The q-Ehrhart Polynomials of Right-Angled Simplices

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Certification of Approval

I certify that I have read The q-Ehrhart Polynomials of Right-Angled Simplices by Florence Elizabeth Lam and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirement for the degree Masters of Art at San Francisco State University.

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Abstract

Given a right-angled lattice d-simplex $\mathcal{P}$, the corresponding Ehrhart polynomial $L_{\mathcal{P}}(t)$ contains Fourier-Dedekind sums as coefficients. We derive an explicit formula for the q-Ehrhart function $\mathrm{ehr}^{\lambda}_{\mathcal{P}}(q,t)$ of $\mathcal{P}$, which was introduced by Frédéric Chapoton in 2016 as a q-analog of the classical Ehrhart polynomial. Using generating functions combined with Lagrange interpolation, we prove that $\mathrm{ehr}^{\lambda}_{\mathcal{P}}(q,t)$ contains sums analogous to Fourier-Dedekind sums as coefficients, starting from the $d=2$ case and extending to arbitrary dimension. From the $d=2$ case, we obtain a q-analog of Dedekind's reciprocity law. Additionally, we analyze the recovery of the classical Ehrhart polynomial by evaluating the limit of $\mathrm{ehr}^{\lambda}_{\mathcal{P}}(q,t)$ as $q \to 1$. We show that this is straightforward for $d=2$ but computationally challenging for higher dimensions because of the appearance of multiple poles.

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Introduction

This thesis focuses on Ehrhart theory, which studies the problem of counting lattice points in polytopes. In 1962, Eugène Ehrhart proved that if $\mathcal{P}$ is an integral polytope of dimension d, then the number of lattice points in the t^{th} dilate of $\mathcal{P}$ is a polynomial in t of degree d (Theorem 1.1.1). Because of this result, the number of lattice points in the t^{th} dilate of $\mathcal{P}$, denoted $L_{\mathcal{P}}(t)$ (Definition 1.1.9), is called the Ehrhart polynomial. In 2016, Frédéric Chapoton introduced the q-Ehrhart function $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$, a q-analog to the Ehrhart polynomials which assigns weights (via an integral linear form λ) to each lattice point in $\mathcal{P}$ (Definition 1.2.2). He proved a result analogous to Ehrhart's theorem: if $\mathcal{P}$ is an integral polytope of dimension d and λ is an integral form that is generic and positive with respect to $\mathcal{P}$, then $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ is a polynomial in $[t]_q$ with coefficients in $\mathbb{Q}(q)$ (Theorem 1.2.1).

Ehrhart theory also has surprising connections to number theory. In particular, the Ehrhart polynomial of a right-angled integral simplex contains Fourier-Dedekind sums (Definition 1.3.5) as coefficients. The Fourier-Dedekind sum is a generalization of the Dedekind sum (Definition 1.3.4), which appears in the study of the Dedekind eta function as well as in the Todd classes of a toric variety.

The goal of this paper is to derive an explicit formula for the q-Ehrhart polynomial of a right-angled simplex of arbitrary dimension. To do this, we begin by introducing some fundamental concepts in Ehrhart theory in Section 1.1. In Section 1.2, we present an overview of a q-analog of the Ehrhart polynomials introduced by Chapoton, and in Section 1.3, we provide background information on Dedekind sums and Fourier-Dedekind sums along with a few results, such as Dedekind's reciprocity law, that we will apply in Section 2.2.

Section 2 presents the primary findings of this paper. In Section 2.1, we focus on right triangles. We use a generating function for the q-Ehrhart polynomial and apply Lagrange interpolation to arrive at an expression for $\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ in Theorem 2.1.1. With this result, we obtain a result analogous to Dedekind's reciprocity law (Corollary 2.1.2), and in Section 2.2, we recover $L_{\mathcal{P}}(t)$ by taking the limit of $\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ as $q \to 1$ using Sage. The code we use can be found in the Appendix. In Section 2.3, we turn our attention to right-angled tetrahedra. Following the techniques established in Section 2.1, we derive an expression for $\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t)$, as presented in Theorem 2.3.1. In Section 2.4, we show that recovering $L_{\mathcal{P}}(t)$ is more computationally challenging in the $d = 3$ case because of the number of poles involved in computing the limit of $\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ as $q \to 1$. Finally, we repeat the same techniques in previous sections to derive a formula for $\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ for general right-angled d-simplices in Section 2.5.

Chapter 1

Background

In this section, we will give some background information on Ehrhart theory, including some definitions and theorems we will need to better understand q-Ehrhart polynomials.

1.1 Basic Definitions and Theorems

One of the most important tools in combinatorics is the generating function, which represents a sequence of numbers as coefficients in a power series. Informally, it is an object that carries information of a sequence of numbers neatly into a sum. For an introduction to generating functions, we recommend *Generatingfunctionology* by Herbert Wilf [11].

Definition 1.1.1. The *generating function* for a sequence $(a_n)_{n=0}^{\infty}$ is

$$F(z) := \sum_{n\geq 0} a_n z^n.$$

A polytope is a generalization of a polygon to arbitrary dimensions, and we provide a few basic definitions needed for this thesis. For comprehensive introductions to polytopes, we recommend [3, 7, 12].

Definition 1.1.2. Let $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^d$ be a finite set of points. The *convex polytope* $\mathcal{P}$ is the smallest convex set containing all points in this set. In other words,

$$\mathcal{P} := \{\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n \in \mathbb{R}^d : \lambda_k \geq 0 \text{ for all } k \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

Throughout the rest of this thesis, we use the term *polytope* to mean *convex polytope*.

Definition 1.1.3. The *dimension* d of a polytope $\mathcal{P}$ is the dimension of the affine space $\text{span}(\mathcal{P}) = \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in \mathcal{P} \text{ and } \lambda \in \mathbb{R}\}$, and we say $\mathcal{P}$ is a d-polytope.

Definition 1.1.4. A *hyperplane* H is a set $H \subset \mathbb{R}^d$ such that $H = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$ for some nonzero vector $\mathbf{a} \in \mathbb{R}^d$ and some constant $b \in \mathbb{R}$.

For the following definitions, suppose that $\mathcal{P} \subset \mathbb{R}^d$ is a polytope.

Definition 1.1.5. We say a hyperplane $H = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$ is a *supporting hyperplane* of $\mathcal{P}$ if $\mathcal{P}$ lies entirely on one side of H. In other words,

$$\mathcal{P} \subset \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \leq b\} \text{ or } \mathcal{P} \subset \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \geq b\}.$$

Definition 1.1.6. A *face* of $\mathcal{P}$ is a set of the form $\mathcal{P} \cap H$, where H is a supporting hyperplane of $\mathcal{P}$. A *vertex* of a polytope $\mathcal{P}$ is a 0-dimensional face of $\mathcal{P}$. An *edge* of a polytope $\mathcal{P}$ is a

1-dimensional face of $\mathcal{P}$. Finally, a *facet* of a d-polytope $\mathcal{P}$ is a $(d-1)$-dimensional face of $\mathcal{P}$.

Definition 1.1.7. A d-polytope with exactly $d+1$ vertices is called a *d-simplex*.

Fundamental to Ehrhart theory is the concept of dilation, which is the following:

Definition 1.1.8. Let $t \in \mathbb{Z}_{>0}$. The t^{th} dilate of $\mathcal{P}$, written as $t\mathcal{P}$, is the polytope

$$t\mathcal{P} := \{(tx_1, \ldots, tx_d) : (x_1, \ldots, x_d) \in \mathcal{P}\}.$$

Definition 1.1.9. A *lattice polytope* or *integral polytope* $\mathcal{P}$ is a convex polytope such that all of its vertices have integer coordinates.

Definition 1.1.10. Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope and $t \in \mathbb{Z}_{>0}$. The *lattice-point enumerator* of $\mathcal{P}$ is $L_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$.

The lattice-point enumerator $L_{\mathcal{P}}(t)$ is also called the *discrete volume* or *Ehrhart polynomial* of $\mathcal{P}$ in honor of Eugène Ehrhart, whose seminal 1962 paper established that $L_{\mathcal{P}}(t)$ is a polynomial in t of degree d for any lattice d-polytope $\mathcal{P}$.

Theorem 1.1.1 (Ehrhart, [6])**.** *Let $\mathcal{P}$ be a lattice d-polytope. Then $L_{\mathcal{P}}(t)$ is a polynomial in t of degree d with rational coefficients.*

Theorem 1.1.2. *[2, Corollary 3.15] Let $\mathcal{P}$ be a lattice d-polytope. Then the constant term of $L_{\mathcal{P}}(t)$ is* 1.

Looking back at Definition 1.1.10, we can think of $L_{\mathcal{P}}(t)$ as a sequence $(L_{\mathcal{P}}(t))_{t=1}^{\infty}$, and define a generating function for this sequence.

Definition 1.1.11. Let $\mathcal{P}$ be a lattice polytope. The *Ehrhart series* of $\mathcal{P}$ is the following generating function of $L_{\mathcal{P}}(t)$:

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t.$$

If we consider the special case where $\mathcal{P}$ is any lattice polygon in $\mathbb{R}^2$, then the following theorem gives us information on the coefficients of $L_{\mathcal{P}}(t)$:

Theorem 1.1.3 (Pick, [9])**.** *Let $\mathcal{P} \subset \mathbb{R}^2$ be a lattice polygon with area A and B lattice points on its boundary. Then $L_{\mathcal{P}}(t) = At^2 + \frac{1}{2}Bt + 1$.*

1.2 The q-Ehrhart Polynomials

In 2016, Frédéric Chapoton [4] introduced a q-analog of classical Ehrhart polynomials, which replaces the number of lattice points in a polytope $\mathcal{P}$ by a weighted sum.

Definition 1.2.1. An *integral linear form* $\lambda\colon \mathbb{Z}^d \to \mathbb{Z}$ is a function defined by

$$\lambda(\mathbf{m}) = \lambda_1 m_1 + \cdots + \lambda_d m_d,$$

where $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$.

For the purposes of Chapoton's q-Ehrhart function, which we will define shortly, we require that our linear forms λ satisfy the following properties:

- We say that λ is *positive* with respect to $\mathcal{P}$ if $\lambda(\mathbf{x}) \geq 0$ for every vertex $\mathbf{x} \in \mathcal{P}$.

- We say that λ is *generic* with respect to $\mathcal{P}$ if $\lambda(\mathbf{x}) \neq \lambda(\mathbf{y})$ for any edge $\mathbf{x}$—$\mathbf{y}$ of $\mathcal{P}$.

Chapoton's q-analog of the Ehrhart polynomials is the following:

Definition 1.2.2. Let $\lambda \colon \mathbb{Z}^d \to \mathbb{Z}$ be a fixed integral form and $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope. The q-Ehrhart function is

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{m})}.$$

In Chapoton's paper, he proved something similar to Ehrhart's theorem, that the q-Ehrhart function is a polynomial in $[t]_q$, which is the following:

Definition 1.2.3. The q-integer of t is given by

$$[t]_q = \frac{q^t - 1}{q - 1} = 1 + q + \cdots + q^{t-1}.$$

Theorem 1.2.1 (Chapoton, [4])**.** *Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope and λ be an integral linear form that is generic and positive with respect to $\mathcal{P}$. Then there exists a polynomial $cha_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Q}(q)[x]$ such that $cha_{\mathcal{P}}^{\lambda}(q,[t]_q) = ehr_{\mathcal{P}}^{\lambda}(q,t)$ for all $t \in \mathbb{Z}_{>0}$.*

Furthermore, Thomas Kunze and Matthias Beck proved an analogous result to Theorem 1.1.2 for the q-Ehrhart function:

Theorem 1.2.2 (Beck and Kunze, [1])**.** *Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope and λ be an integral linear form that is generic and positive with respect to $\mathcal{P}$. Then the constant term of $cha_{\mathcal{P}}^{\lambda}(q,x)$ is* 1*.*

From now on, when we use the term *integral form*, we assume that it is generic and positive with respect to $\mathcal{P}$. Thus, following Chapoton's theorem, we will use the terms *q-Ehrhart function* and *q-Ehrhart polynomial* interchangeably.

1.3 Dedekind Sums

We continue by introducing more background information, with this section discussing Dedekind sums and their generalization to Fourier–Dedekind sums.

Definition 1.3.1. A function $a\colon \mathbb{Z} \to \mathbb{C}$ is *periodic* on $\mathbb{Z}$ if there exists $b \in \mathbb{Z}_{>0}$ such that $a(n) = a(n+b)$ for any $n \in \mathbb{Z}$. The smallest value of b is the *period* of a.

These functions are special in that they can be expressed as a polynomial in the b^{th} root of unity $\xi_b := e^{2\pi i/b}$.

Theorem 1.3.1. *[2, Theorem 7.2] Let $a\colon \mathbb{Z} \to \mathbb{C}$ be any periodic function on $\mathbb{Z}$ with period b. Then we can express $a(n)$ as the following finite Fourier series expansion:*

$$a(n) = \sum_{k=0}^{b-1} \hat{a}(k)\xi_b^{nk},$$

where the Fourier coefficients are

$$\hat{a}(n) = \frac{1}{b}\sum_{k=0}^{b-1} a(k)\xi_b^{-nk},$$

with $\xi_b = e^{2\pi i/b}$.

Before we define the Dedekind sum, we introduce the fractional part and the sawtooth function:

Definition 1.3.2. The *floor* of $x \in \mathbb{R}$, written as $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The *fractional part* of x is given by $\{x\} = x - \lfloor x \rfloor$.

Definition 1.3.3. Let $x \in \mathbb{R}$. The *sawtooth function* is defined by

$$((x)) = \begin{cases} \{x\} - \frac{1}{2} & x \notin \mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

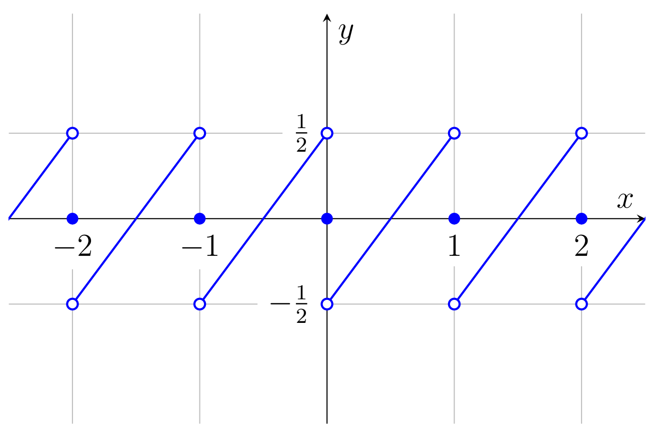

Figure 1.1: The sawtooth function $y = ((x))$.

Notice from the graph of $y = ((x))$ in Figure 1.1 that the sawtooth function is a periodic function with period 1. Now consider the function $\left(\left(\frac{a}{b}\right)\right) : \mathbb{Z} \to \mathbb{C}$, where $a \in \mathbb{Z}$ is a variable and $b \in \mathbb{Z}_{>0}$ is fixed. This function is periodic on the integers with period b, so we can express it as a polynomial in ξ_b by Theorem 1.3.1:

Lemma 1.3.1. *[2, Lemma 7.3] The finite Fourier series for the discrete sawtooth function* $\left(\left(\frac{a}{b}\right)\right) : \mathbb{Z} \to \mathbb{C}$ *is given by*

$$\left(\left(\frac{a}{b}\right)\right) = \frac{1}{2b}\sum_{k=1}^{b-1}\frac{1+\xi_b^k}{1-\xi_b^k}\xi_b^{ak} = \frac{i}{2b}\sum_{k=1}^{b-1}\cot\frac{\pi k}{b}\xi_b^{ak}.$$

Now that we have the definition of the sawtooth function and the fact that $\left(\left(\frac{a}{b}\right)\right)$ is periodic with period b, we can define the Dedekind sum:

Definition 1.3.4. The *Dedekind sum* of two relatively prime integers a and $b > 0$ is given by

$$s(a,b) = \sum_{k=0}^{b-1}\left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right).$$

If we apply Lemma 1.3.1, then we can express the Dedekind sum as the following finite sum:

Lemma 1.3.2. *[2, Lemma 7.4], Let a and b be relatively prime positive integers. Then*

$$s(a,b) = \frac{1}{4b}\sum_{\mu=1}^{b-1}\frac{1+\xi_b^{\mu}}{1-\xi_b^{\mu}}\frac{1+\xi_b^{-\mu a}}{1-\xi_b^{-\mu a}}.$$

The Dedekind sum, named after Richard Dedekind [5], appears in the study of the behavior of the Dedekind eta function

$$\eta(\omega) = e^{\pi i\omega/12}\prod_{k=1}^{\infty}(1-e^{2\pi i\omega k}),$$

where $\omega \in \mathbb{C}$ has positive imaginary part. He also proved the reciprocity law for Dedekind sums:

Theorem 1.3.2 (Dedekind's reciprocity law, [5])**.** *If a and b are relatively prime positive integers, then*

$$s(a,b) + s(b,a) = \frac{1}{12}\left(\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right) - \frac{1}{4}.$$

Hans Rademacher and Don Zagier studied and defined generalizations of the Dedekind sums, and in 1951, L.J. Mordell [8] showed that Dedekind sums appear when counting the number of lattice points in the tetrahedron given by

$$\mathcal{P} = \left\{ (x, y, z) \in \mathbb{R}^3 \colon x, y, z \geq 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1 \right\},$$

where a, b, c are positive coprime integers. Then in 1993, James Pommersheim [10] proved a formula for the Todd class of a toric variety and applied the result to derive a formula for the Ehrhart polynomial for an arbitrary lattice tetrahedron, generalizing Mordell's result and showing a connection between algebraic geometry and Dedekind sums.

We now present a generalization of the Dedekind sum:

Definition 1.3.5. Suppose $\xi_b = e^{2\pi i/b}$ is a b^{th} root of unity, $n \in \mathbb{Z}$, and $a_1, a_2, \ldots, a_m$ are pairwise relatively prime positive. The *Fourier-Dedekind sum* is given by

$$s_n(a_1, a_2, \ldots, a_m; b) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{\left(1 - \xi_b^{ka_1}\right)\left(1 - \xi_b^{ka_2}\right) \cdots \left(1 - \xi_b^{ka_m}\right)}.$$

Here is a property of Fourier-Dedekind sums that we will use in the next chapter. The proof is omitted but can be found in Chapter 8 of [2].

Lemma 1.3.3. *[2, page 150] Let a and b be relatively prime positive integers. Then*

$$s_0(a, 1; b) = -s(a, b) + \frac{b - 1}{4b}.$$

Chapter 2

Main Results

In this chapter, we begin by deriving a formula for the q-Ehrhart polynomials of right triangles, then right-angled tetrahedra, and finally arbitrary right-angled d-simplices.

2.1 The $d = 2$ Case

Consider the right triangle $\mathcal{P} = \{(x, y) \in \mathbb{R}^2_{\geq 0} \colon \frac{x}{a} + \frac{y}{b} \leq 1\}$, where a and b are relatively prime positive integers, and let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2_{>0}$ be an integral linear form. The Ehrhart polynomial of $\mathcal{P}$ is

$$L_{\mathcal{P}}(t) = \# \left\{ (k, \ell) \in \mathbb{Z}^2_{\geq 0} \colon \frac{k}{a} + \frac{\ell}{b} \leq t \right\}.$$

Upon multiplying both sides of the inequality above by ab, we introduce a slack variable $m \in \mathbb{Z}_{\geq 0}$ to convert the inequality to equality, so that

$$L_{\mathcal{P}}(t) = \#\{(k, \ell, m) \in \mathbb{Z}^3_{\geq 0} \colon bk + a\ell + m = abt\}.$$

Thus, $L_{\mathcal{P}}(t)$ counts the number of non-negative integer solutions to the equation $bk+a\ell+m = abt$, and we can view $L_{\mathcal{P}}(t)$ as the coefficient of z^{abt} in the Taylor expansion of a product of three geometric series, each representing one of the variables k, ℓ, and m. More specifically, the generating function for the solutions to the equation above is given by

$$\sum_{t\geq 0} L_{\mathcal{P}}(t) z^{abt} = \left(\sum_{k\geq 0} z^{bk}\right)\left(\sum_{\ell\geq 0} z^{a\ell}\right)\left(\sum_{m\geq 0} z^{m}\right) = \frac{1}{(1-z^b)(1-z^a)(1-z)}.$$

This generating function enumerates all tuples $(k,\ell,m) \in \mathbb{Z}^3_{\geq 0}$ that satisfy the equation $bk + a\ell + m = abt$.

Following the definition of the q-Ehrhart polynomial with an arbitrary integral form $\lambda(\mathbf{m}) = \lambda_1 m_1 + \lambda_2 m_2$, we have the following generating function for the q-Ehrhart polynomial:

$$\begin{aligned}\sum_{k\geq 0} \operatorname{ehr}^{\lambda}_{\mathcal{P}}(q,t) z^{abt} &= \left(\sum_{k\geq 0} (q^{\lambda_1} z^b)^k\right)\left(\sum_{\ell\geq 0} (q^{\lambda_2} z^a)^{\ell}\right)\left(\sum_{m\geq 0} z^{m}\right) \\ &= \frac{1}{(1-q^{\lambda_1} z^b)(1-q^{\lambda_2} z^a)(1-z)}.\end{aligned}$$

This generating function gives us the coefficient of z^{abt}, which is $\operatorname{ehr}^{\lambda}_{\mathcal{P}}(q,t)$. On the other hand, $\operatorname{ehr}^{\lambda}_{\mathcal{P}}(q,t)$ is also the constant term with respect to z of the function

$$f(q,z) = \frac{1}{(1-q^{\lambda_1} z^b)(1-q^{\lambda_2} z^a)(1-z) z^{abt}}.$$

To find the constant term of f, we express f in terms of partial fractions:

$$f(q,z) = \frac{A(q,z)}{1-q^{\lambda_2} z^a} + \frac{B(q,z)}{1-q^{\lambda_1} z^b} + \frac{C}{1-z} + \frac{D_1}{z} + \cdots + \frac{D_{abt}}{z^{abt}},$$

where $C = \lim_{z \to 1} f(q,z) \cdot (1-z) = \frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})}$. Since our goal is to find the constant term of f, we ignore the terms corresponding to the $z = 0$ pole because they have trivial constant term when expanded as a Laurent series..

To find $A(q,z)$, we view q as a constant and use Lagrange interpolation: since $A(q,z)$ is a polynomial in z with degree less than a, if we know the values of $A(q,z)$ at a points, then we can determine what $A(q,z)$ is. The denominator corresponding to $A(q,z)$, $1 - q^{\lambda_2} z^a$, is equal to zero at

$$\alpha_k = q^{\frac{-\lambda_2}{a}} \exp\left(\frac{2\pi i k}{a}\right), \text{ for } 0 \leq k \leq a-1.$$

Each zero of the denominator is a simple pole of the rational function $\frac{A(q,z)}{1-q^{\lambda_2} z^a}$, so we can find the value of $A(q,\alpha_k)$ by taking the limit of $A(q,z)$ as z approaches α_k:

$$\begin{aligned}
A(q,\alpha_k) &= \lim_{z \to \alpha_k} f(q,z) \cdot (1 - q^{\lambda_2} z^a) \\
&= \lim_{z \to \alpha_k} \frac{1}{(1-q^{\lambda_1} z^b)(1-q^{\lambda_2} z^a)(1-z) z^{abt}} \cdot (1 - q^{\lambda_2} z^a) \\
&= \frac{q^{\lambda_2 bt}}{(1-q^{\lambda_1}\alpha_k^b)(1-\alpha_k)}. \qquad (2.1)
\end{aligned}$$

Therefore,

$$A(q,z) = \sum_{k=0}^{a-1} A(q,\alpha_k) \prod_{j \neq k} \frac{z - \alpha_j}{\alpha_k - \alpha_j}.$$

We are interested in finding $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \mathrm{const}(f)$, which means that we only care about the value of

$$A(q,0) = \sum_{k=0}^{a-1} A(q,\alpha_k) \prod_{j \neq k} \frac{-\alpha_j}{\alpha_k - \alpha_j}. \qquad (2.2)$$

The product can be simplified if we write down what α_k is equal to:

$$\prod_{\substack{j\neq k\\0\leq j\leq a-1}}\frac{\alpha_j}{\alpha_j-\alpha_k}=\prod_{\substack{j\neq k\\0\leq j\leq a-1}}\frac{e^{\frac{2\pi ij}{a}}}{e^{\frac{2\pi ij}{a}}-e^{\frac{2\pi ik}{a}}}\tag{2.3}$$
$$=\prod_{\substack{j\neq k\\0\leq j\leq a-1}}\frac{1}{1-e^{\frac{2\pi i(k-j)}{a}}}.$$

Since $j\neq k$, we have $1\leq k-j\leq a-1$. Applying the change of index variable $m=k-j$,

$$\prod_{\substack{j\neq k\\0\leq j\leq a-1}}\frac{1}{1-e^{\frac{2\pi i(k-j)}{a}}}=\prod_{m=1}^{a-1}\frac{1}{1-e^{\frac{2\pi im}{a}}}=\frac{1}{\prod_{m=1}^{a-1}(1-e^{\frac{2\pi im}{a}})}.$$

To evaluate $\prod_{m=1}^{a-1}(1-e^{\frac{2\pi im}{a}})$, consider the polynomial z^a-1. The zeros of this polynomial are the a^{th} roots of unity, so

$$z^a-1=\prod_{m=0}^{a-1}(z-e^{\frac{2\pi im}{a}})=(z-1)\prod_{m=1}^{a-1}(z-e^{\frac{2\pi im}{a}}).$$

After rearranging, we have

$$\prod_{m=1}^{a-1}(z-e^{\frac{2\pi im}{a}})=\frac{z^a-1}{z-1},$$

and so

$$\prod_{m=1}^{a-1}(1-e^{\frac{2\pi im}{a}})=\lim_{z\to 1}\prod_{m=1}^{a-1}(z-e^{\frac{2\pi im}{a}})=\lim_{z\to 1}\frac{z^a-1}{z-1}=\lim_{z\to 1}az^{a-1}=a.$$

Therefore, the product in (2.3) is equal to $\frac{1}{a}$, and plugging in the values of $A(q,\alpha_k)$ from (2.1) into (2.2), we get a concrete expression for $A(q,0)$:

$$A(q,0)=\frac{q^{\lambda_2bt}}{a}\sum_{k=0}^{a-1}\frac{1}{(1-q^{\lambda_1}\alpha_k^b)(1-\alpha_k)}.$$

To find $B(q,0)$, apply the same methods as above to obtain

$$B(q,0) = \frac{q^{\lambda_1 at}}{b} \sum_{\ell=0}^{b-1} \frac{1}{(1-q^{\lambda_2}\beta_\ell^a)(1-\beta_\ell)},$$

where $\beta_\ell = q^{\frac{-\lambda_1}{b}} \exp(\frac{2\pi i \ell}{b})$, $0 \le \ell \le b-1$, is a zero of $1-q^{\lambda_1}z^b$, the denominator corresponding to $B(q,z)$.

Now that we have the expressions for $A(q,0)$, $B(q,0)$, and C, we have the following formula for the q-Ehrhart polynomial and a reciprocity property:

Theorem 2.1.1. *Suppose* $\mathcal{P} = \{(x,y) \in \mathbb{R}^2_{\ge 0} \colon \frac{x}{a} + \frac{y}{b} \le 1\}$ *is a right triangle,* a *and* b *are relatively prime positive integers, and* $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2_{>0}$ *is an integral linear form. Then*

$$\begin{aligned} \mathrm{ehr}^\lambda_{\mathcal{P}}(q,t) = {} & \frac{q^{\lambda_2 bt}}{a} \sum_{k=0}^{a-1} \frac{1}{(1-q^{\lambda_1}\alpha_k^b)(1-\alpha_k)} + \frac{q^{\lambda_1 at}}{b} \sum_{\ell=0}^{b-1} \frac{1}{(1-q^{\lambda_2}\beta_\ell^a)(1-\beta_\ell)} \\ & + \frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})}, \end{aligned} \tag{2.4}$$

equivalently,

$$\begin{aligned} \mathrm{cha}^\lambda_{\mathcal{P}}(q,x) = {} & \frac{((q-1)x+1)^{\lambda_2 b}}{a} \sum_{k=0}^{a-1} \frac{1}{(1-q^{\lambda_1}\alpha_k^b)(1-\alpha_k)} \\ & + \frac{((q-1)x+1)^{\lambda_1 a}}{b} \sum_{\ell=0}^{b-1} \frac{1}{(1-q^{\lambda_2}\beta_\ell^a)(1-\beta_\ell)} + \frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})} \end{aligned} \tag{2.5}$$

where $\alpha_k = q^{\frac{-\lambda_2}{a}} \exp\left(\frac{2\pi i k}{a}\right)$ *and* $\beta_\ell = q^{\frac{-\lambda_1}{b}} \exp(\frac{2\pi i \ell}{b})$.

To obtain $\mathrm{cha}^\lambda_{\mathcal{P}}(q,x)$ from $\mathrm{ehr}^\lambda_{\mathcal{P}}(q,t)$, first apply the substitution $q^{kt} = ((q-1)[t]_q + 1)^k$ from Definition 1.2.3. By Theorem 1.2.1, $\mathrm{ehr}^\lambda_{\mathcal{P}}(q,t) = \mathrm{cha}^\lambda_{\mathcal{P}}(q,[t]_q)$, where $\mathrm{cha}^\lambda_{\mathcal{P}}$ is a polynomial in $[t]_q$. Finally, set $x = [t]_q$.

If we examine the constant term of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$, we obtain a reciprocity law analogous to Dedekind's reciprocity law:

Corollary 2.1.2.

$$\frac{1}{a}\sum_{k=0}^{a-1}\frac{1}{(1-q^{\lambda_1}\alpha_k^b)(1-\alpha_k)}+\frac{1}{b}\sum_{\ell=0}^{b-1}\frac{1}{(1-q^{\lambda_2}\beta_\ell^a)(1-\beta_\ell)}+\frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})}=1.$$

Proof. By the binomial theorem,

$$((q-1)x+1)^{\lambda_2 b}=\sum_{\hat{k}=0}^{\lambda_2 b}\binom{\lambda_2 b}{\hat{k}}((q-1)x)^{\hat{k}}$$

and

$$((q-1)x+1)^{\lambda_1 a}=\sum_{\hat{\ell}=0}^{\lambda_1 a}\binom{\lambda_1 a}{\hat{\ell}}((q-1)x)^{\hat{\ell}}.$$

Thus, the constant term of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ is

$$\frac{1}{a}\sum_{k=0}^{a-1}\frac{1}{(1-q^{\lambda_1}\alpha_k^b)(1-\alpha_k)}+\frac{1}{b}\sum_{\ell=0}^{b-1}\frac{1}{(1-q^{\lambda_2}\beta_\ell^a)(1-\beta_\ell)}+\frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})},$$

which is equal to 1 by Theorem 1.2.2. □

2.2 Recovering $L_{\mathcal{P}}(t)$ from $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ for the $d=2$ case

A natural question that follows from these computations is whether we can recover the classical Ehrhart polynomial of $\mathcal{P}$ by taking the limit of $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ as q approaches one. This is possible, and we show the derivation here. Notice that $\lim\limits_{q\to 1}\alpha_k=\xi_a^k=\exp\left(\frac{2\pi i k}{a}\right)$ and $\lim\limits_{q\to 1}\beta_\ell=\xi_b^\ell=\exp\left(\frac{2\pi i\ell}{b}\right)$. All terms in $A(q,0)$ and $B(q,0)$, except for the $k=0$ and $\ell=0$

terms, have well-defined limits as q approaches 1. Therefore, finding the limit reduces to isolating the terms with singularities at $q = 1$, namely the $k = 0$ and $\ell = 0$ terms, and C, and computing the limit of their sum. Let $\tilde{A}(q,0)$ and $\tilde{B}(q,0)$ be the $k = 0$ and $\ell = 0$ terms respectively. Then

$$\begin{aligned}
&\lim_{q\to 1} \tilde{A}(q,0) + \tilde{B}(q,0) + C \\
&= \lim_{q\to 1} \frac{q^{\lambda_2 bt}}{a(1-q^{\lambda_1}\alpha_0^b)(1-\alpha_0)} + \frac{q^{\lambda_1 at}}{b(1-q^{\lambda_2}\beta_0^a)(1-\beta_0)} + \frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})} \qquad (2.6)\\
&= \lim_{q\to 1} \frac{q^{\lambda_2 bt}}{a(1-q^{\lambda_1-(\lambda_2 b/a)})(1-q^{-\lambda_2/a})} + \frac{q^{\lambda_1 at}}{b(1-q^{\lambda_2-(\lambda_1 a/b)})(1-q^{-\lambda_1/b})} + \frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})}.
\end{aligned}$$

If we make the substitution $q = e^\epsilon$, then the limit as $q \to 1$ is equivalent to the limit as $\epsilon \to 0$. Every term in (2.6) is of the form $T = \frac{ce^{r\epsilon}}{(1-e^{u\epsilon})(1-e^{v\epsilon})}$, so we define the following variables to express $\tilde{A}(q,0), \tilde{B}(q,0)$, and C as Laurent series centered at $\epsilon = 0$:

- $\tilde{A}(q,0)$: $c_1 = \frac{1}{a}$, $r_1 = b\lambda_2 t$, $u_1 = \frac{a\lambda_1 - b\lambda_2}{a}$, $v_1 = -\frac{\lambda_2}{a}$
- $\tilde{B}(q,0)$: $c_2 = \frac{1}{b}$, $r_2 = a\lambda_1 t$, $u_2 = -\frac{a\lambda_1 - b\lambda_2}{b}$, $v_2 = -\frac{\lambda_1}{b}$
- C: $c_3 = 1$, $r_3 = 0$, $u_3 = \lambda_1$, $v_3 = \lambda_2$.

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n}$, where B_n is the n^{th} Bernoulli number, T has the following Laurent series expansion centered at $\epsilon = 0$:

$$T = \left[\frac{c}{uv}\right]\frac{1}{\epsilon^2} + \left[\frac{c}{uv}\left(r - \frac{u+v}{2}\right)\right]\frac{1}{\epsilon} + \left[\frac{c}{uv}\left(\frac{r^2}{2} - r\frac{u+v}{2} + \frac{u^2+v^2+3uv}{12}\right)\right]\epsilon^0 + \cdots$$

Using Sage (see Appendix for code), we can compute the Laurent series of the sum $\tilde{A}(q,0) + \tilde{B}(q,0) + C$. Consistent with our expectations, the coefficients of the ϵ^{-2} and ϵ^{-1} terms are both zero. Therefore, the limit of the sum as $\epsilon \to 0$ is equal to the constant term of the Laurent series, which is

$$\lim_{q\to 1} \tilde{A}(q,0) + \tilde{B}(q,0) + C = \frac{ab}{2}t^2 + \frac{a+b+1}{2}t + \frac{a^2+b^2+3ab+3a+3b+1}{12ab}.$$

Next, combine the other terms in $A(q,0)$ and $B(q,0)$ to obtain

$$\begin{aligned} L_{\mathcal{P}}(t) &= \lim_{q\to 1} \mathrm{ehr}^{\lambda}_{\mathcal{P}}(q,t) \\ &= \frac{ab}{2}t^2 + \frac{a+b+1}{2}t + \frac{a^2+b^2+3ab+3a+3b+1}{12ab} \\ &\quad + \frac{1}{a}\sum_{k=1}^{a-1} \frac{1}{(1-\xi_a^{kb})(1-\xi_a^{k})} + \frac{1}{b}\sum_{\ell=1}^{b-1} \frac{1}{(1-\xi_b^{\ell a})(1-\xi_b^{\ell})} \\ &= \frac{ab}{2}t^2 + \frac{a+b+1}{2}t + \frac{a^2+b^2+3ab+3a+3b+1}{12ab} + s_0(b,1;a) + s_0(a,1;b). \end{aligned} \tag{2.7}$$

We can further simplify the Fourier-Dedekind sums above using Lemma 1.3.3 and Theorem 1.3.2:

$$\begin{aligned} s_0(b,1;a) + s_0(a,1;b) &= -s(b,a) + \frac{a-1}{4a} - s(a,b) + \frac{b-1}{4b} \\ &= -\frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) + \frac{1}{4} + \frac{a-1}{4a} + \frac{b-1}{4b}. \end{aligned} \tag{2.8}$$

Combining expressions (2.7) and (2.8), the constant term simplifies to one as promised by Theorem 1.1.2, and

$$L_{\mathcal{P}}(t) = \frac{ab}{2}t^2 + \frac{a+b+1}{2}t + 1,$$

Observe that the area of a right triangle with sides of length a and b is $A = \frac{ab}{2}$, and if we apply the inclusion-exclusion principle, the number of lattice points on the boundary is $B = a+b+1$ as there are $a+b+1$ points along the legs and vertices, and no lattice points along the hypotenuse because a and b are coprime. Thus, the result here matches our expectation from Theorem 1.1.3.

2.3 The $d = 3$ Case

Now consider the right-angled tetrahedron $\mathcal{P} = \{(x, y, z) \in \mathbb{R}^3_{\geq 0} \colon \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1\}$, where a, b, and c are pairwise relatively prime positive integers, and let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3_{>0}$ be some integral linear form. We will follow the techniques used in the $d = 2$ case. The Ehrhart polynomial of $\mathcal{P}$ is

$$\begin{aligned} L_{\mathcal{P}}(t) &= \# \left\{ (k, \ell, m) \in \mathbb{Z}^3_{\geq 0} \colon \frac{k}{a} + \frac{\ell}{b} + \frac{m}{c} \leq t \right\} \\ &= \#\{(k, \ell, m, n) \in \mathbb{Z}^4_{\geq 0} \colon bck + ac\ell + abm + n = abct\}. \end{aligned}$$

Since $L_{\mathcal{P}}(t)$ counts the number of non-negative integer solutions to the equation

$$bck + ac\ell + abm + n = abct,$$

the generating function for the solutions to this equation is given by

$$\sum_{t \geq 0} L_{\mathcal{P}}(t) z^{abct} = \left(\sum_{k \geq 0} z^{bck} \right) \left(\sum_{\ell \geq 0} z^{ac\ell} \right) \left(\sum_{m \geq 0} z^{abm} \right) \left(\sum_{n \geq 0} z^{n} \right).$$

The generating function for the q-Ehrhart polynomial with an arbitrary integral form $\lambda(\mathbf{m}) = \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3$ is

$$\begin{aligned} &\left(\sum_{k \geq 0} (q^{\lambda_1} z^{bc})^k \right) \left(\sum_{\ell \geq 0} (q^{\lambda_2} z^{ac})^\ell \right) \left(\sum_{m \geq 0} (q^{\lambda_3} z^{ab})^m \right) \left(\sum_{n \geq 0} z^{n} \right) \\ &= \frac{1}{(1 - q^{\lambda_1} z^{bc})(1 - q^{\lambda_2} z^{ac})(1 - q^{\lambda_3} z^{ab})(1 - z)}. \end{aligned}$$

This generating function gives the coefficient of z^{abct}, and the coefficient of z^{abct} is the constant term (with respect to z) of the function

$$f(q,z) = \frac{1}{(1-q^{\lambda_1}z^{bc})(1-q^{\lambda_2}z^{ac})(1-q^{\lambda_3}z^{ab})(1-z)z^{abct}}.$$

This means that $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \mathrm{const}(f)$. To find this limit, we express f in partial fractions:

$$f(q,z) = \frac{A(q,z)}{1-q^{\lambda_1}z^{bc}} + \frac{B(q,z)}{1-q^{\lambda_2}z^{ac}} + \frac{C(q,z)}{1-q^{\lambda_3}z^{ab}} + \frac{D}{1-z} + \sum_{j=1}^{abct}\frac{E_j}{z^j},$$

where $D = \lim_{z\to 1} f(q,z)\cdot(1-z) = \frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})(1-q^{\lambda_3})}$. Like in the $d=2$ case, the terms corresponding to the $z=0$ pole are not relevant when evaluating $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$, and we use Lagrange interpolation to find $A(q,z), B(q,z)$, and $C(q,z)$. First, we find $A(q,z)$. The rational function $\frac{A(q,z)}{1-q^{\lambda_1}z^{bc}}$ has simple poles at

$$\alpha_k = q^{\frac{-\lambda_1}{bc}}\exp\left(\frac{2\pi i k}{bc}\right), 0\le k\le bc-1,$$

and the value of $A(q,\alpha_k)$ is

$$A(q,\alpha_k) = \lim_{z\to\alpha_k} f(q,z)\cdot(1-q^{\lambda_2}z^{bc}) = \frac{q^{\lambda_1 at}}{(1-q^{\lambda_2}\alpha_k^{ac})(1-q^{\lambda_3}\alpha_k^{ab})(1-\alpha_k)}.$$

Thus,

$$A(q,z) = \sum_{k=0}^{bc-1} A(q,\alpha_k)\prod_{j\neq k}\frac{z-\alpha_j}{\alpha_k-\alpha_j},$$

and

$$A(q,0) = \sum_{k=0}^{bc-1} A(q,\alpha_k)\prod_{j\neq k}\frac{\alpha_j}{\alpha_j-\alpha_k}.$$

Following the same techniques as above, the product equals $\frac{1}{bc}$, so

$$A(q,0) = \frac{q^{\lambda_1 at}}{bc}\sum_{k=0}^{bc-1}\frac{1}{(1-q^{\lambda_2}\alpha_k^{ac})(1-q^{\lambda_3}\alpha_k^{ab})(1-\alpha_k)}.$$

Similarly, let $\beta_\ell = q^{\frac{-\lambda_2}{ac}} \exp(\frac{2\pi i\ell}{ac})$, $0 \le \ell \le ac-1$, be the simple poles of the rational function $\frac{B(z)}{1-q^{\lambda_2} z^{ac}}$. And, let $\gamma_m = q^{\frac{-\lambda_3}{ab}} \exp\left(\frac{2\pi i m}{ab}\right)$, $0 \le m \le ab-1$, be the simple poles of the rational function $\frac{C(z)}{1-q^{\lambda_3} z^{ab}}$. Then

$$B(q,0) = \frac{q^{\lambda_2 bt}}{ac} \sum_{\ell=0}^{ac-1} \frac{1}{(1-q^{\lambda_1}\beta_\ell^{bc})(1-q^{\lambda_3}\beta_\ell^{ab})(1-\beta_\ell)}$$

and

$$C(q,0) = \frac{q^{\lambda_3 ct}}{ab} \sum_{m=0}^{ab-1} \frac{1}{(1-q^{\lambda_1}\gamma_m^{bc})(1-q^{\lambda_2}\gamma_m^{ac})(1-\gamma_m)},$$

which gives us the following expression for the q-Ehrhart polynomial:

Theorem 2.3.1. *Suppose $\mathcal{P} = \{(x,y,z) \in \mathbb{R}^3_{\ge 0} \colon \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \le 1\}$ is a right-angled tetrahedron, $a, b,$ and c are pairwise relatively prime positive integers, and $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3_{>0}$ is some integral linear form. Then*

$$\begin{aligned}
\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = {} & \frac{q^{\lambda_1 at}}{bc} \sum_{k=0}^{bc-1} \frac{1}{(1-q^{\lambda_2}\alpha_k^{ac})(1-q^{\lambda_3}\alpha_k^{ab})(1-\alpha_k)} \\
& + \frac{q^{\lambda_2 bt}}{ac} \sum_{\ell=0}^{ac-1} \frac{1}{(1-q^{\lambda_1}\beta_\ell^{bc})(1-q^{\lambda_3}\beta_\ell^{ab})(1-\beta_\ell)} \\
& + \frac{q^{\lambda_3 ct}}{ab} \sum_{m=0}^{ab-1} \frac{1}{(1-q^{\lambda_1}\gamma_m^{bc})(1-q^{\lambda_2}\gamma_m^{ac})(1-\gamma_m)} + \frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})(1-q^{\lambda_3})},
\end{aligned}$$

equivalently,

$$
\begin{aligned}
\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) = {} & \frac{((q-1)x+1)^{\lambda_1 a}}{bc}\sum_{k=0}^{bc-1}\frac{1}{(1-q^{\lambda_2}\alpha_k^{ac})(1-q^{\lambda_3}\alpha_k^{ab})(1-\alpha_k)} \\
& + \frac{((q-1)x+1)^{\lambda_2 b}}{ac}\sum_{\ell=0}^{ac-1}\frac{1}{(1-q^{\lambda_1}\beta_\ell^{bc})(1-q^{\lambda_3}\beta_\ell^{ab})(1-\beta_\ell)} \\
& + \frac{((q-1)x+1)^{\lambda_3 c}}{ab}\sum_{m=0}^{ab-1}\frac{1}{(1-q^{\lambda_1}\gamma_m^{bc})(1-q^{\lambda_2}\gamma_m^{ac})(1-\gamma_m)} \\
& + \frac{1}{(1-q^{\lambda_1})(1-q^{\lambda_2})(1-q^{\lambda_3})},
\end{aligned}
$$

where $\alpha_k = q^{\frac{-\lambda_1}{bc}}\exp\left(\frac{2\pi ik}{bc}\right)$, $\beta_\ell = q^{\frac{-\lambda_2}{ac}}\exp(\frac{2\pi i\ell}{ac})$, *and* $\gamma_m = q^{\frac{-\lambda_3}{ab}}\exp\left(\frac{2\pi im}{ab}\right)$.

2.4 Recovering $L_{\mathcal{P}}(t)$ from $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ for the $d=3$ case

To recover $L_{\mathcal{P}}(t)$, we take the limit of $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ as $q\to 1$. Notice that

$$
\begin{aligned}
\lim_{q\to 1}\alpha_k &= \xi_{bc}^{k} = \exp\left(\frac{2\pi ik}{bc}\right), \\
\lim_{q\to 1}\beta_\ell &= \xi_{ac}^{\ell} = \exp\left(\frac{2\pi i\ell}{ac}\right), \\
\lim_{q\to 1}\gamma_m &= \xi_{ab}^{m} = \exp\left(\frac{2\pi im}{ab}\right).
\end{aligned}
$$

All terms in $A(q,0)$, $B(q,0)$, and $C(q,0)$, except for the $k=0,b,c$ and $\ell=0,a,c$ and $m=0,a,b$ terms, respectively, have well-defined limits as $q\to 1$. Finding $L_{\mathcal{P}}(t)$ amounts to bookkeeping these terms along with D, computing the limit

$$
\lim_{q\to 1}(A_{k=0}+A_{k=b}+A_{k=c}+B_{\ell=0}+B_{\ell=a}+B_{\ell=c}+C_{m=0}+C_{m=a}+C_{m=b}+D),
$$

and adding the limits of the remaining terms that have well-defined limits as $q \to 1$. This can be done using Sage, but is computationally tedious.

2.5 The General Dimension Case

Finally, consider the right-angled d-simplex

$$\mathcal{P} = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d_{\geq 0} \colon \sum_{i=1}^{d} \frac{x_i}{a_i} \leq 1 \right\},$$

where $a_1, \ldots, a_d$ are pairwise relatively prime positive integers, and let $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d_{>0}$ be some integral linear form. Upon introducing a slack variable, we have

$$L_{\mathcal{P}}(t) = \# \left\{ (k_1, \ldots, k_d, m) \in \mathbb{Z}^{d+1}_{\geq 0} \colon \sum_{i=1}^{d} \widehat{a_i} k_i + m = At \right\},$$

where $A = \prod_{i=1}^{d} a_i$, and $\widehat{a_i} = \frac{A}{a_i}$. Repeating the methods we used above, since $L_{\mathcal{P}}(t)$ counts the number of non-negative integer solutions to the equation $\sum_{i=1}^{d} \widehat{a_i} k_i + m = At$, the generating function for the solutions to this equation is given by

$$\sum_{t \geq 0} L_{\mathcal{P}}(t) z^{At} = \left(\sum_{m \geq 0} z^m \right) \prod_{i=1}^{d} \left(\sum_{k_i \geq 0} \left(z^{\widehat{a_i}} \right)^{k_i} \right),$$

so the generating function for the q-Ehrhart polynomial is

$$\left(\sum_{m \geq 0} z^m \right) \prod_{i=1}^{d} \left(\sum_{k_i \geq 0} \left(q^{\lambda_i} z^{\widehat{a_i}} \right)^{k_i} \right) = \frac{1}{(1-z) \prod_{i=1}^{d} \left(1 - q^{\lambda_i} z^{\widehat{a_i}}\right)}.$$

Similar to the previous cases, this generating function gives the coefficient of z^{At}, which is the constant term (with respect to z) of the function

$$f(q,z) = \frac{1}{z^{At}(1-z) \prod_{i=1}^{d} \left(1 - q^{\lambda_i} z^{\widehat{a_i}}\right)}.$$

In other words, $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \mathrm{const}(f)$. To find this limit, we express f in partial fractions:

$$f(q,z) = \frac{B}{1-z} + \sum_{i=1}^{d} \frac{A_i(q,z)}{1-q^{\lambda_i} z^{\widehat{a_i}}} + \sum_{j=1}^{A} \frac{C_j}{z^j},$$

where $B = \lim_{z\to 1} f(q,z)\cdot(1-z) = \prod_{i=1}^{d} \frac{1}{1-q^{\lambda_i}}$. Again, the terms corresponding to the $z=0$ pole are not relevant in calculating $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$. Following the methods from the previous cases, we interpolate to find $A_i(q,z)$. The rational function with $A_i(q,z)$ as its numerator has simple poles at

$$\alpha_{i,k} = q^{\frac{-\lambda_i}{\widehat{a_i}}} \exp\left(\frac{2\pi i k}{\widehat{a_i}}\right), 0 \le k \le \widehat{a_i} - 1,$$

which means

$$A_i(q,\alpha_{i,k}) = \lim_{z\to\alpha_{i,k}} f(q,z)\cdot(1-q^{\lambda_i} z^{\widehat{a_i}}) = \frac{q^{\lambda_i a_i t}}{1-\alpha_{i,k}} \prod_{\substack{\ell\ne i\\ 1\le\ell\le d}} \frac{1}{1-q^{\lambda_\ell}\alpha_{i,k}^{\widehat{a_i}}}.$$

After interpolating, we obtain

$$A_i(q,z) = \sum_{k=0}^{\widehat{a_i}-1} A_i(q,\alpha_{i,k}) \prod_{j\ne k} \frac{z-\alpha_{i,k}}{\alpha_{i,j}-\alpha_{i,k}}$$

and

$$\begin{aligned} A_i(q,0) &= \frac{q^{\lambda_i a_i t}}{\widehat{a_i}} \sum_{k=0}^{\widehat{a_i}-1} \left(\frac{1}{1-\alpha_{i,k}}\right) \left(\prod_{\substack{\ell\ne i\\ 1\le\ell\le d}} \frac{1}{1-q^{\lambda_\ell}\alpha_{i,k}^{\widehat{a_\ell}}}\right) \\ &= \frac{q^{\lambda_i a_i t}}{\widehat{a_i}} \sum_{k=0}^{\widehat{a_i}-1} \frac{1}{(1-\alpha_{i,k})\prod_{\substack{\ell\ne i\\ 1\le\ell\le d}}(1-q^{\lambda_\ell}\alpha_{i,k}^{\widehat{a_\ell}})}. \end{aligned}$$

With these terms, we have an expression for $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ for a right-angled d-simplex:

Theorem 2.5.1. *Suppose*

$$\mathcal{P} = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d_{\geq 0} \colon \sum_{i=1}^{d} \frac{x_i}{a_i} \leq 1 \right\}$$

is a right-angled d-simplex, $a_1, \ldots, a_d$ *are pairwise relatively prime positive integers, and* $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d_{>0}$ *is an integral linear form. Then*

$$\mathrm{ehr}^{\lambda}_{\mathcal{P}}(q,t) = \prod_{i=1}^{d} \frac{1}{1-q^{\lambda_i}} + \sum_{i=1}^{d} \frac{q^{\lambda_i a_i t}}{\widehat{a_i}} \sum_{k=0}^{\widehat{a_i}-1} \frac{1}{(1-\alpha_{i,k}) \prod_{\substack{\ell \neq i \\ 1 \leq \ell \leq d}} (1 - q^{\lambda_\ell} \alpha_{i,k}^{\widehat{a_\ell}})},$$

equivalently,

$$\mathrm{cha}^{\lambda}_{\mathcal{P}}(q,x) = \prod_{i=1}^{d} \frac{1}{1-q^{\lambda_i}} + \sum_{i=1}^{d} \frac{((q-1)x+1)^{\lambda_i a_i}}{\widehat{a_i}} \sum_{k=0}^{\widehat{a_i}-1} \frac{1}{(1-\alpha_{i,k}) \prod_{\substack{\ell \neq i \\ 1 \leq \ell \leq d}} (1 - q^{\lambda_\ell} \alpha_{i,k}^{\widehat{a_\ell}})},$$

where $\alpha_{i,k} = q^{\frac{-\lambda_i}{\widehat{a_i}}} \exp\left(\frac{2\pi i k}{\widehat{a_i}}\right)$.

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Appendix

```
# Declare the symbolic variables

# L1 = lambda_1 and L2 = lambda_2

var('a b L1 L2 t eps')

assume(a > 0, b > 0)


# Substitute q with exp(eps)

# As eps approaches 0, q approaches 1

q = exp(eps)


# Define the three fractional terms

A = (q^(b * L2 * t))/(a * (1 - q^(L1 - (L2 * b)/a)) * (1 - q^(-L2/a)))

B = (q^(a * L1 * t))/(b * (1 - q^(L2 - (L1 * a)/b)) * (1 - q^(-L1/b)))

C = 1/((1 - q^L1) * (1 - q^L2))
```

```
# Sum the terms into the full expression
F = A + B + C


# Calculate the Laurent series expansion around eps = 0
# The '1' tells Sage to compute terms up to Order(eps^1)
# .truncate() gets rid of Big-O notation
expansion = F.series(eps == 0, 1).truncate()


# Use full_simplify to algebraically simplify the constant term
limit = expansion.full_simplify()
print("The limit as q -> 1 is:")
show(limit)
```