

DECOMPOSITIONS OF
BIVARIATE ORDER POLYNOMIALS

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CERTIFICATION OF APPROVAL

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DECOMPOSITIONS OF
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We study bivariate order polynomials on bicolored posets. Richard Stanley first introduced univariate order polynomials in 1970. Recently, Sandra Zuniga Ruiz extended his work to bicolored posets and took a look at strict order polynomials on those posets. We continue her work by exploring what happens with weak as well as strict order polynomials on bicolored posets. Our main theorem decomposes bivariate weak order polynomials into a sum of type- w order polynomials. We also prove a reciprocity theorem between weak and strict order polynomials.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee

Date

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Chapter 1

Introduction

In 1970 Richard Stanley introduced the notion of counting order preserving maps on partially ordered sets, or posets. A map φ is (weakly) order preserving if $u \preceq v$ in P implies $\varphi(u) \leq \varphi(v)$. Similarly, a map φ is strictly order preserving if $u \preceq v$ in P implies $\varphi(u) < \varphi(v)$.

Posets can be used to describe real world scheduling problems. Order polynomials help us count the number of ways we can re-order the poset, or the scheduling according to certain constraints. These polynomials are an invariant of a poset P and encode arithmetic information about P .

Stanley studied (weak) order polynomials $\Omega_P(x)$ as well as strict order polynomials $\Omega_P^\circ(x)$, both of which count the total number of order preserving maps from a given poset P to $[x] := \{1, 2, \dots, x\}$. Linear extensions and all other terms will be defined later in Chapter 2

Theorem 1.1 (Stanley, [4]). *Let P be a partially ordered set with n elements and*

a fixed natural labeling. Then

$$\Omega_P(x) = \sum_L \binom{x - \text{des}(w(L))}{n}$$

where the sum is over all possible linear extensions L of P .

Note that the binomial coefficients in Theorem 1.1 come from order polynomials of type- w maps on chains. The following theorem gives us reciprocity between weak and strict order polynomials.

Theorem 1.2 (Stanley, [3]). *Let P be a finite poset with n elements. Then $\Omega_P(x)$ and $\Omega_P^\circ(x)$ are polynomials in x with rational coefficients and they satisfy*

$$\Omega_P(x) = (-1)^n \Omega_P^\circ(-x).$$

Sandra Zuniga Ruiz introduced bivariate order polynomials on a bicolored poset, i.e., a poset that is a disjoint union of celeste elements C and silver elements S . A map φ is a (weakly) order preserving (x, y) -map if $u \preceq v$ in P implies $\varphi(u) \leq \varphi(v)$ and $c \in C$ gives us $\varphi(c) \geq y$. Similarly, a map φ is a strictly order preserving (x, y) -map if $u \preceq v$ in P implies $\varphi(u) < \varphi(v)$ and $c \in C$ gives us $\varphi(c) > y$. The next theorem is the strict bivariate version of Theorem 1.1.

Theorem 1.3 (Zuniga Ruiz, [2]). *Let P be a bicolored poset with a fixed natural*

reverse labeling ω . Then

$$\Omega_P^\circ(x, y) = \sum_L \Omega_{w(L)}^\circ(x, y)$$

where the sum is over all possible linear extensions L of P .

Chapter 2 introduces the concepts and tools needed for computing order polynomials. We formally define bicolored posets, order preserving maps, and order polynomials. We will look at examples of these definitions and many more to gain a better understanding of them.

Chapter 3 presents work on bicolored chains. We will be looking at (weak) order polynomials, strict order polynomials, and type- w versions of both. We will also go over some corollaries that tie the different types of order polynomials together. We are going to see what happens to the type- w maps when we are looking at a word w versus its inverse \bar{w} .

Zuniga Ruiz showed a way to decompose a strict bivariate order polynomial of a given poset. We will provide a similar decomposition for weak bivariate order polynomials in Chapter 4. This chapter contains our main results. Here we are looking at a general bicolored poset rather than a chain. We find the order polynomial of a bicolored poset by looking at type- w maps of all possible linear extensions of the poset, yielding the weak bivariate version of Theorem 1.1.

Theorem 1.4. *Let P be a bicolored poset with a fixed natural labeling ω . Then*

$$\Omega_P(x, y) = \sum_L \Omega_{\omega(L)}(x, y)$$

where the sum is over all possible linear extensions L of P .

The following theorem is the bivariate version of Stanley's Theorem 1.2.

Theorem 1.5. *Let P be a bicolored poset with n elements. Then*

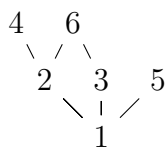
$$\Omega_P^\circ(-x, -y) = (-1)^n \Omega_P(x, y + 1)$$

Finally, Chapter 5 looks at ways this work can be extended.

Chapter 2

Background

A **partially ordered set** or **poset** is a set P along with a binary relation \preceq . For any $a, b, c \in P$ the relation must be reflexive ($a \preceq a$), antisymmetric ($a \preceq b$ and $b \preceq a$ means $a = b$), and transitive ($a \preceq b$ and $b \preceq c$ means $a \preceq c$).



Example 2.1. Consider the set $\{1, 2, 3, 4, 5, 6\}$ with the relation of divisibility. We get the poset depicted on the left. Note, this set truly has a partial order since 2 and 5 are incomparable elements with the chosen relation.

Example 2.2. Now consider the set $\{1, 2, 3, 4, 5, 6\}$ with the regular \leq relation. We get the poset depicted below on the right. Every element in this poset is comparable and thus has total order. We call such a poset a **chain**. If we have any chain of x elements, we can think of it as the set $\{1, 2, \dots, x\} =: [x]$.

A map $\varphi : P \rightarrow [x]$ is **(weakly) order preserving** if $u \preceq v$ in P implies $\varphi(u) \leq \varphi(v)$ in $[x]$. The map φ is **strictly order preserving** if $u \prec v$ in P implies

$\varphi(u) < \varphi(v)$. The univariate **weak order polynomial** $\Omega_P(x)$ counts the number of order preserving maps while the **strict order polynomial** $\Omega_P^\circ(x)$ counts the number of strictly order preserving maps.

A **bicolored** poset $P = C \uplus S$ is a poset that is a disjoint union of celeste elements C and silver elements S . A map $\varphi : P \rightarrow [x]$ is a **weakly order preserving (x, y) -map** if $u \preceq v$ in P implies $\varphi(u) \leq \varphi(v)$ and $c \in C$ implies $\varphi(c) \geq y$. The map φ is a **strictly order preserving (x, y) -map** if $u \prec v$ in P implies $\varphi(u) < \varphi(v)$ in $[x]$ and $c \in C$ implies $\varphi(c) > y$. The bivariate **order polynomial** $\Omega_P(x, y)$ counts the number of order preserving (x, y) -maps while the **strict order polynomial** $\Omega_P^\circ(x, y)$ counts the number of strictly order preserving (x, y) -maps.

The following theorem came from Zuniga Ruiz's work, along with some minor corrections— the binomial coefficients were slightly off.

Theorem 2.1 (Zuniga Ruiz, [2]). *If $P = \{v_n \prec v_{n-1} \prec \cdots \prec v_{k+1} \prec v_k \prec \cdots \prec v_2 \prec v_1\}$ is a chain of length n and $v_{k+1} \in C$ is the minimal celeste element in P , then*

$$\Omega_P^\circ(x, y) = \sum_{a=k+1}^n \binom{x-y}{a} \binom{y}{n-a}.$$

Proof. We want to count the number of strictly order preserving maps $\varphi : P \rightarrow [x]$ with $y < \varphi(v_{k+1})$. This gives us a set of inequalities for φ with $n - k - 1$ cases. For

the first case we have

$$1 \leq \varphi(v_n) < \cdots < \varphi(v_{k+2}) \leq y < \varphi(v_{k+1}) < \cdots < \varphi(v_1) \leq x.$$

There are $\binom{x-y}{k+1} \binom{y}{n-k-1}$ possible ways to do this. For the second case,

$$1 \leq \varphi(v_n) < \cdots < \varphi(v_{k+3}) \leq y < \varphi(v_{k+2}) < \cdots < \varphi(v_1) \leq x$$

which gives us $\binom{x-y}{k+2} \binom{y}{n-k-2}$ possibilities. We proceed until the last case

$$1 \leq y < \varphi(v_n) < \cdots < \varphi(v_1) \leq x.$$

This gives us $\binom{x-y}{n} \binom{y}{0}$ possibilities. Thus, the total number of strictly order preserving maps is

$$\Omega_P^\circ(x, y) = \sum_{a=k+1}^n \binom{x-y}{a} \binom{y}{n-a}. \quad \square$$

Before we can compute $\Omega_P^\circ(x, y)$ for a general poset, we introduce some important tools. Let P be a bicolored poset with n elements. A **natural labeling** of P is a bijection $\omega : P \rightarrow [n]$ where $u \prec v$ in P implies $\omega(u) < \omega(v)$. Similarly, a **natural reverse labeling** of P gives us $\omega(u) > \omega(v)$.

Example 2.3. Depicted below is a poset along with two different labellings (in red). Figure 1 shows a natural reverse label where $\omega(v_4) = 4$, $\omega(v_3) = 2$, $\omega(v_2) = 3$, $\omega(v_1) = 1$. While Figure 2 shows a natural label where $\omega(v_4) = 1$, $\omega(v_3) = 2$, $\omega(v_2) = 3$, $\omega(v_1) = 4$. Note how $v_4 \prec v_1$ gives us $\omega(v_4) > \omega(v_1)$ in Figure 1 and $v_4 \prec v_1$ gives us $\omega(v_4) < \omega(v_1)$ in Figure 2.

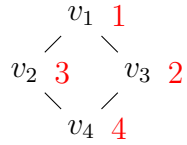


Figure 1

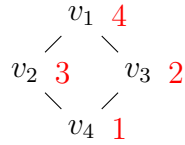
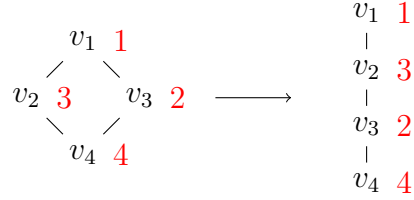


Figure 2

A **linear extension** L is a chain that **refines** P so that $u \preceq v$ in P gives us $u \preceq v$ in L . We find all possible linear extensions of P by considering two cases between each pair of incomparable elements: $u \prec v$ or $v \prec u$.

From any given linear extension $L = \{v_n \prec v_{n-1} \prec \cdots \prec v_1\}$ we find the associated **word** $w(L) = \omega(v_n)\omega(v_{n-1})\cdots\omega(v_1)$. We say that i is an **ascent** of $w(L)$ if $\omega(v_i) < \omega(v_{i-1})$ and i is a **descent** of $w(L)$ if $\omega(v_i) > \omega(v_{i-1})$.

Example 2.4. Here we have the same poset from Example 2.3 with a fixed natural reverse labeling and a linear extension $L = \{v_4 \prec v_3 \prec v_2 \prec v_1\}$. From L we get the word $w(L) = \omega(v_4)\omega(v_3)\omega(v_2)\omega(v_1) = 4231$. This word has one ascent at $i = 3$ and descents at $i = 4, 2$.



An order preserving (x, y) -map φ of L is of **type- $w(L)$** if $\varphi(v_i) \leq \varphi(v_{i-1})$ for every ascent i of $w(L)$ and $\varphi(v_i) < \varphi(v_{i-1})$ for every descent i . Let $\Omega_{w(L)}(x, y)$ count the number of type- $w(L)$ order preserving (x, y) -maps where $\varphi(c) \geq y$ for the minimal celeste element $c \in L$. Similarly, $\Omega_{w(L)}^\circ(x, y)$ counts the number of type- $w(L)$ order preserving (x, y) -maps where $\varphi(c) > y$.

Theorem 2.2 (Zuniga Ruiz, [2]). *Consider the linear extension $L = \{v_n \prec v_{n-1} \prec \cdots \prec v_2 \prec v_1\}$ of a bicolored poset P with a fixed labeling and associated word $w(L) = \omega(v_n)\omega(v_{n-1})\cdots\omega(v_1)$. Let v_{k+1} be the minimal celeste element in L and let $\tilde{w}(L) = \omega(v_n)\omega(v_{n-1})\cdots\omega(v_{k+1})$. Then*

$$\Omega_{w(L)}^\circ(x, y) = \sum_{a=k+1}^n \binom{x - y + \text{asc}(w(L)) - \text{asc}(\tilde{w}(L))}{a} \binom{y + \text{asc}(\tilde{w}(L))}{n - a}.$$

Proof. Consider $\varphi : L \rightarrow [x]$, a type- w order preserving (x, y) -map. We want to count the number of such maps. The map φ gives us a string of inequalities where $y < \varphi(v_{k+1})$, $\varphi(v_i) \leq \varphi(v_{i-1})$ if i is an ascent of $w(L)$, and $\varphi(v_i) < \varphi(v_{i-1})$ if i is a descent of $w(L)$. We get rid of weak inequalities by creating a bijection $\varphi \mapsto \bar{\varphi}$ defined by

$$\begin{aligned}
\bar{\varphi}(v_n) &= \varphi(v_n) \\
\bar{\varphi}(v_{n-1}) &= \varphi(v_{n-1}) + \text{asc}(\omega(v_n)\omega(v_{n-1})) \\
\bar{\varphi}(v_{n-2}) &= \varphi(v_{n-2}) + \text{asc}(\omega(v_n)\omega(v_{n-1})\omega(v_{n-2})) \\
&\vdots \\
\bar{\varphi}(v_k) &= \varphi(v_k) + \text{asc}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_k)) \\
&\vdots \\
\bar{\varphi}(v_2) &= \varphi(v_2) + \text{asc}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_2)) \\
\bar{\varphi}(v_1) &= \varphi(v_1) + \text{asc}(w(L)).
\end{aligned}$$

If i is a descent of $w(L)$, $\text{asc}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_i)) - \text{asc}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_{i-1})) = 0$ so that $\varphi(v_i) < \varphi(v_{i-1})$ gives us $\bar{\varphi}(v_i) < \bar{\varphi}(v_{i-1})$. If i is an ascent of $w(L)$ we have $\text{asc}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_i)) - \text{asc}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_{i-1})) = 1$. Then $\varphi(v_i) \leq \varphi(v_{i-1})$ gives us $\bar{\varphi}(v_i) < \bar{\varphi}(v_{i-1})$. Hence, $\bar{\varphi}$ is a strictly increasing function with $y + \text{asc}(\tilde{w}(L)) < \bar{\varphi}(v_{k+1})$ and

$$1 \leq \bar{\varphi}(v_n) < \bar{\varphi}(v_{n-1}) < \cdots < \bar{\varphi}(v_2) < \bar{\varphi}(v_1) \leq x + \text{asc}(w(L)).$$

By Theorem 2.1 the total number of type- w order preserving (x, y) -maps is thus

$$\Omega_{w(L)}^\circ(x, y) = \sum_{a=k+1}^n \binom{x - y + \text{asc}(w(L)) - \text{asc}(\tilde{w}(L))}{a} \binom{y + \text{asc}(\tilde{w}(L))}{n - a}. \quad \square$$

The following corollary shows how Theorem 2.1 is a special case of Theorem 2.2.

Corollary 2.3. *Let P be a bicolored chain of length n with a natural reverse labeling $\omega = w(P)$ and minimal celeste element v_{k+1} . Then*

$$\Omega_P^\circ(x, y) = \Omega_\omega^\circ(x, y).$$

Proof. Since P has a natural reverse label, we get the words $\omega = n(n-1)(n-2)\cdots 1$ and $\tilde{\omega} = n(n-1)(n-2)\cdots(k+1)$. Then $\text{asc}(\omega) = \text{asc}(\tilde{\omega}) = 0$ so that we have $\Omega_\omega^\circ(x, y) = \sum_{a=k+1}^n \binom{x-y}{a} \binom{y}{n-a} = \Omega_P^\circ(x, y)$. \square

The next lemma is a fact from combinatorics that we will use in later proofs.

Lemma 2.4. *Consider the sets $\{x \in \mathbb{Z}^d : a \leq x_1 < x_2 < \cdots < x_d \leq b\}$ and $\{x \in \mathbb{Z}^d : a \leq x_1 \leq x_2 \leq \cdots \leq x_d \leq b\}$. The cardinality of these sets are $\binom{b-a+1}{d}$ and $\binom{b-a+d}{d}$ respectively.*

Chapter 3

Bicolored Chains

The following theorem and Theorem 3.3 below are the weak order polynomial versions of Zuniga Ruiz's work.

Theorem 3.1. *If $P = \{v_n \prec v_{n-1} \prec \cdots \prec v_{k+1} \prec v_k \prec \cdots \prec v_2 \prec v_1\}$ is a chain of length n and $v_{k+1} \in C$ is the minimal celeste element in P , then*

$$\Omega_P(x, y) = \sum_{a=k+1}^n \binom{x-y+a}{a} \binom{y+n-a-2}{n-a}.$$

Proof. We want to count the number of weakly order preserving maps $\varphi : P \rightarrow [x]$ with $y \leq \varphi(v_{k+1})$. This gives us a set of inequalities for φ with $n - k - 1$ cases. For the first case we have

$$1 \leq \varphi(v_n) \leq \cdots \leq \varphi(v_{k+2}) < y \leq \varphi(v_{k+1}) \leq \cdots < \varphi(v_1) \leq x.$$

Then there are $\binom{x-y+k+1}{k+1} \binom{y+n-k-3}{n-k-1}$ possible ways to do this by Lemma 2.4. For the second case,

$$1 \leq \varphi(v_n) \leq \cdots \leq \varphi(v_{k+3}) < y \leq \varphi(v_{k+2}) \leq \cdots \leq \varphi(v_1) \leq x$$

So there are $\binom{x-y+k+2}{k+2} \binom{y+n-k-4}{n-k-2}$ possibilities. We proceed until the last case

$$1 \leq y \leq \varphi(v_n) \leq \cdots \leq \varphi(v_1) \leq x.$$

This gives us $\binom{x-y+n}{n}$ possibilities. Thus, the total number of order preserving maps is

$$\Omega_P(x, y) = \sum_{a=k+1}^n \binom{x-y+a}{a} \binom{y+n-a-2}{n-a}. \quad \square$$

The following corollary gives us reciprocity between $\Omega_P^\circ(x, y)$ and $\Omega_P(x, y)$ in the case that P is a chain. Theorem 1.5 generalizes this to an arbitrary poset P .

Corollary 3.2. *If $P = \{v_n \prec v_{n-1} \prec \cdots \prec v_{k+1} \prec v_k \prec \cdots \prec v_2 \prec v_1\}$ is a chain of length n and $v_{k+1} \in C$ is the minimal celeste element in P , then*

$$\Omega_P^\circ(-x, -y) = (-1)^n \Omega_P(x, y + 1).$$

Proof. We will use that $(-1)^d \binom{-n}{d} = \binom{n+d-1}{d}$, by definition of binomial coefficients.

By Theorems 2.1 and 3.1 we see that

$$\begin{aligned}
\Omega_P^{\circ}(-x, -y) &= \sum_{a=k+1}^n \binom{-x+y}{a} \binom{-y}{a} \\
&= \sum_{a=k+1}^n (-1)^a \binom{x-y+a-1}{a} (-1)^{n-a} \binom{y+n-a-1}{n-a} \\
&= (-1)^n \sum_{a=k+1}^n \binom{x-y+a-1}{a} \binom{y+n-a-1}{n-a} \\
&= (-1)^n \Omega_P(x, y+1). \quad \square
\end{aligned}$$

Theorem 3.3. *Consider the linear extension $L = \{v_n \prec v_{n-1} \prec \cdots \prec v_2 \prec v_1\}$ of a bicolored poset P with a fixed labeling and associated word $w(L) = \omega(v_n)\omega(v_{n-1})\cdots\omega(v_1)$. Let v_{k+1} be the minimal celeste element in L and let $\tilde{w}(L) = \omega(v_n)\omega(v_{n-1})\cdots\omega(v_{k+1})$. Then*

$$\Omega_{w(L)} = \sum_{a=k+1}^n \binom{x-y+\text{des}(\tilde{w}(L))-\text{des}(w(L))+a}{a} \binom{y-\text{des}(w(L))+n-a-2}{n-a}.$$

Proof. This proof proceeds like the proof of Theorem 2.2 except that we will use Theorem 3.1 to deduce our results. Consider $\varphi : L \rightarrow [x]$, a type- w order preserving (x, y) -map. We want to count the number of such maps with $y \leq \varphi(v_{k+1})$. The map φ gives us a string of inequalities where $\varphi(v_i) \leq \varphi(v_{i-1})$ if i is an ascent of $w(L)$ and $\varphi(v_i) < \varphi(v_{i-1})$ if i is a descent of $w(L)$. We get rid of strict inequalities by using the bijection $\varphi \mapsto \bar{\varphi}$ defined by

$$\begin{aligned}
\bar{\varphi}(v_n) &= \varphi(v_n) \\
\bar{\varphi}(v_{n-1}) &= \varphi(v_{n-1}) - \text{des}(\omega(v_n)\omega(v_{n-1})) \\
&\vdots \\
\bar{\varphi}(v_k) &= \varphi(v_k) - \text{des}(\omega(v_n)\cdots\omega(v_k)) \\
&\vdots \\
\bar{\varphi}(v_2) &= \varphi(v_2) - \text{des}(\omega(v_n)\cdots\omega(v_2)) \\
\bar{\varphi}(v_1) &= \varphi(v_1) - \text{des}(w(L)).
\end{aligned}$$

If i is a descent of $w(L)$, $\text{des}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_i)) - \text{des}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_{i-1})) = 1$ so that $\varphi(v_i) < \varphi(v_{i-1})$ gives us $\bar{\varphi}(v_i) \leq \bar{\varphi}(v_{i-1})$. If i is an ascent of $w(L)$ we have $\text{des}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_i)) - \text{des}(\omega(v_n)\omega(v_{n-1})\cdots\omega(v_{i-1})) = 0$. Then $\varphi(v_i) \leq \varphi(v_{i-1})$ gives us $\bar{\varphi}(v_i) \leq \bar{\varphi}(v_{i-1})$. Hence, $\bar{\varphi}$ is a weakly increasing function with $y + \text{des}(\tilde{w}(L)) \leq \bar{\varphi}(v_{k+1})$ and

$$1 \leq \bar{\varphi}(v_n) \leq \bar{\varphi}(v_{n-1}) \leq \cdots \leq \bar{\varphi}(v_2) \leq \bar{\varphi}(v_1) \leq x + \text{des}(w(L)).$$

By Theorem 3.1 the total number of type- w order preserving (x, y) -maps is thus

$$\Omega_{w(L)}(x, y) = \sum_{a=k+1}^n \binom{x - y + \text{des}(\tilde{w}(L)) - \text{des}(w(L)) + a}{a} \binom{y - \text{des}(\tilde{w}(L)) + n - a - 2}{n - a}.$$

□

Theorem 3.1 is a special case of Theorem 3.3, as we will see in the next corollary.

Corollary 3.4. *Let P be a bicolored chain of length n with a natural labeling $\omega = w(P)$ and minimal celeste element v_{k+1} . Then*

$$\Omega_P(x, y) = \Omega_\omega(x, y).$$

Proof. From our natural labeling we have the words $\omega = 123 \cdots n$ and $\tilde{\omega} = 123 \cdots (n-k)$. Note, $\text{asc}(\omega) = n - 1$ and $\text{asc}(\tilde{\omega}) = n - k - 1$. Thus

$$\begin{aligned} \Omega_{w(L)}(x, y) &= \sum_{a=k+1}^n \binom{x - y + n - 1 - (n - k - 1) + 1}{a} \binom{y + n - k - 1 - 1}{n - a} \\ &= \sum_{a=k+1}^n \binom{x - y + k + 1}{a} \binom{y + n - k - 2}{n - a} \\ &= \Omega_P(x, y). \end{aligned}$$

□

The **inverse** of $w = w_1 w_2 \cdots w_n$ is the word \bar{w} defined by $w_j \mapsto w_{n+1-j}$. This changes all ascents to descents and descents to ascents in the original word w . In particular, $\text{asc}(w(L)) = \text{des}(\bar{w}(L))$ and $\text{des}(w(L)) = \text{asc}(\bar{w}(L))$.

The following corollary gives us reciprocity between Theorem 2.2 and Theorem 3.3.

Corollary 3.5. *Consider the linear extension $L = \{v_n \prec v_{n-1} \prec \cdots \prec v_2 \prec v_1\}$ of a bicolored poset P with a fixed labeling and associated word $w(L) = \omega(v_n)\omega(v_{n-1})\cdots\omega(v_1)$. Then*

$$\Omega_{w(L)}^\circ(-x, -y) = (-1)^n \Omega_{\bar{w}(L)}(x, y + 1).$$

Proof. Let v_{k+1} be the minimal celeste element in L and let $\tilde{w}(L) = \omega(v_n)\omega(v_{n-1})\cdots\omega(v_{k+1})$. Since $\text{asc}(\tilde{w}(L)) = \text{des}(\bar{\tilde{w}}(L))$ we see that

$$\begin{aligned} \Omega_{w(L)}^\circ(-x, -y) &= \sum_{a=k+1}^n \binom{-x + y + \text{asc}(w(L)) - \text{asc}(\tilde{w}(L))}{a} \binom{-y + \text{asc}(\tilde{w}(L))}{n - a} \\ &= (-1)^n \sum_{a=k+1}^n \binom{x - y + \text{asc}(\tilde{w}(L)) - \text{asc}(w(L)) + a - 1}{a} \\ &\quad \binom{y - \text{asc}(\tilde{w}(L)) + n - a - 1}{n - a} \end{aligned}$$

$$\begin{aligned}
&= (-1)^n \sum_{a=k+1}^n \binom{x - y + \text{des}(\tilde{w}(L)) - \text{des}(\bar{w}(L)) + a - 1}{a} \\
&\qquad\qquad\qquad \binom{y - \text{des}(\tilde{w}(L)) + n - a - 1}{n - a} \\
&= (-1)^n \Omega_{\bar{w}(L)}(x, y+1).
\end{aligned}$$

□

If a poset P is a chain with a reverse natural labeling, we can see that Corollary 3.2 follows directly from Corollary 3.5. Note that Corollary 3.2 is also a special case of Theorem 1.5 since a bicolored chain is also a bicolored poset.

Chapter 4

Bicolored Posets

The previous chapter dealt with bicolored chains. In this chapter, we take a look at general bicolored posets. We will start by proving the decomposition theorems for strict and weak order polynomials.

Proof of Theorem 1.3. We will show a bijection between strictly order preserving maps φ and type- $w(L)$ maps on our poset P .

Suppose φ is a strictly order preserving map. We will build a linear extension of P from this map. If $\varphi(v_i) < \varphi(v_j)$ for $v_i, v_j \in P$, then let $v_i \prec v_j$ in our linear extension. Note that $\omega(v_i) > \omega(v_j)$ due to the natural reverse labeling. If $\varphi(v_i) = \varphi(v_j)$ with $\omega(v_i) < \omega(v_j)$, then let $v_i \prec v_j$ in our linear extension. If $\varphi(v_i) = \varphi(v_j)$ with $\omega(v_i) > \omega(v_j)$, then let $v_i \succ v_j$. This gives us the linear extension L with associated word $w(L)$. By construction, our strictly order preserving map φ is also type- $w(L)$.

Now let L be a linear extension of P and let $\hat{\varphi}$ be a type- $w(L)$ map. Type- $w(L)$ maps agree with strictly order preserving maps between comparable elements in

posets by definition. The map $\hat{\varphi}$ may give us weak or strict inequalities between incomparable elements in P . Strictly order preserving maps place no restrictions on what happens between incomparable elements in the poset. Thus, $\hat{\varphi}$ is also a strictly order preserving map on P .

Let L_1 and L_2 be two different linear extensions of P . Then there must be at least one pair $v_i, v_j \in P$ where $v_i < v_j$ in L_1 and $v_i > v_j$ in L_2 . This means we have an ascent in one of the associated words from the linear extensions and a descent in the same spot in the other word. Hence, the type- $w(L)$ maps are distinct. \square

Proof of Theorem 1.4. The proof of this theorem is similar to that of Theorem 1.3 with one major difference. Here we have $\omega(v_i) < \omega(v_j)$ when $v_i \prec v_j$ in our linear extension due to the natural labeling. \square

Proof of Theorem 1.5. Fix a natural reverse labeling of P . Using Theorem 1.3 and Corollary 3.5 we have

$$\begin{aligned} \Omega_P^\circ(-x, -y) &= \sum_L \Omega_{w(L)}^\circ(-x, -y) \\ &= (-1)^n \sum_L \Omega_{\bar{w}(L)}(x, y + 1) \\ &= (-1)^n \Omega_P(x, y + 1). \end{aligned}$$

Note that the last equation holds because \bar{w} is a natural labeling of P . \square

Chapter 5

Further Work

Richard Stanley showed that we can get the chromatic polynomial $P(G, x)$ for a graph G by decomposing it into strict order polynomials.

Theorem 5.1 (Stanley, [3]). *Let G be a graph. Then*

$$P(G; x) = \sum_{\sigma} \Omega_{\sigma}^{\circ}(x)$$

where the sum is over all possible acyclic orientations σ of G .

In 2003 Klaus Dohmen, André Pönitz, and Peter Tittmann introduced bivariate chromatic polynomials [1]. Our work can be continued by taking a look at the connection between bivariate chromatic polynomials and bivariate order polynomials. One might look for a decomposition analogous to Stanley's Theorem 5.1.

Bibliography

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