analysis*. Then we will show how MacMahon's methods can be applied to the problem of enumerating lattice points in polyhedra. Corteel, Lee and Savage have developed five guidelines that provide a simplification of MacMahon's partition analysis for integral, linear, homogeneous systems of inequalities. We will discuss these guidelines and then expand on them to include linear systems of equalities in the effort to find the Ehrhart polynomial for faces of the Birkoff polytope.

I certify that the Abstract is a correct representation of the content of this thesis.

Matthias Beck
Chairperson, Advisory Committee

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Contents

1. A Beginning

In the introduction to the second volume of *C*ombinatory Analysis [9], Major Percy A. MacMahon states:

"In conclusion, I would say that I am aware that the reader will probably find imperfections in the volumes, but I shall be satisfied if they are found to contain ideas which are new and fresh, and such as are likely to prove starting-points for further investigations in an exceedingly interesting field of pure mathematics."

The original problem for this thesis was to find the Ehrhart polynomial of the Chan-Robbins-Yuen polytope. While attempting to apply the traditional methods of Ehrhart theory (outlined in chapter 2) to the given problem, the author realized the need of a different method of computation. This necessity lead to the study of MacMahon's partition analysis (outlined in chapter 3).

In the spirit of the above quote the purpose of this paper is to show means of merging Ehrhart theory with MacMahon's partition analysis. Even though the theories are decades apart and arose independently, there is a significant amount of interplay between the two in which this paper only begins to show.

2. Ehrhart Theory

2.1. Introduction. Let $a \in R^n$ and $b \in \mathbb{R}$. Then a *hyperplane* consists of the set $\{x \in \mathbb{R}^n | a^T x = b\}$ and a *halfspace* consists of the set $\{x \in \mathbb{R}^n | a^T x \geq b\}$. If A is an $m \times n$ matrix and $b \in \mathbb{R}^m$ then a *polyhedron* P consists of the set

$$P := \{x \in \mathbb{R}^n | Ax \leq b\}.$$

In other words, a polyhedron is the intersection of finitely many halfspaces.

If $x, y \in \mathbb{R}^n$ and $y \neq 0$ then the set $\{x+ty | t \geq 0\}$ is called a *ray* and the set $\{x+ty | t \in \mathbb{R}\}$ is called a *line*. We say a polyhedron is bounded if it does not contain a ray. A bounded polyhedron is called a *polytope*. This definition is known as the *hyperplane description* of a polytope.

Let $P \subset \mathbb{R}^n$ be a polyhedron and suppose $x \in P$. If there exists some $c \in R^n$ such that for all $y \in P$, $c^T x < c^T y$ where $y \neq x$ we say x is a *vertex* of P. It can be shown (refer to [4]) that every polytope is the convex hull of it's vertices. This leads to an alternate definition of a polytope: A polytope is the convex hull of finitely many points. This is known as the *vertex description* of a polytope.

A polytope is said to be *integral* if all of its vertices are integer points. If all of the vertices of a polytope are rational points, the polytope is said to be *rational*.

The *dimension* of a polyhedron is the dimension of the affine space formed by the span of the polyhedron. If $P = \{x \in \mathbb{R}^n | Ax \leq b\}$ and A' be some collection of rows of A, then the set $F := P \cap \{x \in \mathbb{R}^n | A'x = b\}$ is itself a polyhedron and called a *face* of P. The dimension of a face F is the dimension of the span of F. If P is an n dimensional polyhedron then the $n-1$ dimensional faces are called *facets*, 1 dimensional faces are called *edges* and the 0 dimensional faces are the vertices.

Two common questions involving a polytope, P, are: How many integer points does P contain? For a given positive integer, t, how many integer points does $t \cdot P$ contain? This is due largely in part to the connections between discrete volume and continuous volume.

A discrete set of points in $\mathbb{R}^n$ such as $\mathbb{Z}^n$ forms a *lattice*. The *discrete volume* of an n dimensional polytope P is the number of lattice points contained in the polytope, i.e., $\#(P \cap \mathbb{Z}^n)$. We can obtain the continuous volume from the discrete by applying the usual Riemann techniques for integration. In doing so we obtain

$$\mathrm{Vol}(P) = \lim_{t\to\infty} \# \left(P \cap \frac{1}{t}\mathbb{Z}^n\right)\frac{1}{t^n}.$$

Note, if $P \subset \mathbb{R}^n$ has dimension m less than n, then the above $\mathrm{Vol}(P) = 0$. To resolve this issue, we can find the *relative volume* of P by computing the volume in m dimensional space to obtain

$$\mathrm{RelVol}(P) = \lim_{t\to\infty} \# \left(P \cap \frac{1}{t}\mathbb{Z}^m\right)\frac{1}{t^m}.$$

When a polytope is not full-dimensional, relative volume is often referred to as being the volume of the polytope unless otherwise stated. Since $\#\left(P \cap \frac{1}{t}\mathbb{Z}^n\right) = \#(tP \cap \mathbb{Z}^n)$, the need to count the lattice points in the t^{th} dilation of P arises.

For an n dimensional polytope P let $e_p(t) := \#(tP \cap \mathbb{Z}^n)$. In 1962 Eugène Ehrhart, then a high school teacher, published the following theorem:

Theorem 2.1. *If P is an n dimensional, integral polytope then $e_p(t)$ is a polynomial in t of degree n.*

This polynomial is referred to as the Ehrhart polynomial. In Section 3 we provide a rough sketch of the proof of Ehrhart's theorem. For the detailed proof of Ehrhart's theorem and more detail of the volume of polytopes refer to [3].

2.2. Generating Functions**.** Let $\{a_k\}_{k=0}^{\infty}$ be an infinite sequence and suppose we wish to manipulate the sequence to perhaps find a closed formula. Then we can embed the sequence into a *generating function* of the form

$$F(x) = \sum_{n=0}^{\infty} a_k x^k.$$

Generating functions are very useful in that the degree of each monomial keeps track of the position in the sequence while the coefficient provides the actual value of the term. Then if we play by the rules of either formal or analytic power series we may be able to derive the desired result. The following is one of many nice introductory examples of the power of generating functions provided in [3].

Consider the Fibonacci sequence $f_0 = 0$, $f_1 = 1$ and $f_{k+2} = f_{k+1} + f_k$ for $k \geq 0$. Let

$$F(x) = \sum_{k=0}^{\infty} f_k x^k.$$

Then we have

$$\sum_{k=0}^{\infty} f_{k+2} x^k = \frac{1}{x^2} \sum_{k=0}^{\infty} f_{k+2} x^{k+2} = \frac{1}{x^2}[F(x) - x],$$

$$\sum_{k=0}^{\infty} f_{k+1} x^k = \frac{1}{x} \sum_{k=0}^{\infty} f_{k+1} x^{k+1} = \frac{1}{x} F(x).$$

Since

$$\sum_{k=0}^{\infty} f_{k+2} x^k = \sum_{k=0}^{\infty} (f_{k+1} + f_k) x^k = \sum_{k=0}^{\infty} f_{k+1} x^k + \sum_{k=0}^{\infty} f_k x^k,$$

we have

$$\frac{1}{x^2}[F(x) - x] = \frac{1}{x} F(x) + F(x).$$

Solving for $F(x)$ we obtain

$$F(x) = \frac{x}{1-x-x^2} = \frac{x}{\left(1 - \frac{1+\sqrt{5}}{2}x\right)\left(1 - \frac{1-\sqrt{5}}{2}x\right)},$$

which has the following partial fraction expansion

$$\frac{\frac{1}{\sqrt{5}}}{1 - \frac{1+\sqrt{5}}{2}x} - \frac{\frac{1}{\sqrt{5}}}{1 - \frac{1-\sqrt{5}}{2}x}.$$

Now we will make use of the well known geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

to obtain

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \left(\frac{1+\sqrt{5}}{2}x\right)^k - \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \left(\frac{1-\sqrt{5}}{2}x\right)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right) x^k. \end{aligned}$$

This provides the desired closed form:

$$f_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right).$$

The rational function obtained using the properties of geometric series is called a *rational generating function*. Often we will jump back and forth from generating functions to rational generating functions. A natural question to ask about these generating functions is, when are formal series appropriate as opposed to analytic series? In many cases, we expand rational functions into their corresponding geometric series. As long as we are living in the world of power series, we know these expansions are unique. However, as we will see in section 4.1 when we begin to deal with Laurent series convergence becomes an issue. In the meantime, however, we have no need to consider such things and will proceed accordingly.

The main goal of the next section is to show means of finding the rational generating function corresponding to

$$1 + \sum_{t \geq 1} \#(tP \cap \mathbb{Z}^n)x^t,$$

where P is an n dimensional integral polytope. We can safely consider only positive solutions because we can always shift a polytope into the positive orthant without changing the lattice count. We will then show how to extract the Ehrhart polynomial and volume of P from its rational generating function.

2.3. Overview of Current Methods. Suppose $P \subset \mathbb{R}^n$ is n dimensional. Then we say P is a *simplex* if P has $n+1$ vertices.

Let T be a finite collection of n dimensional simplices such that:

(1) $\bigcup_{S \in T} S = P$
(2) For all $S_i, S_j \in T$ either $S_i \cap S_j = \varnothing$ or $S_i \cap S_j$ is a face of both S_i and S_j.
(3) Any vertex of S, where $S \in T$, is a vertex of P.

Then T forms a *triangulation* of P. A polytope P is said to be *triangulated* if P is decomposed into a triangulation. Many proofs in Ehrhart theory rely on the following theorem.

Theorem 2.2. *Every polytope can be triangulated.*

Let $v, w_1, w_2, \ldots, w_k \in R^n$. Then a *cone* is a set of the form

$$K := \{v + \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_k w_k | \lambda_i \geq 0\}.$$

If K does not contain a line, then K is said to be a *pointed cone* where v is called the *apex* and each w_i is called a *generator*. K is said to be *rational* if all of its generators and apex are rational. A pointed cone $K \subset \mathbb{R}^n$ is said to be *simple* if it has exactly n linearly independent generators.

Suppose we have an n dimensional polytope $P \subset \mathbb{R}^n$ with vertices $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$. Now consider the points in $\mathbb{R}^{n+1}$ given by $w_1 = (v_1, 1), w_2 = (v_2, 1), \ldots, w_k = (v_k, 1)$. If we use these points as generators with the origin as the apex we obtain a cone $K_P \subset \mathbb{R}^{n+1}$. Notice all we are doing is lifting the polytope into a higher dimension but are not affecting its relative volume. In other words the integer point count of the lifted polytope will remain the same as the integer point count of the polytope. Moreover, if we slice the cone with the hyperplane $x_{n+1} = t$, we obtain the t^{th} dilate of P. Picture contour maps, in the topographical sense. The process of forming a cone via the vertices of a polytope is known as *coning over* the polytope. This process is very useful in finding Ehrhart polynomials as the rational generating functions of simple cones have nice properties and we can obtain information about our original polytope from them.

The definition for a triangulation of a pointed cone K is essentially the same as the definition of a triangulation of a polytope. The only differences we triangulate into simple cones instead of simplices, and instead of having no new vertices, we have no new generators.

Theorem 2.3. *Every pointed cone can be triangulated into simple cones.*

The triangulation theorems allow us to prove things only for simplices or simple cones. We can always decompose a polyhedron into simplices (or simple cones), extract information about the pieces, and then construct the desired information for the whole polyhedron via inclusion-exclusion.

Now we are ready to state the following theorems, which lead to Ehrhart's theorem. The proofs of these theorems along with the triangulation theorems are provided in [3].

For a simple cone $K \subset \mathbb{R}^n$, let

$$\sigma_K(z) = \sum_{m \in K \cap \mathbb{Z}^d} z^m$$

with the usual monomial notation $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, and $z^m = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$.

Theorem 2.4. *Let K be an n dimensional, rational, simple cone with generators, $w_1, \ldots, w_n \in \mathbb{Z}^n$. Then*

$$\sigma_K(z) = \frac{\sigma_{\Pi_K}(z)}{(1 - z^{w_1})(1 - z^{w_2}) \cdots (1 - z^{w_n})},$$

where $\Pi_K := \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_n w^n | 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n < 1\}$.

Lemma 2.5. *Let P be an n dimensional rational simplex and K_P be the cone over P. Then*

$$1 + \sum_{t \geq 1} \#(tP \cap \mathbb{Z}^n) z_{n+1}^t = \sigma_{K_P}(1, \ldots, 1, z_{n+1}).$$

Lemma 2.6. *Let P be an n dimensional integral simplex and K_P be the cone over P. Then*

$$1 + \sum_{t \geq 1} \#(tP \cap \mathbb{Z}^n) z_{n+1}^t = \sigma_{K_P}(1, \ldots, 1, z_{n+1}) = \frac{\sigma_{\Pi_{K_P}}(1, \ldots, 1, z_{n+1})}{(1 - z_{n+1})^{n+1}}.$$

Note this only holds for integral polytopes. If P is rational then the generators of K_P, $w_k = (v_k, 1)$ will have rational entries. In order to have

$$\sigma_{K_P}(z) = \frac{\sigma_{\Pi_{K_P}}(z)}{(1 - z^{w_1})(1 - z^{w_2}) \cdots (1 - z^{w_n})},$$

we need to scale the generators to integer points, which will cause at least one exponent of z_{n+1} to no longer be 1 in the rational generating function.

Lemma 2.7. *If P is an n dimensional, integral, simplex then $\sigma_{\Pi_{K_P}}(1,\ldots,1,z_{n+1}) = g(z_{n+1})$ is a polynomial of degree at most n and $g(1) \neq 0$.*

Lemma 2.8. *f is a polynomial of degree n if and only if*

$$\sum_{t\geq 0} f(t)x^t = \frac{g(x)}{(1-x)^{n+1}},$$

where g is a polynomial of degree at most n and $g(1) \neq 0$.

Ehrhart's theorem follows by piecing the above lemmas together. We also can obtain even more information about integral polytopes from their rational generating functions.

Lemma 2.9. *Let P be an n dimensional integral polytope and suppose*

$$1 + \sum_{t\geq 1} e_P(t)x^t = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0}{(1-x)^{n+1}},$$

then $a_0 = 1$.

Theorem 2.10. *Let P be an n dimensional integral polytope and suppose*

$$1 + \sum_{t\geq 1} e_P(t)x^t = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + 1}{(1-x)^{n+1}},$$

then

$$e_p(t) = \binom{t+n}{n} + a_1\binom{t+n-1}{n} + \cdots + a_{n-1}\binom{t+1}{n} + a_n\binom{t}{n}.$$

This theorem is extremely useful in that if we have the rational generating function for an integral polytope, we automatically obtain the Ehrhart polynomial. Also notice

$$e_p(0) = \binom{n}{n} + a_1\binom{n-1}{n} + \cdots + a_{n-1}\binom{1}{n} + a_n\binom{0}{n} = \binom{n}{n} = 1.$$

Once we have the Ehrhart polynomial of an integral polytope in hand we can easily obtain the relative volume of the polytope.

Theorem 2.11. *If P is an n dimensional integral polytope and*

$$e_P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 x + 1,$$

then

$$\mathrm{RelVol}(P) = \lim_{t\to\infty} \#\left(tP \cap \mathbb{Z}^n\right) \frac{1}{t^n} = \lim_{t\to\infty} \frac{a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + 1}{t^n} = a_n.$$

If we expand the polynomial in the numerator of the rational generating function in Lemma 1.10, the leading coefficient of this polynomial is $\frac{1}{n!}(a_n + a_{n-1} + \cdots + a_1 + 1)$. This combined with Theorem 1.11 gives

Corollary 2.12. *If P is an n dimensional integral polytope and*

$$1 + \sum_{t\geq 1} e_P(t) x^t = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 + 1}{(1-x)^{n+1}},$$

then $\mathrm{RelVol}(P) = \frac{1}{n!}(a_n + a_{n-1} + \cdots + a_1 + 1)$.

For example let

$$P := \{(x_1, x_2) \in \mathbf{R}^2 \mid x_2 + 2x_1 \geq 2,\ 2 \geq x_2,\ \text{and } 1 \geq x_1\}.$$

Then P is an integral 2 dimensional simplex with vertices $(0,2), (1,0), (1,2)$. The generators for the cone over P are $(0,2,1), (1,0,1), (1,2,1)$ and we have

$$\sigma_{K_P}(z) = \frac{\sigma_{\Pi_{K_P}}(z)}{(1 - z_2 z_3)(1 - z_1 z_3)(1 - z_1 z_2^2 z_3)}.$$

To find $\sigma_{\Pi_{K_P}}(z)$, we will use a little bit of linear algebra. If (n_1, n_2, n_3) is a lattice point in Π_{K_P}, then

$$n_1 = \lambda_2 + \lambda_3$$

$$n_2 = 2\lambda_1 + 2\lambda_3$$

$$n_3 = \lambda_1 + \lambda_2 + \lambda_3,$$

where $0 \leq \lambda_1, \lambda_2, \lambda_3 < 1$. This implies

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

Inverting our matrix we obtain

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -\frac{1}{2} & 1 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}.$$

This gives $-n_1 + n_3 = \lambda_1$, which implies $\lambda_1 = 0$ and thus $n_1 = n_3$. This implies $\frac{n_2}{2} = \lambda_3$. Therefore $n_2 = 0$ or $n_2 = 1$. If $n_2 = 1$ then $\lambda_2 = n_3 - \frac{1}{2}$, which implies $n_1 = n_3 = 1$. If $n_2 = 0$ then $n_3 = \lambda_2$, which implies $n_1 = n_3 = 0$. Therefore the only integer points of Π_{K_P} are $(0, 0, 0)$ and $(1, 1, 1)$. This gives

$$\sigma_{\Pi_{K_P}}(z) = z_1^0 z_2^0 z_3^0 + z_1^1 z_2^1 z_3^1 = 1 + z_1 z_2 z_3$$

and

$$\sigma_{K_P}(z) = \frac{1 + z_1 z_2 z_3}{(1 - z_2 z_3)(1 - z_1 z_3)(1 - z_1 z_2^2 z_3)}.$$

Therefore the desired rational generating function for the Ehrhart polynomial of P is

$$\sigma_{K_P}(1, 1, z_3) = \frac{1 + z_3}{(1 - z_3)^3},$$

which yields the desired Ehrhart polynomial,

$$e_P(t) = \binom{t+2}{2} + \binom{t+1}{2}.$$

Very little is known about polytopes with irrational vertices. However, we can use methods similar to above to show Ehrhart's theorem for rational polytopes:

Theorem 2.13. *If P is an n dimensional rational polytope then $e_P(t) = \#(tP \cap \mathbb{Z}^n)$ is quasi-polynomial in t of degree n. The period of $e_P(t)$ divides the least common multiple of the denominators of the coordinates of the vertices of P.*

Not much will be said about rational polytopes in this paper. Regardless, the previous theorem is definitely worth mentioning.

In 1994 Alexander Barvinok showed for a fixed dimension there exists a polynomial time algorithm to compute the number of integer points in rational polyhedra [2]. His method involved cleverly triangulating cones into unimodular cones via valuations. This algorithm has been implemented by Jesus De Loera et.al. [7] into a program called LattE (Lattice point Enumeration).

The main drawbacks of most current methods of obtaining the Ehrhart polynomial of a polytope P are we must have the vertex description of P and if P is not simple we must perform some sort of triangulation. If we are only given the hyperplane description of P, we can find its vertices, however, if P consists of the intersection of many halfspaces, this could be time consuming. If P has many vertices, triangulation will be very time consuming as would the inclusion-exclusion required to piece P back together. In the following section we will illustrate the need for alternative methods.

2.4. The Birkoff and Chan-Robbins-Yuen Polytopes**.** The n^{th} Birkoff Polytope B_n is defined to be the convex hull of the set of $n \times n$ permutation matrices. In other words, B_n consists of $n \times n$ matrices with row and column sums equal to 1. These matrices are also known as doubly stochastic matrices and are useful in statistics and probability theory.

For $n \geq 2$, the n^{th} Chan-Robbins-Yuen Polytope (CYR_n) is defined to be the convex hull of the set of $n \times n$ permutation matrices, A_n, where if $j \geq i + 2$, then $a_{ij} = 0$. CYR_n has 2^{n-1} vertices and is $\binom{n}{2}$ dimensional. The fact that the row and column sums are 1, provide the hyperplane description.

CYR_n is a face of the n^{th} Birkoff polytope. While working on the Birkoff polytope, Chan and Robbins conjectured that the relative volume of CYR_n is

$$\mathrm{RelVol}(CYR_n) = \prod_{i=0}^{n-2} \frac{1}{i+1}\binom{2i}{i},$$

the product of the first $n-1$ Catalan numbers. In the effort to prove this conjecture Chan, Robbins and Yuen did extensive work on CYR. The conjecture was later proved by Doron Zeilberger in 1998 [11]. The closed form for the volume of CYR_n suggests a possible recursion of the Ehrhart polynomials with respect to the dimensions. Such a recursion was presented in [6].

Theorem 2.14. *Let $e_{CYR_n}(t)$ denote the Ehrhart polynomial for CYR_n evaluated at t. Then for every nonnegative integer t, the sequence*

$$e_{CYR_1}(t), \ldots, e_{CYR_n}(t), \ldots$$

satisfies a linear recursion of degree $p(t)$ with integer coefficients, where $p(t)$ is the number of partitions of t.

Chan et.al. determined for $t = 1, \ldots, 12$, there is no lower-degree recursion for the sequence. However, it may be possible to find a lower-degree recursion for larger t. This would be necessary as counting the number of partitions of t increases in difficulty as t increases. One possible means finding a recursion of the Ehrhart polynomials is to find a corresponding recursion between the rational generating functions for the sequence

$e_{CYR_1}, \ldots, e_{CYR_n}, \ldots$. In the effort to find such recursions, we will explore the study of partitions provided by Percy A. MacMahon.

3. MacMahon's Partition Analysis

3.1. Introduction. In the early 1900s, Percy A. MacMahon developed the Ω Operator as a tool for enumerating partitions via their corresponding diophantine relations. In this chapter, we will give an introduction to MacMahon's techniques provided in his now classic *Combinatory Analysis* [9].

3.2. Definition of $\Omega_{\geq}$ and Identities. Define a partition of a positive integer t to be a list of positive integers $(a_1, \ldots, a_k)$ arranged in descending order whose sum is t. The order of a partition gives rise to a system of diophantine relations; $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_k$, which can be altered to produce various forms of partitions. For example, we can have partitions of at most k parts, partitions into odd parts, partitions into distinct parts, etc.

For example the following generating function gives the number $N(t)$ of ways three nonnegative integers can sum to a positive integer t without taking into account the aforementioned ordering, $a_1 \geq a_2 \geq a_3 \geq 0$.

$$\sum_{t\geq 0} N(t)x^t = \sum_{a_i \geq 0} x^{a_1+a_2+a_3} = \sum_{a_1\geq 0} x^{a_1} \sum_{a_2 \geq 0} x^{a_2} \sum_{a_3\geq 0} x^{a_3} = \frac{1}{(1-x)^3}.$$

Notice that the previous diophantine relations are equivalent to the following:

$$a_1 - a_2 \geq 0$$
$$a_2 - a_3 \geq 0$$
$$a_3 \geq 0.$$

Considering this system, introduce a new variable for each inequality to obtain

$$\sum_{a_i\geq 0} \lambda_1^{a_1-a_2} \lambda_2^{a_2-a_3} \lambda_3^{a_3} x^{a_1+a_2+a_3}.$$

Now if we expand this series, eliminate all terms with negative powers, and set $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then we will obtain the generating function for partitions into at most 3 parts. This is precisely what the operator $\Omega_{\geqslant}$ does.

Definition 3.1. *For a multiple Laurent series,*

$$\sum_{\nu_1,\ldots\nu_k=-\infty}^{\infty} A_{\nu_1,\ldots\nu_k}\lambda_1^{\nu_1}\ldots\lambda_k^{\nu_k},$$

define the operator $\Omega_{\geqslant}$ by:

$$\Omega_{\geqslant}\sum_{\nu_1,\ldots\nu_k=-\infty}^{\infty} A_{\nu_1,\ldots\nu_k}\lambda_1^{\nu_1}\ldots\lambda_k^{\nu_k} := \sum_{\nu_1,\ldots\nu_k=0}^{\infty} A_{\nu_1,\ldots\nu_k}.$$

Three of the many identities presented in [9] are:

Id 3.2.

$$\Omega_{\geqslant}\frac{1}{(1-\lambda x)(1-\frac{y}{\lambda})} = \frac{1}{(1-x)(1-xy)}$$

Id 3.3.

$$\Omega_{\geqslant}\frac{1}{(1-\lambda x)(1-\frac{y}{\lambda^2})} = \frac{1}{(1-x)(1-x^2y)}$$

Id 3.4.

$$\Omega_{\geqslant}\frac{1}{(1-\lambda^2 x)(1-\frac{y}{\lambda})} = \frac{1+xy}{(1-x)(1-xy^2)}$$

MacMahon leaves the verification of many of his identities to the reader. Therefore we will now present them.

Proof. For Id 3.2, consider the crude generating function:

$$\frac{1}{(1-\lambda x)(1-\frac{y}{\lambda})} = \sum_{a_i\geq 0}\lambda^{a_1-a_2}x^{a_1}y^{a_2}.$$

If $a_2 > a_1$, then λ will have a negative power. To prevent this from happening, let $a_1 - a_2 = b$, force the restriction, $b \geq 0$ and make the appropriate substitutions into the crude generating function to obtain

$$\sum_{a_2,b\geq 0}\lambda^b x^{a_2+b}y^{a_2}=\sum_{a_2,b\geq 0}(\lambda x)^b(xy)^{a_2}=\frac{1}{(1-\lambda x)(1-xy)}.$$

In doing so, we have removed the possibilities for λ to have a negative power. Now if we set $\lambda=1$, we have the desired identity.

For Id 3.3, we similarly let $a_1-2a_2=b$, restrict $b\geq 0$ and make the appropriate substitutions into

$$\sum_{a_i\geq 0}\lambda^{a_1-2a_2}x^{a_1}y^{a_2}=\frac{1}{(1-\lambda x)(1-\frac{y}{\lambda^2})}.$$

For Id 3.4, consider the following sum split into two parts:

$$\sum_{a_i\geq 0}\lambda^{2a_1-a_2}x^{a_1}y^{a_2}=\sum_{\substack{a_1\geq 0\\ a_2=2k_1+1\\ k_1\geq 0}}\lambda^{2a_1-a_2}x^{a_1}y^{a_2}+\sum_{\substack{a_1\geq 0\\ a_2=2k_1\\ k_1\geq 0}}\lambda^{2a_1-a_2}x^{a_1}y^{a_2}.$$

Eliminate the terms in which λ has negative powers in the odd part by letting $2a_1=2k_2+a_2=2k_1+2k_2+2$ for $k_2\geq 0$. Then $a_1=k_1+k_2+1$, and

$$\sum_{\substack{a_1\geq 0\\ a_2=2k_1+1\\ k_1\geq 0}}\lambda^{2a_1-a_2}x^{a_1}y^{a_2}=\sum_{k_i\geq 0}\lambda^{2k_2+1}x^{k_1+k_2+1}y^{2k_1}=\lambda xy\sum_{k_i\geq 0}\lambda^{2k_2}x^{k_1+k_2}y^{2k_1}.$$

Eliminate the terms in which λ has negative powers in the even part by letting $2a_1=2k_2+a_2=2k_1+2k_2$ for $k_2\geq 0$. Then $a_1=k_1+k_2$, and

$$\sum_{\substack{a_1\geq 0\\ a_2=2k_1\\ k_1\geq 0}}\lambda^{2a_1-a_2}x^{a_1}y^{a_2}=\sum_{k_i\geq 0}\lambda^{2k_2}x^{k_1+k_2}y^{2k_1}.$$

Therefore,

$$\sum_{a_i\geq 0}\lambda^{2a_1-a_2}x^{a_1}y^{a_2}=\lambda xy\sum_{k_i\geq 0}\lambda^{2k_2}x^{k_1+k_2}y^{2k_1}+\sum_{k_i\geq 0}\lambda^{2k_2}x^{k_1+k_2}y^{2k_1}$$

$$= (1+\lambda xy)\sum_{k_i\geq 0}\lambda^{2k_2}x^{k_1+k_2}y^{2k_1}$$

Setting $\lambda = 1$ we have Id 3.4. $\square$

Returning to the example, regroup the variables of our generating function to obtain

$$\sum_{a_i\geq 0}\lambda_1^{a_1-a_2}\lambda_2^{a_2-a_3}\lambda_3^{a_3}x^{a_1+a_2+a_3} = \sum_{a_i\geq 0}(\lambda_1 x)^{a_1}\left(\frac{\lambda_2}{\lambda_1}x\right)^{a_2}\left(\frac{\lambda_3}{\lambda_2}x\right)^{a_3},$$

which leads to the following crude generating function:

$$\sum_{a_i\geq 0}(\lambda_1 x)^{a_1}\left(\frac{\lambda_2}{\lambda_1}x\right)^{a_2}\left(\frac{\lambda_3}{\lambda_2}x\right)^{a_3} = \frac{1}{(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x\right)\left(1-\frac{\lambda_3}{\lambda_2}x\right)}.$$

This equality allows us to conclude:

$$\sum_{a_1\geq a_2\geq a_3\geq 0}x^{a_1+a_2+a_3} = \Omega_{\geqslant}\frac{1}{(1-\lambda_1 x)(1-\frac{\lambda_2}{\lambda_1}x)(1-\frac{\lambda_3}{\lambda_2}x)}.$$

Now apply Id 3.2 to obtain

$$\Omega_{\geqslant}^{\lambda_1}\frac{1}{(1-\lambda_1 x)(1-\frac{\lambda_2}{\lambda_1}x)(1-\frac{\lambda_3}{\lambda_2}x)} = \frac{1}{(1-x)(1-\lambda_2 x^2)(1-\frac{\lambda_3}{\lambda_2}x)}$$

$$\Omega_{\geqslant}^{\lambda_2}\frac{1}{(1-x)(1-\lambda_2 x^2)(1-\frac{\lambda_3}{\lambda_2}x)} = \frac{1}{(1-x)(1-x^2)(1-\lambda_3 x^3)}.$$

Now since there are no terms in which λ_3 has a negative power we have

$$\Omega_{\geqslant}^{\lambda_3}\frac{1}{(1-x)(1-x^2)(1-\lambda_3 x^3)} = \frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

Thus, the rational generation function for the number of ways to partition a positive integer t into at most 3 parts is:

$$\Omega_{\geqslant}\frac{1}{(1-\lambda_1 x)(1-\frac{\lambda_2}{\lambda_1}x)(1-\frac{\lambda_3}{\lambda_2}x)} = \frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

Similarly it can be shown in general that the rational generating function for a positive integer t into k parts is:

$$\Omega_{\geqslant}\frac{1}{(1-\lambda_1 x)(1-\frac{\lambda_2}{\lambda_1}x)\dots(1-\frac{\lambda_k}{\lambda_{k-1}}x)} = \frac{1}{(1-x)(1-x^2)\dots(1-x^k)}.$$

3.3. Definition of $\Omega_=$ and Identities.

Definition 3.5. *For a multiple Laurent series,*

$$\sum_{\nu_1,\dots\nu_k=-\infty}^{\infty} A_{\nu_1,\dots\nu_k}\lambda_1^{\nu_1}\dots\lambda_k^{\nu_k},$$

define the operator $\Omega_=$ *by:*

$$\Omega_= \sum_{\nu_1,\dots\nu_k=-\infty}^{\infty} A_{\nu_1,\dots\nu_k}\lambda_1^{\nu_1}\dots\lambda_k^{\nu_k} := A_0,\dots,A_0.$$

As with $\Omega_{\geq}$, MacMahon presents some identities for $\Omega_=$, two of which are:

Id 3.6.

$$\Omega_=\frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)} = \frac{1}{(1-xy)}.$$

Id 3.7.

$$\Omega_=\frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)} = \frac{1}{(1-xy)(1-xz)}$$

These lead to more general results:

Id 3.8.

$$\Omega_=\frac{1}{(1-A_1\lambda)(1-A_2\lambda)\cdots(1-A_n\lambda)\left(1-\frac{y}{\lambda}\right)} = \frac{1}{(1-A_1 y)(1-A_2 y)\cdots(1-A_n y)}.$$

Id 3.9.

$$\Omega_=\frac{1}{(1-A\lambda)\left(1-\frac{y_1}{\lambda}\right)\left(1-\frac{y_1}{\lambda}\right)\cdots\left(1-\frac{y_n}{\lambda}\right)} = \frac{1}{(1-Ay_1)(1-Ay_2)\cdots(1-Ay_n)}$$

Proof. To obtain Id 3.6, consider

$$\frac{1}{(1-\lambda x)(1-\frac{y}{\lambda})} = \sum_{a_i \geq 0} (\lambda x)^{a_1} \left(\frac{y}{\lambda}\right)^{a_2}$$

$$= \sum_{a_1 \geq 0} \lambda^{a_1 - a_2} x^{a_1} y^{a_2}.$$

To force the exponent of λ to be zero, we must have $a_1 = a_2$. Therefore,

$$\Omega_{=} \sum_{a_i \geq 0} \lambda^{a_1 - a_2} x^{a_1} y^{a_2} = \sum_{a_1 \geq 0} (xy)^{a_1},$$

which leads to the desired result.

For Id 3.8, consider

$$\frac{1}{(1-A_1\lambda)(1-A_2\lambda)\cdots(1-A_n\lambda)\big(1-\frac{y}{\lambda}\big)}$$

$$= \sum_{a_i \geq 0} (A_1\lambda)^{a_1} (A_2\lambda)^{a_2} \cdots (A_n\lambda)^{a_n} \left(\frac{y}{\lambda}\right)^{b}$$

$$= \sum_{a_i \geq 0} \lambda^{a_1 + \cdots + a_n} A_1^{a_1} \cdots A_n^{a_n} y^b.$$

Similarly, to obtain the zero exponent of λ, we must have $b = a_1 + a_2 + \cdots + a_n$, which gives:

$$\Omega_{=} \sum_{a_1 \geq 0} \lambda^{a_1 + \cdots + a_n} A_1^{a_1} \cdots A_n^{a_n} y^b$$

$$= \sum_{a_i \geq 0} A_1^{a_1} A_2^{a_2} \cdots A_n^{a_n} y^{a_1 + \cdots + a_n}$$

$$= \sum_{a_i \geq 0} (A_1 y)^{a_1} (A_2 y)^{a_2} \cdots (A_n y)^{a_n}.$$

Finally to obtain Id 3.9 we have:

$$\frac{1}{(1 - A\lambda)\left(1 - \frac{y_1}{\lambda}\right)\left(1 - \frac{y_1}{\lambda}\right) \cdots \left(1 - \frac{y_n}{\lambda}\right)}$$

$$= \sum_{b, a_i \geq 0} (A\lambda)^b \left(\frac{y_1}{\lambda}\right)^{a_1} \left(\frac{y_2}{\lambda}\right)^{a_2} \cdots \left(\frac{y_n}{\lambda}\right)^{a_n}$$

$$= \sum_{b, a_i \geq 0} A^b \lambda^{b - a_1 - \cdots - a_n} y_1^{a_1} \cdots y_2^{a_2},$$

and $b - a_1 - \cdots - a_n = 0$ if and only if $b = a_1 + \cdots + a_n$, therefore,

$$\Omega_{=} \sum_{a_i \geq 0} A^b \lambda^{b - a_1 - \cdots - a_n} y_1^{a_1} \cdots y_2^{a_2}$$

$$= \sum_{a_i \geq 0} A^{a_1 + \cdots + a_n} y_1^{a_1} \cdots y_n^{a_n}$$

$$= \sum_{a_i \geq 0} (A y_1)^{a_1} (A y_2)^{a_2} \cdots (A y_n)^{a_n}.$$

$\square$

In this paper we make use of two packages developed to do Omega calculus on computer algebra systems. Andrews Et. Al. developed an package for Mathematica called the Omega package [1] and and Xin developed a package for Maple called the Ell package [12].

3.4. Formal vs. Analytic**.** One question involving these crude generating functions is, are we dealing with analytic or formal Laurent series? MacMahon was not explicit on this matter, however Andrews et. al. [1] provide an argument as to why we should restrict ourselves to the analytic. As rational functions we have

$$\sum_{t\geq 0}\lambda^t x^t + \sum_{t<0}\lambda^t x^t = 0$$

and thus

$$-\Omega_{\geq}\sum_{t<0}\lambda^t x^t = \Omega_{\geq}\sum_{t\geq 0}\lambda^t x^t = \sum_{t\geq 0} x^t = \frac{1}{1-x}.$$

However,

$$\Omega_{\geq}\sum_{t\geq 0}\lambda^t x^t = \sum_{t\geq 0} x^t = \frac{1}{1-x}$$

and

$$-\Omega_{\geq}\sum_{t<0}\lambda^t x^t = 0.$$

Therefore, considering formal series does not permit the Omega operators to be well-defined. As a result, we must restrict MacMahon's Omega Calculus to the rational functions corresponding to converging Laurent series.

4. Partition Analysis and Polytopes

4.1. Introduction. Applying MacMahon's Partition analysis to polytopes can be done in a manner quite similar to the process of coning over a polytope. As with the coning process we obtain some sort of an embedding of the polytope's generating function into a grander sort of generating function, in this case the crude generating functions. Then we apply the appropriate Omega operator to recover the desired generating function. The main advantages are that we no longer require the use of triangulation and if we are only given the hyperplane description of a polytope, we do not have to convert to the vertex description.

4.2. The General Idea. Consider a rational polytope's hyperplane description, $P := \{x \in \mathbb{R}^n | Ax \geq b\}$. Note if P is rational we can assume all the entries of A and b are integral. For each defining halfspace $a_i x - tb_i \geq 0$ embed $\lambda_i^{(a_i x - tb_i)}$ into a crude generating function. Since it only makes sense to consider positive dilations, we have the constraint $t \geq 0$. Embed y^t in the crude generating function as well to obtain

$$F(\lambda, y) = \sum_{x_i, t \geq 0} \lambda_1^{(a_1 x - tb_1)} \lambda_2^{(a_2 x - tb_2)} \cdots \lambda_m^{(a_m x - tb_m)} y^t.$$

Then the corresponding crude rational generating function is

$$\frac{1}{(1 - \lambda_1^{a_{11}} \lambda_2^{a_{21}} \cdots \lambda_m^{a_{m1}}) \cdots (1 - \lambda_1^{a_{1n}} \lambda_2^{a_{2n}} \cdots \lambda_m^{a_{mn}}) \left(1 - \frac{y}{\lambda_1^{b_1} \lambda_2^{b_2} \cdots \lambda_m^{b_m}}\right)},$$

where a_{ij} is the $(ij)^{th}$ entry of A.

Theorem 4.1. *If $P := \{x \in \mathbb{R}^n | Ax \geq b\}$ is an integral polytope with Ehrhart polynomial $e_P(t)$ and*

$$F(\lambda, y) = \frac{1}{(1 - \lambda_1^{a_{11}} \lambda_2^{a_{21}} \cdots \lambda_m^{a_{m1}}) \cdots (1 - \lambda_1^{a_{1n}} \lambda_2^{a_{2n}} \cdots \lambda_m^{a_{mn}}) \left(1 - \frac{y}{\lambda_1^{b_1} \lambda_2^{b_2} \cdots \lambda_m^{b_m}}\right)}$$

where a_{ij} is the $(ij)^{th}$ entry of A then

$$\sum_{t\geq 0} e_P(t)y^t = \Omega_{\geq} F(\lambda, y).$$

Proof. If $P := \{x \in \mathbb{R}^n | Ax \geq b\}$ then $\#(tP \cap \mathbb{Z}^n) = \#(\{x \in \mathbb{R}^n | Ax \geq tb\} \cap \mathbb{Z}^n)$. A point $x \in \mathbb{R}^n$ is a solution for a halfspace inequality $a_i x \geq tb_i$ if and only if $a_i x - tb_i \geq 0$.

Let

$$F^*(\lambda, y, z) :=$$
$$\frac{1}{(1 - z_1\lambda_1^{a_{11}}\lambda_2^{a_{21}}\cdots\lambda_m^{a_{m1}})\cdots(1 - z_n\lambda_1^{a_{1n}}\lambda_2^{a_{2n}}\cdots\lambda_m^{a_{mn}})\left(1 - \frac{y}{\lambda_1^{b_1}\lambda_2^{b_2}\cdots\lambda_m^{b_m}}\right)}$$

Note $F^*(\lambda, y, 1) = F(\lambda, y)$ and

$$F^*(\lambda, y, z) =$$
$$\sum_{x_i, t\geq 0} (z_1\lambda_1^{a_{11}}\lambda_2^{a_{21}}\cdots\lambda_m^{a_{m1}})^{x_1}\cdots(z_n\lambda_1^{a_{1n}}\lambda_2^{a_{2n}}\cdots\lambda_m^{a_{mn}})^{x_n}\left(\frac{y}{\lambda_1^{b_1}\lambda_2^{b_2}\cdots\lambda_m^{b_m}}\right)^t$$
$$= \sum_{x_i, t\geq 0} \lambda_1^{(a_1x - tb_1)}\lambda_2^{(a_2x - tb_2)}\cdots\lambda_m^{(a_mx - tb_m)} z_1^{x_1} z_2^{x_2}\cdots z_n^{x_n} y^t.$$

Then

$$\Omega_{\geq} F^*(\lambda, y, z) =$$
$$\Omega_{\geq} \sum_{x_i, t\geq 0} \lambda_1^{(a_1x - tb_1)}\lambda_2^{(a_2x - tb_2)}\cdots\lambda_m^{(a_mx - tb_m)} z_1^{x_1} z_2^{x_2}\cdots z_n^{x_n} y^t$$
$$= \sum_{\substack{a_ix - b_i \geq 0 \\ x_i, t\geq 0}} z_1^{x_1} z_2^{x_2}\cdots z_n^{x_n} y^t$$
$$= \sum_{\substack{t\geq 0 \\ x\in(t\cdot P\cap\mathbb{Z}^n)}} z_1^{x_1} z_2^{x_2}\cdots z_n^{x_n} y^t.$$

Therefore

$$\Omega_{\geq} F(\lambda, y) = \Omega_{\geq} F^*(\lambda, y, 1) = \sum_{t\geq 0} e_p(t)y^t.$$

$\square$

Similarly we have

Theorem 4.2. *If $P\{x \in \mathbb{R}^n | Ax \geq b\}$ is a rational polytope with Ehrhart quasi-polynomial $Q_p(t)$ and*

$$F(\lambda, y) = \frac{1}{(1-\lambda_1^{a_{11}}\lambda_2^{a_{21}}\cdots\lambda_m^{a_{m1}})\cdots(1-\lambda_1^{a_{1n}}\lambda_2^{a_{2n}}\cdots\lambda_m^{a_{mn}})\left(1-\frac{y}{\lambda_1^{b_1}\lambda_2^{b_2}\cdots\lambda_m^{b_m}}\right)}$$

where a_{ij} is the $(ij)^{th}$ entry of A then

$$\sum_{t\geq 0} Q_P(t)y^t = \Omega_{\geq} F(\lambda, y).$$

Let t be a positive integer and recall the polytope:

$$P := \{(x_1, x_2) \in \mathbf{R}^2 \mid x_2 + 2x_1 \geq 2,\ 2 \geq x_2,\ \text{and } 1 \geq x_1\},$$

then the number of integer points contained in $t \cdot P$ is equivalent to the number of integer solutions of the following system:

$$\begin{gathered} x_2 + 2x_1 - 2t \geq 0 \\ 2t - x_2 \geq 0 \\ t - x_1 \geq 0 \\ t \geq 0. \end{gathered}$$

Encode this system into a crude generating function to obtain

$$\sum_{x_i, t \geq 0} \lambda_1^{x_2+2x_1-2t}\lambda_2^{2t-x_2}\lambda_3^{t-x_1}y^t,$$

which corresponds to the following crude rational generating function:

$$\frac{1}{(1-\frac{\lambda_1^2}{\lambda_3})(1-\frac{\lambda_1}{\lambda_2})(1-\frac{\lambda_2^2\lambda_3}{\lambda_1^2}y)}.$$

To find the desired rational generating function, we use the three identities provided above, and we have

$$\Omega_{\geqslant}^{\lambda_3} \frac{1}{(1-\frac{\lambda_1^2}{\lambda_3})(1-\frac{\lambda_1}{\lambda_2})(1-\frac{\lambda_2^2\lambda_3}{\lambda_1^2}y)} = \frac{1}{(1-\frac{\lambda_1}{\lambda_2})(1-\frac{\lambda_2^2}{\lambda_1^2}y)(1-\lambda_2^2 y)}$$

$$\Omega_{\geqslant}^{\lambda_1} \frac{1}{(1-\frac{\lambda_1}{\lambda_2})(1-\frac{\lambda_2^2}{\lambda_1^2}y)(1-\lambda_2^2 y)} = \frac{1}{(1-y)(1-\lambda_2^2 y)(1-\frac{1}{\lambda_2})}$$

$$\Omega_{\geqslant}^{\lambda_2} \frac{1}{(1-y)(1-\lambda_2^2 y)(1-\frac{1}{\lambda_2})} = \frac{1+y}{(1-y)^3}.$$

Therefore we can conclude:

$$\Omega_{\geqslant} \frac{1}{(1-\frac{\lambda_1^2}{\lambda_3})(1-\frac{\lambda_1}{\lambda_2})(1-\frac{\lambda_2^2\lambda_3}{\lambda_1^2}y)} = \frac{1+y}{(1-y)^3}.$$

This is the same rational generation function we obtained above via conventional methods.

For another example let

$$P := \left\{ \mathbf{x} \in \mathbb{R}^3 \middle| Ax \leq b \right\},$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then the number of lattice points in $t \cdot P$ is equivalent to the number of integer solutions to the system

$$\begin{gathered} 2t - x_1 - x_2 - x_3 \geq 0 \\ -x_1 + x_2 + x_3 \geq 0 \\ x_1 - x_2 + x_3 \geq 0 \\ x_1 + x_2 - x_3 \geq 0 \\ t \geq 0. \end{gathered}$$

Encoding this system into a crude generating function we have

$$\sum_{x_1 \geq 0} \lambda_1^{2t-x_1-x_1-x_3} \lambda_2^{-x_1+x_2+x_3} \lambda_3^{x_1-x_2+x_3} \lambda_4^{x_1+x_2-x_3} y^t = \frac{1}{(1-\frac{\lambda_3\lambda_4}{\lambda_1\lambda_2})(1-\frac{\lambda_2\lambda_4}{\lambda_1\lambda_3})(1-\frac{\lambda_3\lambda_2}{\lambda_1\lambda_4})(1-\lambda_1^2 y)}.$$

Using the Omega package [1] to apply $\Omega_{\geqslant}$ we obtain:

$$\Omega_{\geqslant}^{\lambda_1} \frac{1}{(1-\frac{\lambda_3\lambda_4}{\lambda_1\lambda_2})(1-\frac{\lambda_2\lambda_4}{\lambda_1\lambda_3})(1-\frac{\lambda_3\lambda_2}{\lambda_1\lambda_4})(1-\lambda_1^2 y)}$$

$$= \frac{1 + y\lambda_2^2 + y\lambda_3^2 + \frac{y\lambda_2\lambda_3}{\lambda_4} + \frac{y\lambda_2\lambda_4}{\lambda_3} + \frac{y\lambda_4\lambda_3}{\lambda_2} + y^2\lambda_2\lambda_3\lambda_4 + 4\lambda_4^2}{(1-y)(1-\frac{y\lambda_2^2\lambda_3^2}{\lambda_4^2})(1-\frac{y\lambda_2^2\lambda_4^2}{\lambda_3^2})(1-\frac{y\lambda_4^2\lambda_3^2}{\lambda_2^2})}$$

$$\Omega_{\geqslant}^{\lambda_2} \frac{1 + y\lambda_2^2 + y\lambda_3^2 + \frac{y\lambda_2\lambda_3}{\lambda_4} + \frac{y\lambda_2\lambda_4}{\lambda_3} + \frac{y\lambda_4\lambda_3}{\lambda_2} + y^2\lambda_2\lambda_3\lambda_4 + 4\lambda_4^2}{(1-y)(1-\frac{y\lambda_2^2\lambda_3^2}{\lambda_4^2})(1-\frac{y\lambda_2^2\lambda_4^2}{\lambda_3^2})(1-\frac{y\lambda_4^2\lambda_3^2}{\lambda_2^2})}$$

$$= \frac{1 - y^3\lambda_3\lambda_4 + y(1 + \frac{\lambda_3}{\lambda_4} + \frac{\lambda_4}{\lambda_3}) + y^2(-\lambda_3^2 - \lambda_3\lambda_4 - \lambda_4^2)}{(1-y)(1-y\lambda_3^2)(1-\frac{y\lambda_3^2}{\lambda_4^2})(1-y\lambda_4^2)(1-\frac{y\lambda_4^2}{\lambda_3^2})}$$

$$\Omega_{\geqslant}^{\lambda_3} \frac{1 - y^3\lambda_3\lambda_4 + y(1 + \frac{\lambda_3}{\lambda_4} + \frac{\lambda_4}{\lambda_3}) + y^2(-\lambda_3^2 - \lambda_3\lambda_4 - \lambda_4^2)}{(1-y)(1-y\lambda_3^2)(1-\frac{y\lambda_3^2}{\lambda_4^2})(1-y\lambda_4^2)(1-\frac{y\lambda_4^2}{\lambda_3^2})}$$

$$= \frac{1 + y(1 + \frac{1}{\lambda_4}) + \frac{y^2}{\lambda_4}}{(1-y)^2(1-\frac{y}{\lambda_4^2})(1-y\lambda_4^2)}$$

$$\Omega_{\geqslant}^{\lambda_4} \frac{1 + y(1 + \frac{1}{\lambda_4}) + \frac{y^2}{\lambda_4}}{(1-y)^2(1-\frac{y}{\lambda_4^2})(1-y\lambda_4^2)}$$

$$= \frac{1+y^2}{(1-y)^4}.$$

Therefore the rational generating function for the number of lattice points in $t \cdot P$ is

$$\frac{1+y^2}{(1-y)^4},$$

which gives rise to the desired Ehrhart polynomial

$$e(t) = \binom{t+3}{3} + \binom{t+1}{3}.$$

4.3. Partition Analysis and CYR. CYR_n consists of $n \times n$ matrices of the form

$$\begin{bmatrix} x_{11} & x_{12} & 0 & 0 & \cdots \\ x_{21} & x_{22} & x_{23} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix},$$

where the row and columns both sum to 1.

Since CYR_n has 2^{n-1} vertices, triangulation will be very time consuming, however, the fact that the row and columns of CYR_n both sum to 1 provide us with the hyperplane description. Therefore we can use partition analysis to find $e_{CYR_n}(t)$.

For example, the number of lattice points in CYR_3 is equivalent to the number of integer solutions to the following system:

$$x_{11} + x_{12} = t$$
$$x_{21} + x_{22} + x_{23} = t$$
$$x_{31} + x_{32} + x_{33} = t$$
$$x_{11} + x_{21} + x_{31} = t$$
$$x_{12} + x_{22} + x_{32} = t$$

$$x_{23} + x_{33} = t.$$

Encoding this into a crude generating function we have

$$CYR_3(\lambda, y) =$$

$$\frac{1}{(1-\lambda_1\lambda_4)(1-\lambda_1\lambda_5)(1-\lambda_2\lambda_4)(1-\lambda_2\lambda_5)} \cdots$$

$$\frac{1}{(1-\lambda_2\lambda_6)(1-\lambda_3\lambda_4)(1-\lambda_3\lambda_5)(1-\lambda_3\lambda_6)\left(1-\frac{y}{\lambda_1\cdots\lambda_6}\right)}.$$

Using the aforementioned identities for $\Omega_{=}$ we obtain:

$$\Omega_{=}^{\lambda_1} CYR_3(\lambda, y) =$$

$$\frac{1}{\left(1-\frac{y}{\lambda_2\lambda_3\lambda_5\lambda_6}\right)\left(1-\frac{y}{\lambda_2\lambda_3\lambda_4\lambda_6}\right)} \cdots$$
$$\frac{1}{(1-\lambda_2\lambda_4)(1-\lambda_2\lambda_5)(1-\lambda_2\lambda_6)(1-\lambda_3\lambda_4)(1-\lambda_3\lambda_5)(1-\lambda_3\lambda_6)}$$

$$\Omega_{=}^{\lambda_5}\Omega_{=}^{\lambda_1} CYR_3(\lambda, y) =$$

$$\frac{1}{(1-\frac{y}{\lambda_3\lambda_6})(1-\frac{y}{\lambda_2\lambda_6})(1-\frac{y}{\lambda_2\lambda_3\lambda_4\lambda_6})} \cdots$$
$$\frac{1}{(1-\lambda_2\lambda_4)(1-\lambda_2\lambda_6)(1-\lambda_3\lambda_4)(1-\lambda_3\lambda_6)}$$

$$\Omega_{=}^{\lambda_4}\Omega_{=}^{\lambda_5}\Omega_{=}^{\lambda_1} CYR_3(\lambda) =$$

$$\frac{1}{(1-\frac{y}{\lambda_3\lambda_6})(1-\frac{y}{\lambda_2\lambda_6})(1-\frac{y}{\lambda_3\lambda_6})(1-\frac{y}{\lambda_2\lambda_6})(1-\lambda_2\lambda_6)(1-\lambda_3\lambda_6)}$$

$$\Omega_{=}^{\lambda_3}\Omega_{=}^{\lambda_4}\Omega_{=}^{\lambda_5}\Omega_{=}^{\lambda_1} CYR_3(\lambda, y) =$$

$$\frac{1}{(1-y)^2\left(1-\frac{y}{\lambda_2\lambda_6}\right)^2(1-\lambda_2\lambda_6)}$$

$$\Omega_=^{\lambda_6}\Omega_=^{\lambda_3}\Omega_=^{\lambda_4}\Omega_=^{\lambda_5}\Omega_=^{\lambda_1} CYR_3(\lambda, y) =$$

$$\frac{1}{(1-y)^4}.$$

Therefore

$$\Omega_= CYR_3(\lambda, y) = \frac{1}{(1-y)^4}.$$

Also we can easily obtain

$$\Omega_= CYR_2(\lambda, y) = \frac{1}{(1-y)^2}.$$

Using the Omega Package we obtain:

$$\Omega_= CYR_4(\lambda, y) = \frac{y+1}{(1-y)^7},$$

$$\Omega_= CYR_5(\lambda, y) = \frac{4y^2+5y+1}{(y-1)^{11}}.$$

Using the Ell package we obtain:

$$\Omega_= CYR_6(\lambda, y) = \frac{9y^4+56y^3+58y^2+16y+1}{(1-y)^{16}}.$$

These rational functions give rise to the desired Ehrhart polynomials:

$$e_{CYR_2}(t) = t+1$$

$$e_{CYR_3}(t) = \binom{t+3}{3}$$

$$e_{CYR_4}(t) = \binom{t+6}{6} + \binom{t+5}{6}$$

$$e_{CYR_5}(t) = \begin{pmatrix} t+10 \\ 10 \end{pmatrix} + 5 \begin{pmatrix} t+9 \\ 10 \end{pmatrix} + 4 \begin{pmatrix} t+8 \\ 10 \end{pmatrix}$$

$$e_{CYR_6}(t) = \begin{pmatrix} t+15 \\ 14 \end{pmatrix} + 16 \begin{pmatrix} t+14 \\ 15 \end{pmatrix} + 58 \begin{pmatrix} t+13 \\ 15 \end{pmatrix}$$

$$+56 \begin{pmatrix} t+12 \\ 15 \end{pmatrix} + 9 \begin{pmatrix} t+11 \\ 15 \end{pmatrix}.$$

5. On the Hunt for a New Algorithm

5.1. Introduction. The packages developed by Xin and Andrews et. al. have been very helpful, however the author's personal computer could not process the computations required to go beyond e_{CYR_6}. This may be due to the fact that both packages require simplifying sums of rational functions, which as Xin points out in [12] both Maple and Mathematica have difficulties performing. Therefore it would be useful to find a more efficient algorithm by finding possible means of reducing the simplifications required and merging them with the techniques used to develop the Omega and Ell packages. In this chapter we describe possible avenues leading to such an algorithm.

5.2. Symmetric Functions. In this section, we present the possible beginnings of a more algebraic algorithm to compute $\Omega_{=}F(\lambda, x, y)$, where

$$F(\lambda, x, y) := \frac{1}{(1-x_1\lambda)(1-x_2\lambda)\cdots(1-x_n\lambda)\left(1-\frac{y_1}{\lambda}\right)\left(1-\frac{y_2}{\lambda}\right)\cdots\left(1-\frac{y_m}{\lambda}\right)}.$$

Let t be a positive integer. A *composition* of t is a sequence of integers, whose sum is t. Let $C_{k,t}$ be the set of all compositions of t into at most k parts. Then using terminology provided in [10] define a *weighting* of C_n to be the function $\mathrm{wt} : C_n \to \mathbb{C}\,[[x]]$ defined by

$$\mathrm{wt}\, c := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

where $\mathbb{C}\,[[x]]$ is the ring of formal power series in x over $\mathbb{C}$ and $c = (a_1, \ldots, a_n)$. Define

$$f_{C_{n,t}}(x) := \sum_{c \in C_{n,t}} \mathrm{wt}\, c.$$

Note for fixed n

$$\sum_{t \geq 0} f_{C_{n,t}}(x) y^t = \Omega_{=}F^*(\lambda, y, x),$$

where

$$F^*(\lambda, y, x) := \sum_{a_i, t \geq 0} \lambda^{a_1+\cdots+a_n-t} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} y^t$$
$$= \frac{1}{(1-x_1\lambda)(1-x_2\lambda)\cdots(1-x_n\lambda)\left(1-\frac{y}{\lambda}\right)}.$$

Therefore

$$\sum_{t\geq 0} f_{C_{n,t}}(x) y^t = \frac{1}{(1-x_1 y)(1-x_2 y)\cdots(1-x_n y)}.$$

Theorem 5.1. *If*

$$F(\lambda, x, y) := \frac{1}{\prod_{i=1}^n (1-x_i\lambda)\prod_{i=1}^m \left(1-\frac{y_i}{\lambda}\right)}$$

then

$$\Omega_= F(\lambda, x, y) =$$
$$\sum_{t\geq 0} f_{C_{n+m,2t}}(x,y) z^t - \Omega_=^{\lambda_2}\Omega_\geq^{\lambda_1} \frac{\frac{1}{\lambda_1}}{\prod_{i=1}^n (1-\lambda_1\lambda_2 x_i)\prod_{i=1}^m (1-\lambda_2 y_i)\left(1-\frac{z}{\lambda_1\lambda_2^2}\right)}$$
$$-\Omega_=^{\lambda_2}\Omega_\geq^{\lambda_3} \frac{\frac{1}{\lambda_3}}{\prod_{i=1}^m (1-\lambda_3\lambda_2 y_i)\prod_{i=1}^n (1-\lambda_2 x_i)\left(1-\frac{z}{\lambda_3\lambda_2^2}\right)}.$$

evaluated at $z = 1$.

Proof. Let

$$F(\lambda, x, y) := \frac{1}{\prod_{i=1}^n (1-x_i\lambda)\prod_{i=1}^m \left(1-\frac{y_i}{\lambda}\right)}.$$

Then

$$F(\lambda, x, y) = \sum_{a_i, b_i \geq 0} (x_1\lambda)^{a_1}(x_2\lambda)^{a_2}\cdots(x_n\lambda)^{a_n}\left(\frac{y_1}{\lambda}\right)^{b_1}\left(\frac{y_2}{\lambda}\right)^{b_2}\cdots\left(\frac{y_m}{\lambda}\right)^{b_m}$$

$$= \sum_{a_i, b_i \geq 0} x_1^{a_1} x_2^{a_2}\cdots x_n^{a_n} y_1^{b_1} y_2^{b_2}\cdots y_m^{b_m} \lambda^{a_1+\cdots+a_n-b_1-\cdots-b_m}.$$

Applying $\Omega_=$ to $F(\lambda, x, y)$ is equivalent to eliminating all terms except for those with exponents satisfying the condition

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} b_i = t \text{ for some } t \in \mathbb{Z}_{\geq 0}.$$

Therefore

$$\Omega_= F(\lambda, x, y) = F^*(\lambda, x, y, 1),$$

where

$$F^*(\lambda, x, y, z) = \sum_{t\geq 0} f_{C_{n,t}}(x) f_{C_{m,t}}(y) z^t.$$

If $c_1 = (a_1, ...a_n) \in C_{n,t}$ and $c_2 = (b_1, \ldots, b_m) \in C_{m,t}$ then $c_1c_2 := (a_1, \ldots, a_n, b_1, \ldots, b_m) \in C_{n+m,2t}$, however, $f_{C_{n,t}}(x) \cdot f_{C_{m,t}}(y) \neq f_{C_{n+m,2t}}(x, y)$ because we can have a composition of $2t$ where the first n terms do not sum to t. Therefore we can conclude

$$\sum_{t\geq 0} f_{C_{n,t}}(x) f_{C_{m,t}}(y) z^t =$$

$$\sum_{t\geq 0} f_{C_{n+m,2t}}(x,y) z^t - \sum_{t\geq 0} \left[\sum_{k>t} f_{C_{n,k}}(x) f_{C_{m,2t-k}(y)} \right] z^t$$

$$- \sum_{t\geq 0} \left[\sum_{k<t} f_{C_{n,k}}(x) f_{C_{m,2t-k}}(y) \right] z^t$$

$$= \sum_{t\geq 0} f_{C_{n+m,2t}}(x,y) z^t - \sum_{t\geq 0} \left[\sum_{k>t} f_{C_{n,k}}(x) f_{C_{m,2t-k}}(y) \right] z^t$$

$$- \sum_{t\geq 0} \left[\sum_{k>t} f_{C_{n,2t-k}}(x) f_{C_{m,k}}(y) \right] z^t.$$

Notice

$$\sum_{t\geq 0} \left[\sum_{k>t} f_{C_{n,k}}(x) f_{C_{m,2t-k}}(y) \right] z^t =$$

$$\Omega_=^{\lambda_2} \Omega_\geq^{\lambda_1} \sum_{\substack{a_i, b_i \geq 0 \\ t\geq 0}} x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_m^{b_m} z^t \lambda_1^{a_1+\cdots+a_n-t-1} \lambda_2^{b_1+\cdots+b_m+a_1+\cdots+a_n-2t}$$

$$= \Omega_{=}^{\lambda_2}\Omega_{\geq}^{\lambda_1}\frac{\frac{1}{\lambda_1}}{\prod_{i=1}^{n}(1-\lambda_1\lambda_2 x_i)\prod_{i=1}^{m}(1-\lambda_2 y_i)\left(1-\frac{z}{\lambda_1\lambda_2^2}\right)}.$$

Similarly we have

$$\sum_{t\geq 0}\left[\sum_{k>t} f_{C_{n,2t-k}}(x) f_{C_{m,k}}(y)\right] z^t = \Omega_{=}^{\lambda_2}\Omega_{\geq}^{\lambda_3}\frac{\frac{1}{\lambda_3}}{\prod_{i=1}^{m}(1-\lambda_3\lambda_2 y_i)\prod_{i=1}^{n}(1-\lambda_2 x_i)\left(1-\frac{z}{\lambda_3\lambda_2^2}\right)}.$$

□

Notice that

$$\sum_{t\geq 0} f_{C_{n+m,2t}}(x,y) z^t =$$

$$= \sum_{\substack{a_i,b_i\geq 0\\ t\geq 0}} x_1^{a_1}\cdots x_n^{a_n} y_1^{b_1}\cdots y_m^{b_m} z^t \lambda^{a_1+\cdots+a_n+b_1+\cdots+b_m-2t}$$

$$\Omega_{=}\frac{1}{\prod_{i=1}^{n}(1-x_i\lambda)\prod_{i=1}^{m}(1-y_i\lambda)\left(1-\frac{z}{\lambda^2}\right)}.$$

If we can find the rational generating function for

$$\sum_{t\geq 0} f_{C_{n+m,2t}}(x,y) z^t$$

then we can apply the following theorem provided by G.N. Han [8] to produce a finalized result:

Theorem 5.2. *If $n \in Z_{\geq}$ and $U(\lambda)$ be a Laurent polynomial with degree less than $n-1$ then*

$$\Omega_{\geq}\frac{U(\lambda)}{\prod_{i=1}^{n}(1-x_i\lambda)\prod_{i=1}^{m}\left(1-\frac{y_i}{\lambda}\right)} = \sum_{i=1}^{n}\frac{x_i^{n-1}U\left(\frac{1}{x_i}\right)}{(1-x_i)\prod_{j=1}^{m}(1-x_i y_j)\prod_{j\neq i}(x_i-x_j)}.$$

The formula holds even when the x_i's are not all distinct.

5.3. Switching Gears**.** One can argue that applying partition analysis to polytopes is a generalization of MacMahon's original intent in that every form of partition given by Diophantine relations has a polyhedra interpretation. Therefore, it makes sense to switch roles and apply Ehrhart theory to partition analysis. Making use of Theorem 2.4 we provide alternate proofs for some of MacMahon's identities, which we restate for convenience.

Id 5.3.

$$\Omega_{\geqslant}\frac{1}{(1-\lambda z_1)(1-\frac{z_2}{\lambda})} = \frac{1}{(1-z_1)(1-z_1z_2)}$$

Id 5.4.

$$\Omega_{\geqslant}\frac{1}{(1-\lambda z_1)(1-\frac{z_2}{\lambda^2})} = \frac{1}{(1-z_1)(1-z_1^2z_2)}$$

Id 5.5.

$$\Omega_{\geqslant}\frac{1}{(1-\lambda^2 z_1)(1-\frac{z_2}{\lambda})} = \frac{1+z_1z_2}{(1-z_1)(1-z_1z_2^2)}$$

Proof. For Id 5.3 Let

$$K := \{a_1 - a_2 \geq 0 | a_i \in \mathbb{R}^+\}.$$

Then one can easily show that K is a simple cone with $(1,1)$ and $(1,0)$ as generators. Therefore by Theorem 2.4 we have

$$\sigma_K(z) = \frac{\sigma_{\Pi_K}(z)}{(1-z^{(1,1)})(1-z^{(1,0)})},$$

where $\Pi_K := \{\lambda_1(1,1) + \lambda_2(1,0) | 0 \leq \lambda_1, \lambda_2 < 1\}$. Since the only lattice point in Π_K is $(0,0)$ we have,

$$\sigma_K(z) = \frac{z_1^0 z_2^0}{(1-z_1^1z_2^1)(1-z_1^1z_2^0)}.$$

Thus we have the desired identity.

Similarly we can obtain Id 5.4 by considering the cone

$$K := \{a_1 - 2a_2 \geq 0 | a_i \in \mathbb{R}^+\},$$

which has generators $(1,0)$ and $(2,1)$ with $\Pi_K := \{\lambda_1(1,0)+\lambda_2(2,1)|0 \le \lambda_1, \lambda_2 < 1\}$. Once again $\Pi_K \cap \mathbb{Z} = \{(0,0)\}$ so we have

$$\sigma_K(z) = \frac{z_1^0 z_2^0}{(1-z_1^1 z_2^0)(1-z_1^2 z_2^1)}.$$

For Id 5.5 consider

$$K := \{2a_1 - a_2 \ge 0 | a_i \in \mathbb{R}^+\},$$

which has generators $(0,1)$ and $(1,2)$ with $\Pi_K := \{\lambda_1(0,1) + \lambda_2(1,2)|0 \le \lambda_1, \lambda_2 < 1\}$. We have $\Pi_K \cap \mathbb{Z} = \{(0,0),(1,1)\}$ and thus

$$\sigma_K(z) = \frac{z_1^0 z_2^0 + z_1^1 z_2^1}{(1-z_1^0 z_2^1)(1-z_1^1 z_2^2)}.$$

□

5.4. Five Guidelines for $\Omega_{\ge}$. Corteel, Lee and Savage [5] have developed five guidelines that provide a simplification of MacMahon's partition analysis for integral, linear, homogenous systems of inequalities. We will discuss these guidelines and then expand on them to include linear systems of equalities. To do so we will first cover the terminology presented in [5].

A *constraint* c is a linear inequality of the form,

$$a_0 + \sum_{i=1}^{n} a_i x_i \ge 0,$$

The *negation* of constraint c denoted by $\neg c$ is the linear inequality

$$-a_0 - \sum_{i=1}^{n} a_i x_i > 0.$$

Let C denote a set of constraints, and $C_{x_i \leftarrow x_i + a x_j}$ denote the set of constraints formed by substituting the variable x_i with $x_i + a x_j$ in all constraints contained in C. S_C denotes the set of integer points satisfying all constraints in C, and

$$F_C(\lambda) = \sum_{x \in S_C} \lambda_1^{x_1} \lambda_2^{x_2} \cdots \lambda_n^{x_n}$$

denotes the full generating function for S_C. A constraint c is said to be *implied by* a set of constraints C if $S_{C \cup \{\neg c\}} = \varnothing$. As we are often only interested in nonnegative solutions, we will assume that any set of constraints on variables x_i contain the constraints $x_i \geq 0$ for all i regardless of equality conditions. Before we state the five guidelines, we will first present a slightly more generalized version of a lemma provided in [5].

Lemma 5.6. *Let C be a set of linear homogenous inequalities and suppose the constraint, $x_i - \phi(x) \geq 0$ is implied by C, where*

$$\phi(x) := \sum_{j \neq i} b_j x_j$$

and $b_j \in \mathbb{Z}$ for all j. Let $C' = C_{x_i \leftarrow x_i + \phi(x)}$. Then,

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_C$$

if and only if

$$\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - \phi(\lambda), \lambda_{i+1}, \ldots, \lambda_n) \in S_{C'}.$$

Proof. Let

$$c(x) := a_0 + \sum_{k=1}^{n} a_k x_k$$

and

$$c'(x) := a_0 + \sum_{k=1}^{i-1} a_k x_k + a_i[x_i + \phi(x)] + \sum_{k=i+1}^{n} a_k x_k.$$

Suppose C is a set of linear homogenous equalities and C contains the constraint $c(x) \geq 0$. Then C' contains the constraint $c'(x) \geq 0$, where $C' = C_{x_i \leftarrow x_i + \phi(x)}$.

If $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - \phi(\lambda), \lambda_{i+1}, \ldots, \lambda_n)$ then

$$c'(\lambda') = a_0 + \sum_{k=1}^{i-1} a_k \lambda_k + a_i \left[(\lambda_i - \phi(\lambda)) + \phi(\lambda)\right] + \sum_{k=i+1}^{n} a_k \lambda_k$$
$$= a_0 + \sum_{k=1}^{n} a_k \lambda_k = c(\lambda).$$

Therefore $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfies $c \in C$ if and only if $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - \phi(\lambda), \lambda_{i+1}, \ldots, \lambda_n)$ satisfies $c' \in C'$. Now we need to verify nonnegativity conditions.

If the constraint, $x_i - \phi(x) \geq 0$ is implied by C, where

$$\phi(x) := \sum_{j \neq i} b_j x_j$$

and $b_j \in \mathbb{Z}$ for all j then $\lambda \in S_C$ implies $\lambda_i - \phi(\lambda) \geq 0$ and $\lambda_j \geq 0$ for all j. Therefore $\lambda' \in S_{C'}$. The constraint, $x_i \geq 0 \in C$ implies the constraint, $x_i + \phi(x) \geq 0 \in C'$. Therefore, $\lambda' \in S_{C'}$ implies $[\lambda_i - \phi(\lambda)] + \phi(\lambda) \geq 0$, which implies $\lambda_i \geq 0$. Therefore, $\lambda \in S_C$. $\square$

The Five Guidelines [5]

Theorem 5.7. *Let t be a fixed, nonnegative integer. If $C = \{x \geq t\}$, then*

$$F_C(\lambda) = \frac{\lambda^t}{(1 - \lambda)}.$$

This follows from geometric series since

$$F_C(\lambda) = \sum_{s \geq 0} \lambda^{t+s}.$$

Theorem 5.8. *If C_1 is a set of constraints on variables $x_1, \ldots, x_j$ and C_2 is a set of constraints on variables $x_{j+1}, \ldots, x_n$, then*

$$F_{C_1 \cup C_2}(\lambda) = F_{C_1}(\lambda_1, \ldots, \lambda_j) F_{C_2}(\lambda_{j+1}, \ldots, \lambda_n).$$

This is true because $(x_1, \ldots, x_n, y_1, \ldots y_m) \in S_{C_1 \cup C_2}$ if and only if $(x_1, \ldots, x_n) \in S_{C_1}$ and $(y_1, \ldots, y_m) \in S_{C_1}$.

Theorem 5.9. *Let C be a set of linear homogenous inequalities of variables $x_1, \ldots, x_n$ and with the constraints $x_i \geq 0$. Suppose $x_i - \phi(x) \geq 0$ is implied by C, where*

$$\phi(x) := \sum_{j \neq i} b_j x_j$$

and $b_j \in \mathbb{Z}$ for all j. If $C' = C_{x_i \leftarrow x_i + \phi(x)}$ then,

$$F_C(\lambda) = F_{C'}(\lambda; \lambda_j \leftarrow \lambda_j \lambda_i^{b_j},\ \forall j \neq i).$$

Proof. Lemma 5.6 gives, $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_{C'}$ if and only if $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \phi(\lambda), \lambda_{i+1}, \ldots, \lambda_n) \in S_C$. Therefore we have:

$$F_{C'}(\lambda; \lambda_j \leftarrow \lambda_j \lambda_i^{b_j},\ \forall j \neq i)$$

$$= \sum_{x \in S_{C'}} (\lambda_1 \lambda_i^{b_1})^{x_1} \cdots (\lambda_{i-1} \lambda_i^{b_{i-1}})^{x_{i-1}} \lambda_i (\lambda_{i+1} \lambda_i^{b_{i+1}})^{x_{i+1}} \cdots (\lambda_n \lambda_i^{b_n})^{x_n}$$

$$= \sum_{x \in S_{C'}} \lambda_1^{x_1} \cdots \lambda_{i-1}^{x_{i-1}} \lambda_i^{x_i + \phi(x)} \lambda_{i+1}^{x_{i+1}} \cdots \lambda_n^{x_n}$$

$$= \sum_{x \in S_C} \lambda_1^{x_1} \lambda_2^{x_2} \cdots \lambda_i^{x_i} \cdots \lambda_n^{x_n} = F_C(\lambda).$$

□

Theorem 5.10. *Let c be a constraint with the same variables as the ones occurring in the set C. Then*

$$F_C(\lambda_n) = F_{C \cup \{c\}}(\lambda_n) + F_{C \cup \{\neg c\}}(\lambda_n).$$

This holds because $(x_1, \ldots, x_n) \in S_C$ satisfies either c or $\neg c$ but not both.

Theorem 5.11. *Let $c \in C$. Then*

$$F_C(\lambda_n) = F_{C \setminus \{c\}}(\lambda_n) - F_{\{C \setminus \{c\}\} \cup \{\neg c\}}(\lambda_n).$$

This holds because $\{C \setminus \{c\}\} \cup \{c\} = C$ and by the previous guideline we have

$$F_{C\setminus\{c\}}(\lambda) = F_{\{C\setminus\{c\}\}\cup\{c\}}(\lambda) + F_{\{C\setminus\{c\}\}\cup\{\neg c\}}(\lambda).$$

5.5. Guidelines for $\Omega_=$. We are now going to consider integral systems of homogeneous linear equations. Let c be a constraint of the form

$$a_0 + \sum_{k=1}^{n} a_k x_k = 0.$$

Define the negation of c to be

$$\neg c : \begin{cases} \neg c_1 : \ a_0 + \sum_{k=1}^{n} a_k x_k \geq 1 \\ \neg c_2 : \ -a_0 - \sum_{k=1}^{n} a_k x_k \geq 1. \end{cases}$$

Then we say a constraint c is *implied by C* if

$$S_{C\cup\{\neg c_1\}} = S_{C\cup\{\neg c_2\}} = \varnothing.$$

Lemma 5.12. *Suppose the constraint, $x_i - \phi(x) = 0$ is implied by C, where*

$$\phi(x) := \sum_{j\neq i} b_j x_j$$

and $b_j \in \mathbb{Z}$ for all j.

Let $C' = C_{x_i \leftarrow x_i + \phi(x)}$. Then,

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_C$$

if and only if

$$\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - \phi(\lambda), \lambda_{i+1}, \ldots, \lambda_n) \in S_{C'}.$$

Proof. The proof of this lemma is identical to the analogous lemma for inequalities except of the verification of nonnegativity. If the constraint, $x_i - \phi(x) = 0$ is implied by C, where

$$\phi(x) := \sum_{j \neq i} b_j x_j$$

and $b_j \in \mathbb{Z}$ for all j then $\lambda \in S_C$ implies $\lambda_i - \phi(\lambda) = 0$ and $\lambda_i \geq 0$ for all i. Therefore $\lambda' \in S_{C'}$. The constraint, $x_i \geq 0 \in C$ implies the constraint, $x_i + \phi(x) \geq 0 \in C'$. Therefore, $\lambda' \in S_{C'}$ implies $\lambda_i - \phi(\lambda) \geq 0$ and $\lambda_i + \phi(\lambda) \geq 0$, which implies $\lambda_i \geq 0$. Therefore, $\lambda \in S_C$. $\square$

Theorem 5.13. *Let C be a set of linear homogenous equalities of variables $x_1, \ldots, x_n$ and with the constraints $x_i \geq 0$. Suppose $x_i - \phi(x) = 0$ is implied by C, where*

$$\phi(x) := \sum_{j \neq i} b_j x_j$$

and $b_j \in \mathbb{Z}$ for all j. If $C' = C_{x_i \leftarrow x_i + \phi(x)}$ then,

$$F_C(\lambda) = F_{C'}(\lambda; \lambda_j \leftarrow \lambda_j \lambda_i^{b_j},\ \forall j \neq i).$$

Once again, using Lemma 5.12 instead of Lemma 5.6, the proof of this theorem is identical to the analogous theorem for inequalities.

6. Future Work

This paper is the mere beginnings of a much larger project. In this paper we have presented possible leads for combining Ehrhart theory with partition analysis that have yet reach to their potential. It is the intent of the author to explore these routes in hopes to find a more computational friendly algorithm for Omega calculus, implement the algorithm into a computer algebra system and apply these methods to high-dimensional polytopes such as CYR_n and the Birkoff polytope.

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