Master Thesis

Ehrhart Polynomials of Lattice Zonotopes

Discrete Geometry
Master of Mathematics
Freie Universität Berlin

handed in by
Ellinor Birgitta Janssen
email address: ellinorbij96@zedat.fu-berlin.de
matriculation number: 5280005

Supervisor Prof. Matthias Beck
Second Corrector Prof. Katharina Jochemko

Berlin, August 6, 2021

Table of Contents

1 Introduction

In 1976 Ehrhart proved in [5] and [6] that the map that associates a natural number n with the number of integer points in the n-th dilate of a lattice polytope P is a polynomial, called the Ehrhart polynomial of P. Here, the question arises what information about the corresponding polytope the Ehrhart polynomial encodes and which polynomials are Ehrhart polynomials of lattice polytopes. In dimension 2, this question was solved by Beck et al. in [2, pp. 4-5], which we will elaborate on in Section 2.3. But for higher dimensions this problem is still open. So it is for the class of zonotopes, i.e., polytopes that are the Minkowski sum of finitely many line segments.
The aim of this thesis is to classify the Ehrhart polynomials of lattice zonotopes of dimensions 2 and of degree 2. This constitutes a first step to solve the open problem of characterizing the Ehrhart polynomials and h^*-polynomials of d-dimensional parallelepipeds and zonotopes, as raised in [3, p. 15] by Beck, Jochemko, and McCullough.

First, I will introduce definitions and preliminary knowledge about Ehrhart polynomials of lattice polytopes, h^*-polynomials, and zonotopes.
Chapter 3 will be a discussion of the 2-dimensional case, which is special in the sense that the set of 2-dimensional lattice zonotopes equals the set of centrally symmetric lattice polytopes of dimension 2. We can give a precise characterization of the Ehrhart polynomials of 2-dimensional lattice zonotopes:

Theorem 1.1. *The set of the Ehrhart polynomials* $\mathrm{ehr}(n) = 1 + c_1 n + c_2 n^2$ *of all* 2*-dimensional lattice zonotopes is the set of integer points in the polyhedral complex comprised of the rays*

$$\begin{aligned} S' &= \{(c_2, c_1) : c_1 = c_2 + 1, c_2 \geq 1\}, \\ T \ &= \{(c_2, c_1) : c_1 = \tfrac{c_2}{2} + 2, c_2 \geq 2\}, \\ R' &= \mathbb{R}_{\geq 1} \times \{2\} \end{aligned}$$

and the area enclosed by T and R', where the pair $(c_2, c_1) \in \mathbb{Z}^2$ represents the polynomial $\mathrm{ehr}(n) = 1 + c_1 n + c_2 n^2$. *See Figure 3.4.*

In the proof of this theorem, we will proceed by restricting the set of possible Ehrhart polynomials of 2-dimensional lattice zonotopes using Theorem 2.18 from [3, Theorem 1.3, p. 2] as well as Scott's Theorem 2.5 from [12]. Next, we will give an explicit construction of zonotopes associated with the polynomials in that set and, thereby, obtain the above classification of the Ehrhart and h^*-polynomials of lattice zonotopes of dimension 2.
The natural next step will be to consider degree-2 zonotopes, which we will dedicate Chapter 4 to. We will observe that there exist lattice zonotopes of degree 2 only in dimensions 2 or 3. The main body of this chapter will consist of working towards Theorem 4.9, which provides us with a classification of the h^*-polynomials of 3-dimensional lattice zonotopes of degree 2. Theorem 2.18 will give us a simplicial cone of dimension 3 which equals the convex hull of the h^*-polynomials of all 3-dimensional lattice zonotopes. Using the restriction of the lattice width of 3-dimensional zonotopes of degree 2, we will be able to reduce the question which polynomials of degree 2 are h^*-polynomials of 3-dimensional zonotopes without interior lattice points to the following two cases: On the one hand, we get the set of h^*-polynomials corresponding to the Ehrhart polynomials from Theorem 1.1. We achieve this by constructing 3-dimensional lattice zonotopes from 2-dimensional lattice zonotopes considering them in the (x, y)-plane and adding the unit vector of the z-coordinate as a generating vector and showing that every 3-dimensional lattice zonotope of degree 2 and lattice width 1 is unimodularly equivalent to a zonotope constructed in this way. On the other hand, there are the h^*-polynomials of 3-dimensional lattice parallelepipeds of lattice width bigger than 1. We will show that the latter set is included in the first one and, thereby, obtain Theorem 4.9.
We will conclude with a brief chapter about 3-dimensional zonotopes of degree 3, which is meant as a stimulus to further study the Ehrhart polynomials of lattice zonotopes and develop a classification for higher or even general dimensions. Most importantly, it is to be noted here that the number of lattice points contained in a d-dimensional lattice polytope with a fixed number $k \in \mathbb{Z}_{\geq 1}$ of interior lattice points is bounded from above. Thus, there is only a finite number of corresponding Ehrhart polynomials for each d and each k.

2 Preliminary Knowledge

2.1 Ehrhart Polynomials

In this thesis, we will write $P \subset \mathbb{R}^d$ for a **lattice polytope**, i.e., a convex polytope whose vertices all lie in $\mathbb{Z}^d$. Our object of investigation is the **Ehrhart polynomial** of P, that is, the function that counts the integer points in the n-th dilate of P:

$$\mathrm{ehr}_P : \mathbb{N} \to \mathbb{N},\ n \mapsto \mathrm{ehr}_P(n) := |nP \cap \mathbb{Z}^d|.$$

Eugène Ehrhart showed in [5] and [6] that this function is a polynomial in n. Therefore, the domain of ehr_P can be extended to the set of complex numbers.
The coefficients of the Ehrhart polynomial contain information about the polytope itself. For instance, the leading coefficient of ehr_P equals the normalized volume of P with respect to the sublattice $\mathbb{Z}^d \cap \mathrm{aff}(P)$. Similarly, the second highest coefficient can be interpreted as half the surface area of P, which is the sum over the volumes of each facet of P normalized with respect to the corresponding sublattice. The constant term of the Ehrhart polynomial of P is the Euler characteristic of P, which is 1 since P is a closed convex polytope.
Furthermore, the following reciprocity law provides an interpretation for the values of the Ehrhart polynomial evaluated at negative integers:

Theorem 2.1. (Ehrhart $-$ Macdonald Reciprocity Theorem [9, Theorem 4.6, p. 192])

$$\mathrm{ehr}_P(-n) = (-1)^{\dim(P)}\mathrm{ehr}_{P^\circ}(n) \textit{ for all } n \in \mathbb{N},$$

where ehr_{P° *denotes the function counting the interior lattice points of the n-th dilate of P.*

2.2 h^*-Polynomials

The **h^*-polynomial** of a d-dimensional lattice polytope $P \subset \mathbb{R}^d$ is defined via $h^*(P) : \mathbb{C} \to \mathbb{C},\ t \mapsto h_0(P) + \cdots + h_d(P)t^d$, where

$$\mathrm{ehr}_P(n) = h_0(P)\binom{n+d}{d} + h_1(P)\binom{n+d-1}{d} + \cdots + h_d(P)\binom{n}{d} \tag{2.1}$$

or equivalently using the notion of the Ehrhart series

$$\mathrm{Ehr}_P(t) := \sum_{n\geq 0} \mathrm{ehr}_P(n)t^n = \frac{h^*(P)(t)}{(1-t)^{d+1}}. \tag{2.2}$$

Here, we interpret $\binom{k}{l} := 0$ whenever $k < l$.
Stanley proved in [14, Theorem 2.1, pp. 336-337] that the coefficients of the h^*-polynomial of lattice polytopes are all non-negative integers. The constant coefficient is $h_0(P) = 1$, and $h_1(P) = |P \cap \mathbb{Z}^d| - (d+1)$. The sum of all coefficients of the h^*-polynomial equals the normalized volume of the polytope, i.e., $d! \cdot \mathrm{vol}(P)$.
The degree $\deg(h^*)$ of the h^*-polynomial of a d-dimensional polytope P is smaller or equal to d and is called the **degree of the lattice polytope** P. It is known that the degree of P is the largest number $k \in \mathbb{Z}_{\geq 1}$ such that there is an interior lattice point in the $(d+1-k)$-th dilate of P, i.e.,

$$\begin{aligned}\deg(h^*) &= \max\Big\{k \in \{1,\dots,d\} :\ h_k(P) \neq 0\Big\}\\ &= \max\Big\{k \in \{1,\dots,d\} :\ (d+1-k)P^\circ \cap \mathbb{Z}^d \neq \emptyset\Big\}.\end{aligned}$$

In particular, due to the Reciprocity Theorem 2.1

$$\begin{aligned} h_{\deg(h^*)}(P) &= (-1)^d \left(h_0(P)\cdot 0 + h_1(P)\cdot 0 + \cdots + h_{\deg(h^*)}(P)\binom{-1}{d}\right) \\ &= (-1)^d \mathrm{ehr}_P\left(\deg(h^*) - d - 1\right) \\ &= \mathrm{ehr}_{P^\circ}\left(d+1-\deg(h^*)\right) = \left|\left(d+1-\deg(h^*)\right)P^\circ \cap \mathbb{Z}^d\right|. \end{aligned} \tag{2.3}$$

2.3 The 2-dimensional Case

Pick's Theorem [10] relates the area of a lattice polygon with the number of integer points in the polygon and the number of integer points in its boundary. Thereby, we get an explicit description for the coefficients of the Ehrhart polynomial ehr_P of a 2-dimensional lattice polytope P:

Theorem 2.2. (Pick's Theorem [10])
Let P be a 2-dimensional lattice polytope. Then

$$\mathrm{ehr}_P(n) = An^2 + \tfrac{b}{2}n + 1\,,$$

where A is the area of P and b is the number of lattice points on the boundary of P.

Raman and Öhman [11] use the following two well-known lemmas, which I will state without proof, to show this statement.

Lemma 2.3. *Any 2-dimensional lattice polytope admits a triangulation into elementary triangles, i.e., triangles with lattice vertices and no other lattice points on the boundary and no interior lattice points.*

Lemma 2.4. *The area of any elementary triangle in $\mathbb{Z}^2$ is $1/2$.*

The following proof of Pick's Theorem from [11, pp. 200-202] is based on the consideration of angles:

Proof. We know that $\mathrm{ehr}_P(n) = c_2n^2 + c_1n + 1$ with $c_2 = A$.
Let $\Sigma := \mathrm{ehr}_{P^\circ}(1)$ denote the number of interior points of P. By the Ehrhart-Macdonald Reciprocity Theorem 2.1,

$$\Sigma = \mathrm{ehr}_{P^\circ}(1) = \mathrm{ehr}_P(-1) = A - c_1 + 1 \text{ and, thus, } A = \Sigma + c_1 - 1. \tag{2.4}$$

By Lemma 2.3, it is possible to triangulate P into k elementary triangles. Next, we consider the sum of the internal angles of all these triangles using two distinct approaches.
First, the sum of the three internal angles of any triangle is π. Hence, the sum of the internal angles of all triangles of the triangulation equals $k \cdot \pi$.
On the other hand, we can compute the sum by considering the sum of the internal angles

at each vertex of the triangulation separately. At interior points the internal angles sum up to 2π. At boundary points that are not vertices of P the sum of the internal angles equals π. The sum of internal angles at the vertices of P cannot be determined in such a general way. However, it is known that adding up the internal angles at all the vertices gives $m \cdot \pi - 2\pi$ where m denotes the number of vertices of P.
Then the sum of angles at interior points is $\Sigma \cdot 2\pi$ and the sum of angles at boundary points is $b \cdot \pi - 2\pi$, where b is the number of lattice points on the boundary of P. Double counting yields $k \cdot \pi = \Sigma \cdot 2\pi + b\pi - 2\pi$ and, thus, $k = 2 \cdot \Sigma + b - 2$.
Lemma 2.4 tells us that the area of P is $A = 1/2 \cdot k = \Sigma + b/2 - 1$. Therefore, by (2.4), $c_1 = b/2$. ☐

The following relation established by Scott in 1976 [12] provides us with further information about the coefficients of ehr_P in the 2-dimensional case:

Theorem 2.5. (Scott′s Theorem [12])
Let $\mathrm{ehr}_P(n) = c_2 n^2 + c_1 n + 1$ be the Ehrhart polynomial of a lattice 2-polytope P. If $P^\circ \cap \mathbb{Z}^2 \neq \emptyset$, and P is not unimodularly equivalent to $\Delta := \mathrm{conv}\{(0,0),(3,0),(0,3)\}$, then

$$c_1 \leq \tfrac{1}{2} c_2 + 2. \tag{2.5}$$

Remark 2.6. *Two lattice polytopes P and Q are called* ***unimodularly equivalent*** *if we can obtain one from the other by applying an integral unimodular transformation or translation, that is, if there exist a square integer matrix M with determinant -1 or 1 and a t with integral coordinates such that $Q = MP + t$. The property of convexity, as well as the Ehrhart polynomial are invariant under unimodular equivalence.*

Remark 2.7. *By Pick's Theorem 2.2, we have $2c_1 = \mathrm{ehr}_P(1) = c_2 + c_1 + 1$ and, thus, $c_1 = c_2 + 1$ for a 2-dimensional lattice polytope P with no interior lattice points.*
For $\Delta = \mathrm{conv}\{(0,0),(3,0),(0,3)\}$,

$$\mathrm{ehr}_\Delta(n) = \tfrac{9}{2} n^2 + \tfrac{9}{2} n + 1.$$

Remark 2.8. *By Pick's Theorem 2.2, the following inequality is equivalent to inequality (2.5) in Scott's Theorem:*

$$c_2 = \mathrm{vol}(P) \leq 2 \cdot \mathrm{vol}(P) - b + 4 = 2\left(|P^\circ \cap \mathbb{Z}^2| + 1\right)$$

for $b = |P \cap \mathbb{Z}^2| - |P^\circ \cap \mathbb{Z}^2|$ *and* $|P \cap \mathbb{Z}^2| = \mathrm{ehr}_P(1) = \mathrm{vol}(P) + b/2 + 1$.

Now we may raise the question which polynomials of degree 2 can be realized as Ehrhart polynomials of a 2-dimensional lattice polytope. As mentioned in Section 2.1, it is known that the constant term has to be 1. So we identify a polynomial $c_2n^2 + c_1n + 1$ with the pair (c_2, c_1) and consider the set Q of all such pairs corresponding to Ehrhart polynomials in the plane.
From Pick's Theorem 2.2, it is clear that c_1 is half-integral, because $c_1 = b/2$ where b counts the integer points in the boundary of P and, thus, $b \in \mathbb{Z}$. The same holds true for c_2 as we can deduce from Lemma 2.3 and Lemma 2.4, since c_2 equals the area of P and we can write P as the union of elementary triangles, each of whose area is $1/2$.
Since we can describe the Ehrhart polynomial either in the standard basis or with the coefficients of the h^*-polynomial as in (2.1), we can calculate that the h^*-polynomial corresponding to the Ehrhart polynomial $c_2n^2 + c_1n + 1$ is

$$h_2t^2 + h_1t + 1 = (c_2 - c_1 + 1)t^2 + (c_2 + c_1 - 2)t + 1 \tag{2.6}$$

by solving a system of linear equations with two unknowns. As the coefficients h_2 and h_1 are non-negative integers, c_2 and c_1 are either both integral or both non-integral.

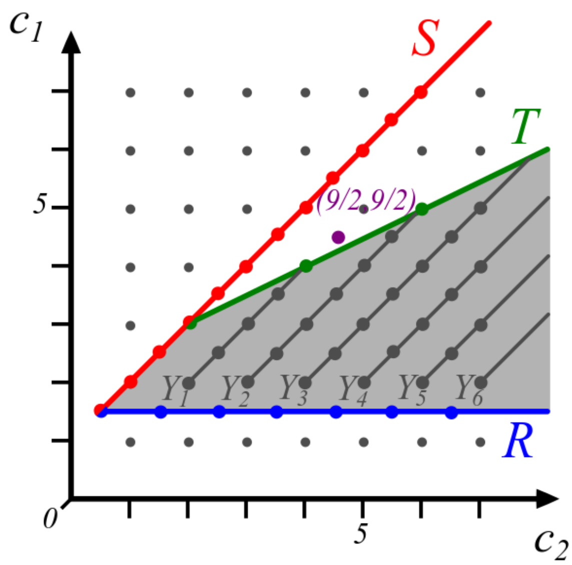


Figure 2.1: The polyhedral complex containing all pairs corresponding to Ehrhart polynomials of 2-dimensional lattice polytopes.

Trivially, we get the bound $c_1 \geq 3/2$ as any 2-dimensional lattice polytope has at least 3

vertices, which are integral points in the boundary. In particular, this is a sharp bound, because there are polygons with exactly three boundary lattice points and arbitrarily large area. Every half-integer point on the ray $R := [1/2, \infty) \times \{3/2\}$ can be realized by a triangle of the following form: $\bigtriangledown_y := \operatorname{conv}\{(0,1),(1,0),(y,y)\}$ for $y \in \mathbb{Z}_{>0}$.
From Remark 2.7, we know that the pairs that correspond to the Ehrhart polynomials of 2-dimensional lattice polytopes with no interior lattice points are all contained in the ray $S := \{(c_2, c_1) : c_1 = c_2 + 1,\ c_2 \geq 1/2\}$. For $y \in \mathbb{Z}_{>0}$, the point $(y/2, y/2+1)$ is associated with the Ehrhart polynomial of the triangle $\triangle_y := \operatorname{conv}\{(0,0),(1,0),(0,y)\}$, which has area $y/2 = A = c_2$ and $2(y/2+1) = y+2 = b = 2c_1$ boundary lattice points. In the point $(1/2, 3/2)$ the rays S and R meet.
Moreover, we get the exceptional point $(9/2, 9/2)$, which represents the Ehrhart polynomial of polytopes unimodularly equivalent to Δ.
Finally, Scott's Theorem 2.5 provides us with a further bound for the pairs that stand for the Ehrhart polynomials of those 2-dimensional lattice polytopes with at least one lattice point in the interior, excepting the special point $(9/2, 9/2)$. In particular, every point of the ray $T := \{(c_2, c_1) : c_1 = c_2/2 + 2,\ c_2 \geq 2\}$ with integral coordinates corresponds to an Ehrhart polynomial, because the rectangle $\square_y := \operatorname{conv}\{(0,0),(2,0),(2,y),(0,y)\}$, where y is a positive integer, has an area of $2y = c_2$ and the number of the lattice points on its boundary is $2(c_2/2+2) = 2(y+2) = 2y+4 = 2(y+1)+2 = b = 2c_1$.
Let $Y_i := \{(c_2, c_1) : c_1 = c_2 + 1, c_1 \leq c_2/2 + 2,\ c_2 \geq i+1\}$ denote the line segment parallel to the ray S enclosed by the rays R and T for $i \in \mathbb{Z}_{\geq 1}$. We order the half-integer points $v_i^0, \dots, v_i^{n_i}$ of Y_i such that $v_i^{k+1} = v_i^k + (1/2, 1/2)$ for all $k \in \{0, \dots, n_i - 1\}$ and $i \in \mathbb{Z}_{\geq 1}$. Then, the point v_i^0 can be realized by the triangle $V_i^0 := \operatorname{conv}\{(-1,0),(1,0),(0,i+1)\}$. We iteratively define the polytopes

$$\begin{aligned} V_i^k &:= \operatorname{conv}\left(V_i^{k-1} \cup \left\{(-1, \tfrac{k+1}{2})\right\}\right) \quad \text{for odd } k \in \{1, \dots, n_i\}, \\ V_i^k &:= \operatorname{conv}\left(V_i^{k-1} \cup \left\{(1, \tfrac{k}{2})\right\}\right) \quad \text{for even } k \in \{2, \dots, n_i\}. \end{aligned} \tag{2.7}$$

The Ehrhart polynomial of the polytope V_i^k corresponds to the point v_i^k for all $k \in \{0, \dots, n_i\}$ and $i \in \mathbb{Z}_{\geq 1}$.

Figure 2.1 shows the region containing all half-integral pairs corresponding to degree-2 Ehrhart polynomials, which consists of the component bounded by the rays S, T, and R, as well as the ray S and the point $(9/2, 9/2)$. Every point in this region with either both coordinates being integers or both coordinates being half-integers but not integers can be

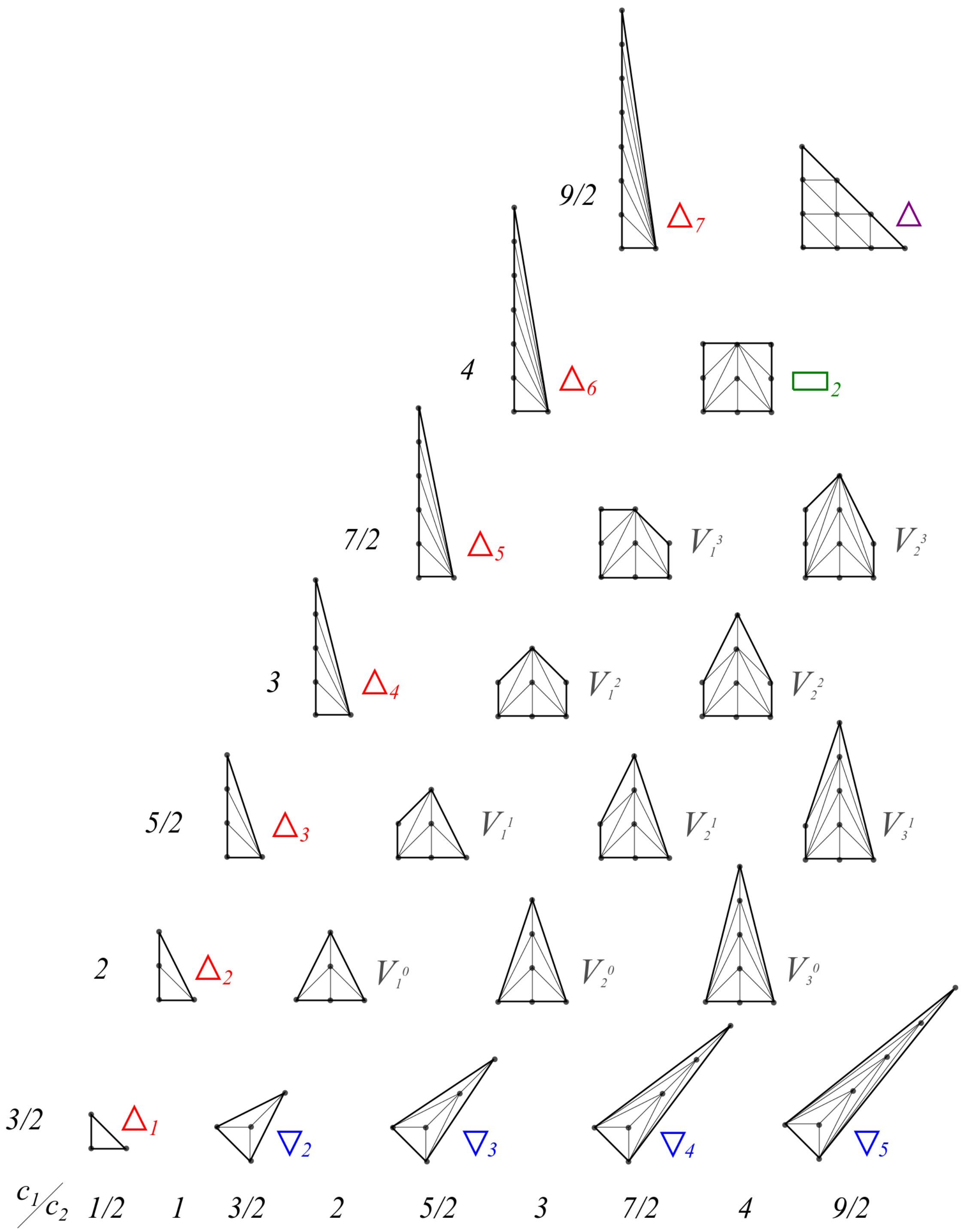


Figure 2.2: Polytopes $\bigtriangledown_y, \triangle_y, \square_y, \triangle, V_i^k$ realizing the points of the polyhedral complex shown in Figure 2.1 whose half-integral coordinates are either both integral or both non-integral and half-integral.

realized as a 2-dimensional lattice polytope. We demonstrate our constructions by way of some examples in Figure 2.2.

2.4 The degree-2 Case

Similarly, we want to study the set of degree-2 lattice polytopes in order to better understand how the classification of the subclass of degree-2 zonotopes which we will develop in Chapter 4 relates. Just as we considered pairs of coefficients of Ehrhart polynomials in the Section 2.3, we will now consider pairs of the form $(h_2, h_1) \in \mathbb{Z}^2_{\geq 0}$ corresponding to the h^*-polynomial $h_2t^2 + h_1t + 1$.

Due to Treutlein [15], we have a generalized version of Scott's Theorem 2.5 for lattice polytopes of degree 2 of general dimension, which restricts the set of possible h^*-polynomials:

Theorem 2.9. [15, Theorem 2, p. 355] *Let $d \in \mathbb{Z}_{\geq 2}$. Let P be a d-dimensional lattice polytope of degree 2 with $h^*(P)(t) = 1 + h_1(P)t + h_2(P)t^2$. Then*

$$h_1(P) \leq \begin{cases} 7 & \text{if } \ h_2(P) = 1, \\ 3h_2(P) + 3 & \text{if } \ h_2(P) \geq 2. \end{cases} \tag{2.8}$$

In order to calculate the h^*-polynomial $h^*(P)(t) = h_2(P)t^2 + h_1(P)t + 1$ of a d-dimensional polytope P of degree 2 we only need to determine the number of lattice points in P as well as its volume, because according to Section 2.2, the coefficients of $h^*(P)$ are $h_1(P) = |P \cap \mathbb{Z}^d| - (d+1)$ and $h_2(P) = d! \cdot \mathrm{vol}(P) - h_1(P) - 1$ for $d \geq 2$.
We can obtain degree-2 lattice polytopes of dimension 3 by constructing a lattice pyramid $\mathcal{P}(Q)$ of height 1 over a 2-dimensional lattice polytope Q that we constructed in the previous section. Then equality holds:

$$\begin{aligned} &h_1(\mathcal{P}(Q)) = |\mathcal{P}(Q) \cap \mathbb{Z}^3| - 4 = |Q \cap \mathbb{Z}^2| - 3 = h_1(Q) \quad \text{and} \\ &h_2(\mathcal{P}(Q)) = 6 \cdot \mathrm{vol}(\mathcal{P}(Q)) - h_1(\mathcal{P}(Q)) - 1 = 2 \cdot \mathrm{vol}(Q) - h_1(Q) - 1 = h_2(Q). \end{aligned} \tag{2.9}$$

Clearly, the polytope $\mathcal{P}(\bigtriangledown_y) := \mathrm{conv}\{(0,1,0), (1,0,0), (y,y,0), (0,0,1)\}$ is a 3-dimensional lattice polytope without interior lattice points. It has $y + 3$ integer points and its volume is

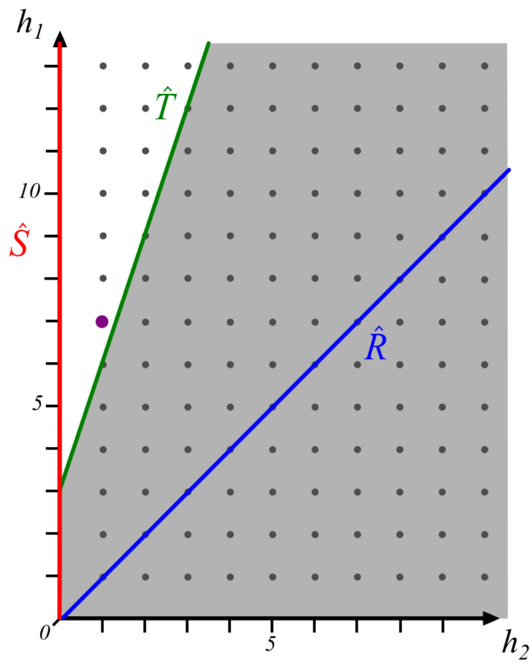


Figure 2.3: The set of integer points in the above polyhedral complex equals the set of pairs corresponding to h^*-polynomials of 3-dimensional lattice polytopes of degree 2. The points (h_2, h_1) satisfying $h_1 \geq h_2$ are the ones that can also be realized in the 2-dimensional case.

$(2y-1)/6$ for all $y \in \mathbb{Z}_{\geq 1}$. So $\mathcal{P}(\bigtriangledown_y)$ has h^*-polynomial $(y-1)t^2 + (y-1)t + 1$. Thereby, all integer points in the diagonal ray $\hat{R} := \{(y,y) : y \geq 1\}$ are realized.
The pyramid $\mathcal{P}(\triangle_y) := \operatorname{conv}\{(0,0,0),(1,0,0),(0,y,0),(0,0,1)\}$ has $y+3$ lattice points and volume $y/6$ for all $y \in \mathbb{Z}_{\geq 1}$. Hence, it corresponds to the point $(h_2,h_1) = (0, y-1)$, which lies in $\hat{S} := \{0\} \times \mathbb{R}_{\geq 0}$.
There are 11 integer points in the pyramid $\mathcal{P}(\Delta) := \operatorname{conv}\{(0,0,0),(3,0,0),(0,3,0),(0,0,1)\}$ and $\operatorname{vol}(\mathcal{P}(\Delta)) = 3/2$. Therefore, $\mathcal{P}(\Delta)$ corresponds to the point $(7,1)$.
The polytope $\mathcal{P}(\sqsubset\!\sqsupset_y) := \operatorname{conv}\{(0,0,0),(2,0,0),(2,y,0),(0,y,0),(0,0,1)\}$ has $3y+4$ lattice points and volume $2y/3$ for all $y \in \mathbb{Z}_{\geq 1}$. Thus, the h^*-polynomial of $\mathcal{P}(\sqsubset\!\sqsupset_y)$ is $(y-1)t^2 + 3yt + 1$. So each integer point in $\hat{T} := \{(y, 3y+3) : y \geq 0\}$ is realized in this way. The ray $\hat{T}$ together with the point $(7,1)$ is the upper bound (2.8) attained by Treutlein's Theorem 2.9.

The polytope $\mathcal{P}(V_i^0) := \operatorname{conv}\{(-1,0,0),(1,0,0),(0,i+1,0),(0,0,1)\}$ has $i+4$ integer points and volume $i/3$ for all $i \in \mathbb{Z}_{\geq 1}$. So it corresponds to the point $(i-1,i)$ for all $i \in \mathbb{Z}_{\geq 1}$. Let $\mathcal{P}(V_i^k)$ be the pyramid with added vertex $(0,0,1)$ over V_i^k as defined in (2.7) for all $i \in \mathbb{Z}_{\geq 1}$ and $k \in \{0,\ldots,n_i\}$. The polytope $\mathcal{P}(V_i^k)$ has one more lattice point than $\mathcal{P}(V_i^{k-1})$ and $\mathrm{vol}(\mathcal{P}(V_i^k)) = \mathrm{vol}(\mathcal{P}(V_i^{k-1})) + 1/2$ for each $i \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, n_i\}$. Its h^*-polynomial is $(i-1)t^2 + (i+k)t + 1$. So all integer points in the region framed by the rays $\hat{T}$ and $\hat{R}$ can be realized as a 3-dimensional degree-2 lattice polytope.

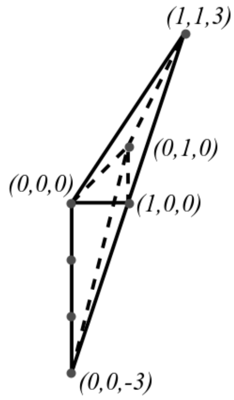


Figure 2.4: The polytope $P_{k,l}$ for $k = 3$ and $l = 2$.

In [7, Proof of Proposition 1.10, p. 80], Henk and Tagami gave a construction for lattice polytopes that realize the points (h_2, h_1) with $h_1 \leq h_2$: Let $k, l \in \mathbb{Z}$ with $0 \leq k \leq m$, and $P_{k,l} := \mathrm{conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,-k), (1,1,l+1)\}$. The lattice polytope $P_{k,l}$ has $4+k$ lattice points and $\mathrm{vol}(P_{k,l}) = k/6 + (l+1)/6 = (k+l+1)/6$. Therefore, $h_1(P_{k,l}) = k$ and $h_2(P_{k,l}) = l$.

Similarly as in (2.9), the equality of the h^*-polynomials $h^*(Q) = h^*(P)$ holds for any d-dimensional lattice polytope P of height 1 with a $(d-1)$-dimensional lattice polytope Q as its base for general dimensions $d \in \mathbb{Z}_{\geq 3}$. Hence, for $d \in \mathbb{Z}_{>3}$, every polynomial of the form $h_2t^2 + h_1t + 1$ with $h_2, h_1 \in \mathbb{Z}_{\geq 0}$ satisfying the upper bound (2.8) can be realized as a d-dimensional lattice pyramid over the 3-dimensional realization constructed above.

2.5 Zonotopes

In this paper, we are interested in a particular class of polytopes, namely the zonotopes. A zonotope is the Minkowski sum of finitely many line segments, i.e., given line segments $L_1, \ldots, L_n \subset \mathbb{R}^d$,

$$L_1 + \cdots + L_n := \{x_1 + \cdots + x_n : x_i \in L_i\}$$

is a zonotope. An equivalent definition is given by a projection of the unit cube $[0,1]^n$: Given a set of vectors $V = \{v_1, \dots, v_n\} \subset \mathbb{R}^d$,

$$Z(v_1, \dots, v_n) := \left\{ \sum_{i=1}^{n} \lambda_i v_i : 0 \le \lambda_i \le 1 \right\}$$

is the **zonotope** generated by V. Up to translation, every zonotope is of this form. We call a zonotope a **lattice zonotope** if its generating set V satisfies $V \subset \mathbb{Z}^d$.

Given a zonotope Z generated by vectors $v_1, \dots, v_n$ as above, we consider its translate by half the sum of all its generating vectors

$$\begin{aligned} Z' &:= Z - \tfrac{1}{2} \sum_{i=1}^{n} v_i \\ &= \left\{ \sum_{i=1}^{n} (\lambda_i - \tfrac{1}{2}) v_i : 0 \le \lambda_i \le 1 \right\} \\ &= \left\{ \tfrac{1}{2} \sum_{i=1}^{n} \lambda_i v_i : -1 \le \lambda_i \le 1 \right\} = Z\left(\pm\tfrac{1}{2} v_1, \dots, \pm\tfrac{1}{2} v_n \right). \end{aligned}$$

The translate Z' is generated by the vectors $v_1/2, \dots, v_n/2, -v_1/2, \dots, -v_n/2$. This zonotope is **symmetric about the origin**, i.e., $x \in Z'$ if and only if $-x \in Z'$. Hence, all zonotopes are **centrally symmetric**, that is, each zonotope has a translate that is symmetric about the origin. We can differentiate between lattice zonotopes that are centrally symmetric about a lattice point and those that are centrally symmetric about a point with at least one half-integer coordinate.
The highest coefficient h_d of the h^*-polynomial equals the number of integer points in the interior of the corresponding zonotope. Lattice zonotopes that are centrally symmetric about an integer point have an odd number of interior lattice points. So their h^*-polynomials will have an odd highest coefficient. On the other hand, there is an even number of interior integer points in lattice zonotopes that are centrally symmetric about a half-integer point, wherefore their h^*-polynomials have even h_d.

Lemma 2.10. *Every translate of a lattice parallelepiped contains an integer point.*

Proof. Let

$$\Pi_{v_1, \dots, v_d} := \left\{ \sum_{i=1}^{d} \lambda_i v_i : 0 \le \lambda_i \le 1 \right\}$$

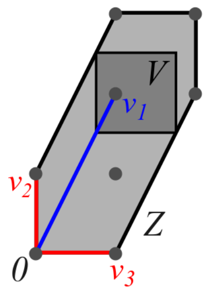
be the d-dimensional parallelepiped spanned by the integer vectors $v_1, \ldots, v_d$, and $t \in \mathbb{R}^d$. Since $v_1, \ldots, v_d$ form a basis of $\mathbb{R}^d$, there are unique $\omega_1, \ldots, \omega_d \in \mathbb{R}$ such that $t = \sum_{i=1}^{d} \omega_i v_i$. For $i \in \{1, \ldots, d\}$ define $\mu_i := \lceil \omega_i \rceil - \omega_i \in [0, 1)$. Then $t + \sum_{i=1}^{d} \mu_i v_i = \sum_{i=1}^{d} \lceil \omega_i \rceil v_i$ is an integer point lying in the translate $t + \Pi_{v_1,\ldots,v_d}$ of the lattice parallelepiped. □

Theorem 2.11. *Every translate of a 2-dimensional lattice zonotope generated by three pairwise linearly independent vectors contains an interior lattice point.*

Proof. Let $Z(v_1, v_2, v_3) := \{\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 : \lambda_i \in [0, 1] \text{ for all } i \in \{1, 2, 3\}\}$ be the lattice zonotope generated by the pairwise linearly independent vectors $v_1, v_2, v_3 \in \mathbb{Z}^2$. Then there exist $\alpha_2, \alpha_3 \in \mathbb{R}_{>0}$ and $\sigma_2, \sigma_3 \in \{1, -1\}$ such that $v_1 = \sigma_2 \alpha_2 v_2 + \sigma_3 \alpha_3 v_3$. Define

$$\epsilon := \frac{1}{2 \cdot \max\{1, \alpha_2, \alpha_3\}}.$$

Then $\epsilon \in (0, 1)$ and $\epsilon \alpha_i \in (0, 1)$ for all $i \in \{2, 3\}$. We are interested in the translate $t + Z$ for a fixed $t \in \mathbb{R}^2$.



$\sigma_2 = \sigma_3 = 1$

Figure 2.5: The 2-dimensional lattice zonotope Z generated by the three vectors $v_1 = (1, 2)$, $v_2 = (0, 1)$, and $v_3 = (1, 0)$. Here $\epsilon = 1/4$.

Consider the case that $\sigma_2 = \sigma_3 = 1$. The set $V := (1 - \epsilon)v_1 + Z(v_2, v_2)$ is a subset of the zonotope Z and the translate of a lattice parallelepiped. Then, according to Lemma 2.10, the translate $t + V$ contains an integer point x. If x lies in the interior of $t + V$, it is an interior lattice point of $t + Z$. If x is a vertex of $t + V$, all vertices of $t + V$ are lattice points, in particular $(1 - \epsilon)v_1 + t$. Since we can write

$$(1 - \epsilon)v_1 + t = \left((1 - \tfrac{3}{2}\epsilon)v_1 + \tfrac{\epsilon\alpha_2}{2} v_2 + \tfrac{\epsilon\alpha_3}{2} v_3\right) + t,$$

it is clear that $(1 - \epsilon)v_1 + t$ lies in the interior of $t + Z$. If x lies in the relative interior of one of the facets of $t + V$, then there exists another integer point $\bar{x}$ in the relative interior of the opposing facet. So w.l.o.g. we can assume that $x = ((1-\epsilon)v_1 + \lambda v_2 + 0 \cdot v_3) + t$ for some $\lambda \in (0,1)$. Choose a

$$\delta \in \left(0, \min\{1-\epsilon, \tfrac{1-\lambda}{\alpha_2}\}\right).$$

Then

$$x = ((1-\epsilon)v_1 + \lambda v_2 + 0 \cdot v_3) + t = ((1-\epsilon-\delta)v_1 + (\lambda + \delta\alpha_2)v_2 + \delta\alpha_3 v_3) + t$$

is an interior lattice point of $t + Z$.
Now consider that $\sigma_2 = -1$ or $\sigma_3 = -1$. Since

$$Z(v_1, v_2, v_3) = Z(v_1, \sigma_2 v_2, \sigma_3 v_3) + \sum_{\substack{i=2 \\ \sigma_i = -1}}^{3} v_i$$

and $v_1 = (-\sigma_2)\alpha_2(-v_2) + (-\sigma_3)\alpha_3(-v_3)$, this case is already covered by the first case. $\square$

Theorem 2.12. [13, p. 319] *Every (lattice) zonotope has a subdivision into (lattice) parallelepipeds.*

Suppose that $w_1, \ldots, w_m \in \mathbb{R}^d$ are linearly independent, and let $\sigma_1, \ldots, \sigma_m \in \{\pm 1\}$. Then

$$\Pi_{w_1,\ldots,w_m}^{\sigma_1,\ldots,\sigma_m} := \left\{ \sum_{i=1}^{m} \lambda_i w_i : \begin{array}{c} 0 \le \lambda_i < 1 \text{ if } \sigma_i = -1 \\ 0 < \lambda_i \le 1 \text{ if } \sigma_i = 1 \end{array} \right\}$$

is the half-open parallelepiped generated by the vectors $w_1, \ldots, w_m$, where the signs $\sigma_1, \ldots, \sigma_m$ refer to which facets are included and which are excluded.

We can refine the statement of Theorem 2.12:

Lemma 2.13. [4, Lemma 9.1, p. 171] *The zonotope $Z(v_1, \ldots, v_n)$ can be written as the disjoint union of translates of $\Pi_{w_1,\ldots,w_m}^{\sigma_1,\ldots,\sigma_m}$, where $\{w_1, \ldots, w_m\}$ ranges over all linearly independent subsets of $\{v_1, \ldots, v_n\}$ with suitable $\sigma_1, \ldots, \sigma_m$.*

Lemma 2.14. [4, Lemma 9.2, p. 172] *Suppose that $w_1, \ldots, w_d \in \mathbb{R}^d$ are linearly independent. Let*

$$\Pi := \left\{ \sum_{i=1}^{d} \lambda_i w_i : 0 \le \lambda_i < 1 \right\}.$$

Then the volume of Π is $\operatorname{vol}(\Pi) = |\det(A)|$, where A is the square matrix whose columns are the vectors $w_1, \ldots, w_d$, and its Ehrhart polynomial is $\operatorname{ehr}_\Pi(n) = \operatorname{vol}(\Pi) n^d$.

Combining the results of the last two lemmas, we can first decompose a lattice zonotope into half-open lattice parallelepipeds and calculate the coefficients of the Ehrhart polynomial by summing up the respective relative volumes:

Theorem 2.15. [4, Theorem 9.9, p. 177] *Let $Z := Z(v_1, \ldots, v_n)$ be the zonotope generated by the integer vectors $v_1, \ldots, v_n$. Then the Ehrhart polynomial of Z is*

$$\mathrm{ehr}_Z(n) = \sum_I g(I) n^{|I|},$$

where I ranges over all linearly independent subsets of $\{v_1, \ldots, v_n\}$, and $g(I)$ denotes the greatest common divisor of all minors of size $|I|$ of the matrix that has the elements of I as its columns, and $g(\emptyset) = 1$.

Another theorem by Beck, Jochemko and McCullough [3] will come in handy later on, for which we first need to introduce the following notation:
Let S_d be the set of all permutations on $[d] := \{1, \ldots, d\}$. The **descent set** of a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_d \in S_d$ is defined as $\mathrm{Des}(\sigma) := \{i \in [d-1] : \sigma_i > \sigma_{i+1}\}$, and for the number of descents of σ we write $\mathrm{des}(\sigma) := |\mathrm{Des}(\sigma)|$.
The **Eulerian number** $a(d,k) := |\{\sigma \in S_d : \mathrm{des}(\sigma) = k\}|$ counts the permutations on $[d]$ with precisely k descents. The (A, j)**-Eulerian number**, defined by

$$a_j(d,k) := |\{\sigma \in S_d : \sigma_d = d + 1 - j \text{ and } \mathrm{des}(\sigma) = k\}|,$$

has the added condition that only permutations with last letter $d + 1 - j$ are included.
Finally, the (A, j)**-Eulerian polynomial** is given by

$$A_j(d,t) := \sum_{k=0}^{d-1} a_j(d,k) t^k.$$

Theorem 2.16. [3, Theorem 4.2, p. 10] *The h^*-polynomial of the half-open unit cube*

$$C_j^d := [0,1]^d \setminus \{x \in \mathbb{R}^d : x_d = x_{d-1} = \cdots = x_{d+1-j} = 1\},$$

where j indicates the number of removed facets, is $h^*(C_j^d) = A_{j+1}(d+1,t)$ for $j \in \{0, \ldots, d\}$.

Theorem 2.17. [3, Proposition 4.10, p. 13] *Every d-dimensional (lattice) zonotope can be* *partitioned into d-dimensional half-open (lattice) parallelepipeds of the following form:*

$$\Pi(I) := \left\{ \sum_{i=1}^{m} \lambda_i v_i : 0 \leq \lambda_i < 1 \text{ for all } i \in I,\ 0 \leq \lambda_i \leq 1 \text{ for all } \ i \notin I \right\},$$

where $v_1, \dots, v_m \in \mathbb{R}^d$ are linearly independent vectors and $I \subseteq \{1, \dots, m\}$.

We know that the h^*-polynomial of a d-dimensional lattice zonotope is the sum of the h^*-polynomials of the half-open d-dimensional lattice parallelepipdes into which the zonotope can be partitioned. Beck, Jochemko and McCullough [3, pp. 13-14] showed that the h^*-polynomial of a half-open lattice parallelepiped of the form $\Pi(I)$ is a nonnegative linear combination of the polynomials $A_1(d+1,t), \dots, A_d(d+1,t)$. Moreover, in their construction of the half-open decomposition of a lattice zonotope in the proof of Theorem 2.17 there is at most one closed parallelepiped. They obtained the following theorem that will be a useful tool for our further considerations:

Theorem 2.18. [3, Theorem 1.3, p. 2] *Let $d \geq 1$. The convex hull of the h^*-polynomials of all d-dimensional lattice zonotopes (viewed as points in $\mathbb{R}^{d+1}$) is equal to the d-dimensional simplicial cone*

$$A_1(d+1,t) + \mathbb{R}_{\geq 0} A_2(d+1,t) + \cdots + \mathbb{R}_{\geq 0} A_{d+1}(d+1,t). \tag{2.10}$$

Remark 2.19. [3, p. 14] *The polynomials $A_1(d+1,t), \dots, A_{d+1}(d+1,t)$ form a basis of the space of polynomials of degree d and smaller.*

Remark 2.20. *Due to Theorem 2.18 we can narrow our search for suitable h^*-polynomials. However, it does not provide a classification of the h^*-polynomials of all d-dimensional lattice zonotopes, because far from every integer point contained in the cone (2.10) can be realized by a d-dimensional lattice zonotope.*

3 Classification of 2-dimensional Lattice Zonotopes

Theorem 1.1 states that the set of Ehrhart polynomials of 2-dimensional lattice zonotopes equals the set of integer points lying in the three rays $S' = \{(c_2, c_1) : c_1 = c_2 + 1, c_2 \geq 1\}$, $T = \{(c_2, c_1) : c_1 = c_2/2 + 2, c_2 \geq 2\}$, $R' = \mathbb{R}_{\geq 1} \times \{2\}$ and the area enclosed by T and R', where the pair $(c_2, c_1) \in \mathbb{Z}^2$ represents the polynomial $\mathrm{ehr}(n) = 1 + c_1 n + c_2 n^2$, as shown in Figure 3.4. Here, we will give the proof:

Proof. From Theorem 2.18, we know that the convex hull of the h^*-polynomials of all 2-dimensional lattice zonotopes (viewed as points in $\mathbb{R}^3$) is equal to the simplicial cone $C := 1 + 1t + \mathbb{R}_{\geq 0}(2t) + \mathbb{R}_{\geq 0}(t + t^2)$, which has facets $\tilde{S} := \{0\} \times [1, \infty)$ and $\tilde{R} := \{(h_2, h_1) : h_1 = h_2 + 1, h_2 \geq 0\}$.

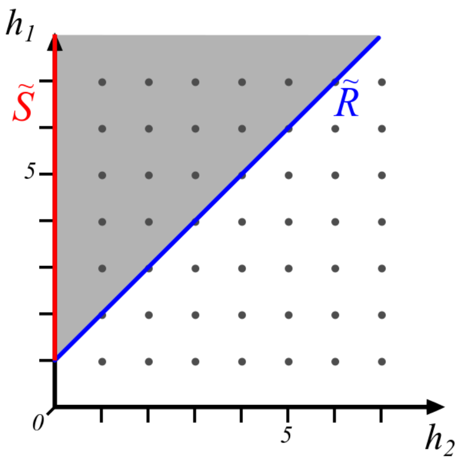


Figure 3.1: The convex hull of the h^*-polynomials of all 2-dimensional lattice zonotopes viewed as points (h_2, h_1) with facets $\tilde{S}$ and $\tilde{R}$, where $h^*(t) = 1 + h_1 t + h_2 t^2$.

The question is for which pairs $(h_2, h_1) = (B, B + N)$ with $B \in \mathbb{Z}_{\geq 0}$ and $N \in \mathbb{Z}_{\geq 1}$ the associated h^*-polynomial $h^*(t) = 1 + h_1 t + h_2 t^2$ corresponds to the Ehrhart polynomial of a lattice zonotope. Using equation (2.6), we calculate that the point $(B, B+N)$ is associated with the potential Ehrhart polynomial

$$f_{(B,N)}(n) = 1 + \frac{3+N}{2}n + \left(B + \frac{1+N}{2}\right)n^2. \tag{3.1}$$

Assuming that $f_{(B,N)}$ is the Ehrhart polynomial of a lattice zonotope Y, the number of lattice points on the boundary of Y is $3+N$ according to Pick's Theorem 2.2. For even N, the zonotope Y would have an odd number of lattice points of the boundary, which is a contradiction, as zonotopes are centrally symmetric. Thus, all possible h^*-polynomials will have

$$(h_2, h_1) = (B, B+N) \text{ with } B, N \in \mathbb{Z}_{\geq 0} \text{ and } N \text{ odd.} \tag{3.2}$$

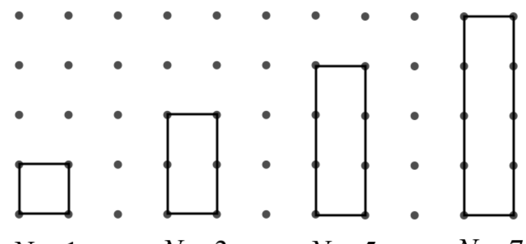


Figure 3.2: The 2-dimensional lattice zonotopes Z_N generated by the two vectors $(1,0), (0,(N+1)/2)$ for $N \in \{1,3,5,7\}$.

The pairs $(0, N)$ with $N \in \mathbb{Z}_{\geq 1}$ are the integer points in the facet $\tilde{S}$. According to (3.2), we need only consider odd N. The lattice zonotope Z_N generated by the two vectors $(1,0)$ and $(0,(N+1)/2)$ has the Ehrhart polynomial $f_{(0,N)}$ because this zonotope has the $3+N$ integer points $(0,i), (1,i)$ for all $i \in \{0,1,\ldots,(N+1)/2\}$ on its boundary and an area of $(1+N)/2$. See Figure 3.2.

The points $(B, B+1)$ with $B \in \mathbb{Z}_{\geq 0}$, lying on the facet $\tilde{R}$ of the simplicial cone C, correspond to the polynomial $f_{(B,1)}(n) = 1 + 2n + (B+1)n^2$. The zonotope Z_B generated by the two vectors $(1,0)$ and $(1, B+1)$ has precisely $f_{(B,1)}$ as its Ehrhart polynomial since this zonotope has the 4 integer points $(0,0), (1,0), (1, B+1)$, and $(2, B+1)$ on its boundary; moreover, Z_B admits a triangulation into $2(B+1)$ elementary triangles and, thus, has an area of $(B+1)$ by Lemma 2.4. Consider Figure 3.3 for examples.

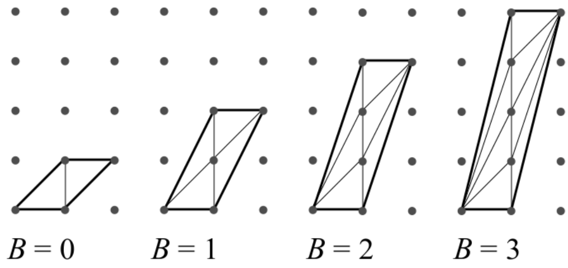


Figure 3.3: The 2-dimensional lattice zonotopes Z_B generated by the two vectors $(1, 0), (1, B + 1)$ for $B \in \{0, 1, 2, 3\}$ with a triangulation into elementary triangles.

Now that we understand the facets of the cone C, we will take a closer look at its interior. Using the description from (3.1), we can translate our picture in Figure 3.1 for the coefficients (h_2, h_1) of h^*-polynomials into one for coefficients (c_2, c_1) of an Ehrhart polynomial $1 + c_1 n + c_2 n^2$. Taking into account Scott's Theorem 2.5, we obtain Figure 3.4.

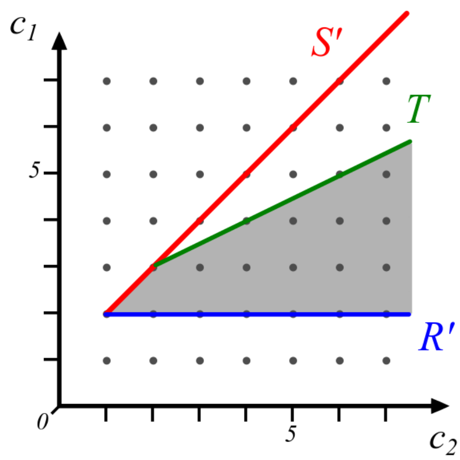


Figure 3.4: The polyhedral complex containing all pairs corresponding to Ehrhart polynomials of 2-dimensional lattice zonotopes.

An integer point $(2y, y + 2)$, $y \in \mathbb{Z}_{\geq 1}$, on the ray $T = \{(c_2, c_1) : c_1 = c_2/2 + 2\}$ can be realized by the zonotope Y_y generated by the vectors $(2, 0)$ and $(0, y)$, because it has an area of $2y$ and $2(y + 1) + 2 = 2(y + 2)$ integer points on the boundary.

Now we consider the rays parallel to T of the form $X_m := \{(2p + m, p + 2) : p \in \mathbb{Z}_{\geq 1}\}$ for some $m \in \mathbb{Z}_{\geq 1}$; $X_0 = T$. See Figure 3.5.
For each integer point (x_2, x_1) that lies in the interior of the area enclosed by the rays T and R', there exists an $m \in \mathbb{Z}_{\geq 1}$ such that the ray X_m contains (x_2, x_1). Furthermore, each integer point is associated with a lattice zonotope, because, by Theorem 2.15, the zonotope

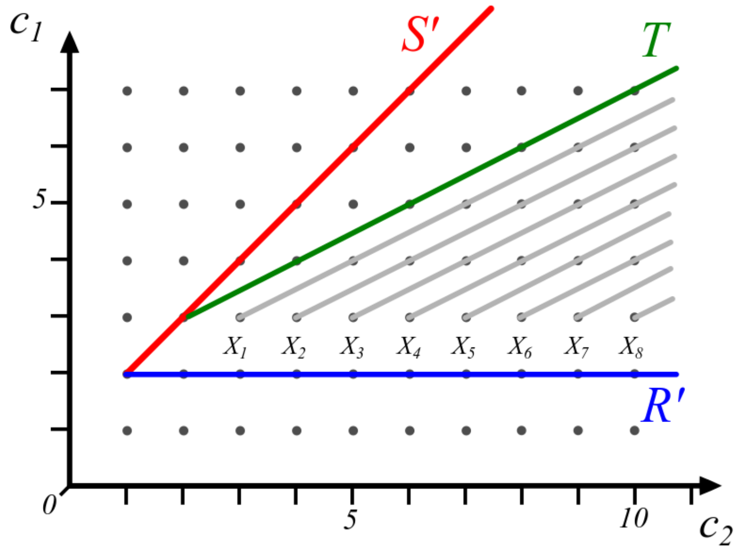


Figure 3.5: The rays X_m.

generated by the three vectors $(0, p)$, $(1, 0)$ and $(1, m)$ has the Ehrhart polynomial

$$\begin{aligned}&1 + \Big(\gcd(0, p) + \gcd(1, 0) + \gcd(1, m)\Big)n+\\ &\left(\left|\det\begin{pmatrix}0 & 1\\ p & 0\end{pmatrix}\right| + \left|\det\begin{pmatrix}0 & 1\\ p & m\end{pmatrix}\right| + \left|\det\begin{pmatrix}1 & 1\\ 0 & m\end{pmatrix}\right|\right) n^2\\ =&1 + (p + 2)n + (2p + m)n^2.\end{aligned}$$

See Figure 3.6 for examples.

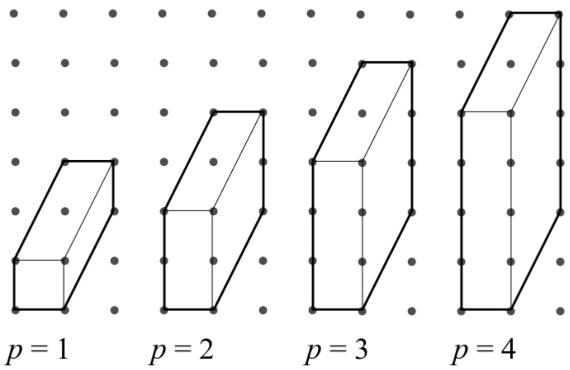


Figure 3.6: The lattice zonotopes generated by the three vectors $(0, p)$, $(1, 0)$, $(1, m)$ for $m = 2$ and $p \in \{1, 2, 3, 4\}$ partitioned into parallelepipeds.

□

Remark 3.1. *We can translate the picture in Figure 3.4 back to the context of h^*-polynomials. Note that by (3.2), we can realize only integer points (h_2, h_1) with $h_1 - h_2$ odd.*

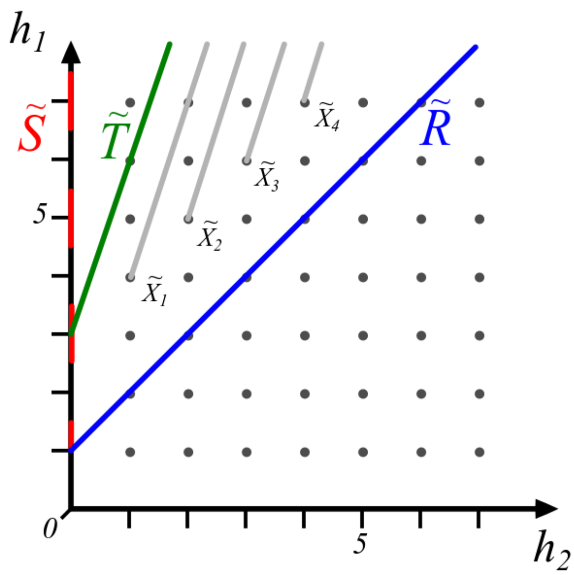


Figure 3.7: The marked integer points correspond to the h^*-polynomials of 2-dimensional lattice zonotopes.

Define

$$\begin{aligned}
\tilde{S} :=& \{0\} \times \{h_1 \in \mathbb{Z}_{\geq 0} \text{ odd}\}, \\
\tilde{T} :=& \{(h_2, h_1) : h_1 = 3h_2 + 3 \geq 3\}, \\
\tilde{R} :=& \{(h_2, h_1) : h_1 = h_2 + 1 \geq 1\}, \\
\tilde{X}_i :=& \{(h_2, h_1) : h_1 = 3h_2 - 2i + 3, h_2 \geq i\} \text{ for all } i \in \mathbb{Z}_{\geq 1}, \text{ and } \tilde{X}_0 := \tilde{T}.
\end{aligned}$$

The set

$$\left(\tilde{S} \cup \tilde{T} \cup \tilde{R} \cup \bigcup_{i \geq 1} \tilde{X}_i\right) \cap \mathbb{Z}^2$$

is precisely the set of points corresponding to h^*-polynomials of 2-dimensional lattice zonotopes.

4 Classification of Lattice Zonotopes of Degree 2

In this chapter, we are interested in the Ehrhart polynomials of a lattice zonotope P of dimension $d \geq 3$ and degree 2. We note that by (2.3) the highest coefficient of the h^*-polynomial of P is $h_2(P) = |(d-1)P^\circ \cap \mathbb{Z}^2|$ and $|m \cdot P^\circ \cap \mathbb{Z}^2| = 0$ for all $m \in \{1, \ldots, d-2\}$. At first we look at parallelepipeds, the "building blocks" of zonotopes, and show that their second dilate will contain at least one interior integer point:

Theorem 4.1. *Let $d \in \mathbb{Z}_{\geq 1}$, and let $v_1, \ldots, v_d \in \mathbb{Z}^d$ be linearly independent vectors. Let $\Pi_{v_1,\ldots,v_d} := \{\sum_{i=1}^d \lambda_i v_i : 0 \leq \lambda_i \leq 1\}$ be the parallelepiped spanned by $v_1, \ldots, v_d$. Then the second dilate of $\Pi_{v_1,\ldots,v_d}$ contains an interior lattice point.*

Proof. Define $v := \sum_{i=1}^d v_i$. Clearly, v is an integer point.
Since $v_1, \ldots, v_d$ are linearly independent vectors spanning $\mathbb{R}^d$, the interior of $\Pi_{v_1,\ldots,v_d}$ is not empty and

$$\left\{ \sum_{i=1}^{d} \lambda_i v_i : -\tfrac{1}{2} < \lambda_i < \tfrac{1}{2} \right\}$$

is an open neighborhood of the origin. Therefore,

$$\begin{aligned} v + \left\{ \sum_{i=1}^{d} \lambda_i v_i : -\tfrac{1}{2} < \lambda_i < \tfrac{1}{2} \right\} &= \left\{ \sum_{i=1}^{d} \lambda_i v_i : \tfrac{1}{2} < \lambda_i < \tfrac{3}{2} \right\} \\ &\subseteq \left\{ \sum_{i=1}^{d} \lambda_i v_i : 0 \leq \lambda_i \leq 2 \right\} = 2 \cdot \Pi_{v_1,\ldots,v_d}. \end{aligned}$$

is an open neighborhood of v contained in $2 \cdot \Pi_{v_1,\ldots,v_d}$. So v is an interior integer point in the second dilate of $\Pi_{v_1,\ldots,v_d}$. □

By Theorem 2.12, every lattice zonotope can be subdivided into lattice parallelepipeds. It follows that any lattice zonotope with no interior lattice points has at least one integer point in its second dilate. Taking into consideration the relation between the highest coefficient of the h^*-polynomial and the number of integer points in the dilates of the zonotope, it is clear that there can only be d-dimensional zonotopes of degree d or $d-1$. In particular, there are no d-dimensional lattice zonotopes of degree smaller or equal to 2 for $d \in \mathbb{Z}_{\geq 4}$ and no 3-dimensional lattice zonotopes of degree 1. Hence, only 3-dimensional lattice zonotopes of degree 2 will be of interest to us in the following.

In order to study this subset of 3-dimensional zonotopes, we proceed similarly as in the previous chapter by applying Theorem 2.18 for the 3-dimensional case. We obtain that the convex hull of the h^*-polynomials of all 3-dimensional lattice zonotopes is equal to the 3-dimensional simplicial cone

$$\begin{aligned}&(1+4t+t^2)+\mathbb{R}_{\geq 0}(4t+2t^2)+\mathbb{R}_{\geq 0}(2t+4t^2)+\mathbb{R}_{\geq 0}(t+4t^2+t^3)\\ =&\Big\{1+(4+4\alpha+2\beta+\gamma)t+(1+2\alpha+4\beta+4\gamma)t^2+\gamma t^3:\ \ \alpha,\beta,\gamma\in\mathbb{R}_{\geq 0}\Big\}.\end{aligned}\tag{4.1}$$

For the case of 3-dimensional lattice zonotopes of degree 2, the parameter γ is 0. So the 2-dimensional simplicial cone

$$\Big\{1+(4+4\alpha+2\beta)t+(1+2\alpha+4\beta)t^2:\ \alpha,\beta\in\mathbb{R}_{\geq 0}\Big\}\tag{4.2}$$

equals the convex hull of the corresponding h^*-polynomials. We represent each h^*-polynomial $h^*(t)=1+h_1t+h_2t^2$ by a point in $\mathbb{Z}^2$ of the form (h_2,h_1). So we consider $D := \{(1+2\alpha+4\beta, 4+4\alpha+2\beta) : \ \alpha,\beta\in\mathbb{R}_{\geq 0}\}$. See Figure 4.1. The facets of D are

$$\begin{aligned}\bar{S} &:= \{(h_2,h_1) : h_1 = 2h_2+2\geq 1\} \text{ and}\\ \bar{R} &:= \{(h_2,h_1) : h_1 = h_2/2+7/2\geq 1\}.\end{aligned}\tag{4.3}$$

We want to find out for which of the integer points inside this cone there is a 3-dimensional lattice zonotope with a matching h^*-polynomial. We can immediately exclude a lot of points by scrutinizing the properties of the coefficients.

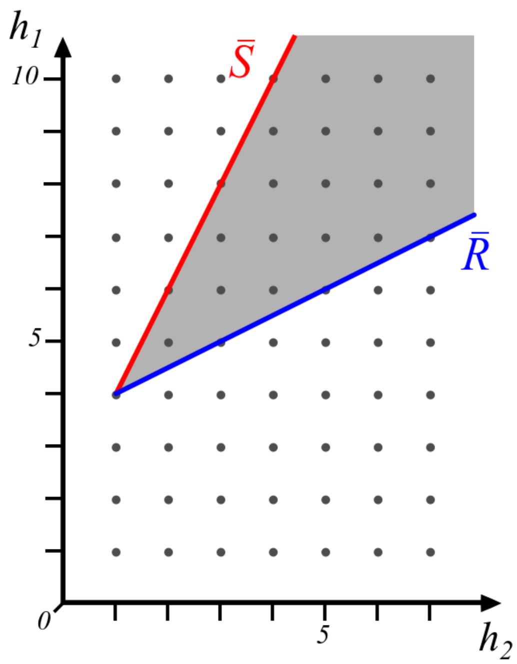


Figure 4.1: The convex hull D of the h^*-polynomials of all 3-dimensional lattice zonotopes of degree 2 viewed as points (h_2, h_1) with facets $\bar{S}$ and $\bar{R}$, where $h^*(t) = 1 + h_1 t + h_2 t^2$.

Inserting $n = 1$ in the Ehrhart polynomial using the expression (2.1), we get

$$\mathrm{ehr}_P(1) = h_0(P)\binom{4}{3} + h_1(P)\binom{3}{3} + h_2(P)\binom{2}{3} = 4 + h_1(P). \tag{4.4}$$

Since P does not contain any interior lattice points, all the integer points in the first dilate lie in the boundary. As discussed before, the number of integer points in a zonotope's boundary is always even because zonotopes are centrally symmetric. Hence, $\mathrm{ehr}_P(1)$ is even and, consequently, so is $h_1(P)$.

We can derive a similar result about the coefficient $h_2(P)$. The second dilate of a lattice zonotope is centrally symmetric about a lattice point. Such lattice zonotopes have an odd number of interior lattice points. Since $h_2(P)$ equals the number of integer points in the interior of the second dilate of the lattice zonotope P, it is clear that $h_2(P)$ is odd.

Furthermore, we know from Theorem 2.12 that each lattice zonotope has a subdivision into lattice parallelepipeds. Since the volume of each lattice parallelepiped is integral according to Lemma 2.14, so is the volume of the zonotope as the sum of the volumes of those lattice parallelepipeds. The property that the sum of the coefficients of the h^*-polynomial equals the zonotope's normalized volume, that is $1 + h_1(P) + h_2(P) = 6 \cdot \mathrm{vol}(P)$, induces that the sum is divisible by 6. Define

$$D' := \{(h_2, h_1) \in D : \; h_1 \text{ even}, \; h_2 \text{ odd}, \; 1 + h_1 + h_2 \text{ divisible by } 6\}. \tag{4.5}$$

We can build a 3-dimensional lattice zonotope Z' of degree 2 from a 2-dimensional zonotope Z that we constructed in the previous chapter (see Theorem 1.1) by considering it in the (x, y)-plane and adding the unit vector of the z-coordinate as a generating vector. The two zonotopes Z' and Z have the same volume and the number of integer points on the boundary of Z' is twice the number of integer points in the zonotope Z. Let $1 + h_1 t + h_2 t^2$ be the h^*-polynomial of Z and $1 + h'_1 t + h'_2 t^2$ the h^*-polynomial of Z'. Then

$$\frac{1 + h_1 + h_2}{2} = \text{vol}(Z) = \text{vol}(Z') = \frac{1 + h'_1 + h'_2}{6}$$

and $6 + 2h_1 = 2 \cdot \text{ehr}_Z(1) = 2 \cdot |Z \cap \mathbb{Z}^2| = |Z' \cap \mathbb{Z}^2| - |Z'^{\circ} \cap \mathbb{Z}^2| = \text{ehr}_{Z'}(1) - 0 = 4 + h'_1$. We calculate that $h'_1 = 2 + 2h_1$ and $h'_2 = h_1 + 3h_2$. We define the function

$$g : \begin{cases} \{(h_2, h_1) : \; 1 + h_1 t + h_2 t^2 \; h^*\text{-polynomial of a 2-dim. lattice zonotope}\} \to \mathbb{Z}^2 \\ (h_2, h_1) \mapsto (h'_2, h'_1) := (h_1 + 3h_2, 2 + 2h_1). \end{cases} \tag{4.6}$$

Under the function g, the set $\{(m, m+1) : m \in \mathbb{Z}_{\geq 0}\} \subset \tilde{R}$ (see Remark 3.1) is mapped to $\{(4m+1, 2m+4) : m \in \mathbb{Z}_{\geq 0}\}$. This equals precisely the set $\bar{R} \cap D'$. The image of the set $\{(0, m) : m \in \mathbb{Z}_{\geq 1} \text{ odd}\} = \tilde{S}$ under g is $\{(m, 2m+2) : m \in \mathbb{Z}_{\geq 1} \text{ odd}\}$. This is the set $\bar{S} \cap D'$.

Lastly, for each $i \in \mathbb{Z}_{\geq 0}$ the set $\{(i + m, i + 3m + 3) \; : \; m \; \in \; \mathbb{Z}_{\geq 0}\} \; \subset \; \tilde{X}_i$ is mapped to

$$Y_i := \{(4i + 6m + 3, 2i + 6m + 8) : m \in \mathbb{Z}_{\geq 0}\}.$$

Define

$$\bar{X}_i := \{(4i + 6m + 3, 2i + 6m + 8) : m \in \mathbb{R}_{\geq 0}\} \tag{4.7}$$

for $i \in \mathbb{Z}_{\geq 0}$. Then $\bar{X}_i$ is the ray containing the set Y_i. The prerequesite that

$$1 + 2i + 6m + 8 + 4i + 6m + 3 = 12 + 6i + 12m$$

is divisible by 6 is satisfied for $i \in \mathbb{Z}_{\geq 0}$ if $2m \in \mathbb{Z}_{\geq 0}$. However, m cannot be half-integral, because then the first coordinate is even and the second is odd. It follows $Y_i = \bar{X}_i \cap D'$. Using the same argument, we can show that all integer points in the intersection of D' and the region enclosed by the rays $\bar{R}$, $\bar{S}$, and $\bar{X}_0$ lie in $\bar{R} \cup \bigcup_{i \geq 0} \bar{X}_i$.

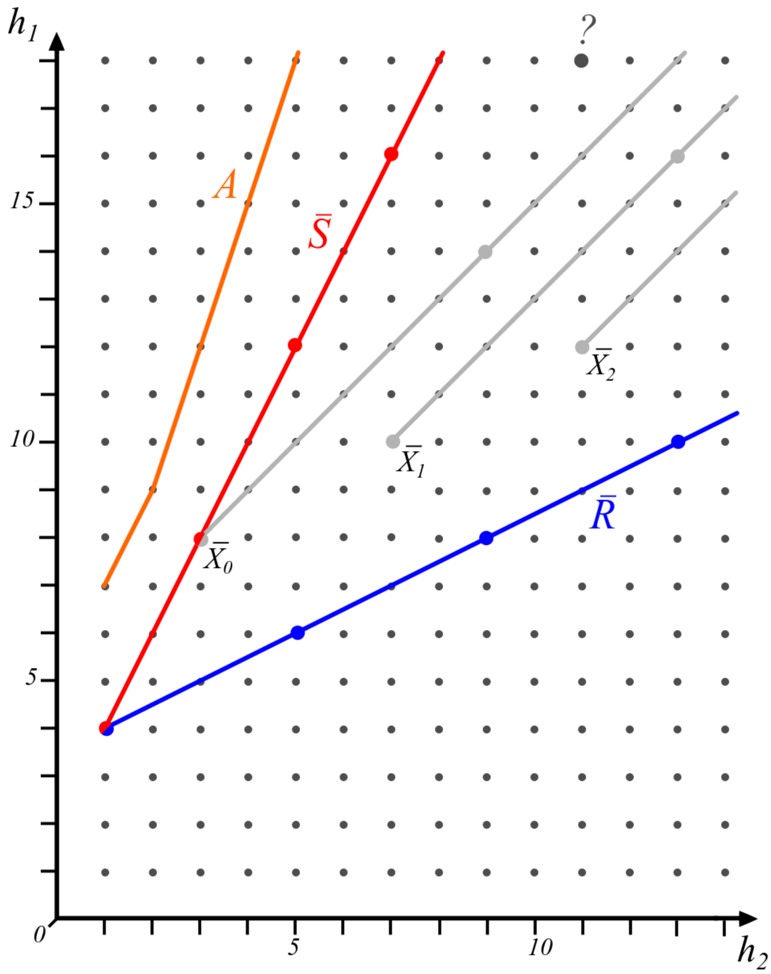


Figure 4.2: The image of the set of points corresponding to the h^*-polynomials of 2-dimensional lattice zonotopes under the function g, as well as the upper bound A for h_1 of h^*-polynomials of degree 2 given by Theorem 2.9.

Now it only remains unclear which of the integer points in the interior of the cone enclosed by the rays $\bar{S}$ and $\bar{X}_0$ correspond to h^*-polynomials of 3-dimensional lattice zonotopes of degree 2. The set of the lattice points in the interior of this cone that satisfy the conditions that h_1 is even, h_2 is odd and $1 + h_1 + h_2$ is divisible by 6, is the image under the function g of the interior integer points of the cone spanned by the rays $\{0\} \times [1, \infty)$ and $\tilde{T}$ in the 2-dimensional case. Scott's Theorem 2.5 shows that there are no 2-dimensional lattice polytopes with h^*-polynomials corresponding to the interior lattice points of that cone. For our study of 3-dimensional lattice zonotopes of degree 2, the upper bound given by Treutlein's Theorem 2.9, a generalized version of Scott's Theorem, is not tight and does not provide us with further information about the points in question as can be seen in Figure 4.2.

In this context, it proves useful to consider the notion of lattice width. See, for example, [1] for the definitions. The **width** of a lattice polytope P in direction $v \in \mathbb{R}^d \setminus \{0\}$ is given by

$$\omega_v(P) := \max\{v^T x : \ x \in P\} - \min\{v^T x : \ x \in P\}.$$

The **lattice width** of P is defined via $\omega(P) := \min\{\omega_v(P) : \ v \in \mathbb{Z}^d \setminus \{0\}\}$. We note that if P is a lattice polytope, $\omega_v(P) \in \mathbb{Z}_{\geq 0}$ for all $v \in \mathbb{Z}^d \setminus \{0\}$ and, thus, $\omega(P)$ takes non-negative integer values.

The property of 3-dimensional degree-2 lattice zonotopes of not having any interior lattice point allows us to deduce information about their lattice width:

Theorem 4.2. *Every 3-dimensional lattice zonotope of degree 2 has lattice width 1 or is a lattice parallelepiped.*

Proof. Let Z be a 3-dimensional lattice zonotope of degree 2 and lattice width bigger than 1. Assume that Z is generated by pairwise linearly independent vectors $v_1, \ldots, v_k \in \mathbb{Z}^3$ for some $k \geq 4$. We will show that Z contains an integer point in its interior.
Since Z is 3-dimensional, there is a set of three generating vectors that forms a basis of the vector space $\mathbb{R}^3$, say $\{v_1, v_2, v_3\}$. Thus, there exist $\alpha_i \in \mathbb{R}_{\geq 0}$ and $\sigma_i \in \{1, -1\}$ for all $i \in \{1, 2, 3\}$ such that $v_4 = \sigma_1\alpha_1 v_1 + \sigma_2\alpha_2 v_2 + \sigma_3\alpha_3 v_3$.

First we consider the case that there is $j \in \{1, 2, 3\}$ such that $\alpha_j = 0$. Then $\alpha_i > 0$ for all $i \in \{1, 2, 3\} \setminus \{j\}$, because the generating vectors are pairwise linearly independent. W.l.o.g. let $\alpha_1 = 0$. Then v_2, v_3, v_4 lie in a hyperplane and $F := Z(v_2, v_3, v_4)$ is a facet of $Z' := Z(v_1, v_2, v_3, v_4)$. In particular, F and $\bar{F} := v_1 + F$ are 2-dimensional lattice zonotopes with three generators each. Therefore, there exist lattice points $x \in \operatorname{relint}(F)$ and $\bar{x} \in \operatorname{relint}(\bar{F})$ due to Theorem 2.11. Since $\omega_v(Z) \geq 2$ for all normal vectors v of F, we know that there are parallel hyperplanes H_1, H_2, H_3 such that $Z \cap \mathbb{Z}^3 \cap H_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$ and there exist $l, m \in \{1, 2, 3\}$ such that $F \subset H_l$ and $\bar{F} \subset H_m$. Let H_2 be a hyperplane that lies between H_1 and H_3. If F or $\bar{F}$ is contained in H_2, the point x or $\bar{x}$ is an

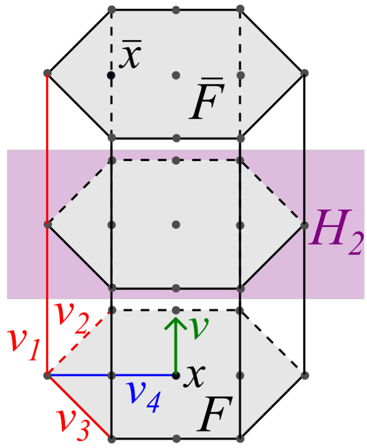


Figure 4.3: The zonotope Z', which coincides in this example with the convex hull of the two opposing facets $\operatorname{conv}(F \cup \bar{F})$.

interior lattice point of Z, respectively. Otherwise, $\operatorname{conv}(F \cup \bar{F}) \subseteq Z$ and $\operatorname{conv}(F \cup \bar{F}) \cap H_2$ is the translate of a 2-dimensional lattice zonotope generated by three vectors and, thereby, contains an interior integer point according to Theorem 2.11.

Now consider that $\alpha_i > 0$ for all $i \in \{1,2,3\}$. We can assume that $\sigma_i = 1$ for all $i \in \{1,2,3\}$, because

$$Z' = Z(v_1, v_2, v_3, v_4) = Z(\sigma_1 v_1, \sigma_2 v_2, \sigma_3 v_3, v_4) + \sum_{\substack{i=1 \\ \sigma_i = -1}}^{3} v_i \quad \text{and}$$

$$v_4 = \sigma_1 \alpha_1 v_1 + \sigma_2 \alpha_2 v_2 + \sigma_3 \alpha_3 v_3 = 1 \cdot \alpha_1(\sigma_1 v_1) + 1 \cdot \alpha_2(\sigma_2 v_2) + 1 \cdot \alpha_3(\sigma_3 v_3).$$

The lattice point v_4 lies in the interior of Z', because for

$$\epsilon := \frac{1}{2 \cdot \max\{1, \alpha_i : i \in \{1,2,3\}\}} \in (0,1)$$

we have $1 - \epsilon \in (0,1)$ and $\epsilon\alpha_i \in (0,1)$ for all $i \in \{1,2,3\}$, and

$$v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \epsilon\alpha_1 v_1 + \epsilon\alpha_2 v_2 + \epsilon\alpha_3 v_3 + (1-\epsilon) v_4.$$

Since $Z' \subseteq Z$, this is a contradiction as Z is of degree 2 and does not contain interior lattice points. $\square$

Lemma 4.3. *Every 3-dimensional lattice zonotope of degree 2 and lattice width 1 is unimodular to a zonotope that can be constructed from a 2-dimensional zonotope Z considering it in the (x,y)-plane* *and adding the unit vector of the z-coordinate as a generating vector.*

Proof. Let Z be a 3-dimensional lattice zonotope of degree 2 and lattice width 1. Let F be a facet of Z such that the width of Z in direction of a normal vector v of F equals 1. Let H and $\bar{H}$ denote the parallel hyperplanes in which F and the symmetric counterpart $\bar{F}$ lie, respectively. Then Z is contained inbetween H and $\bar{H}$. Choose two vectors v_1 and v_2 that generate the lattice structure $H \cap \mathbb{Z}^3$ in the hyperplane H. Since $\omega_v(P) = 1$ for each normal vector v of F, there is an edge of Z, say v_3, with one vertex lying in H and the other in $\bar{H}$ and no integer points in its relative interior. Hence, the three vectors v_1, v_2, v_3 are linearly independent and form a lattice basis of $\mathbb{Z}^3$. We can map v_1, v_2, v_3 to the standard basis of $\mathbb{Z}^3$. The change of basis matrix has the vectors v_1, v_2, v_3 as its columns and determinant 1 or -1. Thus, it is unimodular. □

Now we will consider which polynomials can be realized as Ehrhart polynomials of 3-dimensional lattice parallelpipeds of degree 2 and lattice width bigger than 1. If we can show that no point in $D' \setminus \left(\bigcup_{i\geq 0} \bar{X}_i \cup \bar{S} \cup \bar{R}\right)$ (see (4.3), (4.5), and (4.7) for the notation) can be realized as a 3-dimensional lattice prallelepiped of degree 2, we will have proven that the points in $D' \setminus \left(\bigcup_{i\geq 0} \bar{X}_i \cup \bar{S} \cup \bar{R}\right)$ cannot be realized at all as 3-dimensional lattice zonotopes of degree 2.

Let $u, v, w \in \mathbb{Z}^3$ be linear independent vectors such that $Z := Z(u, v, w)$ is a 3-dimensional lattice parallelepiped of degree 2 and lattice width 2 or bigger. Let $\mathcal{L}(u)$ denote the relative volume of the line segment $Z(u) = \{\lambda u : \lambda \in [0, 1]\}$ w.r.t. to the lattice structure of the ray $\{\lambda u : \lambda \in \mathbb{R}\}$. Then $\mathcal{L}(u) = |Z(u) \cap \mathbb{Z}^3| - 1 \geq 1$, because u is a lattice vector. We call $u \in \mathbb{Z}^3$ **primitive** if $\mathcal{L}(u) = 1$.

Lemma 4.4. *Each 3-dimensional lattice parallelepiped of degree 2 and lattice width bigger than 1 has at most one non-primitive generator.*

Proof. Assume $\mathcal{L}(u) > 1$ and $\mathcal{L}(v) > 1$. Let $\lambda_z := \min\{\lambda \in (0, 1) : \lambda z \in \mathbb{Z}^3\}$ and $\tilde{z} := \lambda_z z$ for all $z \in \{u, v\}$. Then $\tilde{z} \leq z/2$ for all $z \in \{u, v\}$. Consider the facets $F := Z(u, v)$ and $\bar{F} := Z(u, v) + w$ of $Z := Z(u, v, w)$. Let $H_F, H_{\bar{F}}$ be the hyperplanes containing F and $\bar{F}$, respectively. Since $\omega(Z) \geq 2$, there exists a hyperplane H_1 parallel to H and lying between H_F and $H_{\bar{F}}$ such that $H_1 \cap Z \cap \mathbb{Z}^3 \neq \emptyset$. The set $F_1 := H_1 \cap Z$ is a translate of the 2-dimensional lattice zonotope F. Let $\mu \in (0,1)$ such that $F_1 = F + \mu w$. Then

$$\begin{aligned} Y :=& Z(\tilde{u}, \tilde{v}) + \tfrac{1}{2}(\tilde{u} + \tilde{v}) + \mu w \\ =& \left\{ \lambda_1 \tilde{u} + \lambda_2 \tilde{v} : \tfrac{1}{2} \leq \lambda_i \leq \tfrac{3}{2} \text{ for all } i \in \{1,2\} \right\} + \mu w \subseteq \operatorname{rel\,int}(F_1) \subseteq \operatorname{int}(Z) \end{aligned}$$

is the translate of a 2-dimensional lattice parallelepiped. By Lemma 2.10, there exists a lattice point in Y and, thus, in the interior of Z, which is a contradiction to the assumption that Z is of degree 2. □

Next, we will focus on 3-dimensional lattice parallelepipeds of degree 2 with lattice width bigger than 1 and only primitive generators. For that purpose, we refer to solid angles, a generalization of the 2-dimensional angle. The **solid angle** of a point $x \in \mathbb{R}^d$ w.r.t. a convex polytope $P \subset \mathbb{R}^d$ is defined as

$$\alpha_P(x) := \lim_{\epsilon \to 0} \frac{\operatorname{vol}\,(B_\epsilon(x) \cap P)}{\operatorname{vol}\, B_\epsilon(x)}, \tag{4.8}$$

where $B_\epsilon(x)$ denotes the d-dimensional ball of radius $\epsilon > 0$ with center at x. See [4, p. 227] for the definition. Note that if P is not full-dimensional, then $\alpha_P(x) = 0$ for all $x \in \mathbb{R}^d$. Otherwise, $\alpha_P(x) = 0$ if $x \notin P$, and $\alpha_P(x) = 1$ if $x \in P^\circ$. If x lies in the boundary of a full-dimensional polytope P, then $\alpha_P(x) \in (0,1)$. In the following, we will consider the solid angle of a point x w.r.t. a lower-dimensional polytope P within the affine span generated by P and use the same notation $\alpha_P(x)$ for this, i.e., in the definition (4.8) we have the relative volumes w.r.t. the affine span of P instead of the usual volumes in $\mathbb{R}^d$.
To us, the measure

$$A(P) := \sum_{x \in P \cap \mathbb{Z}^d} \alpha_P(x)$$

is of particular interest, because, as shown in [16, Lecture 7],

$$A(P + t) = A(P) = \operatorname{vol}(P) \text{ for a lattice parallelepiped } P \subset \mathbb{R}^d \text{ and } t \in \mathbb{R}^d. \tag{4.9}$$

Lemma 4.5. *Each 3-dimensional lattice parallelepiped Z of degree 2 with lattice width bigger than 1 and only primitive generators satisfies $(h_2(Z), h_1(Z)) \in \{(8,10), (13,10), (3,8), (9,8)\}$.*

Proof. Let u, v, w be primitive. Let $F := Z(u,v)$, $\bar{F} := Z(u,v) + w$, and $H_F, H_{\bar{F}}$ the corre-

sponding hyperplanes. Since Z has lattice width bigger or equal to 2, there is a parallel hyperplane H_1 lying between H_F and $H_{\bar{F}}$ such that $H_1 \cap Z \cap \mathbb{Z}^3 \neq \emptyset$. Let $F_1 := H_1 \cap Z$. Then F_1 is a rational translate of the 2-dimensional lattice zonotope F. Since $Z := Z(u, v, w)$ does not contain interior lattice points, F_1 does not contain integer points in its relative interior. Furthermore, w is a primitive vector. Thus, the vertices of F_1 do not lie in $\mathbb{Z}^3$. Hence, any integer point in F_1 is contained in the relative interior of its facets. As the facets of F_1 are translates of the primitive vectors u and v, each facet can contain at most 1 lattice point in its relative interior. So $A(F_1) \leq 4 \cdot 1/2 = 2$ if we consider F_1 in dimension 2 w.r.t. the lattice structure of $H_1 \cap \mathbb{Z}^3$. Since there is an affine lattice isomorphism mapping $H_1 \cap \mathbb{Z}^3$ to $H_F \cap \mathbb{Z}^3$, we can regard F_1 as a rational translate of F. Due to (4.9), $A(F) = A(F_1) \leq 2$. Therefore, any facet of Z has at most 1 integer point in its relative interior and $\mathrm{ehr}_Z(1) \leq 8 + 6 = 14$.
Assume that the facet F does not contain any integer point in its relative interior, i.e., $|F \cap \mathbb{Z}^3| = 4$. Due to the symmetry of zonotopes, $\bar{F}$ also does not contain a lattice point in its interior. Since Z has width bigger or equal to 2 in the direction of each normal vector of the facet $G := Z(u, w)$, there exists a hyperplane H_2 that is parallel to the hyperplane H_G containing G, lies between G and $\bar{G} := Z(u, w) + v$, and satisfies $H_2 \cap Z \cap \mathbb{Z}^3 \neq \emptyset$. Let $x \in G_1 := H_2 \cap Z$. Then x does not lie in the relative interior of G_1, because Z does not contain interior lattice points, nor is x a vertex of G_1, because v is a primitive vector. Hence, x lies in the relative interior of a facet of G_1. We know that $x \notin G_1 \cap Z(u, v)$ and $x \notin G_1 \cap Z(u, v) + w$, because F and $\bar{F}$ do not have integer points in the relative interior. So x lies in the relative interior of $Z(v, w)$ or $Z(v, w) + u$. It follows that the facets $Z(v, w)$ and $Z(v, w) + u$ each contain exactly 1 lattice point in their respective relative interior because of symmetry. Analogously, we can make the same argument for $Z(v, w)$ and show that also the facets $Z(u, w)$ and $Z(u, w) + v$ each contain exactly 1 lattice point in their respective relative interior. So at most 2 facets of Z cannot contain lattice points in their relative interior, i.e., $\mathrm{ehr}_Z(1) \geq 8 + 4 = 12$.

Due to the symmetric property of zonotopes, any 3-dimensional lattice parallelepiped of degree 2 with only primitive generators and lattice width bigger than 1 contains either 12 or 14 lattice points. The points $(8, 10) \in \bar{X}_i$ and $(13, 10) \in \bar{R}$ are the only two points in D' of the form (h_2, h_1) such that $h_1 + 4 = 14$. The points $(3, 8) \in \bar{S}$ and $(9, 8) \in \bar{R}$ are the only two points in D' satisfying $h_1 + 4 = 12$. □

Remark 4.6. *The point* $(13, 10)$ *can be realized by the parallelepiped* Q *spanned by the vectors* $(1, 1, 0)$, $(-1, 1, 0)$, *and* $(1, 1, 2)$, *which has degree 2 and lattice width 2.*

By Lemma 2.15, the Ehrhart polynomial of Q *is* $\mathrm{ehr}_Q(n) = 4n^3 + 6n^2 + 3n + 1$.

Remark 4.7. *The Ehrhart polynomial corresponding to the* h^**-polynomial* $1 + h_1 t + h_2 t^2$ *is*

$$\mathrm{ehr}(n) = 1 + \frac{11 + 2h_1 - h_2}{6} n + \frac{2 + h_1}{2} n^2 + \frac{1 + h_1 + h_2}{6} n^3.$$

We define the map

$$\psi : \mathbb{Z}^2 \to \mathbb{Z}^3, (h_2, h_1) \mapsto (\tfrac{1+h_1+h_2}{6}, \tfrac{2+h_1}{2}, \tfrac{11+2h_1-h_2}{6}).$$

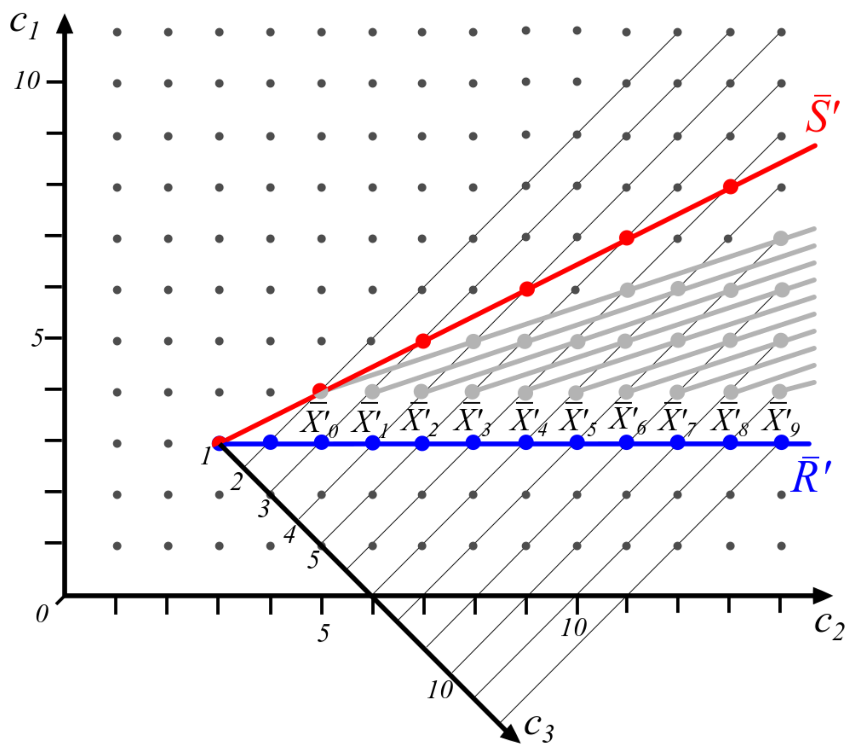


Figure 4.4: The triples (c_3, c_2, c_1) corresponding to the Ehrhart polynomials $1 + c_1 n + c_2 n^2 + c_3 n^3$ of 3-dimensional lattice zonotopes of degree 2.

Conversely, the Ehrhart polynomial $1 + c_1 n + c_2 n^2 + c_3 n^3$ *is associated with the* h^**-polynomial*

$$h^*(t) = 1 + (-3 + c_1 + c_2 + c_3)t + (3 - 2c_1 + 4c_3)t^2.$$

Lemma 4.8. *Each 3-dimensional lattice parallelepiped* Z *of degree 2 with lattice width bigger than 1 and one non-primitive generator satisfies* $(h_2(Z), h_1(Z)) \in \bigcup_{i \geq 0} \bar{X}_i \cup \bar{S} \cup \bar{R}$.

Proof. Let $\mathcal{L}(w) > 1$ and let u, v be primitive. Let $\lambda_w := \min\{\lambda \in (0,1) : \lambda w \in \mathbb{Z}^3\}$ and $\tilde{w} := \lambda_w w$. Then $\tilde{w}$ is a primitive lattice vector. $F_1 := Z(u,v) + \tilde{w}$ has the same lattice structure as the facets $F := Z(u,v)$ and $\bar{F} := Z(u,v) + w$ of $Z := Z(u,v,w)$. Since Z is of degree 2, F_1 does not contain lattice points in its relative interior. Hence, also F and $\bar{F}$ have no integer points in the interior. In particular, $|F \cap \mathbb{Z}^3| = |\bar{F} \cap \mathbb{Z}^3| = 4$, because $\mathcal{L}(u) = \mathcal{L}(v) = 1$.
We can consider Z as a "stack" of translates of the zonotope $\tilde{Z} := Z(u, v, \tilde{w})$, i.e.,

$$Z_i := \tilde{Z} + i\tilde{w} \quad \text{for all } i \in \{0, 1, \ldots, \mathcal{L}(w)\} \quad \text{and} \quad Z = \bigcup_{i=0}^{\mathcal{L}(w)} Z_i.$$

Applying the inclusion-exclusion-law, we obtain

$$\mathrm{ehr}_Z = (\mathcal{L}(w)+1) \cdot \mathrm{ehr}_{\tilde{Z}} - \mathcal{L}(w) \cdot \mathrm{ehr}_F = \mathrm{ehr}_{\tilde{Z}} + \mathcal{L}(w) \cdot (\mathrm{ehr}_{\tilde{Z}} - \mathrm{ehr}_F)\,. \tag{4.10}$$

Since the facet F has relative volume 1 and contains 4 lattice points in its boundary, $\mathrm{ehr}_F(n) = n^2 + 2n + 1$. Moreover, we note that $\tilde{Z}$ has only primitive generators and satisfies $\deg(\tilde{Z}) = 2$ as well as $\omega(\tilde{Z}) \geq 2$.
Hence, any 3-dimensional lattice parallelepiped Z of degree 2 with one non-primitive generator and lattice width bigger than 1 can be described as the union of translates of a lattice parallelepiped $\tilde{Z}$ that corresponds to either one of the points $(3,8) \in \bar{S}$ and $(9,8) \in \bar{R}$.

According to Remark 4.7, the triples of coefficients of the Ehrhart polynomial corresponding to the pairs $(h_2, h_1) = (3,8)$ and $(9,8)$ of coefficients of the h^*-polynomial are $(c_3, c_2, c_1) = (2,5,4)$ and $(3,5,3)$. By (4.10), the sets

$$\begin{aligned} &\{(2,5,4) + k \cdot (2,4,2) : k \in \mathbb{Z}_{\geq 1}\} \subset \bar{S}' \quad \text{and} \\ &\{(3,5,3) + k \cdot (3,4,1) : k \in \mathbb{Z}_{\geq 1}\} \subset \bar{R}' \cup \bigcup_{i \in \mathbb{Z}_{\geq 0}} \bar{X}_i', \end{aligned}$$

where $\bar{S}' := \psi(\bar{S}), \bar{R}' := \psi(\bar{R})$, and $\bar{X}_i{}' := \psi(\bar{X}_i)$ for all $i \in \mathbb{Z}_{\geq 0}$, contain all points corresponding to 3-dimensional lattice parallelepipeds of degree 2 with one non-primitive generator and lattice width bigger than 1. $\square$

We have shown that the Ehrhart polynomial of any 3-dimensional lattice parallelepiped of degree 2 and lattice width bigger than 1 corresponds to a point in $\bigcup_{i\geq 0} \bar{X}_i \cup \bar{S} \cup \bar{R}$. Therefore, the points in consideration, i.e., the points in $D' \setminus \left(\bigcup_{i\geq 0} \bar{X}_i \cup \bar{S} \cup \bar{R}\right)$, cannot be realized as 3-dimensional lattice zonotopes of degree 2.

Theorem 4.9. *The set of the h^*-polynomials $h^*(t) = 1 + h_1 t + h_2 t^2$ of all 3-dimensional lattice zonotopes of degree 2 is the set of integer points $D' \cap \left(\bigcup_{i\geq 0} \bar{X}_i \cup \bar{S} \cup \bar{R}\right)$, where the pair $(h_2, h_1) \in \mathbb{Z}^2$ represents the polynomial $h^*(t) = 1 + h_1 t + h_2 t^2$. See (4.5) and Figure 4.2.*

Remark 4.10. *According to Remark 2.19 and (4.1), we can write the h^*-polynomial of each 3-dimensional lattice zonotope Z of degree 2 as a linear combination of the polynomials $A_1(4,t)$, $A_2(4,t)$, $A_3(4,t)$. There exist $a_2, a_3 \in \mathbb{R}_{\geq 0}$ such that*

$$\begin{aligned} h^*(Z)(t) =& A_1(4,t) + a_2 \cdot A_2(4,t) + a_3 \cdot A_3(4,t) \\ =& (1 + 4t + t^2) + a_2(4t + 2t^2) + a_3(2t + 4t^2). \end{aligned}$$

Under this basis transformation, the ray $\bar{S}$ (see Figure 4.2) is mapped to the positive part of the a_2-axis $\bar{S}''$, $\bar{R}$ to the positive half of the a_3-axis $\bar{R}''$, and the rays $\bar{X}_i$ to the rays $\bar{X}_i{}'' := \{(m+i, m+1) : m \in \mathbb{Z}_{\geq 0}\}$ for $i \in \mathbb{Z}_{\geq 0}$.

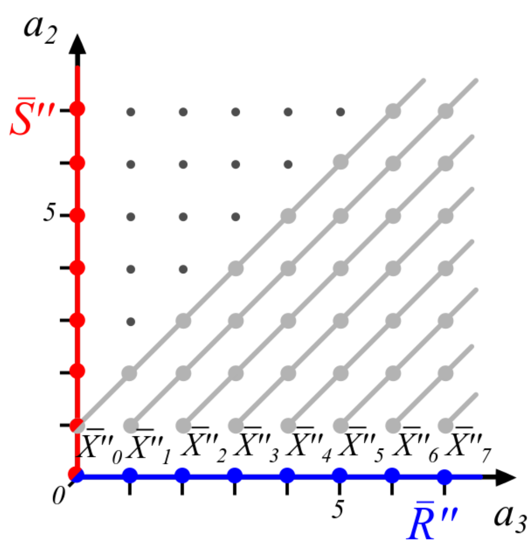


Figure 4.5: The h^*-polynomials of all 3-dimensional lattice zonotopes of degree 2 described in the basis of the polynomials $A_1(4,t), A_2(4,t), A_3(4,t)$.

5 Classification of 3-dimensional Lattice Zonotopes

In the previous chapter, we studied 3-dimensional lattice zonotopes of degree 2. Now we are interested in lattice zonotopes of dimension 3 with highest coefficient of the h^*-polynomial $h_3 \neq 0$. This part of the thesis does not deliver any final results, but is to be understood as a starting point and motivation for further research.

In (4.1), we already found that the simplicial cone

$$\left\{1 + (4 + 4\alpha + 2\beta + \gamma)t + (1 + 2\alpha + 4\beta + 4\gamma)t^2 + \gamma t^3 : \ \alpha, \beta, \gamma \in \mathbb{R}_{\geq 0}\right\}$$

is equal to the convex hull of the h^*-polynomials of all 3-dimensional lattice zonotopes. Let $h_3 = \gamma \in \mathbb{Z}_{\geq 1}$ be fixed.
In [8, Theorem 1, p. 1023], Lagarias and Ziegler showed that the volume of a d-dimensional lattice polytope that contains exactly $k \geq 1$ interior integer points is bounded from above by $k(7k+7)^{d2^{d+1}}$. Inserting $d = 3$ and $k = h_3 = 1$, we obtain the upper bound 14^{48}. Since the sum of all coefficients of the h^*-polynomial is the normalized volume of the corresponding polytope, this also constitutes an upper bound for the coefficients of the h^*- and Ehrhart polynomials of 3-dimensional lattice zonotopes with at least one interior integer point. Hence, there is only a finite number of such polynomials.
By (2.3), h_3 is the number of interior integer points in the corresponding zonotope. So h_3 is odd if and only if h_1 is odd due to (4.4). Moreover, $1 + h_1 + h_2 + h_3$ equals the normalized volume of the zonotope and, thus, is divisble by 6. In particular, the sum is even and, hence, h_2 is odd.
According to [2, Theorem 3.5, p. 9], if $h_3 > 0$, the following inequality is true:

$$h_0 + h_1 \leq h_2 + h_3. \tag{5.1}$$

Say $h_3 = 1$. Then inequality (5.1) becomes $h_1 \leq h_2$.
Figure 3.7 shows that, up to unimodularity, there are three pairwise distinct 2-dimensional lattice zonotopes with $h_2 = 1$, i.e., exactly one interior lattice point. See Figure 5.1. If we consider those zonotopes in the x-y-plane and add the vector $(0,0,2)$ as a generator, we obtain three 3-dimensional zonotopes satisfying $h_3 = 1$.

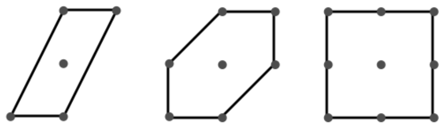


Figure 5.1: Up to unimodularity, the only three pairwise distinct 2-dimensional lattice zonotopes with $h_2 = 1$.

Bibliography

[1] A. Barvinok. *A course in convexity*, volume 54 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle, and R. P. Stanley. Coefficients and roots of Ehrhart polynomials. In *Integer points in polyhedra—geometry, number theory, algebra, optimization*, volume 374 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2005.

[3] M. Beck, K. Jochemko, and E. McCullough. h^*-polynomials of zonotopes. *Trans. Amer. Math. Soc.*, 371(3):2021–2042, 2019.

[4] M. Beck and S. Robins. *Computing the continuous discretely: Integer-point enumeration in polyhedra*. Undergraduate Texts in Mathematics. Springer, New York, 2007.

[5] E. Ehrhart. Sur un problème de géométrie diophantienne linéaire. I. Polyèdres et réseaux. *J. Reine Angew. Math.*, 226:1–29, 1967.

[6] E. Ehrhart. Sur un problème de géométrie diophantienne linéaire. II. Systèmes diophantiens linéaires. *J. Reine Angew. Math.*, 227:25–49, 1967.

[7] M. Henk and M. Tagami. Lower bounds on the coefficients of Ehrhart polynomials. *European J. Combin.*, 30(1):70–83, 2009.

[8] J. C. Lagarias and G. M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canad. J. Math.*, 43(5):1022–1035, 1991.

[9] I. G. Macdonald. Polynomials associated with finite cell-complexes. *J. London Math. Soc. (2)*, 4:181–192, 1971.

[10] G. A. Pick. Geometrisches zur Zahlenlehre. *Sitzungsberichte des deutschen naturwissenschaftlich-medicinischen Vereines für Böhmen "Lotos"*, 19:311–319, 1899.

[11] M. Raman-Sundström and L.-D. Öhman. Mathematical fit: a case study. *Philos. Math. (3)*, 26(2):184–210, 2018.

[12] P. R. Scott. On convex lattice polygons. *Bull. Austral. Math. Soc.*, 15(3):395–399, 1976.

[13] G. C. Shephard. Combinatorial properties of associated zonotopes. *Canadian J. Math.*, 26:302–321, 1974.

[14] R. P. Stanley. Decompositions of rational convex polytopes. *Ann. Discrete Math.*, 6:333–342, 1980.

[15] J. Treutlein. Lattice polytopes of degree 2. *J. Combin. Theory Ser. A*, 117(3):354–360, 2010.

[16] G. M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.