Flow Zonotopes and Cographic Hyperplane Arrangements

Master's Thesis

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1 Introduction

In this thesis we introduce a new concrete constructing of the normal vectors of the cographic hyperplane arrangement associated with the cographic matroid for simple, connected and bridgeless graphs. Furthermore, we introduce the flow zonotope, which is the zonotope associated with the cographic hyperplane arrangement, and determine its Ehrhart polynomial. Before we take a closer look at these results, we first want to provide background information and give some motivation why it might be interesting to study cographic hyperplane arrangements and flow zonotopes.

Because of its remarkable arithmetic behaviour, the graphic zonotope leads to a famous example for applying Ehrhart theory. Zonotopes, in general, are polytopes defined as the Minkowski sums of finitely many line segments. The first systematic investigation on zonotopes was given in 1969 by Bolker in [5]. Zonotopes include many familiar polytopes including cubes, truncated octahedra, and rhombic dodecahedra. A remarkable aspect of zonotopes is that they allow a natural tiling into half-open parallelepipeds as we will see in Lemma 3.1. We will exploit this special natural tiling to deduce a formula for the Ehrhart polynomial of general zonotopes in Theorem 3.3 which is due to Richard Stanley [15].

In 1962 Ehrhart proved in [6] that, for a lattice d-polytope P, the lattice point enumerator $L_p(t) := \#(tP \cap \mathbb{Z}^d)$ is a polynomial in t of degree d. This polynomial is called the *Ehrhart polynomial* and can be written as $L_P(t) = \sum_{i=0}^{d} c_i(P)t^i$ where the coefficients $c_i(P)$, $0 \leq i \leq d$ depend only on P. More about Ehrhart theory is explained in Chapter 3.

The remarkable behaviour mentioned in the first sentence of the second paragraph of this thesis is that the k-th coefficient of the Ehrhart polynomial of the graphic zonotope of a simple graph G, which we will define in a second, is just the number of forests of size k in G as shown by Example 3.5. Thus, it depends only on the independent sets of size k of the graphic matroid induced by G, see Definition 2.21.

Throughout this thesis we assume all graphs to be simple.

Definition 1.1. Let $G = (V, E)$ be a graph. We define the *graphic zonotope* of G as

$$\mathbf{Z}_G := \sum_{e \in E} u_e$$

where $u_e := [0, x_e]$ and x_e is the column vector of the signed vertex-edge incidence matrix M of G corresponding to the edge $e \in E$.

To each graphic zonotope there is an associated hyperplane arrangement, the so-called graphic hyperplane arrangement, whose normal vectors are the x_e for $e \in E$ of Definition 1.1. As we will discover in Chapter 3, in a sense hyperplane arrangement and zonotopes can be considered equivalent. So, there is a bunch of properties that can be stated for both of them. Furthermore, the theory of

zonotopes and hyperplane arrangements was generalized to the theory of oriented matroids, see [23, Chapter 7]. A motivation to study the graphic zonotope resp. the graphic hyperplane arrangement is that they carry quite a trove of information about the underlying graph, as we will encounter.

In 1975, Thomas Zaslavsky started with his Ph.D. thesis [21] the modern theory of hyperplane arrangements. He studied not only properties of general hyperplane arrangements, but in [8] together with Curtis Greene also properties of graphic hyperplane arrangements (for its definition see Example 3.2). Greene and Zaslavsky showed in [8, Lemma 7.1] that the regions of the graphic hyperplane arrangement of a graph G are in one-to-one correspondence with the acyclic orientations of G. Another important property of the graphic hyperplane arrangement of a graph G stemming from the underlying graphic matroid is that its normal vectors are linearly independent if and only if they induce a forest on G, see Proposition 3.1.

Studying the graphic hyperplane arrangement, the question arises if there is also such an arithmetically nice behaviour for the dual case, i.e., the zonotope associated with the cographic hyperplane arrangement. So, these zonotopes, which we call *flow zonotopes*, are the Minkowski sums of line segments starting at 0 and ending with the normal vectors of the cographic hyperplane arrangement. An associated question is, in what way we can understand the cographic hyperplane arrangement to be the arrangement dual to the graphic hyperplane arrangement. Answering these questions are goals of this thesis.

First defined by Greene and Zaslavsky in [8, Chapter 8], it was left unclear how the normal vectors of the cographic hyperplane arrangement look arithmetically. Thus, a first step of answering the first question is to concretely construct the normal vectors of the cographic hyperplane arrangement. Therefore, we use an ansatz differing from the one Greene and Zaslavsky gave.

Beyond Greene and Zaslavsky's result stating that the regions of the cographic hyperplane arrangement are in one-to-one correspondence with the totally cyclic orientations on G, we show that the normal vectors of the cographic hyperplane arrangement, which we constructed, are linearly independent if and only if they induce a complement of a spanning set on G. This relates on the theory of matroids since it shows that the independent sets of the normal vectors of the cographic hyperplane arrangement are exactly the independent sets of the dual of the graphic matroid, the so-called cographic matroid, see Definition 2.22. This completes the picture on the relations between the graphic/cographic zonotopes/hyperplane arrangements and their corresponding matroids. Thus, our answer to the second question is that the cographic hyperplane arrangement can be seen as dual from a matroid resp. graph theoretical perspective; furthermore, we generalize the duality correspondence between acyclic and totally cyclic orientations that was known for planar graphs before. There is a deep relation between the flows on G and its cographic hyperplane arrangement. As we will show, its normal vectors give a basis of the flow space of G with respect to a given orientation on G, i.e., the affine vector space of all functionals satisfying the flow-equations on G, see Definition 2.10. This relation also explains why we assume G to be bridgeless, see Definition 2.5, when we want to construct the cographic hyperplane arrangement

resp. the flow zonotope as we will see in Chapter 4. These deep relations also explain why we call the flow zonotope this way.

The main results of Chapter 4.2 are that in the case that G is Steinitz, i.e., simple, planar and 3-connected, the cographic hyperplane arrangement and thus also the flow zonotope, are the graphic hyperplane arrangement resp. the graphic zonotope of the dual graph of G. For general simple and connected graphs, the construction and results are quite different since we do not have an easy notion of the dual of a general simple and connected graph. We cannot use the convenient way of generalizing the notion of the dual of a graph which is embedding the graph into a two dimensional manifold. The problems of this approach are that if the genus of the manifold does not equal zero, the complement of the dual of a spanning tree is not a spanning tree (see [7]) and that totally cyclic orientations on the primal graph do, in general, not induce acyclic orientations on the dual graph and vice versa. Thus, we cannot use this construction to obtain the desired properties. Instead, in Chapter 4, we construct the cographic hyperplane arrangement using polytopal duality, also known as polarity, see Definition 2.36. We show that the normal vectors of the cographic hyperplane arrangement are the column vectors of a boundary map stemming from the cellular chain complex associated to the CW-complex given by the face-structure of a special polytope satisfying all required properties.

Eventually, we determine the Ehrhart polynomial of flow zonotopes using the knowledge and techniques from Chapter 3 and come to the following main result of this thesis.

Theorem 1.1. *Let $G = (V, E)$ be a simple graph. Then the Ehrhart polynomial $L_{\mathcal{C}_G}$ of $\mathcal{C}_G$ is given by*

$$L_{\mathcal{C}_G}(t) = \sum_{k=0}^{|E|-|V|+1} d_k t^k$$

where the coefficient d_k is the number of (labeled) complements of (labeled) spanning sets of size k.

Thus, the Ehrhart polynomial of flow zonotopes perfectly fits into the drawn picture: Its coefficients are given by the number of independent sets of size k of the cographic matroid. Thus, they give the information that is, on the level of matroid duality, dual to the information given by the graphic zonotope.

2 Preliminaries

2.1 Graphs and flows

This chapter serves as a source of definitions, examples and motivations for the theorems to come and the tools to be developed. Throughout this chapter we follow the notations of [3] and [14].

Definition 2.1. A *simple graph* $G = (V, E)$ is a discrete structure composed of a finite set V of vertices and a collection $E \subseteq \binom{V}{2}$ of unordered pairs of vertices, called edges. This defines a simple graph as it excludes the existence of multiple edges between vertices and, in particular, edges with equal endpoints, i.e., loops.

In this thesis we assume all graphs to be simple. Therefore, we will abbreviate the notation and will just write graph whenever we mean simple graphs.

Definition 2.2. Let $G = (V, E)$ be a graph. A vertex v is *incident* to an edge e if and only if e contains v as vertex. Two vertices of G are called *adjacent* if and only if they are incident to some common edge.

Example 2.1. A common example of graphs are the complete graphs. *Complete graphs* K_n are the graphs on $n \in \mathbb{N}$ vertices which satisfy $E = \binom{V}{2}$, i.e., every two vertices are adjacent. Figure 1 shows the complete graph K_4 on four vertices.
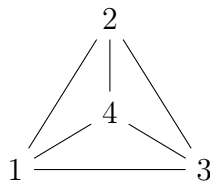

Figure 1: The complete graph K_4.

Definition 2.3. A graph $H = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$. If H is a subgraph of G, we say that G *contains* H.

Definition 2.4. A graph $G = (V, E)$ is called *planar* if and only if there exists an embedding of G into the real plane $\mathbb{R}^2$ such that edges do not cross except possibly at vertices. Such an embedding is called *planar embedding*.

Example 2.2. As we can see in Figure 1, there exists a planar embedding for the complete graph K_4 and therefore, K_4 is a planar graph.

An important definition when dealing with flows, which we will define below, is the following:

Definition 2.5. Let $G = (V, E)$ be a graph. A *bridge* or *isthmus* is an edge of G whose removal would increase the number of connected components of G. A graph is called *bridgeless* if it does not contain any bridge.

Thus, by definition, a bridge is an edge connecting two connected components, call them A and B, as illustrated by Figure 2.
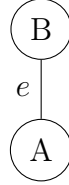

Figure 2: The edge e is a bridge.

Definition 2.6. Let $G = (V, E)$ be a graph. We denote the vertices of G by $v_1, v_2, ..., v_n$. An *orientation on the set of edges of G with respect to vertices* is a subset $\rho \subseteq E$ such that for an edge $e = v_i v_j \in E$ with $i < j$, we direct e from v_i to v_j if $e \in \rho$ and from v_j to v_i otherwise. Then e is called a *directed edge*. Besides, v_i is called the *tail of e* and v_j is called the *head of e* if $v_i v_j$ is directed from v_i to v_j and the other way around if $v_i v_j$ is directed from v_j to v_i.

Definition 2.7. Let G be an oriented graph. A *directed path* in G is a sequence $v_1, v_2, ..., v_s$ of distinct vertices such that $v_{j-1} v_j$ is a directed edge in G for all $j \in \{2, ..., s\}$. If $v_s v_1$ is also a directed edge, then $v_1, v_2, ..., v_s, v_{s+1} := v_1$ is called a *directed cycle*.

An orientation ρ of G is called *acyclic* if and only if there are no directed cycles in G.

An orientation ρ of G is called *totally cyclic* if and only if every directed edge of G is contained in a directed cycle.

Example 2.3. Figure 3 shows an acyclic orientation of a triangle (left) and also a totally cyclic orientation (right). The right shows us a directed cycle whereas the left does not contain any directed cycle.
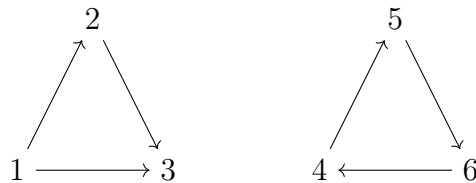
Figure 3: An acyclic orientation (left) and a totally cyclic orientation (right).

Definition 2.8. An *(undirected) trail* P in G is a sequence $v_1, v_2, ..., v_s$ of vertices such that $v_{j-1} v_j$ is an edge in G for all $j \in \{2, ..., s\}$. If $v_s v_1$ is also an edge, then $v_1, v_2, ..., v_s, v_{s+1} := v_1$ is called an *(undirected) circuit*. If all vertices of P are distinct, P is called an *(undirected) path*. Then, if $v_s v_1$ is also an edge, $v_1, v_2, ..., v_s, v_{s+1} := v_1$ is called an *(undirected) cycle*.

Definition 2.9. Let $G = (V, E)$ be a graph. The *degree* $\deg(v)$ of a vertex $v \in V$ is the number of edges incident to v. If G is oriented, the *indegree* $\mathrm{indeg}(v)$ of a vertex $v \in V$ is the number of (directed) edges with head v and the *outdegree* $\mathrm{outdeg}(v)$ the number of (directed) edges with tail v.

Example 2.4. If we look back at Figure 3, we see, for example, that the indegree of vertex 1 is zero and its outdegree is two. For vertex 3 it is the other way around. Vertex $2, 4, 5$ and 6 each has in- and outdegree one.

Definition 2.10. Given a graph $G = (V, E)$ together with an orientation ρ, a *flow on G* is a map $f \colon E \to \mathbb{R}$ that assigns a value $f(e) \in \mathbb{R}$ to each edge $e \in E$ such that there is a conservation of flow at every vertex v:

$$\sum_{\{e \in E \ \mid \ v \text{ is the head of } e \}} f(e) = \sum_{\{e \in E \ \mid \ v \text{ is the tail of } e \}} f(e). \tag{1}$$

That is, what "flows" into the vertex v is precisely what "flows" out of v. The *flow space* $\mathcal{F}_G \subseteq \mathbb{R}^E$ of G with orientation ρ is defined as the affine subspace of all $f \in \mathbb{R}^E$ satisfying condition (1). The flow space depends on the chosen orientation ρ.

Example 2.5. Figure 4 shows a flow on the two triangles that we have encountered in Example 2.3 with the orientations that we have seen in that example.
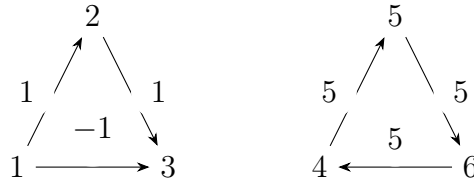
Figure 4: Flows on a triangle.

The flow space $\mathcal{F}_T$ of the left triangle T is given by

$$\mathcal{F}_T = \{f \in \mathbb{R}^3 \mid -f_{13} - f_{12} = 0 \text{ and } f_{12} - f_{23} = 0 \text{ and } f_{23} + f_{13} = 0\}$$

where f_{ij} denotes the flow value on the edge ij.

Remark 2.1. If a graph G has a bridge, then G will not have any nowhere-zero flow or any totally cyclic orientation.

Definition 2.11. A *forest* is a graph that has no cycles. A *tree* is a connected forest. A *spanning forest* is an inclusion-maximal cycle-free subgraph of G, i.e., an inclusion-maximal forest.

The next definition explains why spanning forests are called spanning.

Definition 2.12. Let $G = (V, E)$ be a graph. A subset $S \subseteq E$ of the set of edges of G is called *spanning* if and only if for every vertex $v \in V$ there exists an edge $e \in S$ that is incident to v.

Remark 2.2. The definition of spanning in the context of graphs corresponds to the definition of spanning which we will see in Chapter 2.2 in the context of matroids since every spanning set S of a graph G contains as a subset a spanning forest which gives us a basis of the graphic matroid of G.

Definition 2.13. Let $G = (V, E)$ be a graph. A *leaf* is a vertex whose degree is one.

In the proof of the following proposition we will construct the most popular basis of the flow space, the cycle basis, to obtain the following result:

Proposition 2.1. (See, for example, [3, Proposition 7.6.1]). *Let $G = (V, E)$ be a graph with a fixed, but arbitrary orientation. Then $\dim \mathcal{F}_G = \xi(G) := |E| - |V| + c$ where c denotes the number of connected components of G.*

Proof. We will construct a basis of $\mathcal{F}_G$ with $\xi(G)$ elements. We first observe that, if G is the disjoint union of G_1 and G_2, then $\mathcal{F}_G = \mathcal{F}_{G1} \times \mathcal{F}_{G2}$ and hence $\dim \mathcal{F}_G = \dim \mathcal{F}_{G1} + \dim \mathcal{F}_{G2}$. We will therefore assume that G is connected. Let $T \subseteq G$ be a spanning tree, i.e., $T = (V, E_0)$ for some $E_0 \subseteq E$, that is connected and without cycles. Let $e = uv \in E \setminus E_0$ be oriented from u to v. Since T is connected, there is a path $v =: v_0, v_1, ..., v_k := u$ in T that connects v to u. In particular, $v = v_0, v_1, ..., v_k = u, v$ is a directed cycle C_e in $T \cup e \subseteq G$, called the *fundamental cycle* with respect to T and e.

For a cycle $C \subseteq E$ define a function $f_C \colon E \to \mathbb{Z}$ through

$$f_C(\tilde{e}) := \begin{cases} 1 & \text{if } \tilde{e} = v_{i-1}v_i \text{ is oriented from } v_{i-1} \text{ to } v_i, \\ -1 & \text{if } \tilde{e} = v_iv_{i-1} \text{ is oriented from } v_i \text{ to } v_{i-1}, \end{cases} \tag{2}$$

for an edge $\tilde{e} \in C$ and $f_C(\tilde{e}) = 0$ for $\tilde{e} \in E \setminus C$. The function f_C defined by (2) is a nonzero element of $\mathcal{F}_G$.

We claim that $\{f_{C_e} \mid e \in E \setminus E_0\}$ is a basis of $\mathcal{F}_G$. Note that the elements in this collection are linearly independent since for each edge $e \in E \setminus E_0$, there is just exactly one f_{C_e} with a non-zero entry at position e. Hence, we only need to show that they are spanning. For a given $f \in \mathcal{F}_G$, we can add suitable scalar multiples of the f_{C_e}'s, $e \in E \setminus E_0$, to it and can assume that $f(e') = 0$ for all $e' \in E \setminus E_0$.

Arguing by contradiction, let us assume that $f \neq 0$. Then

$$E_f := \{e' \in E \mid f(e') \neq 0\}$$

is a subset of E_0. However, for (1) to be satisfied at a node v, there have to be either zero or at least two edges in E_f incident to w. Therefore, the graph $(V, E_f) \subseteq T$ contains a cycle which contradicts our assumption on T. Since T is a spanning tree, $|E_0| = |V| - 1$ and so $\mathcal{F}_G$ is of dimension $|E \setminus E_0| = \xi(G)$. $\square$
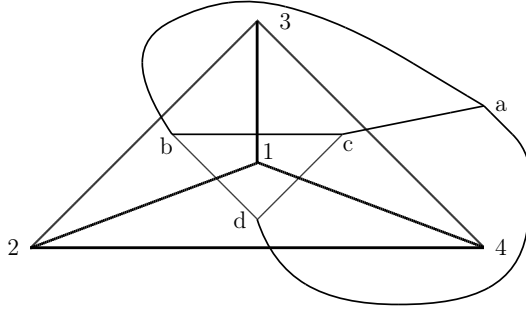
Figure 5: The graph K_4 (fat) and its dual graph (light).

Example 2.6. The flow values for each edge that we assigned in Example 2.5 to the left triangle T give us a cycle basis for its flow space $\mathcal{F}_T$ by Proposition 2.1. The flow values that we assigned to the left triangle also give us a cycle basis over $\mathbb{R}$ since we can use multiplicative inverses over $\mathbb{R}$.

Definition 2.14. We assume that G is a planar bridgeless graph with a given embedding into the plane. The drawing of G subdivides the plane into connected regions in which two points lie in the same region whenever they can be joined by a path in $\mathbb{R}^2$ that does not meet G. Two such regions are neighboring if their topological closures share a proper (i.e., 1-dimensional) part of their boundaries. This induces a graph structure on the subdivision of the plane: for the given embedding of G, we define the *dual graph* $G^\triangle$ as the graph with vertices corresponding to the regions and two regions C_1, C_2 share an edge $e^\triangle$ if an original edge e is properly contained in both their boundaries. Given an orientation on G, an *orientation on* $G^\triangle$ is induced by rotating the edge clockwise. That is, the dual edge will "point" east assuming that the primal edge "points" north.

The dual graph $G^\triangle$ is typically not simple, but it has parallel edges or loops. If G had bridges, $G^\triangle$ would have loops.

Example 2.7. Figure 5 shows us the complete graph K_4 (fat) and its dual graph (light) as defined in Definition 2.14. We can observe that the dual of K_4 is again K_4, i.e., that K_4 is self-dual.

2.2 Matroids

In 1935, matroids were introduced by Whitney in [20] in order to abstract notions of linear algebra and graph theory. Whitney also introduced the notion of matroid duality in the same paper. In the following we want to use this way of abstraction to set up the theory that leads us to our definition of the cographic hyperplane arrangement and also of the flow zonotope. Thus, we will need basic concepts concerning matroids that we will introduce in this chapter following [13, Chapter 39]. In particular, we will give the definitions of the graphic and the cographic matroid.

Definition 2.15. A pair $M = (S, \mathcal{I})$ is called a *matroid* if S is a finite set and $\mathcal{I}$ is a nonempty collection of subsets of S satisfying:

$$\emptyset \in \mathcal{I}, \tag{3}$$

$$\text{if } I \in \mathcal{I} \text{ and } J \subseteq I, \text{ then } J \in \mathcal{I} \tag{4}$$

and

$$\text{if } I, J \in \mathcal{I} \text{ and } |I| < |J|, \text{ then there exists } z \in J \setminus I \text{ such that } I \cup z \in \mathcal{I}. \tag{5}$$

The set S is called the *ground set* of M.

Definition 2.16. Given a matroid $M = (S, \mathcal{I})$, a subset I of S is called *independent* if I belongs to $\mathcal{I}$, and *dependent* otherwise.

Definition 2.17. For $U \subseteq S$, a subset B of U is called a *basis* of U if B is an inclusion-wise maximal independent subset of U.

Under condition (4), condition (5) is equivalent to the fact that for any subset U of S, any two bases of U have the same size.

Definition 2.18. A subset of S is called *spanning* if it contains a basis as a subset.

Definition 2.19. Two matroids $M_1 = (S_1, \mathcal{I}_1)$ and $M_2 = (S_2, \mathcal{I}_2)$ are called *isomorphic* if and only if there exists a bijection $\Phi : S_1 \to S_2$ such that X is contained in the independence set of M_1 if and only if $\Phi(X)$ is contained in the independence set of M_2 for all $X \subseteq S_1$.

Now, we define dual matroids: Let $M = (S, \mathcal{I})$ be a matroid, define

$$I^{\triangle} := \{I \subseteq S \mid S \setminus I \text{ is a spanning set of } M\}. \tag{6}$$

That this definition is stated in a way such that we obtain again a matroid, is ensured by the following theorem.

Theorem 2.1. (See, for example, [13, Theorem 39.2.]). $M^{\triangle} := (S, \mathcal{I}^{\triangle})$ *is a matroid.*

Definition 2.20. The matroid $M^{\triangle}$ is called the *dual matroid* of M.

The bases of $M^{\triangle}$ are the complements of the bases of M. This implies $(M^{\triangle})^{\triangle} = M$ which justifies the name dual.

Lemma 2.1. (See, for example, [13, p. 657]). *Let $G = (V, E)$ be a graph and let $\mathcal{I}$ be the collection of all subsets of E that form a forest. Then $M(G) = (E, \mathcal{I})$ is a matroid.*

Famous examples of matroids are the graphic and cographic matroid. First, Whitney pointed out in [20] that the circuits of any graph G define a matroid. Tutte called in [17] this matroid the circuit-matroid and its dual the bond-matroid of G. Furthermore, Tutte determined in [17] a necessary and sufficient condition, in terms of matroid structure, for a given matroid M to be graphic (cographic). In [17] Tutte uses a reversed terminology, in which he called bond matroids "graphic" and cycle matroids "cographic", but this has not been followed by later authors.

Definition 2.21. The matroid $M(G) = (E, \mathcal{I})$ built in the way described in Lemma 2.1 is called the *cycle matroid* of G. Any matroid obtained this way, or isomorphic to such a matroid, is called a *graphic matroid*.

Definition 2.22. Let $G = (V, E)$ be a graph and $M(G) = (E, \mathcal{I})$ its induced graphic matroid. Then we call its dual $M^{\triangle}(G) = (E, \mathcal{I}^{\triangle})$ the *cocycle (bond) matroid* of G. Every matroid obtained this way, or isomorphic to such a matroid, is called a *cographic matroid*.

Thus, the bases of $M^{\triangle}(G)$ are exactly the complements of spanning forests of G. Hence, the independent sets of $M^{\triangle}(G)$ are those edge sets F for which $E \setminus F$ contains a spanning set of G, i.e., $(V, E \setminus F)$ has the same number of components as G.

2.3 Polytopes, polyhedra and cones

Needing the machinery to handle pointed cones, polyhedra and polytopes, we will introduce the main notion concerning pointed cones, polyhedra and polytopes in this chapter. We will define these objects in a way such that they are convex. We will mostly follow [2, Chapter 2.1 and Chapter 3.2], [14] and [23].

Definition 2.23. A subset $L \subseteq \mathbb{R}^d$ is called an *affine halfspace* if $L = \{x \mid ax \leq b\}$ for some $a \in (\mathbb{R}^d)^*$ with $a \neq 0$ and some $b \in \mathbb{R}$. If $b = 0$, then L is called a *linear halfspace*. Here $(\mathbb{R}^d)^*$ denotes the vector space dual to $\mathbb{R}^d$.

Definition 2.24. A subset $H \subseteq \mathbb{R}^d$ is called an *affine hyperplane* if

$$H = \{x \mid cx = b\}$$

for some $c \in (\mathbb{R}^d)^*$ with $c \neq 0$ and some $b \in \mathbb{R}$. If $b = 0$, then H is called a *linear hyperplane*.

Definition 2.25. A *pointed cone* $\mathcal{K} \subseteq \mathbb{R}^d$ is a set of the form

$$\mathcal{K} = \{w + \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m \mid \lambda_1, \lambda_2, ..., \lambda_m \geq 0\}$$

where $w, v_1, v_2, ..., v_m \in \mathbb{R}^d$ are such that there exists an affine hyperplane H for which $H \cap \mathcal{K} = \{w\}$. The vector w is called the *apex* of $\mathcal{K}$ and the v_k's are the *generators* of $\mathcal{K}$. In this thesis every cone is assumed to be pointed. Therefore, we will abbreviate notation and just write cone instead of pointed cone.
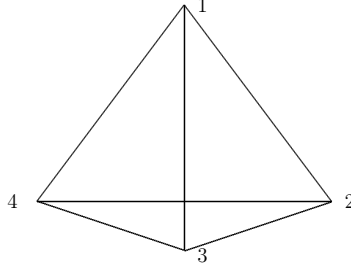
Figure 6: A tetrahedron on the vertex set $\{1, 2, 3, 4\}$.

Definition 2.26. A subset $P \in \mathbb{R}^d$ is called a *polyhedron* if and only if there exists a matrix $A \in \mathbb{R}^{m \times d}$ and a vector $b \in \mathbb{R}^m$ such that

$$P = \{x \mid Ax \leq b\}.$$

Thus, P is a polyhedron if and only if it is the intersection of finitely many affine halfspaces.

Definition 2.27. For a finite point set $\{v_1, v_2, ..., v_n\} \subseteq \mathbb{R}^d$, a polytope P is the smallest convex set containing these points; that is

$$P = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \mid \lambda_1, \lambda_2, ..., \lambda_n \geq 0 \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

This definition is called the vertex description of P, and we use the notation $P = \operatorname{conv}\{v_1, v_2, ..., v_n\}$.

Example 2.8. The simplest example for a polytope is the *d-simplex*, i.e., the convex hull of $d+1$ affinely independent points. A 3-simplex is called *tetrahedron*. Figure 6 shows a tetrahedron on the vertex set $\{1, 2, 3, 4\}$.

The theory of polytopes would not be so rich if there were not the following fundamental result (Minkowski[1896], Steinitz[1916] and Weyl[1935]):

Theorem 2.2. (See, for example, [23, Chapter 1]). *A subset $P \subseteq \mathbb{R}^d$ is a polytope if and only if P is a bounded polyhedron.*

A bunch of different proofs for this theorem and proofs of generalisations concerning, among others, cones, can be found in [23, Chapter 1].

Definition 2.28. Let P be a polyhedron or cone. A hyperplane Q is called a *supporting hyperplane* if and only if P is contained in one of the two closed halfspaces bounded by Q, and the intersection of Q and the boundary of P is nonempty.

Definition 2.29. Let $P \subseteq \mathbb{R}^d$ be a polyhedron or cone. A *face* of P is an intersection of P with a supporting hyperplane. The *dimension* of a face $F \subseteq P$, also for P itself, is the dimension of its affine hull. A polyhedron resp. cone of

dimension d is called a *d-polyhedron* resp. *d-cone*. For any k-dimensional polyhedron resp. k-dimensional cone, for $0 \leq k \leq d$, a 0-dimensional face is called a *vertex*, a 1-dimensional face is called an *edge*, a $(k-2)$-dimensional face is called a *ridge* and a $(k-1)$-dimensional face is called a *facet*. We define the empty set to be the (-1)-dimensional face and the polyhedron resp. the cone itself to be the k-dimensional face.

For a given polytope P, we are going to denote the set of facets of P by F, its set of ridges by R, its set of edges by E and its set of vertices by V.

Definition 2.30. The *face lattice* of a polyhedron or cone P is the poset $L(P)$ of all faces of P, partially ordered by inclusion.

Definition 2.31. The *f-vector* of a d-polytope P is the vector

$$(f_{-1}, f_0, f_1, ..., f_d) \in \mathbb{N}^{d+2},$$

where $f_k = f_k(P)$ denotes the number of k-dimensional faces of P.

As in all geometric disciplines, a fundamental invariant of an object is its dimension.

Definition 2.32. Let P be a d-polytope. Two i-dimensional faces F_1 and F_2, $i \in \{1, ..., d-1\}$, are called *adjacent* if and only if $F_1 \cap F_2$ is an $(i-1)$-dimensional face of P.

Example 2.9. A famous example with rich structure are zonotopes. They naturally appear in many disciplines; see [5, Theorem 3.3] for references.

Suppose we have given n line segments in $\mathbb{R}^d$, such that each line segment has one endpoint at the origin and the other endpoint is located at the vector $u_j \in \mathbb{R}^d$, for $j \in [n]$. Then, the *zonotope* $\mathcal{Z}(u_1, ..., u_n)$ of these line segments is defined as their Minkowski sum, i.e.,

$$\mathcal{Z}(u_1, u_2, ..., u_n) := \{x_1 + x_2 + \cdots + x_n \mid x_j = \lambda_j u_j \text{ with } \lambda_j \in [0, 1]\}.$$

An equivalent definition of d-dimensional zonotopes is the image of a cube under an affine projection, i.e.,

$$\mathcal{Z} = \mathcal{Z}(V) := \left\{ x \in \mathbb{R}^d \mid x = y + \sum_{i=1}^{p} \alpha_i v_i, \ -1 \leq \alpha_i \leq 1 \right\}$$

for some $y \in \mathbb{R}^d$ and some matrix (vector configuration) $V = (v_1, ..., v_p) \in \mathbb{R}^{d \times p}$.

A famous zonotope that we will use later is the *permutahedron* $\mathcal{P}_d$ defined as

$$\mathcal{P}_d := \mathcal{Z}(x_2 - x_1, x_3 - x_1, ..., x_d - x_{d-1}),$$

where $x_1, ..., x_d \in \mathbb{R}^d$ are the unit vectors in $\mathbb{R}^d$. It is more common to define the permuathedron $\mathcal{P}_d$ as the convex hull of all permutations of the numbers $1, ..., d$. This definition and a proof of the equivalence of both definitions can be found in [2, Chapter 9.3].

Definition 2.33. A d-cone is said to be simplicial if and only if it is generated by exactly d linearly independent generators.

Definition 2.34. A d-polytope P is said to be simplicial if and only if its facets are simplices.

There are several equivalent conditions for a polytope to be simplicial, which we will not state in this thesis to prevent loosing track. However, these conditions can be found in [23, Proposition 2.16].

Theorem 2.3. (See, for example, [23, p. 253]). *Let P be a simplicial d-polytope. Then for any $k \in \{-1, 0, 1, ..., d-2\}$ the Dehn-Sommerville equations*

$$(-1)^{d-1} f_k = \sum_{i=k}^{d-1} (-1)^i \binom{i+1}{k+1} f_i$$

hold.

Theorem 2.4. (See, for example, [23, Chapter 8.2]). *Let P be a polytope. We have the Euler-Poincaré formula*

$$\sum_{k=0}^{\dim(P)} (-1)^k f_k(P) = 1.$$

Definition 2.35. Let P be a polytope. The graph whose vertex set is the set of vertices of P and whose edge set is the set of edges of P is called the *graph of the polytope P*. We will denote the graph of a polytope P by $G(P)$.

Example 2.10. The graph given in Figure 1 can be seen as the graph of the tetrahedron which is the 3-simplex.

Definition 2.36. Let $P \subseteq \mathbb{R}^d$ be a d-polytope with 0 in its interior. Its *dual polytope $P^\triangle$* is defined as

$$P^\triangle := \{z \in \mathbb{R}^d \mid x^T z \leq 1 \text{ for all } x \in P\}.$$

Example 2.11. Figure 7 shows the tetrahedron (light) (we assume the tetrahedron contains 0 in its interior) with its dual (fat) which is also a tetrahedron, i.e., the tetrahedron is self-dual. In Example 2.7 we have observed the same behaviour for their graphs.
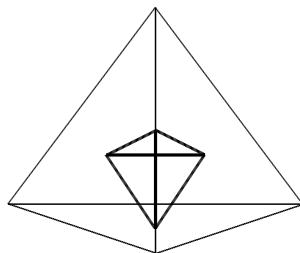
Figure 7: A tetrahedron (light) and its dual (fat).

Definition 2.37. Let $P \subseteq \mathbb{R}^d$ be a d-polytope with 0 in its interior. For a face A of P we call the inclusion-wise maximal face $A^\triangle$ satisfying $x^T z = 1$ for all $x \in A$ and $z \in A^\triangle$ the *face dual to* A.

Example 2.12. Consider the tetrahedron T that we have seen in Example 2.11. For each vertex v of T, the face of $T^\triangle$ dual to v is a facet of $T^\triangle$ since this facet f is the inclusion-wise maximal face satisfying the condition $v^T z = 1$ for all $z \in f$. The vertices and edges of f also satisfy the above condition, but they are not inclusion-wise maximal satisfying it. The faces dual to the edges of T are the edges of $T^\triangle$ and the facets of T are dual to the vertices of $T^\triangle$.

3 The Ehrhart polynomial of the graphic zonotope

3.1 Hyperplane arrangements and zonotopes

We are interested in the relations between the graphic, resp. cographic hyperplane arrangements and their corresponding zonotopes since these relations are the reason we construct both the graphic and the flow zonotope. In this chapter we want to point out the relations that we have for general hyperplane arrangements. Therefore, we want to introduce some notions about hyperplane arrangements and zonotopes so that we can arrive at Theorem 3.1. We partly see in what sense zonotopes and hyperplane arrangements can be considered equivalent. Throughout this chapter we mostly follow [23, Chapter 7] and add some details from [2]. The relation between zonotopes and hyperplane arrangements marks only the beginning of the deep theory of oriented matroids; see [23, Chapter 7] and [22].

Definition 3.1. A *fan* in $\mathbb{R}^d$ is a family $\mathcal{C} = \{C_1, ..., C_n\}$ of non-empty polyhedral cones with the following two properties:

i) Every non-empty face of a cone in $\mathcal{C}$ is also a cone in $\mathcal{C}$.

ii) The intersection of any two cones in $\mathcal{C}$ is a face of both.

A fan $\mathcal{C}$ is called *complete* if $\bigcup_{i=1}^{n} C_i = \mathbb{R}^d$. We will consider only complete fans here. Thus, we will omit the word complete.

Definition 3.2. Let $P \subseteq \mathbb{R}^d$ be a non-empty d-polytope. The *normal fan* $\mathcal{N}(P)$ of P is defined as
$$\mathcal{N}(P) := \{N_F \mid F \in L(P) \setminus \emptyset\}$$
where
$$N_F := \{c \in (\mathbb{R}^d)^* \mid F \subseteq \{x \in P \mid cx = \operatorname*{argmax}_{y \in P} cy\}\},$$
i.e., we take the cones of those linear functions that are maximal on a fixed face of P.

Definition 3.3. A *(linear) hyperplane arrangement* $\mathcal{H} := \{H_1, ..., H_p\}$ is a finite set of (linear) hyperplanes in $\mathbb{R}^d$.

In this thesis we always refer to linear hyperplane arrangements. Therefore, we will leave out the word linear.

Definition 3.4. Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^d$. A *region* of $\mathcal{H}$ is a connected component of $\mathbb{R}^d \setminus \mathcal{H}$.

Example 3.1. To each zonotope $\mathcal{Z} := \mathcal{Z}(u_1, u_2, ..., u_n)$ there is an *associated hyperplane arrangement* $\mathcal{H}(\mathcal{Z}) = \{H_1, ..., H_n\}$ whose hyperplanes $H_1, ..., H_n$ have as normal vectors the vectors $u_1, ..., u_n$.

Example 3.2. Our motivating and governing example is an arrangement stemming from a given simple graph $G = (V, E)$ with $|V| = d$. To an edge $ij \in E$ we associate the hyperplane

$$H_{ij} := \{x \in \mathbb{R}^V \mid x_i = x_j\}.$$

The *graphic hyperplane arrangement* of G is then

$$\mathcal{H}_G := \{H_{ij} \mid ij \in E\}.$$

For this definition we do not have to choose any orientation on G.

Conversely, to a subset

$$S \subseteq \{x_2 - x_1, x_3 - x_1, ..., x_d - x_{d-1}\},$$

where $x_1, ..., x_d \in \mathbb{R}^d$ are the d unit vectors in $\mathbb{R}^d$, we associate the graph G_S with node set $V := [d]$ and edge set

$$E := \{ij \mid x_j - x_i \in S\}.$$

For the converse direction we have chosen an orientation from the smaller to greater endpoint of an edge by just allowing vectors satisfied this condition.

Remark 3.1. Consider the set $\{x_2 - x_1, x_3 - x_1, ..., x_d - x_{d-1}\}$, where $x_1, ..., x_d \in \mathbb{R}^d$, that we have seen in Example 3.2. This set induces a matrix $M \in \mathbb{R}^{d \times E}$ whose columns are given by the elements of $\{x_2 - x_1, x_3 - x_1, ..., x_d - x_{d-1}\}$ by writing a 1 for x_i, $i \in [d]$. This matrix is also known as the *signed vertex-edge incidence matrix* of G.

Our main motivation for considering this special class of hyperplane arrangements is that they geometrically carry quite a trove of information about the underlying graph as we will see in Chapter 3.2.

Example 3.3. Let $\mathcal{H} := \{H_1, ..., H_p\}$ be a finite set of hyperplanes in $\mathbb{R}^d$. The arrangement $\mathcal{H}$ dissects $\mathbb{R}^d$ into regions which are cones. The set of regions of $\mathcal{H}$ and all their faces (considering the regions as cones) is a complete fan $\mathbf{F}_H$ since all regions are non-empty polyhedral cones whose union is the whole space $\mathbb{R}^d$ and the intersection of any two regions is a face of both. The cones of the fan are also referred to as the *faces of the hyperplane arrangement* $\mathcal{H}$. Thus, we can also consider their *face lattice* $L(\mathcal{H})$.

The following two theorems give us important relations between hyperplane arrangements and zonotopes and show why they are interesting from both a geometrical and combinatorial point of view.

Theorem 3.1. (See, for example, [23, Chapter 7]). *Let $\mathcal{Z} = \mathcal{Z}(V) \subseteq \mathbb{R}^d$ be a zonotope given by a vector configuration $V = (v_1, ..., v_p) \in \mathbb{R}^{d \times p}$. Then the normal fan $\mathcal{N}(\mathcal{Z})$ of $\mathcal{Z}$ is the fan $\mathbf{F}_H$ of the hyperplane arrangement $\mathcal{H}_V := \{H_1, ..., H_p\}$ in $\mathbb{R}^d$ given by $H_i := \{c \in (\mathbb{R}^d)^* \mid cv_i = 0\}$.*

Theorem 3.2. (See, for example, [3, Theorem 7.5.5.]). *Let $\mathcal{Z}$ be a zonotope with associated hyperplane arrangement $\mathcal{H}(\mathcal{Z})$. Then the face lattices $L(\mathcal{Z})$ and $L(\mathcal{H}(\mathcal{Z}))$ are anti-isomorphic.*

3.2 Graphic hyperplane arrangements and graphic zonotopes

As mentioned in Chapter 3.1, a graphic hyperplane arrangement geometrically carries a trove of information about the underlying simple graph. In this chapter we want to open this trove and discuss the properties that are important for us. We already came across the definition of the graphic hyperplane arrangement in Chapter 3.1.

The first property that we want to consider is given by the following proposition. It gives the connection between the graphic matroid and the graphic hyperplane arrangement and explains why we call this hyperplane arrangement "graphic":

Proposition 3.1. *The graph G_S is a forest if and only if the corresponding subset $S \subseteq \{x_2 - x_1, x_3 - x_1, ..., x_d - x_{d-1}\}$, where $x_1, ..., x_d \in \mathbb{R}^d$, is linearly independent.*

Proof. [1] Suppose G contains a cycle C. Consider the matrix representation of S. Then there is a reorientation making C a directed cycle. For our vectors in S, reorientation just means multiplying by ± 1, i.e., an operation conserving linear independency. If we consider this directed cycle C, we can observe that for each vertex v contained in C, there is one edge contained in C whose tail is v, so, giving a -1 entry, and one whose head is also v, giving a $+1$ entry. The rest of the entries are zero. Now consider the submatrix S' obtained from the vectors of S corresponding to the edges of C. We observe that in each row of S' there is exactly one -1 for the tail and one $+1$ entry for the head and the rest of the entries is zero. Since each vertex in C is both a tail for one of its two incident edges which are also in C, and a head for the other, we obtain zero by adding up the columns of S'. Thus, these vectors are linearly dependent.

We use induction on the number of edges for the converse direction. For one edge the statement is easily true. So, suppose we have $n \in \mathbb{N}$ many edges. Every forest contains a leaf v. Now, the induction step is to remove the edge e incident to v. The vector corresponding to e was linearly independent from the others since it was the only vector with a non-zero entry at position v. So, removing e and its corresponding vector keeps the other vectors linearly independent. $\square$

The second important property of the graphic hyperplane arrangement is given by the following:

Proposition 3.2. (See, for example, [8, p.112-113]). *There is a one-to-one correspondence between the acyclic orientations of G and the regions of $\mathcal{H}_G$ given by*

$$\mathbf{R}(\mathcal{O}) \coloneqq \{x \in \mathbb{R}^V \mid x_i < x_j \text{ if } ij \text{ is oriented } (i, j) \text{ in } \mathcal{O}\} \tag{7}$$

---

[1]Thanks to Sophie Rehberg, Freie Universität Berlin, who has shown the second direction of this proof during the course Discrete Geometry 2 at FU Berlin, Summer 2021
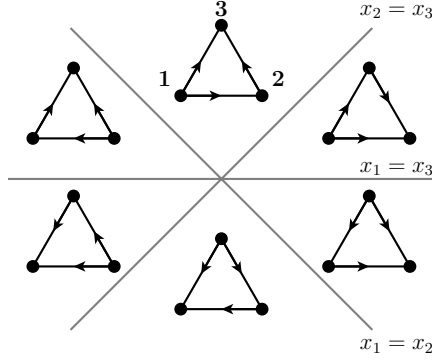
Figure 8: The regions of $\mathcal{H}_{K_3}$ (projected to the plane $x_1 + x_2 + x_3 = 0$) and their corresponding acyclic orientations. Source: [2, p.238]

for each acyclic orientation $\mathcal{O}$ on G, where we denote by (i, j) the edge ij oriented from i to j. Conversely,

$$\mathcal{O}(\mathbf{R}) := \{(i, j) | \ ij \in E \ \ and \ \ x_i < x_j \ if \ x \in \mathbf{R}\} \tag{8}$$

for each region $\mathbf{R}$.

Proof. For each non-empty region $\mathbf{R}$, any $x \in \mathbf{R}$ defines an orientation $\mathcal{O}(x)$ by (8). Suppose that $\mathcal{O}$ is not acyclic. Then there exists a cycle giving a chain of strict inequalities given by the orienting vertices, w.l.o.g. $v_1 < v_2 < \cdots < v_{n-1} < v_n < v_1$ and consequently, $\mathbf{R}$ is empty. Thus, $\mathcal{O}$ is acyclic. This property holds for all $x \in \mathbf{R}$. If we consider any $y \in \mathbf{R}$ then we obtain y from x by moving in $\mathbf{R}$ without crossing any hyperplane of the graphic hyperplane arrangement. Thus, no constraint defining $\mathcal{O}$ gets changed for any point in $\mathbf{R}$ and therefore, $\mathcal{O}$ is an acyclic orientation well-defined on $\mathbf{R}$.

Conversely, given any acyclic orientation $\mathcal{O}$ on G, we will show that $\mathbf{R}(\mathcal{O})$ is non-empty. If follows from the previous direction that $\mathbf{R}(\mathcal{O})$ is a well-defined region of our hyperplane arrangement. We define a partial ordering on $\mathbb{N}$ by $x_i \leq_{\mathcal{O}} x_j$ if and only if $(i, j) \in \mathcal{O}$ (extended by transitivity). In the next step, we extend this partial ordering to a total ordering $x_i <_{\mathcal{O}} x_j$ if and only if $(i, j) \in \mathcal{O}$ and see that any $x \in \mathbb{R}^V$, whose coordinates get ordered by this total ordering, belongs to $\mathbf{R}(\mathcal{O})$. Eventually, we have shown $\mathbf{R}(\mathcal{O}(\mathbf{R})) = \mathbf{R}$ and $\mathcal{O}(\mathbf{R}(\mathcal{O})) = \mathcal{O}$. □

Example 3.4. The one-to-one correspondence between the acyclic orientations on a graph G and the regions of $\mathcal{H}_G$ is illustrated by Figure 8 by using as example the triangle K_3. For each region of H_{K_3} there is an acyclic orientation on K_3 corresponding to it. For each edge we have one corresponding hyperplane. Crossing a hyperplane means to flip the orientation of the corresponding edge to obtain the acyclic orientation corresponding to the next region.

Definition 3.5. Let $G = (V, E)$ be a graph. We define the *graphic zonotope* of G as

$$\mathbf{Z}_G := \sum_{e \in E} u_e$$

where $u_e := [0, x_e]$ and x_e is the column vector of the signed vertex-edge incidence matrix M of G corresponding to the edge $e \in E$.

We have seen three properties of the graphic hyperplane arrangement so far. The next proposition states that at least one of these properties passes on to the graphic zonotope and this motivates a closer look at these special zonotopes.

Proposition 3.3. (See, for example, [3, Proposition 7.5.3.]). *Let G be a graph. The vertices of $\mathbf{Z}_G$ are in one-to-one correspondence with the acyclic orientations on G. Consequently, the vertices of $\mathbf{Z}_G$ are in one-to-one correspondence with the regions of the graphic hyperplane arrangement of G.*

One goal of this work is to construct a hyperplane arrangement for the cographic matroid that has exactly the properties dual to the properties of the graphic hyperplane arrangement:

i) A subset of the normal vectors of that hyperplane arrangement is linearly independent if and only if it induces a complement of a spanning set of G as these sets are exactly the independent sets of the cographic matroid of G.

ii) The regions of that hyperplane arrangement are in one-to-one correspondence with the totally cyclic orientations of G, as this is dual to the statement that the acyclic orientations of G are in a one-to-one correspondence with the regions of the graphic hyperplane arrangement.

Definition 3.6. We call a hyperplane arrangement satisfying the conditions above a *cographic hyperplane arrangement*.

For condition *ii*) it is natural to just consider graphs without bridges because graphs containing bridges do not have any totally cyclic orientations.

3.3 Zonotopal tilings

Another remarkable behaviour of zonotopes is that zonotopes can be decomposed into a disjoint union of half-open parallelepipeds as Lemma 3.1 will show. So, the natural decomposition of zonotopes is into parallelepipeds. In the next chapters we will use this behaviour to first find remarkable formulas for the Ehrhart polynomials of graphic zonotopes and later those of flow zonotopes. We will follow [2, Chapter 9] and [3, Chapter 7.5].

We suppose that $w_1, w_2, ..., w_m \in \mathbb{R}^d$ are linearly independent, and we require $\sigma_1, \sigma_2, ..., \sigma_m \in \{\pm 1\}$. Then we define

$$\Pi_{w_1, w_2, ..., w_m}^{\sigma_1, \sigma_2, ..., \sigma_m} := \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m \mid 0 \le \lambda_j \le 1 \text{ if } \sigma_j = -1 \text{ and}$$
$$0 < \lambda_j \le 1 \text{ if } \sigma_j = 1 \text{ for } j \in [m]\}.$$

So, $\Pi_{w_1, w_2, ..., w_m}^{\sigma_1, \sigma_2, ..., \sigma_m}$ is a half-open parallelepiped for a proper choice of $\sigma_1, \sigma_2, ..., \sigma_m$ generated by $w_1, w_2, ..., w_m$. The signs $\sigma_1, \sigma_2, ..., \sigma_m$ keep track of those facets of the parallelepiped that are included or excluded from the closure of the parallelepiped.
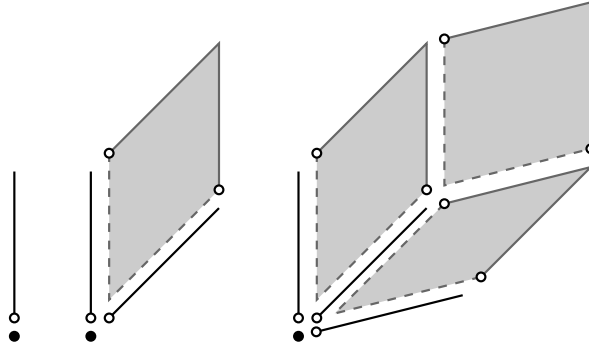
Figure 9: A zonotopal decomposition of $\mathcal{Z}((0,4)^T, (3,3)^T, (4,1)^T)$. Source: [2, p.171]

Lemma 3.1. (See, for example, [2, pp. 171-172]). *The zonotope $\mathcal{Z}(u_1, u_2, ..., u_n)$ can be written as a disjoint union of translates of $\Pi_{w_1,w_2,...,w_m}^{\sigma_1,\sigma_2,...,\sigma_m}$, where $\{w_1, w_2, ..., w_m\}$ ranges over all linearly independent subsets of $\{u_1, u_2, ..., u_n\}$, each equipped with an appropriate choice of signs $\sigma_1, \sigma_2, ..., \sigma_m$.*

Figure 9 illustrates the decomposition of a zonotope as suggested by Lemma 3.1.

3.4 Ehrhart polynomials of (graphic) zonotopes

Definition 3.7. A *lattice polytope* is a polytope $P \subseteq \mathbb{R}^d$ whose vertices are all lattice points of a chosen lattice.

We will work with the lattice $\mathbb{Z}^d \subseteq \mathbb{R}^d$.

Let $P \subseteq \mathbb{R}^d$ be a polytope that spans a subspace $\mathcal{S} \subseteq \mathbb{R}^d$. We denote by $\operatorname{vol} P$ the *(relative) volume* of P, normalized with respect to $\mathcal{S} \cap \mathbb{Z}^d$; that is, we take the volume of a fundamental domain of the integer lattice in $\mathcal{S}$ to be 1. To explain this last, we note that $\mathcal{S} \cap \mathbb{Z}^d$ is linearly equivalent to $\mathbb{Z}^{\dim \mathcal{S}} \subseteq \mathbb{R}^{\dim \mathcal{S}}$; a *fundamental domain* is a domain in $\mathcal{S}$ that corresponds to the unit hypercube $[0,1]^{\dim \mathcal{S}} \subseteq \mathbb{R}^{\dim \mathcal{S}}$, under some invertible linear transformation that carries $\mathcal{S} \cap \mathbb{Z}^d$ to $\mathbb{Z}^{\dim \mathcal{S}}$. When $\mathcal{S} = \mathbb{R}^d$ this is the ordinary volume, see [4, Chapter 2.1] or [2, Chapter 5.4].

Ehrhart proved in [6] that, for a lattice d-polytope P, the lattice point enumerator $L_p(t) := \#(tP \cap \mathbb{Z}^d)$ is a polynomial in t of degree d. This polynomial is called *Ehrhart polynomial* and can be written as $L_P(t) = \sum_{i=0}^{d} c_i(P)t^i$ where the coefficients $c_i(P)$, $0 \leq i \leq d$ depend only on P. If P is a rational polytope, i.e., a polytope whose vertices are all in $\mathbb{Q}^d$, the lattice point enumerator L_P is a quasipolynomial. Ehrhart showed that the leading term of the Ehrhart polynomial is $(\operatorname{vol} P)t^{\dim P}$.

In this thesis we want to apply Ehrhart theory only to lattice polytopes, to be precise, to zonotopes generated by integer vectors. Therefore, we will not

give further details on Ehrhart theory for rational polytopes. For example, [2] can be recommended for the reader interested in this topic, an introduction to quasipolynomials or a more detailed introduction to Ehrhart theory.

Theorem 3.3, which is due to Richard Stanley (see [15]), will give us a remarkable formula for the Ehrhart polynomials of zonotopes generated by integer vectors. To prepare for this theorem, we will first point out how to compute the relative volume of a half-open parallelepiped.

Lemma 3.2. (See, for example, [2, Lemma 9.8]). *Suppose $w_1, w_2, ..., w_n \in \mathbb{Z}^d$ are linearly independent, let*

$$\Pi := \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_n w_n \mid 0 \le \lambda_1, \lambda_2, ..., \lambda_n < 1\},$$

and let V be the greatest common divisor of all $n \times n$ minors of the matrix formed by the column vectors $w_1, w_2, ..., w_n$. Then the relative volume of Π equals V. Furthermore,

$$\#(\Pi \cap \mathbb{Z}^d) = V,$$

and for every positive integer t,

$$\#(t\Pi \cap \mathbb{Z}^d) = V t^n.$$

In other words, for the half-open parallelepiped Π, the discrete relative volume $\#(t\Pi \cap \mathbb{Z}^d)$ coincides with the continuous relative volume $(\operatorname{vol} \Pi) t^n$.

Corollary 3.1. (See, for example, [2, Corollary 9.3]). *Decompose the zonotope $\mathcal{Z} \subseteq \mathbb{R}^d$ into half-open parallelepipeds according to Lemma 3.1. Then the coefficient c_k, for $0 \le k \le d$, of the Ehrhart polynomial*

$$L_{\mathcal{Z}}(t) = c_d t_d + c_{d-1} t_{d-1} + \cdots + c_0$$

equals the sum of the (relative) volumes of the k-dimensional parallelepipeds in the decomposition of $\mathcal{Z}$.

Now, we are finally prepared for Theorem 3.3.

Theorem 3.3. (See, for example, [2, Theorem 9.9]). *Let $\mathcal{Z} := \mathcal{Z}(u_1, ..., u_n)$ be a zonotope generated by the integer vectors $u_1, ..., u_n$. Then the Ehrhart polynomial of $\mathcal{Z}$ is given by*

$$L_{\mathcal{Z}}(t) = \sum_{S} m(S) t^{|S|},$$

where S ranges over all linearly independent subsets of $\{u_1, ..., u_n\}$, and $m(S)$ is the greatest common divisor of all minors of size $|S|$ of the matrix whose columns are the elements of S.

Example 3.5. We apply Theorem 3.3 to compute the Ehrhart polynomial of the graphic zonotope of a (simple and) connected graph $G = (V, E)$. By Definition 3.5 the graphic zonotope is the Minkowski sum of the line segments $u_e := [0, x_e]$ where x_e is are the column vector of the signed vertex-edge incidence matrix M of G corresponding to the edge $e \in E$. We know or can easily show by induction that the signed incidence matrix, also known as the incidence matrix of a directed graph, is totally unimodular, i.e., every minor is 0 or ± 1, see, for example, [19]. Thus, the greatest common divisor of each minor $m(S)$ is 1, where again S ranges over all linearly independent subsets of $\{u_1, ..., u_n\}$. So, using Corollary 3.1, the Ehrhart polynomial of $\mathcal{Z}$ is given by

$$L_{\mathcal{Z}}(t) = \sum_S 1 \cdot t^{|S|} = \sum_{T \text{ a forest of size } k} 1 \cdot t^k = \sum_{k=0}^{|V|-1} c_k t^k$$

where the coefficients c_k are the numbers of labeled forests on G of size k.

We obtain a remarkably easy result for the Ehrhart polynomial of the graphic zonotope bringing together the independent sets of the graphic matroid of G and the volume of its graphic zonotope. This result was shown, for example, by Postnikov (see [12, Proposition 2.4]) and is going back to an Exercise given by Stanley in [16, Exercise 4.32]. It motivated this thesis which started with the question if there is a similar nice arithmetic behaviour for the flow zonotope, which will be introduced in Chapter 4.6.

4 Cographic hyperplane arrangements and zonotopes

4.1 Further definitions

In this chapter we give definitions that partially generalize those of Chapter 2.3 and that are essential for the construction of the cographic hyperplane arrangement.

Definition 4.1. Let $P \subseteq \mathbb{R}^d$ be a d-polytope with 0 in its interior. Let C_i be the set of all i-faces of P, $i \in \{1, ..., d-1\}$. An orientation of the set of i-faces C_i is a sign vector $\epsilon \in \{\pm 1\}^{C_i}$ that labels each face A of C_i with a positive or negative sign.

Definition 4.2. Let P be a d-polytope whose vertices are labeled $1, ..., n$. An *orientation on the set of edges E of P with respect to vertices* is given by a subset $O \subseteq E$ such that for an edge $e = ij \in E$ with $i < j$, we direct e from i to j if $e \in O$ and from j to i otherwise. The *oriented vertex-edge-incidence-matrix* is a matrix whose entries are determined by the vertex-edge incidences, i.e., we have a row for each vertex and a column for each edge. If an edge ij is oriented from i to j, the entry determined by i and ij is -1, the entry determined by j and ij is $+1$ and if a vertex k is not contained in an edge ij, the entry determined by k and ij is 0.

Remark 4.1. The definition of an orientation on the set of edges E of a polytope P with respect to vertices coincides with the definition of an orientation on the set of edges with respect to vertices on the graph of P. Therefore, we can use all notions defined on the set of edges of a graph, e.g., acyclic orientations, directed paths, etc., also on the set of edges of a polytope by considering them as defined on the set of edges of the graph of P.

Example 4.1. Consider the tetrahedron T from Example 2.8 one more time. Let $O = \{13, 23, 34\}$ be a subset of the set of edges of T giving us the orientation $\mathcal{O}$ that can be seen in Figure 10 where we denote that an edge is oriented from vertex i to vertex j by drawing an arrow from i to j.
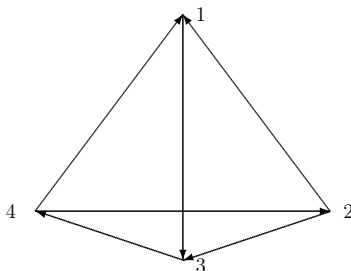

Figure 10: An orientation on the set of edges with respect to vertices.

Definition 4.3. Let P be a d-polytope. If the orientations of two adjacent i-faces of P are the same, then they are *oriented in the same direction*.

Example 4.2. Consider the orientation shown in Figure 10. We see that the orientations of the edges 23 and 34 are the same since they are both oriented from the smaller to the greater endpoint. Thus, 23 and 34 are oriented in the same direction.

Definition 4.4. Let P be a d-polytope whose facets are labeled $f_1, ..., f_m$. An *orientation $\mathcal{O}$ on the set of ridges R with respect to facets* is given by a subset $O \subseteq R$ such that for a ridge $r = f_i \cap f_j$ with $i < j$, we direct r from f_i to f_j if $r \in O$ and the other way around if $r \notin O$. The notation is assigning -1 for f_i and $+1$ for f_j if $r \in O$ and vise versa if $r \notin O$. If P is a 4-polytope, this means geometrically that all edges of a ridge r are oriented in mathematically positive direction going around the boundary of r seen from the facet with smaller label if $r \in O$ and the other way around otherwise. The *oriented ridge-facet-incidence-matrix* is a matrix whose entries are determined by the ridge-facet incidences, i.e., we have a row for each ridge and a column for each facet. If a ridge $r = f_k \cap f_l$ is oriented from f_k to f_l, the entry determined by f_k and r is -1, the entry determined by f_l and r is $+1$ and if a ridge r is not contained in a facet f, the entry determined by r and f is 0.

Definition 4.5. Let P be a 4-polytope. Let the ridges be oriented with respect to facets and the edges with respect to vertices. We denote the *orientation of an edge e of P with respect to a ridge r* by 0 if e is not contained in r, by $+1$ if the orientation of e given by the orientation of r is the same as the one e has with respect to vertices and by -1 otherwise. That means that for the *oriented edge-ridge-incidence-matrix*, we have for the entry determined by an edge e and a ridge r, a $+1$ if the orientation of e given by the orientation of r is the same as the one e has with respect to vertices, a 0 if e is not contained in r and a -1 otherwise.

Example 4.3. Consider again the tetrahedron from Example 2.11. Label the facets with the letters a, b, c, d and not by numbers to prevent having the same labels twice.

Let an orientation of the edges with respect to facets given by $O = \{ac, ad, bd\}$ where an edge ij, $i, j \in \{a, b, c, d\}$ is the intersection of the facets i and j. Now we can orient the edges with respect to facets as defined by Definition 4.4. We denote an edge oriented from lexicographic smaller to greater facet by green color and with orange for the other case as it can be seen in Figure 11. Facet a is the bottom facet, facet c the back one, facet d the left and facet b the right one.
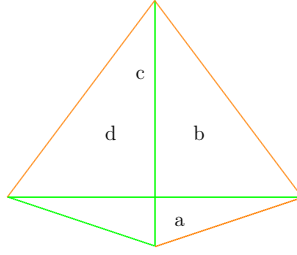
Figure 11: Edges oriented with respect to facets.

Definition 4.6. Let $P^\triangle$ be a 4-polytope. A nonempty sequence S of ridges is called a *ridge cycle* $\mathcal{R}^\triangle$ if and only if it separates the boundary of $P^\triangle$ into two connected pieces each of which contains at least one facet of $P^\triangle$. If $\mathcal{R}^\triangle$ additionally satisfies that each sequence of adjacent ridges of $\mathcal{R}^\triangle$ is directed in the same direction, then $\mathcal{R}^\triangle$ is called an *oriented ridge cycle*.

Remark 4.2. The definition of a ridge cycle implies that for its dual there exists a cut set between the vertices dual to the facets belonging to the first piece of the boundary and the vertices belonging to the second, i.e., the complement of the dual of a ridge cycle is not a spanning edge set.

Example 4.4. Figure 12 shows a ridge cycle on a Schlegel-diagram of a 4-simplex with vertex set $\{1, 2, 3, 4, 5\}$. The ridge cycle contains the ridges 123, 125, 235, 145, 134 and 345, which are all drawn blue. We see that every edge is contained in exactly two ridges of the ridge cycle.
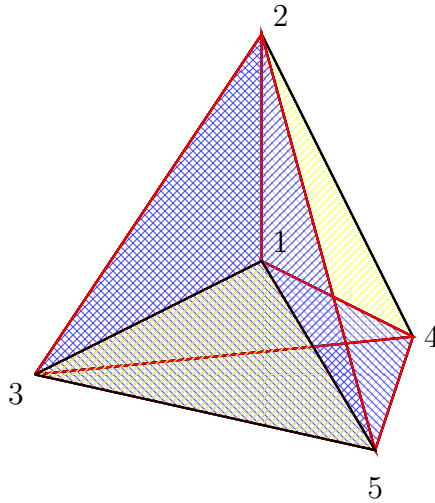
Figure 12: A ridge cycle (blue) on a 4-simplex.

Definition 4.7. Let $P^\triangle$ be a 4-polytope. An orientation $\mathcal{O}^\triangle$ on the set of ridges of $P^\triangle$ is called *totally cyclic* if and only if every ridge is contained in at least one oriented ridge cycle.

Definition 4.8. Let $P^\triangle$ be a 4-polytope. An orientation $\mathcal{O}^\triangle$ on the set of ridges of $P^\triangle$ is called *acyclic* if there does not exist any oriented ridge cycle.

Definition 4.9. A collection of ridges $\mathcal{R}^\triangle$ of a 4-polytope $P^\triangle$ is called a *ridge forest* if and only if it does not contain any ridge cycle. A ridge forest is called a *spanning ridge forest* if and only if it is a ridge forest of maximal cardinality.

Remark 4.3. The definition of a spanning ridge forest implies that for the boundary of every facet $F^\triangle \subseteq P^\triangle$ there is at least one not chosen ridge. Therefore, we know that the dual of the complement of a spanning ridge forest is a spanning subset of the set of edges of P.

Remark 4.4. The previous remark implies that the cardinality of a maximal spanning ridge forest for a polytope $P^\triangle$ is smaller than or equal

$$|R^\triangle| - |F^\triangle| + 1 = |E| - |V| + 1.$$

Definition 4.10. Let $P^\triangle$ be a 4-polytope and let $\mathcal{R}^\triangle$ be a spanning ridge forest. An edge $e^\triangle$ is called a *ridge leaf* of $\mathcal{R}^\triangle$ if and only if $e^\triangle$ is contained in exactly one ridge of $\mathcal{R}^\triangle$.

Example 4.5. Figure 13 shows a ridge forest on a Schlegel-diagram of a 4-simplex with vertex set $\{1, 2, 3, 4, 5\}$. The ridge forest contains the ridges 134 (blue), 145 (black), 124 (red), 123 (yellow), 345 (orange) and 245 (green). For example, the only ridge of the ridge forest containing the edge 23 is 123. Therefore, 23 is a ridge leaf.
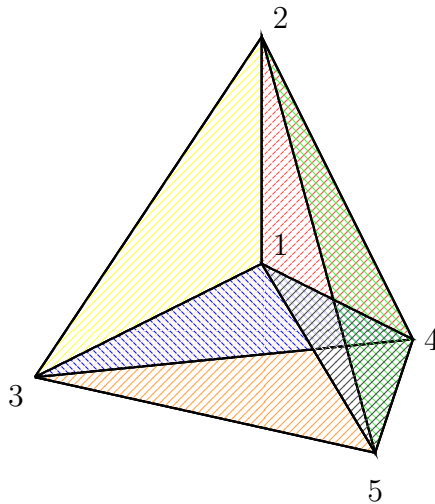

Figure 13: A ridge forest on a 4-simplex.

Let P be a d-polytope, $d = 3$ or $d = 4$, with 0 in its interior. For the construction of the cographic hyperplane arrangement we need the concept of cellular chain complexes. Because of the deep and huge theoretical background needed to set

up cellular chain complexes, we will explain only the notions that are important for us and refer to [10] for the reader interested in the background. The cellular chain complex associated to the CW-complex given by the face-structure of P, is of the form

$$C_{d-1} \xrightarrow{\partial_{d-1}} C_{d-2} \xrightarrow{\partial_{d-2}} \ldots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

The C_i's are free modules whose generators are the i-faces of P for $i \in \{0, \ldots, d-1\}$. The matrix representations of the maps ∂_j for $j \in \{1, \ldots, d-1\}$, called boundary maps, are the oriented $(j-1)$-face-j-face incidence matrices, see [1, Chapter 2]. These matrices encode the orientations of $(j-1)$-faces with respect to j-faces or the orientation of j-faces with respect to $(j-1)$-faces that we have seen in the Definitions 4.2, 4.4 and 4.5 in the way described within these definitions. So, we assume that we have given a *base orientation*, i.e., a labeling of the set of vertices and a labeling of the set of facets required by the Definitions 4.2, 4.4 and 4.5. An advantage of orienting the faces of P by a base orientation is that by re-orientations we can obtain all possible orientations on P. These orientations are encoded by the boundary maps, which are, as said, the oriented $(k-1)$-face-k-face incidence matrices of P for $k \in \{1, 2, \ldots, d-1\}$, i.e., they are encoded by a linear algebra tool. Thus, re-orienting just means multiplying subsets of the columns by -1.

We use the above orientations that are different from the convenient orientations, e.g. see [1, p.4] to obtain our desired properties. For example, using the orientations from [1], we would not have the property that the columns of $\partial_1^\triangle$ are the normal vectors of the graphic hyperplane arrangement of $P^\triangle$.

In the following we do not want to consider only the cellular chain complex associated to the CW-complex given by the face-structure of P, but also the one of its dual $P^\triangle$ and the corresponding chain maps $f_i \colon C_i \to C_{((d-1)-i)}$ that map a face A, which is a generator of C_i, to its dual face $A^\triangle$, which is a generator of $C_{((d-1)-i)}^\triangle$. Of course, there is a map in the other direction, which is induced by duality, too. Thus, the matrix representations of the boundary maps of the cellular chain complex associated to the CW-complex given by the face-structure of $P^\triangle$ are exactly the corresponding transposed of the matrix representations of the cellular chain complex associated to the CW-complex given by the face-structure of P as shown below:

$$\begin{array}{ccccccccccc}
C_{d-1} & \xrightarrow{\partial_{d-1}} & C_{d-2} & \xrightarrow{\partial_{d-2}} & \ldots & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\
\downarrow{\scriptstyle f_{d-1}} & & \downarrow{\scriptstyle f_{d-2}} & & & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
C_0^\triangle & \xleftarrow{\partial_{d-1}^T = \partial_1^\triangle} & C_1^\triangle & \xleftarrow{\partial_{d-2}^T = \partial_2^\triangle} & \ldots & \xleftarrow{\partial_3^T = \partial_{d-3}^\triangle} & C_{d-3}^\triangle & \xleftarrow{\partial_2^T = \partial_{d-2}^\triangle} & C_{d-2}^\triangle & \xleftarrow{\partial_1^T = \partial_{d-1}^\triangle} & C_{d-1}^\triangle
\end{array}$$

So, we can define the dual orientation for a given orientation by using this methodology as follows.

Definition 4.11. Let P be a d-polytope with 0 in its interior and let $P^{\triangle}$ be its dual polytope. For the following we assume that $i \in \{1, 2, ..., d-2\}$ and $j \in \{0, 1, 2, ..., d-1\}$ such that j is either $i+1$ or $i-1$. The orientation of the set of i-faces with respect to the j-faces of $P^{\triangle}$, also called the *dual orientation* of the i-faces with respect to the j-faces of $P^{\triangle}$ is given by the signs of the corresponding oriented i-face-j-face incidence matrix $\partial^{\triangle}_{\max\{i,j\}} = \partial^T_{d-\max\{i,j\}}$ of $P^{\triangle}$ which is induced by duality.

Remark 4.5. The columns of ∂_1 are exactly the normal vectors of the graphic hyperplane arrangement.

Remark 4.6. The column vectors of $\partial^T_{d-1} = \partial^{\triangle}_1$ are, particularly because of the way we have chosen the base orientation, the normal vectors of the graphic hyperplane arrangement of $G(P^{\triangle})$.

4.2 Cographic hyperplane arrangements for graphs that are Steinitz

Our first step for constructing the cographic hyperplane arrangement is to encounter the construction for the special case that $G = (V, E)$ is Steinitz, i.e., that it is 3-connected, planar and simple. Therefore, we assume throughout this chapter that G is Steinitz.

This assumption gives us the following property that we are going to exploit throughout this chapter:

Theorem 4.1. (Steinitz' Theorem for 3-polytopes, see, for example, [23, Chapter 4]). *G is the graph of a 3-polytope if and only if it is simple, planar and 3-connected.*

Thus, it follows that G is the graph of a 3-polytope P.

Fact 4.1. *By construction, $G(P^{\triangle})$ is the graph dual to G.*

Now, we want to use the construction of the cellular chain complex associated to the CW-complex given by the face structure of P. By labeling the facets of P by $1, ..., m$ and the vertices of P by $1, ..., n$ we obtain a base orientation that induces an orientation of the set of edges of P with respect to vertices resp. with respect to facets. Then, the cellular chain complexes associated to the CW-complex given by the face-structure of P and $P^{\triangle}$ are of the following form (see Chapter 4.1).

$$
\begin{array}{ccccc}
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\
\downarrow{f_2} & & \downarrow{f_1} & & \downarrow{f_0} \\
C_0^{\triangle} & \xleftarrow{\partial_2^T = \partial_1^{\triangle}} & C_1^{\triangle} & \xleftarrow{\partial_1^T = \partial_2^{\triangle}} & C_2^{\triangle}
\end{array}
$$

Because P is a 3-polytope, the edges of $P^\triangle$ are dual to the edges of P.

Theorem 4.2. *The columns of the matrix representation of $\partial_1^\triangle = \partial_2^T$ are the normal vectors of the cographic hyperplane arrangement of G.*

To prove this theorem, we need the following two Lemmas, which enable us to prove the two defining properties of the cographic hyperplane arrangement.

Lemma 4.1. (See, for example, [18, p.289]). *A subset $\mathcal{T}$ of the set of edges of $P^\triangle$ is a forest if and only if the set of the edges dual to the edges of $\mathcal{T}$ are complements of spanning sets in P.*

This lemma actually holds in greater generality. It is true for all simple and planar, but not necessarily 3-connected, graphs.

Example 4.6. Figure 14 shows a tetrahedron T on the vertex set $\{1, 2, 3, 4\}$ and its dual $T^\triangle$ which is also a tetrahedron. W.l.o.g. we assume that 0 is contained in the interior of T. This example illustrates Lemma 4.1, showing a forest on the set of edges of $G(T^\triangle)$ (green) inducing a complement of a spanning set on the set of edges of $G(T)$ (orange).
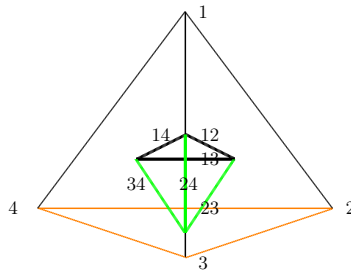
Figure 14: A forest on the set of edges of the dual of a polytope induces a complement of a spanning set on the set of edges of the primal polytope.

Furthermore, an orientation $\mathcal{O}$ on $G(P)$ induces an orientation $\mathcal{O}^\triangle$ on $G(P^\triangle)$ given by Definition 4.11. $\mathcal{O}^\triangle$ and $\mathcal{O}$ have the following property.

Lemma 4.2. (See, for example, [11]). *An orientation $\mathcal{O}$ on $G(P)$ is totally cyclic if and only if $\mathcal{O}^\triangle$ is acyclic on $G(P^\triangle)$.*

This lemma actually holds in greater generality. It is true for all simple and planar, but not necessarily 3-connected, graphs.

Example 4.7. Figure 15 illustrates Lemma 4.2, again using a tetrahedron. The acyclic orientation on $G(T^\triangle)$ induces by duality a totally cyclic orientation on $G(T)$.
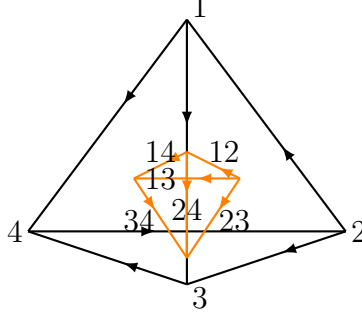
Figure 15: An acyclic orientation on the set of edges of the dual of a polytope induces by duality a totally cyclic orientation on the set of edges of its primal polytope.

The first defining property of the cographic hyperplane arrangement is restated in the following lemma.

Lemma 4.3. *The regions of the hyperplane arrangement whose normal vectors are the column vectors of ∂_2^T are in one-to-one correspondence with the totally cyclic orientations of $G(P)$.*

Proof. The columns of $\partial_2^T = \partial_1^\triangle$ are the normal vectors of the graphic hyperplane arrangement of $G(P^\triangle)$. Consider an acyclic orientation $\mathcal{O}^\triangle$ on $G(P^\triangle)$. It is in one-to-one correspondence with a region of the graphic hyperplane arrangement of $G(P^\triangle)$. Now, by Lemma 4.2, $\mathcal{O}^\triangle$ induces a totally cyclic orientation $\mathcal{O}$ on $G(P)$. ∂_2^T tells us how this dual construction is set up. If we go from one region of the graphic hyperplane arrangement on $G(P^\triangle)$ to another, we have a flip of the orientation of the corresponding edge and obtain a new orientation $(\mathcal{O}^\triangle)'$.

This induces a flip of the orientation of its dual edge in $G(P)$. This way we obtain a new orientation $\mathcal{O}'$ on $G(P)$ which is also totally cyclic since $(\mathcal{O}^\triangle)'$ is still an acyclic orientation on $G(P^\triangle)$, and an orientation on $G(P^\triangle)$ is acyclic if and only if it induces a totally cyclic orientation on $G(P)$ by Lemma 4.2. By considering all regions of this hyperplane arrangement, we obtain all acyclic orientations on $G(P^\triangle)$. Thus, the regions of this hyperplane arrangement are in one-to-one correspondence with the totally cyclic orientations on $G(P)$. $\square$

This is demonstrated by the example shown in Figure 16. On the left side of Figure 16 there are a tetrahedron T on the vertex set $\{1, 2, 3, 4\}$ with a totally cyclic orientation on its set of edges and its dual $T^\triangle$ with an acyclic orientation on its set of edges $E^\triangle$ (orange). On the right side there is the same ensemble with the difference that we obtained a new acyclic orientation on $E^\triangle$ by flipping the orientation of the edge $14^\triangle \in E^\triangle$. As described before, flipping the orientation of the edge $14 \in E$, which is dual to $14^\triangle$, induces, by duality, a new totally cyclic orientation on $G(T)$.

The following lemma gives us the second defining property of the cographic hyperplane arrangement.
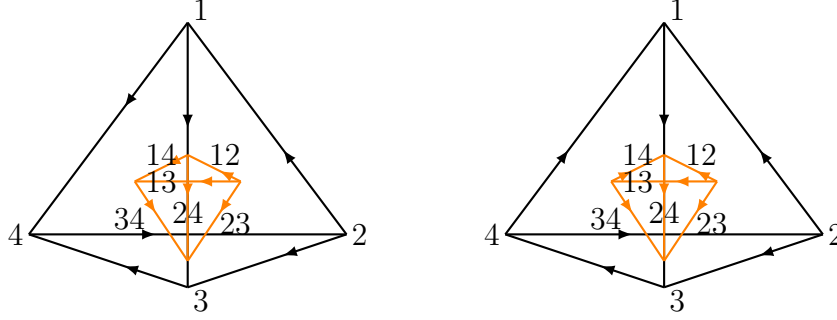
Figure 16: Flipping the orientation of an edge.

Lemma 4.4. *A subset of the set of columns of $\partial_1^\triangle = \partial_2^T$ is linearly independent if and only if it induces a complement of a spanning set on $G(P)$.*

Proof. Choose a subset S of the columns of $\partial_1^\triangle$. By Remark 4.6 and Proposition 3.1 the columns of S are linearly independent if and only if they induce a forest on $G(P^\triangle)$. By Lemma 4.1, forests on $G(P^\triangle)$ induce complements of spanning sets on $G(P)$ and vice versa. $\qquad\square$

Thus, the columns of $\partial_1^\triangle = \partial_2^T$ satisfy exactly the desired properties. Therefore, the columns of ∂_2^T are exactly the normal vectors of the cographic hyperplane arrangement of $G(P)$.

Example 4.8. Consider again the tetrahedron T and its dual $T^\triangle$ of Example 4.6, which are shown by Figure 14.

Orient its faces by the base orientation induced by the labelings shown in Figure 14. Then we obtain the cellular chain complex

$$C_2 \xrightarrow{\quad\partial_2\quad} C_1 \xrightarrow{\quad\partial_1\quad} C_0$$

$$
\begin{array}{c}
 \\
12 \\
13 \\
14 \\
23 \\
24 \\
34
\end{array}
\begin{array}{cccc}
123 & 124 & 134 & 234 \\
\left(\begin{array}{cccc}
1 & -1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1 \\
0 & -1 & 0 & 1 \\
0 & 0 & 1 & -1
\end{array}\right)
\end{array}
\qquad
\begin{array}{c}
 \\
1 \\
2 \\
3 \\
4
\end{array}
\begin{array}{cccccc}
12 & 13 & 14 & 23 & 24 & 34 \\
\left(\begin{array}{cccccc}
-1 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{array}\right)
\end{array}
$$

for T and for its dual $T^\triangle$ we obtain

$$C_0^\triangle \xleftarrow{\quad\partial_1^\triangle\quad} C_1^\triangle \xleftarrow{\quad\partial_2^\triangle\quad} C_2^\triangle$$

$$
\begin{array}{c}
 \\
123 \\
124 \\
134 \\
234
\end{array}
\begin{array}{cccccc}
12 & 13 & 14 & 23 & 24 & 34 \\
\left(\begin{array}{cccccc}
1 & -1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & -1
\end{array}\right)
\end{array}
\qquad
\begin{array}{c}
 \\
12 \\
13 \\
14 \\
23 \\
24 \\
34
\end{array}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\left(\begin{array}{cccc}
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 1
\end{array}\right)
\end{array}
$$

as its cellular chain complex.

The matrix $\partial_2^T = \partial_1^\triangle$ is the vertex-edge incidence matrix of $G(T^\triangle)$. Thus, the columns of $\partial_2^T = \partial_1^\triangle$ are not only the normal vectors of the cographic hyperplane arrangement of $G(T)$, but also the normal vectors of the graphic hyperplane arrangement of $G(T^\triangle)$, i.e., the cographic hyperplane arrangement of $G(T)$ is the graphic hyperplane arrangement of $G(T^\triangle)$. We observe that, as expected, a subset of the set of column vectors of $\partial_2^T = \partial_1^\triangle$ is linearly independent if and only if it induces a complement of a spanning set of $G(T)$ resp. a forest on $G(T^\triangle)$. For example, consider the linearly independent columns of $\partial_2^T = \partial_1^\triangle$ corresponding to the edges $23, 24$ and 34 of $G(T)$ resp. of $G(T^\triangle)$. They induce the complement of a spanning forest in $G(T)$ drawn orange resp. a spanning forest on $G(T^\triangle)$ drawn green in Figure 17.
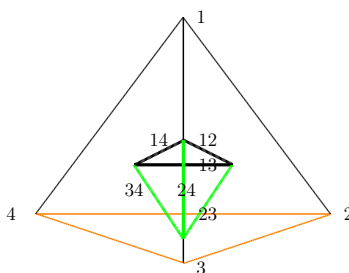
Figure 17: The columns corresponding to the edges $23, 24$ and 34 induce a complement of a spanning forest in $G(T)$.

4.3 Cyclic polytopes

What do we do if our graph G is not Steinitz? Can we extend the above approach to general simple, connected and bridgeless graphs? In the following chapters we are going to develop methods to generalize the construction that we have seen for the case that G is Steinitz. Therefore, we will use cyclic polytopes. The aim of this chapter is to introduce cyclic polytopes and discuss some of their properties, which we will need later in order to construct cographic hyperplane arrangements of those graphs and to compute their Ehrhart polynomials. In particular, we are interested in the properties of cyclic 4-polytopes. The following and other facts about cyclic polytopes can be found, e.g., in [9] or [23].

Definition 4.12. The *moment curve* in $\mathbb{R}^d$ is defined by

$$x \colon \mathbb{R} \to \mathbb{R}^d, \quad t \mapsto x(t) := \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix} \in \mathbb{R}^d.$$

The *cyclic polytope* $C_d(t_1, ..., t_n)$ is the convex hull

$$C_d(t_1, ..., t_n) := \mathrm{conv}\{x(t_1), x(t_2), ..., x(t_n)\}$$

of $n > d$ distinct points $x(t_i)$, with $t_1 < t_2 < \cdots < t_n$, on the moment curve.

Remark 4.7. The combinatorics of the cyclic polytope $C_d(t_1, ..., t_n)$ does not depend on the specific choice of the parameters t_i. This justifies denoting the polytope by $C_d(n)$ and calling it "the" cyclic d-polytope with n vertices.

Example 4.9. Figure 18 shows how the cyclic polytope $C_3(6)$ arises from six points $t_1, ..., t_6$ plugged into the moment curve.
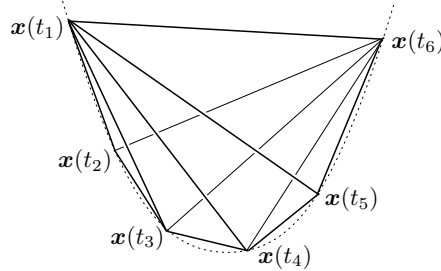
Figure 18: The cyclic polytope $C_3(6)$. Source: [23, p.11]

Definition 4.13. A d-polytope is said to be *k-neighborly* if and only if any subset of k or less vertices forms a face of P.

Theorem 4.3. (See, for example, [23, Theorem 0.7]). *Let $n > d \geq 2$. Choose real parameters $t_1 < t_2 < \cdots < t_n$. Then $C_d(n) := \mathrm{conv}\{x(t_1), ..., x(t_n)\}$ is a simplicial d-polytope. A d-subset $S \subseteq [n]$ forms a facet of $C_d(n)$ if and only if the following "evenness condition", known as Gale's evenness condition, is satisfied. If $i < j$ are not in S, then the number of $k \in S$ between i and j is even:*

$$2 \mid \#\{k \in S \mid i < k < j \text{ for } i, j \notin S\}.$$

Remark 4.8. It follows from Gale's evenness criterion that cyclic polytopes are $\lfloor \frac{d}{2} \rfloor$-neighborly.

Therefore, cyclic polytopes give examples for 2-neighborly 4-polytopes with any number $n \in \mathbb{N}$ of vertices. With Gale's evenness criterion we can determine the whole combinatorics of cyclic polytopes.

Remark 4.9. The number $f_1(C_4(n))$ of edges of the cyclic polytope $C_4(n)$ is

$$f_1(C_4(n)) = \binom{n}{2}$$

since $C_4(n)$ is 2-neighborly, i.e., every vertex is adjacent to every other vertex.

Proposition 4.1. *The f-vector of $C_4(n)$ is $\left(1, n, \binom{n}{2}, 2(\binom{n}{2} - n), \binom{n}{2} - n, 1\right)$.*

Proof. By Remark 4.9, the number of edges of $C_4(n)$ is $\binom{n}{2}$. Denote the number of ridges and facets of $C_4(n)$ by f_2 resp. f_3. The Dehn-Sommerville equations 2.3 give us the following relation:

$$\sum_{j=2}^{3} \binom{j+1}{d-2+1} f_j = (-1)^3 f_2.$$

This is equivalent to

$$f_2 = 2f_3.$$

Using the Euler-Poincaré equation, we obtain

$$n - \binom{n}{2} + f_2 - f_3 = n - \binom{n}{2} + 2f_3 - f_3 = 0,$$

which finally gives us the desired result. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.10. Since cyclic polytopes are simplicial, their duals are simple polytopes.

Remark 4.11. Since cyclic polytopes are simplicial by Theorem 4.3, their ridges are triangles, i.e., every ridge contains exactly three edges. That implies that the signed edge-ridge incidence matrices of cyclic polytopes have three non-zero entries per column that are either 1 or -1.

In the following chapters we will study the cographic hyperplane arrangement of a complete graph using the signed edge-ridge incidence matrices of cyclic polytopes. In this chapter we want to investigate, how, in general, the signed edge-ridge incidence matrices of cyclic 4-polytopes look like. Therefore, we use Gale's evenness criterion as it determines the combinatorics of cyclic polytopes.[2]

Label the vertices of $C_4(n)$ for $n \geq 6$. Consider the vertex figure of any chosen vertex. W.l.o.g. choose vertex 1 since the vertex figures are all the same up to symmetry. Now determine how many facets are containing any edge incident to 1. The edges 12 and $1n$ are contained in $(n-2)$ many facets by Gale's evenness criterion. These are all facets with the vertices $1, 2, i, j$ resp. $1, k, l, n$ where $2 < i, j \leq n$ and $1 < k, l < n$ where i, j resp. k, l are consecutive numbers since these are exactly the cases for which Gale's criterion is satisfied. Thus, 12 and $1n$ are dual to two $(n-2)$-gons, and these two $(n-2)$-gons both share a common edge since 12 and $1n$ are both contained in the triangle $12n$.

The edges 13 and $1(n-1)$ are both contained only in the three facets $1, 3, 4, n$ or $1, 2, 3, n$ or $1, 2, 3, 4$ resp. $1, (n-2), (n-1), n$ or $1, 2, (n-1), n$ or $1, 2, (n-2), (n-1)$. Thus, they are dual to triangles. All other edges $1l$, $l \in \{4, ..., n-2\}$ are contained in exactly four facets that are of the forms $1, (l-1), l, n$ or $1, l, (l+1), n$ or $1, 2, (l-1), l$ or $1, 2, l, (l+1)$. Thus, they are all dual to quadrilaterals. In total, we know that each facet of $C_4(n)^\triangle$ has as facets two $(n-2)$-gons, $(n-5)$

quadrilaterals and two triangles. Applying duality, we obtain that for each vertex of $C_4(n)$ there are two incident edges contained in exactly $(n-2)$ ridges, $(n-5)$ incident edges contained in exactly four ridges and two incident edges contained in exactly three ridges.

Thus, the transpose ∂_2^T of the matrix representation of the second boundary map of the cellular chain complex associated to the CW-complex given by the face-structure of $C_4(n)$ has n columns with $(n-2)$ ± 1 entries and the rest of the entries are zero, $\frac{n(n-5)}{2}$ columns with four ± 1 and rest zero and n columns with three ± 1 and rest zero. Besides, every row of ∂_2^T has three non-zero entries which are ± 1 since every ridge is a triangle.

For $n = 5$ we have the 4-simplex whose edges are all contained in exactly three ridges. Thus, all rows and columns of the matrix representation of ∂_2^T each have three ± 1 entries and the rest is 0.

4.4 Cographic hyperplane arrangements for graphs of 4-polytopes

In this chapter we want to construct the cographic hyperplane arrangement for the graph of a given 4-polytope P. We assume throughout this chapter that P is a 4-polytope with 0 in its interior.

Consider the cellular chain complexes associated to the CW-complex given by the face-structure of P and $P^\triangle$. By labeling the facets of P by $1, ..., m$ and the vertices of P by $1, ..., n$ we obtain a base orientation that induces the orientations given by the Definitions 4.2, 4.4 and 4.5. Then, the cellular chain complexes associated to the CW-complex given by the face-structures of P and $P^\triangle$ are of the following form (see Chapter 4.1).

$$
\begin{array}{ccccccc}
C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\
\downarrow{f_3} & & \downarrow{f_2} & & \downarrow{f_1} & & \downarrow{f_0} \\
C_0^\triangle & \xleftarrow{\partial_3^T = \partial_1^\triangle} & C_1^\triangle & \xleftarrow{\partial_2^T = \partial_2^\triangle} & C_2^\triangle & \xleftarrow{\partial_1^T = \partial_3^\triangle} & C_3^\triangle
\end{array}
$$

Lemma 4.5. *The rank of ∂_2, and thus the rank of $\partial_2^\triangle = \partial_2^T$, is $|E| - |V| + 1$.*

Proof. Since the row rank of a matrix equals its column rank, the rank of $\partial_2^\triangle = \partial_2^T$ equals the rank of ∂_2. The kernel of ∂_1 gives exactly the relations of the hyperplane description of the flow space $\mathcal{F}_{G(P)}$ of the graph $G(P)$ of P. Thus, the dimension of the kernel of ∂_1 equals $|E| - |V| + 1$. Since we have a cellular chain complex, we know that $\text{im}(\partial_2) \subseteq \ker(\partial_1)$ and thus $\text{rank}(\partial_2) \leq |E| - |V| + 1$. On the other hand, we obtain the cycle basis of $G(P)$ by the ridges of P as follows: Select a subset S of $\xi(G(P)) = \binom{|V|}{2} - |V| + 1$ ridges from the $2(\binom{|V|}{2} - |V|)$ ridges of P such that each vertex is contained in at least one ridge. Choose a spanning forest T of $G(P)$ out of the edges of the ridges contained in S. We can choose such a

spanning forest since we have chosen S in such a way that for each vertex v of P there are edges incident to v and contained in S. Then the rows of $\partial_2^\triangle = \partial_2^T$ corresponding to the ridges in S are exactly the vectors of the fundamental cycles of $G(P)$ with respect to T since for each fundamental cycle/ ridge r they have the entry 1 for an edge that is contained in r and oriented from the smaller to the greater endpoint, -1 for an edge that is contained in r and oriented from the greater to the smaller endpoint and 0 for an edge not contained in r. This is exactly how we constructed the cycle basis of $G(P)$ with respect to a spanning forest T in the proof of Proposition 2. $\square$

Corollary 4.1. *By Lemma 4.5, the rows of $\partial_2^\triangle = \partial_2^T$ are an alternative generating system of the flow space where we have given a vector for each edge of $G(P)$. Thus, the columns of $\partial_2^\triangle = \partial_2^T$ are the normal vectors of the hyperplane arrangement $\mathcal{F} \cap \mathcal{A}$ where $\mathcal{A}$ is the arrangement of coordinate hyperplanes in $\mathbb{R}^E$.*

Corollary 4.2. *Corollary 4.1 implies the Euler-Poincaré formula (2.4) for 4-polytopes.*

Proof. Corollary 4.1 implies that

$$\dim(\operatorname{im}(\partial_2)) = \dim(\ker(\partial_1)) = |E| - |V| + 1.$$

On the other hand, we have the same statement for its dual:

$$\dim(\operatorname{im}(\partial_2^\triangle)) = \dim(\ker(\partial_1^\triangle)) = |E^\triangle| - |V^\triangle| + 1 = |R| - |F| + 1.$$

Now it follows from the fact that the row rank of a matrix equals the column rank of a matrix that

$$|E| - |V| + 1 = |R| - |F| + 1.$$

This equals

$$|V| - |E| + |R| - |F| = 0 = 1 - (-1)^4,$$

the well-known form of the Euler-Poincaré formula for 4-polytopes. $\square$

The main goal of this chapter is to prove the following theorem.

Theorem 4.4. *The hyperplane arrangement whose normal vectors are the column vectors of $\partial_2^T = \partial_2^\triangle$ is the cographic hyperplane arrangement of $G(P)$.*

For the proof of Theorem 4.4, we need the following Lemmas.

Lemma 4.6. *Let $\mathcal{R}^\triangle$ be a subset of the set of ridges of $P^\triangle$. Then $\mathcal{R}^\triangle$ is a ridge forest on $P^\triangle$ if and only if it induces by duality a complement of a spanning set $\mathcal{T}$ on the set of edges of P.*

Proof. Suppose $\mathcal{R}^\triangle$ contains a ridge cycle $C^\triangle$. Then, by definition, the ridge cycle separates the boundary of $P^\triangle$ into two connected pieces each of which contains at least one facet of $P^\triangle$. Thus, $C^\triangle$ induces a cut set on the set of edges of P since the facets of $P^\triangle$ correspond to vertices of P and the ridges of $P^\triangle$ to edges of P. Therefore, its complement cannot be spanning.

Suppose $\mathcal{T}$ is not a complement of a spanning set on $G(P)$. Consider the complement $S := E \setminus \mathcal{T}$, which is by definition not a spanning set on $G(P)$. Then there exists a vertex $v \in V$ that is not incident to any edge of S. This implies that all ridges of the facet dual to v are contained in $\mathcal{R}^\triangle$. Thus, $\mathcal{R}^\triangle$ contains a ridge cycle. $\qquad\square$

Lemma 4.7. *A selection of the columns of $\partial_2^\triangle$ is linearly independent if and only if the corresponding set of ridges $\mathcal{R}^\triangle$ of $P^\triangle$ is a ridge forest on $P^\triangle$.*

Proof. Suppose $\mathcal{R}^\triangle$ contains an oriented ridge cycle C. By reorientation we can assume that each closed sequence of ridges of C, i.e., a sequence of ridges without ridge repetition and beginning and ending with the same ridge, is oriented in the same direction.

Then for each edge contained in C, there is an even number of ridges of C containing this edge. Furthermore, by the definition of a ridge cycle, these ridges are contained in a closed sequence of ridges that are oriented in the same direction. Consider such a sequence S and an edge $e^\triangle$ that is the intersection of two ridges contained in S.

We observe that for the position $e^\triangle$ in the column vectors of $\partial_2^\triangle$ corresponding to S there is one non-zero entry each for the two ridges that are both containing $e^\triangle$ and that are contained in S. All other entries for the position $e^\triangle$ are zero. Thus, by adding up all column vectors of $\partial_2^\triangle$ corresponding to S, for each entry $e^\triangle$ the two non-zero entries add up to zero since one entry is one and the other minus one, because both ridges are oriented in the same direction. Since this happens to all entries of all vectors corresponding to the whole ridge cycle by adding up in all directions, eventually we obtain the zero vector.

For the other direction, we argue by induction. For a single ridge the statement is clear. By Remark 4.14 there always exists a ridge leaf. Now we delete the ridge $r^\triangle$ incident to the ridge leaf $e^\triangle$ and apply the induction hypothesis. The edge-ridge incidence vector of $r^\triangle$ was linearly independent to the other vectors since it was the only vector having a non-zero entry for $e^\triangle$. We can proceed by induction since by deleting $r^\triangle$ we do not create a new cycle and, therefore, there has to exist a further ridge leaf. Furthermore, we are decreasing for some edges the number of ridges containing them. $\qquad\square$

Corollary 4.3. *A subset of the set of columns of $\partial_2^\triangle = \partial_2^T$ is linearly independent if and only if it induces a complement of a spanning set on the set of edges of P.*

Proof. A subset of column vectors of $\partial_2^\triangle = \partial_2^T$ is by Lemma 4.7 linearly independent if and only if it induces a ridge forest on $P^\triangle$. By Lemma 4.6, a set of ridges

of $P^\triangle$ is a ridge forest on $P^\triangle$ if and only if it induces by duality a complement of a spanning set $\mathcal{T}$ on the set of edges of P. $\qquad\square$

Lemma 4.8. *An orientation $\mathcal{O}$ on the set of edges of P is totally cyclic if and only if $\mathcal{O}^\triangle$ is acyclic on the set of ridges of $P^\triangle$.*

Proof. Suppose first that $\mathcal{O}^\triangle$ induces an oriented ridge cycle $C^\triangle$ consisting of the ridges $A_1^\triangle, ..., A_m^\triangle$. Then they are either all oriented in one of the two possible directions or in the other. By definition, the edges dual to $A_1^\triangle, ..., A_m^\triangle$ are all oriented towards the vertices dual to the facets contained in the oriented ridge cycle or they are all oriented away from these vertices. In either case, they cannot belong to an oriented cycle of $G(P)$ because the edges dual to the oriented ridge cycle are an oriented cut set. Therefore, $\mathcal{O}$ is not totally cyclic.

Conversely, suppose $\mathcal{O}^\triangle$ is acyclic and let A be an edge of P. Let $F^\triangle$ be one of the two facets of $P^\triangle$ containing the ridge $A^\triangle$. We can choose $F^\triangle$ this way since each ridge of $P^\triangle$ is contained in exactly two facets because this is the dual translation of the fact that each edge contains exactly two vertices. Since $\mathcal{O}^\triangle$ is acyclic, there must be some edge $B^\triangle$ of $P^\triangle$ whose direction is opposite to that of $A^\triangle$ along $F^\triangle$. This implies that A and B have the same direction. This is the case since, by definition of the dual orientation, either both of them are oriented towards the interior or towards the exterior. Furthermore, both A and B are adjacent since they are both incident to the vertex dual to $F^\triangle$. In the same way, considering, e.g., the facet to the "right" of $B^\triangle$ we can find an edge D such that A, B and D are all oriented in the same direction and so on for all directions. Continuing this process, eventually we find a directed cycle C in $\mathcal{O}$ containing A. This process can be applied to each ridge and its dual edge since the directions are induced by duality in each step. Eventually, we obtain a totally cyclic orientation on the set of edges of P. $\qquad\square$

Lemma 4.9. *The column vectors of $\partial_2^\triangle = \partial_2^T$ are the normal vectors of a hyperplane arrangement whose regions are in one-to-one correspondence with the acyclic orientations on the set of ridges of $P^\triangle$.*

Proof. There is a natural association of an acyclic orientation on the set of ridges of $P^\triangle$ with a particular region of the described arrangement. The hyperplane associated with the ridge $r^\triangle$ of $P^\triangle$ breaks up the ambient space into two open half-spaces defined by $r^\triangle$ being oriented positively, i.e., in mathematically positive direction from the facet with smaller label to the one with greater label or negatively, i.e., the other way around. Thus, orienting a ridge selects a half-space, and an orientation on the set of ridges of $P^\triangle$ can be associated with the intersection of this collection of half-spaces. The orientation of each ridge gives a labeling $e_i^\triangle, e_j^\triangle, e_k^\triangle$, $i < j < k$, along its edges in direction of the orientation. We are labeling along those sequences S defined by the proof of Lemma 4.7. Every ridge gets labeled this way since every ridge is contained in an oriented ridge cycle. Whenever an edge was already labeled, we keep this labeling and use it

to determine the others. Thus, we choose only one edge with which the labeling starts. Then the one-to-one correspondence is given by

$$\mathbf{R}(\mathcal{O}^\triangle) := \{x \in \mathbb{R}^{E^\triangle} \mid x_i < x_j < x_k \text{ if } r^\triangle \text{ is} \tag{9}$$

$$\text{oriented from } e_i^\triangle \text{ to } e_j^\triangle \text{ to } e_k^\triangle \text{ in } \mathcal{O}^\triangle\}$$

for each acyclic orientation $\mathcal{O}^\triangle$ on the set of ridges of $P^\triangle$ and conversely,

$$\mathcal{O}^\triangle(\mathbf{R}) := \{r^\triangle \text{ is oriented from } e_i^\triangle \text{ to } e_j^\triangle \text{ to } e_k^\triangle \mid r^\triangle \in R \tag{10}$$

$$\text{and } x_i < x_j < x_k \text{ if } x \in \mathbf{R}\}$$

for each region $\mathbf{R}$. For each non-empty region $\mathbf{R}$, any $x \in \mathbf{R}$ defines an orientation $\mathcal{O}^\triangle(x)$ by (10). Suppose that $\mathcal{O}^\triangle$ is not acyclic. Then there exists an oriented ridge cycle C giving a chain of strict inequalities via the orienting edges of $E^\triangle$ contained in C, w.l.o.g. $x_1 < x_2 < \cdots < x_{l-1} < x_l < x_1$ since, by the definition of an oriented ridge cycle, each sequence of ridges of C is oriented in the same direction. Consequently, $\mathbf{R}$ is empty. Thus, $\mathcal{O}^\triangle$ is acyclic. This property holds for all $x \in \mathbf{R}$: If we consider any $y \in \mathbf{R}$, we obtain y from x by moving in $\mathbf{R}$ without crossing any hyperplane of our hyperplane arrangement. Thus, no constraint defining $\mathcal{O}^\triangle$ gets changed for any point in $\mathbf{R}$ and therefore, $\mathcal{O}^\triangle$ is a well-defined acyclic orientation on $\mathbf{R}$.

Conversely, given any acyclic orientation $\mathcal{O}^\triangle$ on the set of ridges of $P^\triangle$, we will show that $\mathbf{R}(\mathcal{O}^\triangle)$ is non-empty. If follows from the previous direction that $\mathbf{R}(\mathcal{O}^\triangle)$ is a well-defined region of our hyperplane arrangement. We define a partial ordering by $x_i \leq_{\mathcal{O}^\triangle} x_j$ if and only if $r^\triangle$ has the orientation from e_i to e_j in $\mathcal{O}^\triangle$ (extended by transitivity). In the next step, we extend this partial ordering to a total ordering $x_i <_{\mathcal{O}^\triangle} x_j$ if and only if $r^\triangle$ has the orientation from e_i to e_j in $\mathcal{O}^\triangle$. Then any $x \in \mathbb{R}^{E^\triangle}$ whose coordinates get ordered by this total ordering belongs to $\mathbf{R}(\mathcal{O}^\triangle)$. We have shown $\mathbf{R}(\mathcal{O}^\triangle(\mathbf{R})) = \mathbf{R}$ and $\mathcal{O}^\triangle(\mathbf{R}(\mathcal{O}^\triangle)) = \mathcal{O}^\triangle$. $\square$

Corollary 4.4. *The regions of the hyperplane arrangement whose normal vectors are the column vectors of ∂_2^T are in one-to-one correspondence with the totally cyclic orientations on P.*

Proof. By Lemma 4.9, every acyclic orientation $\mathcal{O}^\triangle$ on the set of ridges of $P^\triangle$ is in one-to-one correspondence with a region of the hyperplane arrangement whose normal vectors are the column vectors of $\partial_2^\triangle$.

Now, by Lemma 4.8, $\mathcal{O}^\triangle$ induces a totally cyclic orientation $\mathcal{O}$ on the set of edges of P. Summarizing, the construction is set up as follows. For a ridge $r^\triangle \in P^\triangle$ take the edge e dual to $r^\triangle$ in P and orient e as the edge dual to $r^\triangle$. If we go from one region of the hyperplane arrangement to another, we have to flip the orientation of $r^\triangle$.

This induces to flip the orientation of its dual edge e in P and we still have a totally cyclic orientation $\mathcal{O}'$ in P since the new orientation $(\mathcal{O}^\triangle)'$ is still an acyclic orientation by Lemma 4.9. This way, we obtain a one-to-one correspondence between all acyclic orientations on the ridges of $P^\triangle$ and all totally cyclic orientations on the edges of P. $\square$

Finally, Theorem 4.4 follows from Corollary 4.4 and Corollary 4.3.

4.5 Cographic hyperplane arrangements for general simple and bridgeless graphs

In the previous chapters we discussed how to set up the cographic hyperplane arrangement for the graph of a given 3- or 4-polytope. Now we want to turn around this construction. For a given simple and bridgeless graph $G = (V, E)$ we want to use a polytope P, whose graph is G, to set up the cographic hyperplane arrangement of G. If G is not connected, we can set up the following construction for each of its components to obtain the cographic hyperplane arrangement for the whole graph since the edge sets of the components of G are each disjoint.

So, we can, for simplicity, assume that $G = (V, E)$ is connected and, as said before, bridgeless. However, a problem that arises is that there are graphs G for which there does not exist any polytope whose graph is G, e.g., graphs that are not 3-connected. Another problem is the following.

Remark 4.12. There is in general not a unique polytope whose graph is that of a given polytope whenever there is such a polytope at all. For example, we could choose $C_d(n)$ for any $d \geq 5$ to be a polytope whose graph is the complete graph K_n. This behaviour generalizes to the concept of k-equivalence of polytopes, see [9, Chapter 12]. A d-polytope P is k-equivalent to a d'-polytope P' if the k-skeleta of P and P' are combinatorially equivalent.

However, we can make use of the following observation.

Remark 4.13. By the definition of k-neighborly, the complete graph K_n is the graph of a 2-neighborly polytope P with n vertices since 2-neighborly means that every selection of two vertices forms an edge. This implies that every two vertices of $G(P)$ are adjacent, i.e., $G(P)$ is complete.

So, the idea is to embed G into an appropriate complete graph by adding edges. Then we want to make use of Remark 4.13. Therefore, we need to distinguish the following cases:

1) $|V| = 3$. The only bridgeless graph with three vertices is the triangle. The triangle consists of one cycle. The complement of each edge is a spanning forest of the triangle. Thus, the cographic hyperplane arrangement of the triangle is one-dimensional, having one hyperplane, which is the origin, for each of the three edges and normal vector 1. Therefore, we obtain two regions corresponding to the two totally cyclic orientations of the triangle.

2) $|V| = 4$. Embed G into the complete graph K_4. Then we can consider K_4 as the graph of the 3-dimensional cyclic polytope $C_3(4)$ which is combinatorially equivalent to the tetrahedron. Now we can apply Chapter 4.2: consider the cellular chain complexes of $C_3(4)$ and its dual $C_3(4)^\triangle$ and set up the cographic hyperplane arrangement for $C_3(4)$. Afterwards, erase the added edges as described below.

3) $|V| \geq 5$. Embed G into the complete $K_{|V|}$. Then we can consider $K_{|V|}$ as the graph of the 4-dimensional cyclic polytope $C_4(|V|)$ since $C_4(|V|)$ is 2-neighborly. Now we can apply Chapter 4.4: consider the cellular chain complexes of $C_4(|V|)$ and its dual $C_4(|V|)^\triangle$ and set up the cographic hyperplane arrangement for $C_4(|V|)$. Afterwards, erase the added edges as described below.

Let $|V| \geq 5$. We want to erase the edges that we added when embedding $G = (V, E)$ into $K_{|V|}$. At this point, we exploit again the fact that we can consider re-orientations in terms of linear algebra. Consider an added edge $e \in K_{|V|} \setminus G$. The setminus is considered as an operation on the set of edges of $K_{|V|}$. We want to erase the column belonging to e from the matrix representation of ∂_2^T. Therefore, we consider the ridges containing e, i.e., the rows of ∂_2^T having a 1 or -1 entry on the position of e. Since all non-zero entries are just ± 1, we can just add up pairs of these rows or their negatives to obtain a zero column. This way we also obtain the new basis vectors corresponding to the cycles of the new cycle basis that we obtain when erasing an edge.

Geometrically, this translates into combining each two of the ridges or reorientations of them such that for e one of the possibly re-oriented ridges is oriented in the opposite direction of the other. Figure 19 shows how the edge 13 is erased by uniting the left with the middle triangle that both contain 13 and how the basis vector of the flow space of the graph not containing 13 corresponding to the new cycle is constructed.
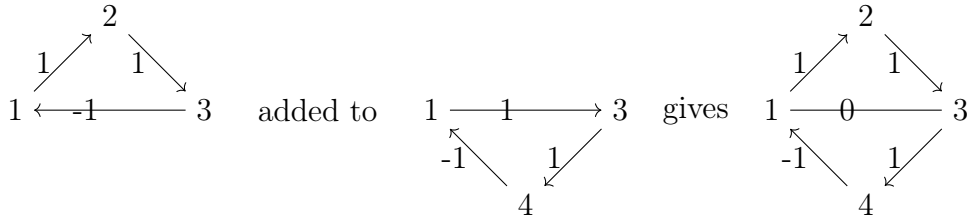

Figure 19: Erasing the edge 13.

Every edge is contained in at least three ridges since this is the dual to the statement that the ridges of a 4-polytope have at least three edges. As long as there are at least two cycles that we obtained from ridges containing the edge we want to erase, this method works. This is the case since we assumed G to be bridgeless and we start with at least three cycles containing each edge of G. We will use the following notation.

Definition 4.14. Let P be a 4-polytope. A cycle obtained from a sequence of ridges $r_1, ..., r_k$ of P by erasing edges contained in $r_1, ..., r_k$ is called a *contracted ridge*.

When we have more than one edge such that all these edges are all contained only in the same (contracted) ridges and some of them are supposed to be erased, but not all, we can only erase all of them or none. However, this would imply that after erasing these edges there would be at least one bridge in G.

If we unite (contracted) ridges having more than one edge in common, we erase all of them to obtain the new (contracted) ridge and thus, we obtain no entry not equal to ± 1 or 0. We can observe that we erase only columns of $\partial_2^T = \partial_2^\triangle$. Thus, when combining (contracted) ridges, we add only ± 1 with 0 or 0 with 0. Therefore, we do not obtain entries other than ± 1 or 0.

In the case that $|V| = 4$, we do not have the property that each edge is contained in at least three facets. However, there are just two possible classes of graphs up to isomorphisms with four vertices that are connected, simple and bridgeless. The first class of graphs are those obtained from the complete graph K_4 by deleting one edge e. In this case we combine the two facets containing e to erase e from the facet-edge incidence matrix. This way we also obtain the basis vectors (of the flow spaces of these graphs) that are corresponding to the (contracted) ridges of the new cycle basis that we obtain when erasing an edge.

The second class is obtained from K_4 by deleting one edge e and the edge e' with the property that both edges have no common incident vertices. Since we have only four vertices, the choice of e' is unique when e was given. Because e and e' have no common vertices, and all facets are triangles, they are contained in two different pairs of facets. Therefore, we can combine the two facets containing e to erase e from the facet-edge incidence matrix without affecting the facets containing e'. Afterwards, we can do the same for e'. This way we also obtain the basis vectors (of the flow spaces of these graphs), that are corresponding to the (contracted) ridges of the new cycle basis that we obtain when erasing an edge.

There are no further connected, simple and bridgeless graphs with four vertices. If we would like to delete more than two edges or two edges that are different from the ones of the second class, we would have to delete an edge of a vertex v that is incident to an edge that we also want to delete. Since the degree of each vertex of the K_4 is three, that means that we would decrease the degree of v to one. Hence, the last edge incident to v would be a bridge.

Definition 4.15. Let $G = (V, E)$ be a simple and bridgeless graph with $|V| \geq 4$. Let ∂_2^T be the transposed of the second boundary map of the cellular chain complex of $C_4(|V|)$ if $|V| \geq 5$ or of $C_3(4)$ if $|V| = 4$. Then let $(\partial_2^T)'$ be the matrix obtained from ∂_2^T by erasing the edges of $K_{|V|} \setminus G$ or if G is a complete graph then $(\partial_2^T)' = (\partial_2^T)$ and call the column vectors of $(\partial_2^T)'$ the normal vectors of the *cographic hyperplane arrangement of G*.

In Remark 4.17 we will argue why calling $(\partial_2^T)'$ the cographic hyperplane arrangement of G is justified.

Definition 4.16. A *contracted ridge forest* is a collection of contracted ridges arising from deleting edges from a ridge forest.

As argued above, the vectors of $((\partial_2^T)')^T$ corresponding to contracted ridge forests stay linearly independent.

Definition 4.17. Let $P^{\triangle}$ be a 4-polytope and let $\mathcal{R}^{\triangle}$ be a contracted ridge forest. An edge $e^{\triangle}$ is called a *contracted ridge leaf* of $\mathcal{R}^{\triangle}$ if and only if $e^{\triangle}$ is contained in exactly one contracted ridge of $\mathcal{R}^{\triangle}$.

Remark 4.14. For every ridge forest and every contracted ridge forest there exists at least one (contracted) ridge leaf since otherwise we would have a ridge cycle.

Example 4.10. As an example consider the graph G given by Figure 20.
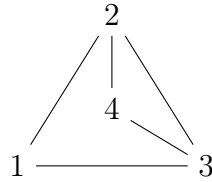

Figure 20: The complete graph K_4 without the edge 14.

In a first step, we embed G into the K_4 that we know from Figure 1 and set up the cographic hyperplane arrangement for K_4 as in Example 4.8. To re-obtain G, we have to erase the edge 14 which is contained in the facets 124 and 134. Therefore, we combine these two facets and obtain the new cycle 1234 (green) as shown by Figure 21.
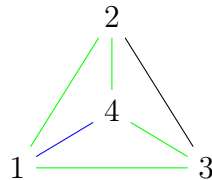

Figure 21: We combine the facets 124 and 134 to erase the edge 14 (blue).

In terms of matrices, we obtain the following matrix which gives us as its columns the cographic hyperplane arrangement of G:

$$
\begin{array}{c}
\\
123 \\
1234 \\
234
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
12 & 13 & 14 & 23 & 24 & 34
\end{array} \\
\left(
\begin{array}{cccccc}
-1 & -1 & 0 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{array}
\right)
\end{array}
$$

We see that 14 is not contained in any cycle anymore, i.e., we do not consider it anymore. We see that the columns are linearly independent if and only if they induce a complement of a spanning set. For example, the columns corresponding to the edges 12 and 13 are not linearly independent and their complement is not spanning.

Eventually, if we erase the edge 23, we obtain the following cographic hyperplane arrangement for the graph arising from the K_4 from Example 4.8 without the edges 14 and 23.

$$
\begin{array}{c}
 \quad 12 \quad 13 \quad 14 \quad 23 \quad 24 \quad 34 \\
1234 \quad \left(\begin{array}{cccccc} 1 & 1 & 0 & 0 & -1 & -1 \end{array} \right)
\end{array}
$$

Corollary 4.5. *Let $G = (V, E)$ be a simple and bridgeless graph. We have seen that the rows of $(\partial_2^T)'$ are an alternative generating system of the flow space where we have given a vector for each edge of G. Thus, the columns of $(\partial_2^T)'$ are the normal vectors of the hyperplane arrangement $\mathcal{F}_G \cap \mathcal{A}$ where $\mathcal{A}$ is the arrangement of coordinate hyperplanes in $\mathbb{R}^E$.*

Remark 4.15. The part of Theorem 4.4 claiming that regions of the cographic hyperplane arrangement are in one-to-one correspondence with the totally cyclic orientations on the edges of G also follows from Greene-Zaslavsky's work, see [8] and Corollary 4.5. Greene and Zaslavsky define the cographic hyperplane arrangement of G to be the induced arrangement, $\mathcal{H}^\perp(G) = \mathcal{A}_{\mathcal{F}_G}$, and write $h(e)$ for the hyperplane corresponding to e, i.e., $\mathcal{A}_{\mathcal{F}_G} = \mathcal{F}_G \cap \mathcal{A}$. Then they show that the regions of the cographic hyperplane arrangement are in one-to-one correspondence with the totally cyclic orientations on the edges of G. Now, Corollary 4.5 states that the columns of $(\partial_2^T)'$ are the normal vectors of the hyperplane arrangement $\mathcal{F}_G \cap \mathcal{A}$.

Remark 4.16. For a graph G that is Steinitz we have shown two different ways of constructing its cographic hyperplane arrangement. In general, these hyperplane arrangements are not equal. However, they are isomorphic by the isomorphism given by the base change between the corresponding flow spaces.

Proposition 4.2. *Let $G = (V, E)$ be a graph with $|V| \geq 4$. Let ∂_2^T be the transposed of the second boundary map of the cellular chain complex of $C_4(|V|)$ if $|V| \geq 5$ or of $C_3(4)$ if $|V| = 4$ and let $(\partial_2^T)'$ be the matrix representation of the normal vectors of the cographic hyperplane arrangement of G. A subset of the set of columns of $(\partial_2^T)'$ is linearly independent if and only if it induces a complement of a spanning set on G.*

Proof. Suppose that K is a complement of a spanning set of G. Then it is also a complement of a spanning set of $K_{|V|}$ since we add only edges by embedding G into $K_{|V|}$ and thus preserve the property that the complement of K is spanning. Thus, by Corollary 4.3, K induces a linearly independent subset of the columns of the cographic hyperplane arrangement of $K_{|V|}$. Since $K \subseteq G$, the erasing procedure means for the columns corresponding to the edges of G only row operations. Row operations preserve linear independency. Thus, the column vectors of the cographic hyperplane arrangement induced by K are linearly independent.

Conversely, suppose that K is not a complement of a spanning set of G. Thus, there are edges of K that are a cut set cutting off a vertex $v \in V$, since otherwise the complement of K were spanning. Every ridge containing v contains exactly two of the edges incident to v. We distinguish two cases.

The first case is that the degree $g := \deg(v)$ of v is even. We first want to consider the case that we have given the following orientations. Embed v and its incident edges into $\mathbb{R}^2$. Go around the edges $e_0, e_1, ..., e_{g-1}$ incident to v clockwise. Then let every second edge be oriented, with respect to vertices, towards v and every other away from v. We assume for a moment that every edge e_i together with its successor $e_{(i+1) \bmod g}$ spans a (contracted) ridge r_i, for $i \in 0, 1, ..., g - 1$, which we obtained from ridges by erasing-operations. Then let these (contracted) ridges be oriented such that e_i and $e_{(i+1) \bmod g}$ are oriented positively with respect to the orientation of these ridges. Since $g := \deg(v)$ is even, at the end of this procedure, the first edge is oriented positively with respect to the orientation of the last (contracted) ridge. Thus, all entries of the (contracted) ridge-edge-incidence matrix are 1. If we use the order of the (contracted) ridges and edges given by this procedure, the (contracted) ridge-edge-incidence matrix looks as follows.

$$
\begin{array}{c}
 \\
r_0 \\
r_1 \\
r_2 \\
\vdots \\
r_{g-2} \\
r_{g-1}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
e_0 & e_1 & e_2 & \cdots & e_{g-2} & e_{g-1}
\end{array} \\
\left(
\begin{array}{cccccc}
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
 & & & \ddots & & \\
0 & 0 & 0 & \cdots & 1 & 1 \\
1 & 0 & 0 & \cdots & 0 & 1
\end{array}
\right)
\end{array}
$$

We see that the columns of this matrix are linearly dependent. Now the claim follows since we can obtain every actual (contracted) ridge C that we obtained from the erasing-operations out of ridges by combining these "dummy" (contracted) ridges. This means only row operations on the above matrix and thus preserves linear independency resp. dependency. Furthermore, re-orientations of (contracted) ridges resp. edges also preserve the linear independency resp. dependency since turning around the orientation of a (contracted) ridge means multiplying the corresponding row by -1 and turning around the orientation of an edge means multiplying the corresponding column by -1. These operations also preserve linear independency resp. dependency.

Now suppose that $g := \deg(v)$ is odd. Embed v and its incident edges into $\mathbb{R}^2$. Go around the edges $e_0, e_1, ..., e_{g-1}$ incident to v clockwise. Then let every second edge be oriented, with respect to vertices, towards v and every other away from v. We assume for a moment that every edge e_i together with its successor $e_{(i+1) \bmod g}$ spans a (contracted) ridge r_i, for $i \in 0, 1, ..., g - 1$, which we obtained from ridges by erasing-operations. Then let these (contracted) ridges be oriented such that e_i and $e_{(i+1) \bmod g}$ are oriented positively with respect to the orientation of these ridges. Since $g := \deg(v)$ is odd, at the end of this procedure, the first edge is oriented negatively with respect to the orientation of the last (contracted) ridge. Thus, all entries of the (contracted) ridge-edge-incidence matrix are $+1$, besides that entry, which is -1. If we use the order of the (contracted) ridges and edges given by this procedure, the (contracted) ridge-edge-incidence matrix

looks as follows.

$$
\begin{array}{c}
\begin{array}{cccccc}
e_0 & e_1 & e_2 & \cdots & e_{g-2} & e_{g-1}
\end{array} \\
\begin{array}{c}
r_0 \\
r_1 \\
r_2 \\
\vdots \\
r_{g-2} \\
r_{g-1}
\end{array}
\left(
\begin{array}{cccccc}
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
& & & \ddots & & \\
0 & 0 & 0 & \cdots & 1 & 1 \\
-1 & 0 & 0 & \cdots & 0 & 1
\end{array}
\right)
\end{array}
$$

We see that the columns of this matrix are linearly dependent. Now the claim follows from the same arguments as in the case that $\deg(v)$ is even. $\qquad\square$

Proposition 4.3. *Let $G = (V, E)$ be a graph with $|V| \geq 4$. Let ∂_2^T be the transposed of the second boundary map of the cellular chain complex of $C_4(|V|)$ if $|V| \geq 5$ or of $C_3(4)$ if $|V| = 4$. Let $(\partial_2^T)'$ be the matrix representation of the normal vectors of the cographic hyperplane arrangement of G. The regions of the cographic hyperplane arrangement of G are in one-to-one correspondence with the totally cyclic orientations on G.*

Proof. By Corollary 4.4 we know that the regions of the cographic hyperplane arrangement of $K_{|V|}$ are in one-to-one correspondence with the totally cyclic orientations on $K_{|V|}$. If we erase an edge e during the erasing-procedure to obtain G from $K_{|V|}$, on the one hand we unite the contracted ridges containing e. On the other hand, we unite the regions corresponding to the contracted ridges that we unite and which were separated by the hyperplane corresponding to e. Thus, we preserve the one-to-one correspondence between the regions of the cographic hyperplane arrangement and the totally cyclic orientations on G. $\qquad\square$

Remark 4.17. By Proposition 4.2 and Proposition 4.3 the cographic hyperplane arrangement, which is given by the column vectors of $(\partial_2^T)'$ of any simple and bridgeless graph G, satisfies the properties of the definition of a cographic hyperplane arrangement, see Definition 3.6. Thus, it is justified to call it the cographic hyperplane arrangement of G.

4.6 Ehrhart polynomials of flow zonotopes

In this Chapter we want to study the Ehrhart polynomials of flow zonotopes. We follow again the case distinction which we introduced in Chapter 4.5. In this chapter we assume all graphs to be bridgeless.

Definition 4.18. Let $G = (V, E)$ be a (simple and bridgeless) graph. The *flow zonotope* $\mathcal{C}_G$ of G is defined as

$$
\mathcal{C}_G := \sum_{e \in E} d_e
$$

where $d_e \coloneqq [0, x_e]$ and x_e is the column vector of the matrix representation of $(\partial_2^T)'$ corresponding to the edge $e \in E$ where we defined $(\partial_2^T)'$ for each simple and bridgeless graph by Definition 4.15.

Remark 4.18. Let $G = (V, E)$ be a simple and bridgeless graph. The zeroth coefficient of the Ehrhart polynomial of the flow zonotope $\mathcal{C}_G$ is 1 because the number of complements of spanning sets of G of size zero is one. These sets are exactly the linearly independent subsets of the set of columns of the cographic hyperplane arrangement of G by Proposition 4.2.

Let T be a triangle. The normal vectors of the cographic hyperplane arrangement of T are three times the vector 1 corresponding to the fact that each edge of T is a complement of a spanning set of T and a complement of more than one edge is not spanning anymore. Thus, there are three linearly independent subsets of the set of normal vectors of the cographic hyperplane arrangement of T of size one whose minor is each 1. Therefore, the Ehrhart polynomial of the flow zonotope $\mathcal{C}_T$ of T is

$$L_{\mathcal{C}_T} = 3t + 1. \tag{11}$$

Proposition 4.4. *Let $G = (V, E)$ be a simple and bridgeless graph with $|V| \geq 4$. Let ∂_2^T be the transposed of the second boundary map of the cellular chain complex of $C_4(|V|)$ if $|V| \geq 5$ or of $C_3(4)$ if $|V| = 4$ and let $(\partial_2^T)'$ be the matrix representation of the normal vectors of the cographic hyperplane arrangement of G. Let S be a linearly independent collection of columns of $(\partial_2^T)'$ of size $1 \leq k \leq |E| - |V| + 1$. Then, there exists a $k \times k$-submatrix of S whose determinant is ± 1.*

Proof. That S consists of linearly independent columns, implies that it induces a (contracted) ridge forest on P. By Remark 4.14, for every (contracted) ridge forest there exists at least one ridge leaf. To prove Proposition 4.4, we use induction. Let $k = 1$, then it is clear that there is a ± 1 entry in one column of $(\partial_2^T)'$ since this corresponds to the fact that the edge corresponding to this column is contained in at least on (contracted) ridge.

Now let $1 \leq l \leq k$ and assume that there exists a $l \times l$- submatrix S' of S with determinant ± 1. Then this also holds for $l - 1$: Since the columns of S' are all linearly independent, they induce a (contracted) ridge forest on P, i.e., there is at least one (contracted) ridge leaf e. This means that there is an edge e induced by S' that is contained only in one of the (contracted) ridges r that we consider for S', i.e., there is one row of S' with a ± 1 entry at the position given by r and e and zero otherwise. Thus, if we apply Laplace expansion along the row corresponding to r, the factor of each minor-summand is zero besides for the submatrix S'' that occurs when removing the column corresponding to e because the entry at the position determined by r and e is ± 1. For the other edges the corresponding entries are zero. Since we multiply the determinant of S'' by ± 1, and obtain by the induction hypothesis as determinant of S' ± 1, the determinant of S'' must be ± 1 as well. Thus, there exists a $(l - 1) \times (l - 1)$-submatrix of S' that has determinant ± 1. $\qquad \square$

Now we are finally prepared for our main theorem.

Theorem 4.5. *Let $G = (V, E)$ be a simple and bridgeless graph. Then the Ehrhart polynomial $L_{\mathcal{C}_G}$ of $\mathcal{C}_G$ is given by*

$$L_{\mathcal{C}_G}(t) = \sum_{k=0}^{|E|-|V|+1} d_k t^k$$

where the coefficient d_k is the number of (labeled) complements of (labeled) spanning sets of size k.

Proof. For $|V| = 3$, the claim was proven by (11). Now let $|V| \geq 4$. By Proposition 4.4, for each linearly independent subset S of the set of columns of $(\partial_2^T)'$ of size k, there exists a $k \times k$ minor of value ± 1 for $1 \leq k \leq |E| - |V| + 1$. Thus, the greatest common divisor $m(S)$ of all $k \times k$ minors of S equals one. By Theorem 3.3, the coefficient d_k of the Ehrhart polynomial $L_{\mathcal{C}_G}$ equals the number of submatrices S of $(\partial_2^T)'$ with non-zero determinant. By Proposition 4.2 the number of submatrices S of $(\partial_2^T)'$ with non-zero determinant is exactly the number of complements of a spanning set of G of size k. By Remark 4.18, the zeroth coefficient of $L_{\mathcal{C}_G}$ equals the number of complements of spanning sets of G of size zero. This number is 1.

In total, we obtain

$$L_{\mathcal{C}_G}(t) = \sum_S m(S) t^{|S|} = \sum_S 1 \cdot t^{|S|} = \sum_{C \text{ a complement of a spanning set of size } k} 1 \cdot t^k$$

$$= \sum_{k=0}^{|E|-|V|+1} d_k t^k$$

where the first sum is over all S that are linearly independent subsets of the set of columns of $(\partial_2^T)'$. $\square$

Example 4.11. Let $C = (V, E)$ by a cycle of length $|E|$. Since each edge of C is a complement of a spanning set of C and a complement of more than one edge is not spanning anymore, we obtain the Ehrhart polynomial

$$L_{\mathcal{C}_C}(t) = |E| \cdot t + 1.$$

Example 4.12. The flow zonotope of the complete graph K_4 is the permutahedron $\mathcal{P}_4$. We can calculate its Ehrhart polynomial $L_{\mathcal{C}_{K_4}}$ as follows. The first coefficient d_1 of $L_{\mathcal{C}_{K_4}}$ is six since every edge of K_4 is a complement of a spanning set. Every choice of two edges of K_4 is a complement of a spanning set. Thus, d_2 equals $\binom{6}{2} = 15$. Every choice of three edges besides the ones incident to the same vertex is a complement of a spanning set. Therefore, d_3 equals $\binom{6}{3} - 4 = 16$. In total, we have

$$L_{\mathcal{C}_{K_4}}(t) = 16t^3 + 15t^2 + 6t + 1.$$

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