TRIANGULATIONS OF GALE DUALS OF ROOT POLYTOPES

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Contents

Chapter 1

Introduction

Our motivation for this paper is based in the paper *Root Polytopes and Growth Series of Root Lattices*, by Federico Ardila, Matthias Beck, Serkan Hoşten, Julian Pfeifle, and Kim Seashore [1]. Here, it was shown that the root polytopes associated to the root lattices A_n, C_n and D_n have an explicitly given unimodular triangulation in every dimension. In general, not all polytopes in dimension three or higher have unimodular triangulations, so finding entire families of polytopes that have unimodular triangulations is rare indeed.

With this in mind, it is conceivable that a polytope that is combinatorially related to the root polytope may also have a unimodular triangulation. Thus, in this paper, we study the structure of the Gale duals of the type-A root polytopes, and determine that in fact, this family does exhibit unimodular triangulations.

We begin, in Chapter 2, with basic definitions of polytopes and triangulations. We define the type-A root polytope and we briefly describe how the unimodular triangulations of this polytope were found in [1]. We define Gale duality, understand how it preserves the combinatorial structure of the original type-A root polytope, and further define the type-A dual polytope, which is the polytope formed by taking the convex hull of the Gale dual configuration. Here we offer a brief first glimpse into the elegance of the structure we will be dealing with.

Chapter 3 takes us into graph theory, offering both background definitions and some insight into the relationship between the type-A root polytope and a particular graph, which we will call the complete digraph. We will see how important this connection is in Corollary 3.3.

Our main results are stated in Chapter 4. In Theorem 4.3 we explicitly find and state the structure of the Gale dual configuration. Through this result, we find not only that a unimodular triangulation of the type-A dual polytope can be found, but that every triangulation (using no new vertices) is unimodular. This is stated and proven in Theorem 4.8. We also find that the type-A dual polytope is equidecomposible in Corollary 4.9 which leads to the natural question: Into how many simplices the type-A dual polytope decomposes. This question is answered in Section 4.3, after we define the chamber complex, which is a decomposition of the affine space of the root polytope into simplicial cones. Through our study of the chamber complex, we find a bijection between simplices in a triangulation and spanning trees of the (single-edged) complete graph. This brings us to our final result, Theorem 4.13, which states that there are exactly n^{n-2} simplices in every triangulation of the type-A dual polytope.

Chapter 5 offers some generalized outcomes from our results, as well as open questions and future work relating to root polytopes and Gale duality.

Chapter 2

Notation

2.1 An Introduction to Polytopes

A **convex polytope** P is a bounded geometric object with flat sides, existing in any dimension. In this paper, our use of "polytope" implies convexity. A polytope can be defined in two equivalent, though different, ways: the vertex description and the hyperplane description. Both are outlined below.

Consider a finite point configuration given by the columns of a $d \times n$ matrix

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} \subseteq \mathbb{R}^d.$$

We define a polytope $P \subset \mathbb{R}^d$ as the **convex hull** of such a point configuration A, which is the set of all convex combinations of $\mathbf{a}_1, \ldots \mathbf{a}_n$. We denote it by

$$P = \operatorname{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{a}_i : \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Geometrically, "taking the convex hull of a finite set of points is like 'shrink wrapping' the points" [6, p. 10]. This is the **vertex description** of a polytope.

Alternatively, a polytope $P \subseteq \mathbb{R}^d$ can be defined by a set of linear constraints on the points in $\mathbb{R}^d$, that is, as the intersection of a finite number of half-spaces. A **half-space** is a subset of $\mathbb{R}^d$ that lies completely on one side of a hyperplane, and a **hyperplane** is a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$. To be more explicit, a hyperplane H has the form

$$H = \{\mathbf{x} \in \mathbb{R}^d : \quad b_1 x_1 + b_2 x_2 + \cdots + b_d x_d = c\},$$

where $b_1, \ldots, b_d, c \in \mathbb{R}$ and $(b_1, \ldots, b_d) \neq \mathbf{0}$. One of the two half-spaces defined by H is

$$H^{\leq} = \{\mathbf{x} \in \mathbb{R}^d : \quad b_1 x_1 + b_2 x_2 + \cdots + b_d x_d \leq c\},$$

and $H^{\geq}$ defines the half-space when using $\geq$ in the inequality instead (defining the space on the "other side" of the hyperplane). If P is defined by the intersection of m half-spaces, then the hyperplane description of P has the form

$$P = \{\mathbf{x} \in \mathbb{R}^d : \quad B\mathbf{x} \leq \mathbf{c}\},$$

where $B \in \mathbb{R}^{m \times d}$ and $\mathbf{c} \in \mathbb{R}^m$. Each row in B gives the variable coefficients of one of the m half-space inequalities, and $\mathbf{c}$ is an m-tuple of the respective constants. This is the **hyperplane description** of a polytope.

Theorem 2.1 (Main Theorem for Polytopes; see, e.g., [8])**.** *A subset $P \subseteq \mathbb{R}^d$ is the convex hull of a finite point set,*

$$P = \operatorname{conv}(A) \quad \textit{for some } A \in \mathbb{R}^{d \times n},$$

if and only if it is a bounded intersection of half-spaces,

$$P = \{\mathbf{x} \in \mathbb{R}^d : \quad B\mathbf{x} \leq \mathbf{c}\} \quad \textit{for some } B \in \mathbb{R}^{m \times d}, \mathbf{c} \in \mathbb{R}^m.$$

This fact is non-trivial to prove (see [2]) but geometrically intuitive.

The **dimension** of a polytope P, denoted $\dim(P)$, is the dimension of the affine space

$$\operatorname{aff}(P) = \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \quad \mathbf{x}, \mathbf{y} \in P, \quad \lambda \in \mathbb{R}\}$$

spanned by P [1]. A d**-polytope** is a polytope of dimension d. We call P a ***d*-simplex** if it is the convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$ (a generalized triangle).

A **face** of P is the intersection of P with a hyperplane H such that P is completely contained in $H^{\leq}$ or $H^{\geq}$. In this case, H is called a **supporting hyperplane** to P. The 0-dimensional faces are the **vertices** of P, and the $(d-1)$-dimensional faces are **facets**, where the dimension of a face comes from viewing the face as a polytope in its own right.

Example 2.1. Consider the points given in the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & 3 \\ 1 & 3 & 1 & 3 \end{pmatrix}. \tag{2.1}$$

Let $P = \operatorname{conv}(A)$. We see that $\dim(P) = 2$, since the four points affinely span $\mathbb{R}^2$, and that P forms a square.

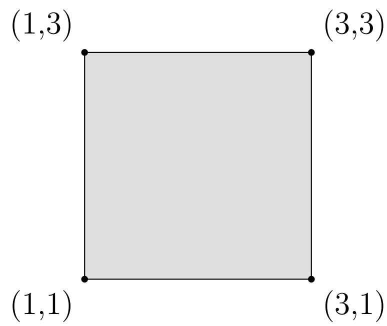


Figure 2.1: The convex hull of A.

The four points in A are the vertices of P. We can alternatively describe this polytope as the intersection of the following four half-spaces:

$$\begin{aligned} x_1 &\geq 1 & & & -x_1 &\leq -1 \\ x_1 &\leq 3 & & & x_1 &\leq 3 \\ x_2 &\geq 1 & &\Longrightarrow & -x_2 &\leq -1 \\ x_2 &\leq 3 & & & x_2 &\leq 3 \end{aligned} \tag{2.2}$$

where points in $\mathbb{R}^2$ are given as (x_1, x_2). We encode the coefficients of x_1 and x_2 in the linear inequalities on the right of (2.2) as rows in the matrix

$$B = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

We see that the set $\{\mathbf{x} \in \mathbb{R}^2 : B\mathbf{x} \leq \mathbf{c}\}$, where $\mathbf{c} = (-1, 3, -1, 3)^T$, defines the same polytope $P = \text{conv}(A)$ from (2.1). The four inequalities from (2.2), when switched to equalities, define the four facets of P.

2.2 The Type-A Root Polytope

We definite the **type-A root system** as the set of vectors

$$A_{n-1} = \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i, j \leq n, i \neq j\},$$

where $\mathbf{e}_k$ denotes the k-th unit vector and n is the dimension of the space. For easy and useful notation, let $\mathbf{a}_{ij} = \mathbf{e}_i - \mathbf{e}_j$, and let us order these vectors using the **positive lexicographic order**; that is, the vectors $\mathbf{a}_{ij}$ for $i < j$ are ordered lexicographically, but then paired with their negative, $\mathbf{a}_{ji}$. Our matrix representation of these vectors is

$$A_{n-1} = \begin{pmatrix} \mathbf{a}_{12} & \mathbf{a}_{21} & \mathbf{a}_{13} & \mathbf{a}_{31} & \cdots & \mathbf{a}_{n-1\ n} & \mathbf{a}_{n\ n-1} \end{pmatrix} \in \{0, 1, -1\}^{n \times 2\binom{n}{2}}.$$

We define the **type-A root polytope** as

$$P_{A_{n-1}} = \mathrm{conv}(A_{n-1}).$$

We make a few observations. First, there is only a single polytope of this type in each dimension, and the polytope is not full dimensional. In fact, $\dim(P_{A_{n-1}}) = n-1$ but it lives in the ambient space $\mathbb{R}^n$. The polytope is embedded in the hyperplane $H_0 = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_n = 0\}$.

Additionally, $P_{A_{n-1}}$ has $2\binom{n}{2}$ vertices (every vector in A_{n-1} is a vertex) and $2\binom{n}{2} + 1$ lattice points, since the vertices and the origin are the only lattice points contained in $P_{A_{n-1}}$. Proposition 8 in [1] contains a more thorough treatment on the structure of this family of polytopes.

Example 2.2. For $n = 3$,

$$A_2 = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_{12} & \mathbf{a}_{21} & \mathbf{a}_{13} & \mathbf{a}_{31} & \mathbf{a}_{23} & \mathbf{a}_{32} \end{array} \\ \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array}$$

is our root configuration, which lies in the hyperplane $H_0 = \{\mathbf{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$. Note our labeling of the vectors in A_2 with the aforementioned choice of $\mathbf{a}_{ij}$.

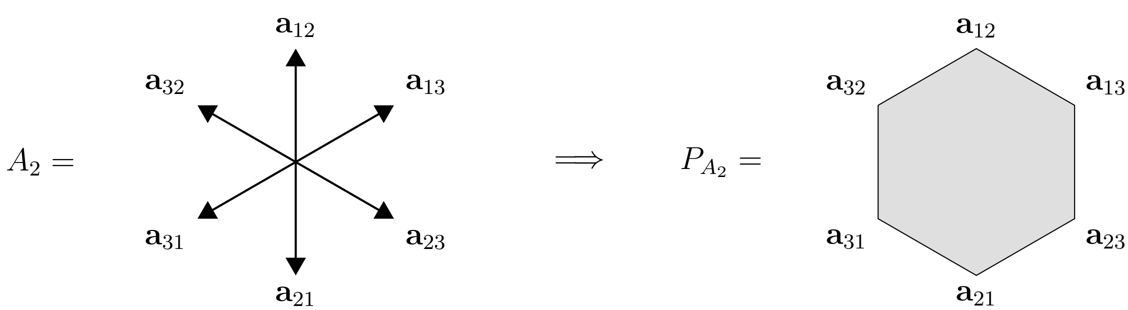


Figure 2.2: The A_2 root vectors and their convex hull.

We will see in the following section that each type-A root polytope has a unimodular triangulation, motivating the questions of this paper.

2.3 Triangulations

Triangulating a polytope is subdividing it into "smaller pieces", namely, into simplices. A brief motivation for this is that simplices are by nature simpler structures, and some questions about polytopes may be reduced instead to questions about simplices.

A **triangulation** of a polytope P is a collection of finitely many simplices $s_1, \ldots, s_k$ such that

(a) $\bigcup_{i=1}^{k} s_i = P$, and

(b) $s_i \cap s_j$ is a common face of both s_i and s_j for all $i \neq j$.

A **unimodular simplex** is a lattice simplex such that if the vertices of the simplex are $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d$, then the vectors $\mathbf{a}_1 - \mathbf{a}_0, \mathbf{a}_2 - \mathbf{a}_0, \ldots, \mathbf{a}_d - \mathbf{a}_0$ form a basis for the sub-lattice $\text{aff}(P) \cap \mathbb{Z}^n$, where $d \leq n$. Geometrically, this requires that in the simplex, if one chooses any of the $d+1$ vertices and then translates the simplex so the chosen vertex is at the origin, the d vectors emanating from that vertex will form a basis for the sub-lattice $\text{aff}(P) \cap \mathbb{Z}^n$.

As a consequence, for full-dimensional simplices ($d = n$), the determinant of a matrix whose columns are the vertices of a unimodular simplex appended by the row $(1, 1, \ldots, 1)$ equals 1 or -1. This is an equivalent definition for a unimodular simplex because we know that a square matrix of basis vectors has determinant 1 or -1. Additionally, the Euclidean volume of a unimodular simplex is $\frac{1}{d!}$, where d is the dimension of the space.

A **unimodular triangulation** of a polytope is a triangulation consisting of all unimodular simplices.

Below are two triangulations of the dimension 2 type-A root polytope.

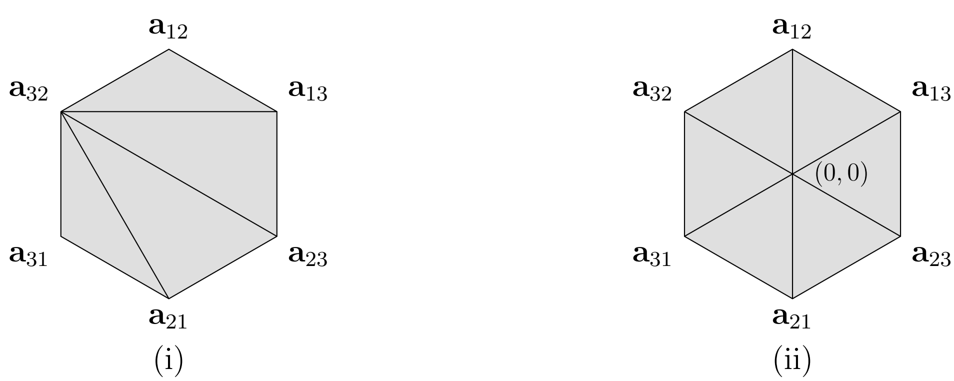


Figure 2.3: Triangulations of P_{A_2}.

Clearly, these are both examples of a triangulation of P_{A_2}, as we have subdivided P_{A_2} into simplices. However, triangulation (ii) is a unimodular triangulation, while triangulation (i) is not. This is a good illustration of the uniformity of unimodular simplices, as we can see the equal volume of the simplices in (ii) and the unequal volume of those in (i) (recall, each unimodular simplex in dimension 2 has volume $\frac{1}{2!} = \frac{1}{2}$).

Also, we see that triangulation (i) uses only the vertices of P_{A_2}, while triangulation (ii) introduces the origin as an additional vertex. It is known that every polytope can be triangulated using no new vertices. However, in the case of the type-A root polytope, the origin is introduced in order to obtain a unimodular triangulation. This can clearly be seen in the $n = 3$ case; any "cutting up" of P_{A_2} that utilizes only the existing vertices will not result in a unimodular triangulation.

It has been shown (see [1]) that for the family of type-A root polytopes, there exist unimodular triangulations in every dimension. These triangulations are obtained precisely as shown in Figure 2.3 — by using the origin (the only interior lattice point in $P_{A_{n-1}}$) and then coning over the boundary.

In an effort to find another such family, we turn our attention to a related polytope—the Gale dual of the root polytope.

2.4 Gale Duality

The **Gale transform** of a polytope P is a vector configuration whose row space (in matrix form) is orthogonal to the row space of the original vector configuration (again in matrix form). This implies that the Gale transform is obtained by finding a basis for the nullspace of a matrix whose columns are vertices of P [6].

This means that Gale transforms are not unique, and different choices of bases can yield different, though combinatorially equivalent, Gale transforms. They are collectively called the Gale dual of P.

The Gale transformation converts our original set of $2\binom{n}{2}$ root vectors in n-dimensional space into a set of $2\binom{n}{2}$ dual vectors in k-dimensional space, where $k = 2\binom{n}{2} - (n-1) = (n-1)^2$.

We use A^*_{n-1} to denote the Gale dual configuration of the root configuration A_{n-1}. Because the Gale transformation preserves the number of vectors in the configuration, we label the columns of A^*_{n-1} with $\mathbf{a}^*_{ij}$, ordered in positive lexicographic order. Our matrix representation of A^*_{n-1} is

$$A^*_{n-1} = \begin{pmatrix} \mathbf{a}^*_{12} & \mathbf{a}^*_{21} & \mathbf{a}^*_{13} & \mathbf{a}^*_{31} & \cdots & \mathbf{a}^*_{n-1\ n} & \mathbf{a}^*_{n\ n-1} \end{pmatrix} \in \{0, 1, -1\}^{(n-1)^2 \times 2\binom{n}{2}}.$$

Our next goal is to explicitly describe the Gale dual of the type-A root polytope. For that, we need a "good" basis for the nullspace of A_{n-1}. We will construct one in the coming chapters. For motivation, let us carry out the computation for A^*_2.

Example 2.3. We illustrate the Gale transformation with the root configuration from Example 2.2. Recall that

$$A_2 = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_{12} & \mathbf{a}_{21} & \mathbf{a}_{13} & \mathbf{a}_{31} & \mathbf{a}_{23} & \mathbf{a}_{32} \end{array} \\ \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array} \tag{2.3}$$

whose convex hull lives in a subspace of $\mathbb{R}^3$ (see Figure 2.2). So,

$$A_2^* = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_{12}^* & \mathbf{a}_{21}^* & \mathbf{a}_{13}^* & \mathbf{a}_{31}^* & \mathbf{a}_{23}^* & \mathbf{a}_{32}^* \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} \end{array} \tag{2.4}$$

where the rows form a basis for the nullspace of (2.3). For instance, the first row in (2.4) is $(1, 1, 0, 0, 0, 0)$. Applying this vector to the columns of (2.3), we see that

$$1 \cdot \mathbf{a}_{12} + 1 \cdot \mathbf{a}_{21} + 0 \cdot \mathbf{a}_{13} + 0 \cdot \mathbf{a}_{31} + 0 \cdot \mathbf{a}_{23} + 0 \cdot \mathbf{a}_{32} = \mathbf{0},$$

the zero vector. By construction, all of the (linearly independent) rows of A_2^* are linear dependence relations on the columns of A_2. Furthermore, we observe that all of the (linearly independent) rows of A_2 are linear dependence relations on the rows of A_2^* as well.

This symmetry generalizes for every n, and so Gale duality is a symmetric relationship [3].

Why is the above A_2^* configuration decided to be "the" dual configuration on which we are choosing to focus? The short answer, for now, is the simplicity of the structure. Notice the first three rows of paired 1's in (2.4). This pattern can always be present in a basis for the nullspace of A_{n-1} for every n, since $\mathbf{a}_{ij} + \mathbf{a}_{ji} = \mathbf{0}$ for every distinct i, j pair between 1 and n, of which there are $\binom{n}{2}$ such pairs.

The structure of A_{n-1}^* with these rows of paired ones will prove to be useful in future theorems. Therefore, moving forward we will keep this property in mind and will refer to this structure as "the" Gale dual configuration.

The convex hull of (2.4) gives rise to a 3-dimensional polytope embedded in 4-dimensional space, as shown in Figure 2.4.

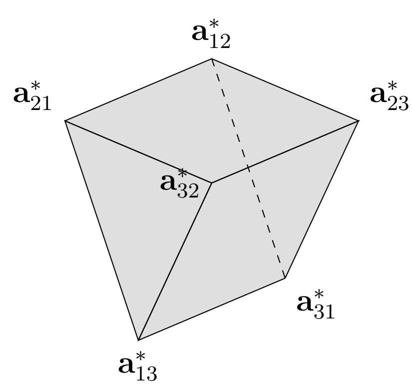


Figure 2.4: The convex hull of A_2^*.

However, before we can generally define the polytope formed by taking the convex hull of the vectors in A_{n-1}^* for every n, we need to formally state the matrix structure of A_{n-1}^*. We have the beginnings of an idea — the rows of paired 1's — but what do the last $(n-1)^2 - \binom{n}{2} = \binom{n-1}{2}$ rows look like?

Let us take a side-step for a moment and work with a new object, namely, the complete digraph. Its relevance to our question will very soon become clear.

Chapter 3

Notions From Graph Theory

3.1 The Complete Digraph

The **complete digraph** $\overleftrightarrow{K_n}$ is a directed graph with vertex set $V = [n] = \{1, \ldots, n\}$ and arc set $E = \{\vec{ij} : 1 \leq i, j \leq n\}$, where i, j are vertices in the graph. We refer to arcs $\vec{ij}$ as **positive arcs** if $i < j$ and as **negative arcs** if $j < i$.

We define the **incidence matrix** of $\overleftrightarrow{K_n}$, denoted $\mathcal{M}(\overleftrightarrow{K_n})$, as the $n \times 2\binom{n}{2}$ matrix with rows indexed by V and columns indexed by E, ordered under positive lexicographic order. The entry in row $v \in V$ and column $e \in E$ is denoted $m_v(e)$ and is given by

$$m_v(e) = \begin{cases} 1 & \text{if } v = \operatorname{init}(e), \\ -1 & \text{if } v = \operatorname{fin}(e), \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{init}(e)$ is the initial vertex (or **tail**) of arc e and $\operatorname{fin}(e)$ is the final vertex (or **head**) of arc e.

Example 3.1. Two such graphs, for $n = 3$ and $n = 4$, are shown in Figure 3.1.

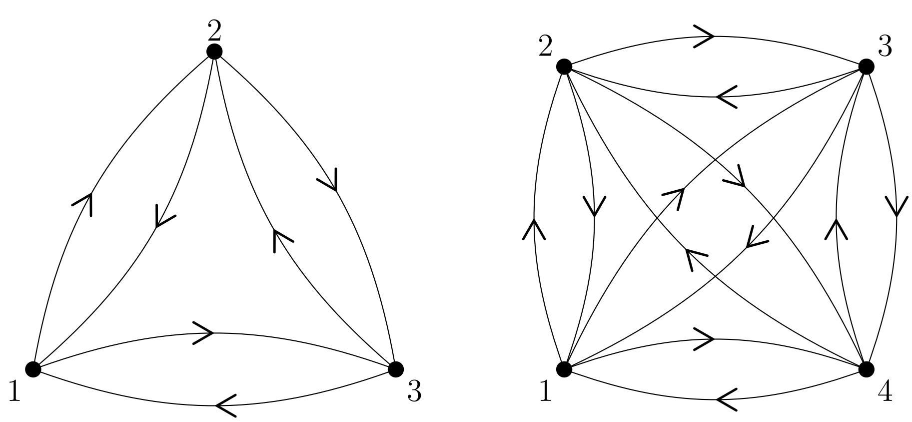


Figure 3.1: The complete digraphs $\overleftrightarrow{K_3}$ and $\overleftrightarrow{K_4}$.

In $\overleftrightarrow{K_3}$, arc $\vec{12}$, which starts at vertex 1 and points towards vertex 2, is said to have 1 as the tail and 2 as the head. Thus,

$$\mathcal{M}(\overleftrightarrow{K_3}) = \begin{array}{c c} & \begin{array}{cccccc} \vec{12} & \vec{21} & \vec{13} & \vec{31} & \vec{23} & \vec{32} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array}. \tag{3.1}$$

If this matrix looks eerily familiar, it is because it should. This matrix is identical to A_2 from Example 2.2, and this is not a coincidence! We can easily see a correspondence between the vector $\mathbf{a}_{12}$ and the column for arc $\vec{12}$. In general, this connection will continue to hold for vectors in A_{n-1} and arcs in $\overleftrightarrow{K_n}$ — the vector $\mathbf{a}_{ij}$ corresponds to the arc $\vec{ij}$.

This connection between our original vectors A_{n-1} and arcs in the complete digraph offers another venue for exploration on the structure of A^*_{n-1}. To investigate this connection further, we familiarize ourselves with two important spaces that arise from the graph, namely, the bond space and the cycle space.

3.2 The Bond Space

Given a digraph D, a **bond** is a set of arcs such that (i) their removal from D disconnects some connected component of D, and (ii) they are a minimal set with this property, i.e., no proper subset of the arcs has property (i). A subset of arcs satisfying (i) is called a **cutset**, so a bond is a minimal cutset [5].

A bond always disconnects a connected digraph into exactly two components, due to the minimality condition of a bond. If V is the vertex set of the graph, then a bond B partitions V into the vertex sets of each component, namely, V_1 and V_2 where $V = V_1 \cup V_2$.

For a fixed bond B, define the function $g_B \in \mathbb{R}^E$ by

$$(g_B)_e = \begin{cases} 1 & \text{if } \operatorname{init}(e) \in V_1, \operatorname{fin}(e) \in V_2, \\ -1 & \text{if } \operatorname{init}(e) \in V_2 \operatorname{fin}(e) \in V_1, \\ 0 & \text{otherwise.} \end{cases}$$

For a given bond, the choice of V_1 and V_2 simply changes the sign of g_B, and does not affect our results.

The **bond space** of $\overleftrightarrow{K_n}$, denoted $\mathcal{B}(\overleftrightarrow{K_n})$, is a subspace of the vector space $\mathbb{R}^E$ and is spanned by the function defined above:

$$\begin{aligned}\mathcal{B}(\overleftrightarrow{K_n}) &= \text{span}\{g_B : B \text{ is a bond of } \overleftrightarrow{K_n}\} \text{ and}\\ \dim(\mathcal{B}(\overleftrightarrow{K_n})) &= \#\text{vertices} - 1\\ &= n-1,\end{aligned}$$

as shown in [5] for general graphs.

Let B_i be the bond that isolates vertex i from the remaining vertices, and let $\mathbf{b}_i$ be the vector g_{B_i}. It turns out that the $\mathbf{b}_i$ for $1 \leq i \leq n$ form a basis for $\mathcal{B}(\overleftrightarrow{K_n})$. Furthermore, the $\mathbf{b}_i$ are exactly the rows of $\mathcal{M}(\overleftrightarrow{K_n})$, as explained through the following theorem.

Theorem 3.1. *The row space of $\mathcal{M}(\overleftrightarrow{K_n})$ is the bond space $\mathcal{B}(\overleftrightarrow{K_n})$.*

From our vertex-to-arc correspondence, we already knew that the row space of the root configuration A_{n-1} is the row space of the incidence matrix $\mathcal{M}(\overleftrightarrow{K_n})$. Theorem 3.1 and transitivity now tells us the bond space $\mathcal{B}(\overleftrightarrow{K_n})$ is the row space of the root configuration A_{n-1}, giving us the following proper labeling of the rows of A_{n-1}.

$$A_{n-1} = \begin{pmatrix}\mathbf{a}_{12} & \mathbf{a}_{21} & \mathbf{a}_{13} & \mathbf{a}_{31} & \cdots & \mathbf{a}_{n-1,n} & \mathbf{a}_{n,n-1}\end{pmatrix} = \begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \vdots\\ \mathbf{b}_{n-1}\\ \mathbf{b}_n\end{pmatrix} \in \{0,1,-1\}^{n\times 2\binom{n}{2}}.$$

The $\mathbf{b}_i$ labels for the rows of A_{n-1} feel natural — there are n rows in A_{n-1} (since the configuration is comprised of vectors in n-space) and there are n bonds of $\overleftrightarrow{K_n}$ which disconnect a single vertex from the rest of the vertices.

Is there an equally appropriate labeling for the rows of A^*_{n-1}, which has $(n-1)^2$ rows? To answer this, we turn to the cycle space of the graph.

3.3 The Cycle Space

A **flow**, also called a circulation, is a function $f \in \mathbb{R}^E$ such that for every $v \in V$,

$$\sum_{\substack{e\in E\\ \text{init}(e)=v}} f_e = \sum_{\substack{e\in E\\ \text{fin}(e)=v}} f_e.$$

This is a labeling on the edges of a digraph such that at each vertex, the total "in" value (or **inflow**) is equal to the total "out" value (or **outflow**).

Example 3.2. Figure 3.2 shows a flow on $\overleftrightarrow{K_3}$. The inflow at vertex 3 is $3 + (-1) = 2$, and the outflow at vertex 3 is $2 + 0 = 2$. They are equal. This is true for each vertex in the graph.

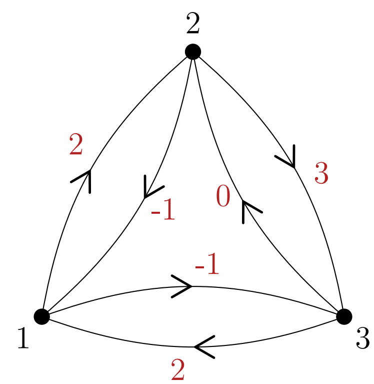


Figure 3.2: A flow of $\overleftrightarrow{K_3}$.

The flow given in Figure 3.2 is expressed as the 6-tuple

$$f = (2, -1, -1, 2, 3, 0),$$

where again we are using the positive lexicographic ordering on the edges. We call $\mathcal{C}(\overleftrightarrow{K_n})$ the **cycle space** of $\overleftrightarrow{K_n}$, and it is another subspace of $\mathbb{R}^E$:

$$\begin{aligned}
\mathcal{C}(\overleftrightarrow{K_n}) &= \mathrm{span}\{f : f \text{ is a flow on } \overleftrightarrow{K_n}\} \text{ and} \\
\dim(\mathcal{C}(\overleftrightarrow{K_n})) &= \#\text{arcs} - (\#\text{vertices} - 1) \\
&= 2\binom{n}{2} - (n-1) \\
&= (n-1)^2,
\end{aligned}$$

as shown in [5] for general graphs.

Consequently, a flow is an element in the nullspace of $\mathcal{M}(\overleftrightarrow{K_n})$. This can be seen by the fact that a row in $\mathcal{M}(\overleftrightarrow{K_n})$ corresponds to a vertex v, with a value of 1 in column $\vec{ij}$ if $\mathrm{init}(v) = 1$, a value of -1 if $\mathrm{fin}(\vec{ij}) = j$ and a value of 0 otherwise. Therefore, a flow is an $|E|$-tuple whose dot product with every row of $\mathcal{M}(\overleftrightarrow{K_n})$ equals 0. We can see this using the flow f from Example 3.2.

$$\mathcal{M}(\overleftrightarrow{K_3}) \cdot f = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

As the rows of $\mathcal{M}(\overleftrightarrow{K_n})$ are representative of bonds, we have the following theorem, adapted from [5]. The corollary is a direct consequence.

Theorem 3.2. *The cycle and bond spaces of* $\overleftrightarrow{K_n}$ *are orthogonal complements in* $\mathbb{R}^{2\binom{n}{2}}$.

Corollary 3.3. *The cycle space of* $\overleftrightarrow{K_n}$ *is the nullspace of* A_{n-1}.

Proof. By Theorems 3.1 and 3.2, we have

$$\text{null}(A_{n-1}) = \text{row}(A_{n-1})^{\perp} = \text{row}(\mathcal{M}(\overleftrightarrow{K_n}))^{\perp} = \mathcal{B}(\overleftrightarrow{K_n})^{\perp} = \mathcal{C}(\overleftrightarrow{K_n}).$$

□

Recall, Gale duality is defined through finding a basis for the nullspace of A_{n-1}. So, an appropriate basis for the cycle space of $\overleftrightarrow{K_n}$ will form the rows of our desired A^*_{n-1} matrix. This may seem like we are simply back where we started, charged with finding a basis matrix for the nullspace of another matrix.

However, the fact that the basis we are looking for now represents something graphically offers a much more tangible and hopeful chance at finding its explicit form. To do so, we examine the cycle space in more detail.

Chapter 4

Main Results

4.1 Constructing an Appropriate Basis

The previous section described the cycle space $\mathcal{C}(\overleftrightarrow{K_n})$ as the set of all flows on the graph $\overleftrightarrow{K_n}$. What do flows have to do with "cycles"? We first officially define a cycle.

Given a digraph D, a **cycle** is a set of arcs that form a closed walk with no repeated vertices, i.e., one can begin at a vertex, traverse each arc in the cycle, and end at the same vertex.

An **undirected cycle** is a cycle where we ignore the direction of the arcs. Such a cycle has two possible orientations, according to the direction in which we traverse the edges.

For such a cycle C and an orientation σ, define a function $f_C \in \mathbb{R}^E$ by

$$(f_C)_e = \begin{cases} 1 & \text{if } e \in C \text{ and } e \text{ agrees with } \sigma, \\ -1 & \text{if } e \in C \text{ and } e \text{ is opposite to } \sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Then f_C is a flow. Note that the opposite orientation gives the vector $-f_C$. Let's illustrate this.

Example 4.1. Consider the cycle $C = \{\vec{12}, \vec{21}\}$ of $\overleftrightarrow{K_n}$, highlighted in Figure 4.1 with gold. The function values for each arc are marked in red, and give the 6-tuple $f_C = (1, 1, 0, 0, 0, 0)$.

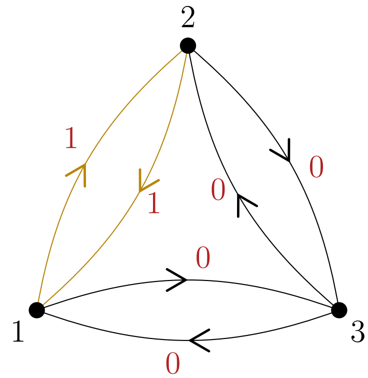


Figure 4.1: A cycle of $\overleftrightarrow{K_3}$ and the associated flow.

We quickly observe that this labeling conserves the flow in and out of each vertex, and so f_C is, in fact, a flow in the cycle space.

Define the **support** $||f||$ of $f \in \mathbb{R}^E$ as the set of arcs $e \in E$ for which $f_e \neq 0$. For instance, the support of $f = (2, -1, -1, 2, 3, 0)$ from Example 3.2 is $\{\vec{12}, \vec{21}, \vec{13}, \vec{31}, \vec{23}\}$. The following is adapted from [5].

Lemma 4.1. *If* $\mathbf{0} \neq \mathbf{f} \in \mathcal{C}(\overleftrightarrow{K_n})$*, then* $||f||$ *contains an undirected cycle.*

Thus, given any flow on $\overleftrightarrow{K_n}$, the set of arcs whose values are nonzero will contain a cycle. In our example, $\{\vec{12}, \vec{21}\}$ and $\{\vec{12}, \vec{23}, \vec{31}\}$ are such cycles. The connection between flows, cycles, and a basis for the cycle space goes even further.

It turns out that we can guarantee a basis for $\mathcal{C}(\overleftrightarrow{K_n})$ via the function $f_C \in \mathbb{R}^E$ defined in (4.1) for particular linearly independent cycles C. This verifies that, as the name implies, the cycle space of the graph is generated by cycles, that is, a flow is a linear combination of the function f_C applied to basis cycles. Before we state this officially, we first need a few more tools.

A **spanning tree** of a connected graph with n vertices is a connected subgraph of $n-1$ edges that contains no cycles. Necessarily, this means that each vertex is incident to some edge in the tree. See Figure 4.2 for examples.

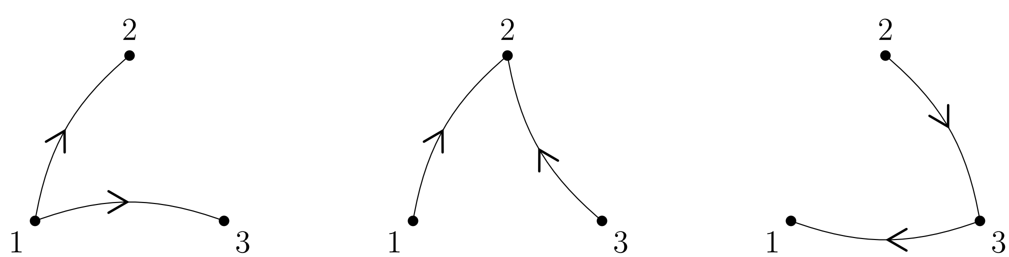


Figure 4.2: Three spanning trees of $\overleftrightarrow{K_3}$.

For a spanning tree T, there are arcs in the graph $\overleftrightarrow{K_n}$ that will not be in the tree. For each arc e such that $e \in E$ but $e \notin T$, adding the arc e to T will create what is called a **fundamental cycle**, denoted C_e, and it is the unique cycle within $T \cup e$.

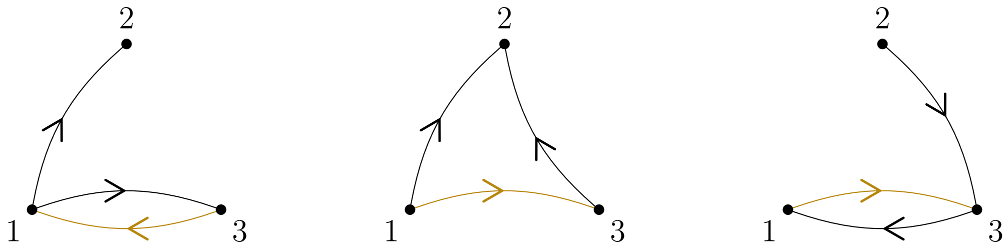


Figure 4.3: Unique fundamental cycles created by $T \cup e$.

Figure 4.3 illustrates different fundamental cycles C_e created by adding an arc e to a spanning tree. These fundamental cycles are the linearly independent cycles needed to guarantee a basis for the cycle space via the flow function f. The following theorem was proven for general graphs in [5].

Theorem 4.2. *Let T be a fixed spanning tree of $\overleftrightarrow{K_n}$. Then the set F of flows f_{C_e} for each C_e as e ranges over all arcs of $\overleftrightarrow{K_3}$ not in T, is a basis for the cycle space $\mathcal{C}(\overleftrightarrow{K_3})$.*

This tells us that given a spanning tree, we can construct a basis for the cycle space whose elements are $2\binom{n}{2}$-tuples given by the function in (4.1). From Corollary 3.3 these basis vectors will constitute a Gale dual of A_{n-1}. Different choices of T will lead to different Gale duals. Recall our desired configuration consisting of $\binom{n}{2}$ rows of paired 1's. What choice of tree T will lead to that structure?

A particularly simple choice gives us our answer: The tree rooted at vertex 1 and consisting of only the positive arcs $\vec{1j}$ for $2 \leq j \leq n$ (such as the one in the left of Figure 4.2). This brings us to our main result on the structure of A^*_{n-1}.

4.2 The Gale Dual A^*_{n-1}

Before we state the result, we need a bit of set-up for notation. The following theorem proves that a good labeling of the rows of A^*_{n-1} will be $\mathbf{b}^*_{ij}$ for $\binom{n}{2}$ rows, and $\mathbf{b}^*_{1ij}$ for the remaining $\binom{n-1}{2}$ rows. A brief reasoning for this is that we index a row with the vertices involved in that row's basis cycle.

When indicating a specific entry in row $\mathbf{b}^*_{ij}$, we will use a second subscript coinciding with the labeling on the column. For example, $b^*_{12,32}$ is the entry in A^*_{n-1} at the intersection of row $\mathbf{b}^*_{12}$ and column $\mathbf{a}^*_{32}$. We will also use the language "entry ij" to mean the entry in some row that coincides with column $\mathbf{a}^*_{ij}$.

Theorem 4.3 (Structure of the type-A dual configuration)**.** *A Gale dual A^*_{n-1} of the root system A_{n-1} is:*

(A) $\binom{n}{2}$ *rows labeled by* $\mathbf{b}^*_{ij}, 1 \leq i < j \leq n$,
where $b^*_{ij,ij} = b^*_{ij,ji} = 1$ *with all other entries* 0;

(B) $\binom{n-1}{2}$ *rows labeled by* $\mathbf{b}^*_{1ij}, 2 \leq i < j \leq n$,
where $b^*_{1ij,1i} = b^*_{1ij,ij} = 1$ *and* $b^*_{1ij,1j} = -1$ *with all other entries* 0.

Proof. Choose the spanning tree T_1 rooted at vertex 1 and consisting of all positive arcs $\vec{1j}$ for $2 \leq j \leq n$. There are three kinds of edges not in T_1:

$$\begin{aligned} &(1) \quad \vec{j1}, \quad j \neq 1; \\ &(2) \quad \vec{ij}, \quad 2 \leq i < j \leq n; \\ &(3) \quad \vec{ji}, \quad 2 \leq i < j \leq n. \end{aligned}$$

(1) The negative arc $\vec{j1}$ in T_1 forms the fundamental 2-cycle $\{\vec{1j}, \vec{j1}\}$. Assign the orientation σ to agree with $\vec{1j}$. Then σ is guaranteed to agree with $\vec{j1}$ as well.

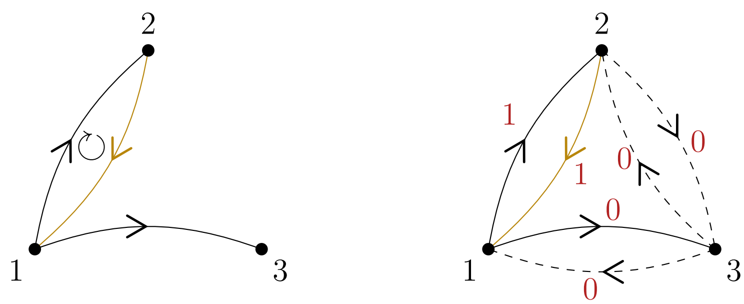


Figure 4.4: Example of the unique 2-cycle created by $T_1 \cup \vec{21}$, and corresponding flow values for all arcs.

The flow $f_{C_{\vec{j1}}}$ from (4.1) associated with the fundamental cycle $C_{\vec{j1}}$ consists of a value of 1 in entries $1j$ and $j1$ and a value of 0 elsewhere. There are $n-1$ vectors of this form in a basis for $\mathcal{C}(\overleftrightarrow{K_n})$, and thus there are $n-1$ of these rows in A^*_{n-1}, which we denote with $\mathbf{b}^*_{1j}$.

$$\begin{array}{c} \\ \mathbf{b}^*_{12} \\ \mathbf{b}^*_{13} \\ \vdots \\ \mathbf{b}^*_{1n} \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccccc} \mathbf{a}^*_{12} & \mathbf{a}^*_{21} & \mathbf{a}^*_{13} & \mathbf{a}^*_{31} & \cdots & \mathbf{a}^*_{1n} & \mathbf{a}^*_{n1} & \cdots & \mathbf{a}^*_{n-1\ n} & \mathbf{a}^*_{n\ n-1} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & & & & & \end{pmatrix} \end{array}$$

This corresponds to (part of) part (A), in that $b^*_{1j,1j} = b^*_{1j,j1} = 1$ with all other entries 0, for $2 \leq j \leq n$. Note that this does not give us the row vectors $\mathbf{b}^*_{ij}$ for $2 \leq i < j \leq n$ (which is the remainder of (A)). We will get those soon.

(2) The positive arc $\vec{ij}$ not in T_1 creates the fundamental 3-cycle $\{\vec{1i}, \vec{1j}, \vec{ij}\}$. Assign an orientation σ to agree with $\vec{1i}$. Then σ agrees with $\vec{ij}$ since $i < j$, but does not agree with $\vec{1j}$.

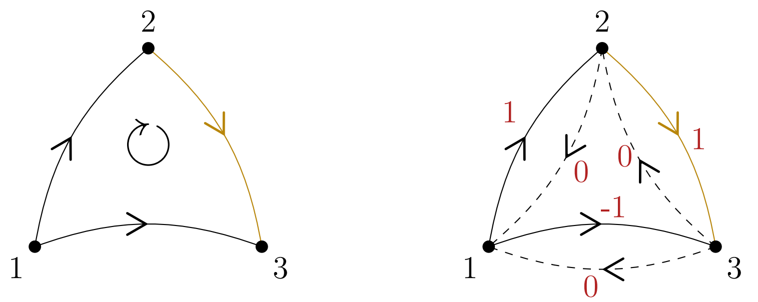


Figure 4.5: Example of the unique 3-cycle created by $T_1 \cup \vec{23}$, and corresponding flow values for all arcs.

Therefore, the flow $f_{C_{\vec{ij}}}$ from (4.1) associated with the fundamental cycle $C_{\vec{ij}}$ has a value of 1 in entries $1i$ and ij, a value of -1 in entry $1j$ and a value of 0 elsewhere. There are $\binom{n}{2} - (n-1) = \binom{n-1}{2}$ vectors of this form, which we denote with $\mathbf{b}^*_{1ij}$ for $2 \leq i < j \leq n$.

$$
\begin{array}{c}
\\
\mathbf{b}^*_{12} \\ \mathbf{b}^*_{13} \\ \vdots \\ \mathbf{b}^*_{1n} \\ \mathbf{b}^*_{123} \\ \vdots \\ \mathbf{b}^*_{1,n-1,n} \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc}
\mathbf{a}^*_{12} & \mathbf{a}^*_{21} & \mathbf{a}^*_{13} & \mathbf{a}^*_{31} & \mathbf{a}^*_{23} & \mathbf{a}^*_{32} & \cdots & \mathbf{a}^*_{1,n-1} & \mathbf{a}^*_{n-1,1} & \mathbf{a}^*_{1n} & \mathbf{a}^*_{n1} & \cdots & \mathbf{a}^*_{n-1,n} & \mathbf{a}^*_{n,n-1}
\end{array} \\
\left(\begin{array}{cccccccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 & 0 & \cdots & 1 & 0 \\
\vdots & & & & & & & & & & & & &
\end{array}\right)
\end{array}
$$

This corresponds to part (B) in the statement of our theorem, in that $b^*_{1ij,1i} = b^*_{1ij,ij} = 1$ and $b^*_{1ij,1j} = -1$ with 0's elsewhere for $2 \leq i < j \leq n$.

(3) Finally, the negative arc $\vec{ji}$ forms the fundamental 3-cycle $\{\vec{1i}, \vec{1j}, \vec{ji}\}$. Assign σ to agree with $\vec{1j}$. Then σ agrees with $\vec{ji}$ since $i < j$, but does not agree with $\vec{1i}$.

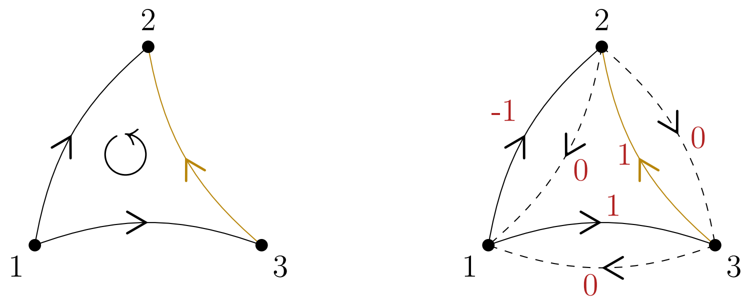


Figure 4.6: Example of the unique 3-cycle created by $T_1 \cup \vec{32}$, and corresponding f values for all arcs.

The flow f from (4.1) associated with the fundamental cycle $C_{\vec{ji}}$ has a value of 1 in entries $1j$ and ji, a value of -1 in entry $1i$ and a value of 0 elsewhere. There are $\binom{n-1}{2}$ vectors of this form, denoted $\mathbf{b}^*_{1ji}$.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccc} \mathbf{a}^*_{12} & \mathbf{a}^*_{21} & \mathbf{a}^*_{13} & \mathbf{a}^*_{31} & \cdots & \mathbf{a}^*_{1,n-1} & \mathbf{a}^*_{n-1,1} & \mathbf{a}^*_{1n} & \mathbf{a}^*_{n1} & \mathbf{a}^*_{23} & \mathbf{a}^*_{32} & \cdots & \mathbf{a}^*_{n-1,n} & \mathbf{a}^*_{n,n-1} \end{array} \\
\begin{array}{c} \mathbf{b}^*_{12} \\ \mathbf{b}^*_{13} \\ \vdots \\ \mathbf{b}^*_{1n} \\ \mathbf{b}^*_{123} \\ \mathbf{b}^*_{132} \\ \vdots \\ \mathbf{b}^*_{1,n-1,n} \\ \mathbf{b}^*_{1,n,n-1} \end{array} &
\begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\end{array}
$$

As we have now found the flows f_{C_e} ranging over all $e \notin T_1$, we have a basis for $\mathcal{C}(\overleftrightarrow{K_n})$, with a total of $2\binom{n}{2} - (n-1) = (n-1)^2$ vectors, as expected. The basis matrix is given above.

However, this is not the column vector form we desire for A^*_{n-1}, nor does it match the statement of the theorem. The matrix above has unmentioned vectors $\mathbf{b}^*_{1ji}$ and does not have all of the vectors $\mathbf{b}^*_{ij}$. This is remedied through a change of basis:

Notice that in $\overleftrightarrow{K_n}$ we have $\binom{n-1}{2}$ basis cycles of the form $\{\vec{1i}, \vec{1j}, \vec{ij}\}$ whose flow vector $\mathbf{b}^*_{1ij}$ has the values $1, -1, 1$ in these exact entries. There are also $\binom{n-1}{2}$ basis cycles of the form $\{\vec{1i}, \vec{1j}, \vec{ji}\}$ whose flow vector $\mathbf{b}^*_{1ji}$ has the values $-1, 1, 1$ in these exact entries.

For example, consider rows $\mathbf{b}^*_{123}$ and $\mathbf{b}^*_{132}$ above. By adding these basis vectors, and replacing $\mathbf{b}^*_{132}$ with the new sum $\mathbf{b}^*_{123} + \mathbf{b}^*_{132}$, we arrive at the missing 2-cycle basis vector $\mathbf{b}^*_{23}$ and eliminate the unmentioned basis vector $\mathbf{b}_{132}$.

In general, this change of basis will give us $\mathbf{b}^*_{ij}$ for $2 \le i < j \le n$, where $b^*_{ij,ij} = b^*_{ij,ji} = 1$ with 0's elsewhere, the remaining rows needed for part (A).

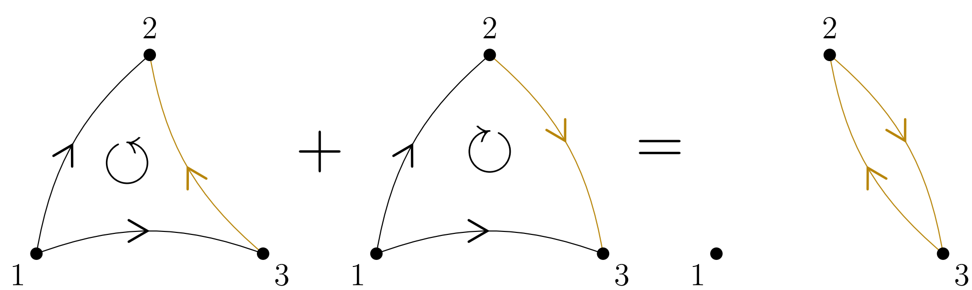
Therefore, our explicit configuration for A^*_{n-1} is as in the statement of the theorem, and given below in general form. $\square$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccc} \mathbf{a}^*_{12} & \mathbf{a}^*_{21} & \mathbf{a}^*_{13} & \mathbf{a}^*_{31} & \cdots & \mathbf{a}^*_{1,n-1} & \mathbf{a}^*_{n-1,1} & \mathbf{a}^*_{1n} & \mathbf{a}^*_{n1} & \mathbf{a}^*_{23} & \mathbf{a}^*_{32} & \cdots & \mathbf{a}^*_{n-1,n} & \mathbf{a}^*_{n,n-1} \end{array} \\
\begin{array}{c} \mathbf{b}^*_{12} \\ \mathbf{b}^*_{13} \\ \vdots \\ \mathbf{b}^*_{1n} \\ \mathbf{b}^*_{23} \\ \vdots \\ \mathbf{b}^*_{n-1,n} \\ \mathbf{b}^*_{123} \\ \vdots \\ \mathbf{b}^*_{1,n-1,n} \end{array} &
\begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 & 0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\end{array}
$$

Condensed, we have

$$
A^*_{n-1} = \begin{pmatrix} \mathbf{a}^*_{12} & \mathbf{a}^*_{21} & \mathbf{a}^*_{13} & \mathbf{a}^*_{31} & \cdots & \mathbf{a}^*_{n-1,n} & \mathbf{a}^*_{n,n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{b}^*_{12} \\ \mathbf{b}^*_{13} \\ \vdots \\ \mathbf{b}^*_{1n} \\ \mathbf{b}^*_{23} \\ \vdots \\ \mathbf{b}^*_{n-1,n} \\ \mathbf{b}^*_{123} \\ \vdots \\ \mathbf{b}^*_{1,n-1,n} \end{pmatrix} \in \{0,1,-1\}^{(n-1)^2 \times 2\binom{n}{2}}.
$$

This change of basis vectors can also be observed in the graph, via cycle addition.



With the dual configuration A^*_{n-1} now precisely determined, we can finally define the polytope it forms and begin to comment on its geometry.

4.3 Vertices and Hyperplanes of A^*_{n-1}

Similar to the root configuration, this dual configuration is not full dimensional — the vectors of A^*_{n-1} are $(n-1)^2$-tuples, but lie in one dimension lower. Specifically, the dual configuration A^*_{n-1} lies in the hyperplane

$$H_1 = \{\mathbf{x} \in \mathbb{R}^{(n-1)^2} : \sum_{i=1}^{\binom{n}{2}} x_n = 1\}.$$

We define the **type-A dual polytope** as

$$P_{A^*_{n-1}} = \mathrm{conv}(A^*_{n-1}),$$

and $\dim(P_{A^*_{n-1}}) = (n-1)^2 - 1$.

Theorem 4.4. *All $2\binom{n}{2}$ vectors in A^*_{n-1} are vertices in the polytope $P_{A^*_{n-1}}$. Furthermore, there exists an edge between each vertex $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$ for each pair $1 \leq i, j, \leq n$.*

Proof. There are $\binom{n}{2}$ rows of paired 1's in our desired construction of A^*_{n-1}. This tells us that for each vector (column) in the configuration, the first $\binom{n}{2}$ components will have exactly one 1 and all the rest of the components will be 0.

$$A^*_{n-1} = \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}^*_{12} & \mathbf{a}^*_{21} & \mathbf{a}^*_{13} & \mathbf{a}^*_{31} & \cdots & \mathbf{a}^*_{n-1\ n} & \mathbf{a}^*_{n\ n-1} \end{array} \\ \left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & & & & & & \end{array} \right) \end{array} \leftarrow \text{the } \binom{n}{2} \text{ row}$$

Define the function $h_k : \mathbb{R}^{(n-1)^2} \to \mathbb{R}$ given by $h_k(\mathbf{x}) = x_k - 1$, for $k = 1, 2, \ldots, \binom{n}{2}$. Apply this function to the column vectors $\mathbf{a}^*_{ij}$ in A^*_{n-1}. For a given value of k, there exists exactly one pair $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$ such that $h_k(\mathbf{a}^*_{ij}) = h_k(\mathbf{a}^*_{ji}) = 1-1 = 0$, and for all other vectors, $h_k(\mathbf{a}^*_{kl}) = 0 - 1 = -1$. This indicates that the one pair of $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$ live on the hyperplane $x_k - 1 = 0$ and all other vectors lie entirely in the half-space $x_k - 1 < 0$. Therefore, $x_k - 1 = 0$ is a supporting hyperplane for $P_{A^*_{n-1}}$ for $k = 1, \ldots, \binom{n}{2}$, and $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$ live on a face of the polytope.

If $\mathbf{a}^*_{ij}$ were not a vertex, then this would imply that there exists another vertex of $P_{A^*_{n-1}}$ "beyond" $\mathbf{a}^*_{ij}$ but still on the hyperplane $x_k - 1 = 0$, coplanar to $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$. But, as $P_{A^*_{n-1}}$ is the smallest convex set containing A^*_{n-1}, and every other vector of A^*_{n-1} lies in the half-space $x_k - 1 < 0$, we have that $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$ are vertices of $P_{A^*_{n-1}}$.

Since this was an arbitrary value of k in $\{1, \ldots, \binom{n}{2}\}$, and each value of k gives two distinct vectors as vertices, the $2\binom{n}{2}$ vectors in A^*_{n-1} are vertices of $P_{A^*_{n-1}}$. This also shows the existence of an edge between $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$ for each pair i, j. □

Corollary 4.5. *There exist $\binom{n}{2}$ facets of $P_{A^*_{n-1}}$ containing the vertex sets $A^*_{n-1} \setminus \{\mathbf{a}^*_{ij}, \mathbf{a}^*_{ji}\}$ for every pair $(i, j), i \neq j$. In other words, each edge $\{\mathbf{a}^*_{ij}, \mathbf{a}^*_{ji}\}$ from Theorem 4.4 lies "opposite" a facet comprised of all other vertices.*

Proof. We know that the vectors $\mathbf{a}^*_{ij}$ in A^*_{n-1} live in $\mathbb{R}^{(n-1)^2}$ but their affine hull is of codimension 1, as the vectors are embedded in the hyperplane H_1 with equation $x_1 + \cdots + x_{\binom{n}{2}} = 1$.

Consider $A^*_{n-1} \setminus \{\mathbf{a}^*_{ij}, \mathbf{a}^*_{ji}\}$ for some $i < j$. This forms an $(n-1)^2 \times (2\binom{n}{2} - 2)$ matrix, and the columns still live in $\mathbb{R}^{(n-1)^2}$. However, one row of $A^*_{n-1} \setminus \{\mathbf{a}^*_{ij}, \mathbf{a}^*_{ji}\}$ is now comprised entirely of 0's, because the vector $\mathbf{b}^*_{ij}$ only has nonzero entries exactly in columns $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$ (Theorem 4.3).

Geometrically, this embeds the configuration $A^*_{n-1} \setminus \{\mathbf{a}^*_{ij}, \mathbf{a}^*_{ji}\}$ into an even lower-dimensional space, namely, a coordinate plane $x_k = 0$ intersected with the affine hyperplane H_1. We will denote this intersection H^k_1, where

$$H^k_1 = \{\mathbf{x} \in \mathbb{R}^{(n-1)^2} : x_1 + \cdots + x_{\binom{n}{2}} = 1 \text{ and } x_k = 0\}$$

and k is a value between 1 and $\binom{n}{2}$ referring to the row number of vector $\mathbf{b}^*_{ij}$.

Clearly, H^k_1 is a supporting hyperplane in H_1 to $P_{A^*_{n-1}}$ because $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$ both live in the open half-space $H^{k<}_1$, with all other vertices living in H^k_1.

We wish to show that the vectors in $A^*_{n-1} \setminus \{\mathbf{a}^*_{ij}, \mathbf{a}^*_{ji}\}$ actually span H^k_1, forming a facet of $P_{A^*_{n-1}}$. A general fact of Gale duality tells us that a set S is a hyperplane in the Gale dual A^*_{n-1} if and only if the complement of S is a positive linear dependence relation of the vectors in the original configuration A_{n-1}.

Since $\mathbf{a}_{ij} + \mathbf{a}_{ji} = \mathbf{0}$ for all i, j, the complement of $\{\mathbf{a}_{ij}, \mathbf{a}_{ji}\}$, which is precisely $A^*_{n-1} \setminus \{\mathbf{a}^*_{ij}, \mathbf{a}^*_{ji}\}$, forms a hyperplane in H_1, i.e., they affinely span H^k_1. From this we see that all vectors in $A^*_{n-1} \setminus \{\mathbf{a}^*_{ij}, \mathbf{a}^*_{ji}\}$ form a facet of $P_{A^*_{n-1}}$.

With $\binom{n}{2}$ such pairs $\mathbf{a}^*_{ij}$ and $\mathbf{a}^*_{ji}$, we will have $\binom{n}{2}$ facets of this form. $\square$

4.4 Unimodular Triangulations

The construction of the dual configuration A^*_{n-1} via the cycle space of $\overleftrightarrow{K_n}$ gives us more than its explicit form. The following theorem is taken from [5], though its proof is credited to Tutte [7]. In the theorem, M_T denotes the basis matrix obtained via a spanning tree T, where the flow function f_{C_e} is taken over all edges not in T, just as was done in our proof of Theorem 4.3.

Theorem 4.6. *Let T be a spanning tree of $\overleftrightarrow{K_n}$. Then the basis matrix M_T of $\mathcal{C}(\overleftrightarrow{K_n})$ is unimodular.*

A **unimodular matrix** is an $m \times n$ matrix $(m \leq n)$ such that every every $m \times m$ submatrix has determinant $1, -1$, or 0. Theorem 4.6 says that the basis matrix obtained via a spanning tree is unimodular.

Corollary 4.7. *The vector configuration A^*_{n-1} is a unimodular matrix.*

Proof. Our desired A^*_{n-1} configuration is a basis matrix for $\mathcal{C}(\overleftrightarrow{K_n})$ after a change of basis, which does not change the determinant of any maximal square submatrix. $\square$

Unimodular triangulations of the original root polytope $P_{A_{n-1}}$ were found in [1] by utilizing the only interior lattice point, the origin (see Figure 2.3). However, $P_{A^*_{n-1}}$ does not have any interior lattice points, and so any unimodular triangulation will only consist of the existing vertices. Therefore, moving forward, we will only consider triangulations of $P_{A^*_{n-1}}$ using no new vertices.

Theorem 4.8. *Every triangulation of $P_{A^*_{n-1}}$ that use only the vertices of $P_{A^*_{n-1}}$ is unimodular.*

Proof. Recall that every vector in A^*_{n-1} is a vertex of $P_{A^*_{n-1}}$ from Theorem 4.4. Let Δ be a triangulation of $P_{A^*_{n-1}}$. Then Δ consists of t simplices, we will call them $s^*_1, \ldots, s^*_t$, where s^*_i is an $((n-1)^2-1)$-dimensional simplex with $(n-1)^2$ vertices. Fix an s^*_i. Necessarily, the $(n-1)^2$ vertices of s^*_i are some collection of $(n-1)^2$ columns of A^*_{n-1}.

Since A^*_{n-1} is an $(n-1)^2 \times 2\binom{n}{2}$ matrix, the submatrix whose columns are the vertices of s^*_i is an $(n-1)^2 \times (n-1)^2$ maximal submatrix of A^*_{n-1}. Corollary 4.7 tells us that this submatrix has determinant $1, -1$, or 0. The determinant of the submatrix of the vertices of s^*_i cannot equal zero, as the vectors are affinely independent in $\mathbb{R}^{(n-1)^2-1}$ but embedded in $\mathbb{R}^{(n-1)^2}$, making them linearly independent. Thus, the determinant of s^*_i is 1 or -1, and it forms a unimodular simplex by definition.

Since this was an arbitrary simplex s^*_i in Δ, every simplex in Δ is unimodular, i.e., Δ is a unimodular triangulation. And as Δ was arbitrary, we have our result that every triangulation of $P_{A^*_{n-1}}$ is unimodular. $\square$

Recall, unimodular simplices have Euclidean volume $\frac{1}{d!}$, where d is the dimension of the simplex. Therefore, as a direct result of Theorem 4.8 we have the following corollary.

Corollary 4.9. *The type-A dual polytope $P_{A^*_{n-1}}$ is equidecomposible, that is, every triangulation of $P_{A^*_{n-1}}$ using no new vertices contains the same number of simplices.*

Proof. Let V be the volume of $P_{A^*_{n-1}}$, and let Δ be a triangulation of $P_{A^*_{n-1}}$. Then Δ is a unimodular triangulation by Theorem 4.8, and so each simplex $s^*_i \in \Delta$ has volume $\frac{1}{((n-1)^2-1)!}$. Therefore,

$$
\begin{aligned}
& \frac{\#\text{ of simplices in }\Delta}{((n-1)^2-1)!} = V \\
\implies \quad & \#\text{ of simplices in }\Delta = V \cdot ((n-1)^2-1)! \qquad (4.2)
\end{aligned}
$$

As both V and n are fixed, and this volume equivalence must be true for any triangulation of $P_{A^*_{n-1}}$, every triangulation of $P_{A^*_{n-1}}$ contains the same number of simplices. $\square$

Example 4.2. All of the unimodular triangulations of $P_{A^*_2}$ are represented in Figure 4.7 and listed below.

$$\Delta_1 = \{\{\mathbf{a}^*_{21}, \mathbf{a}^*_{31}, \mathbf{a}^*_{23}, \mathbf{a}^*_{32}\}, \{\mathbf{a}^*_{21}, \mathbf{a}^*_{13}, \mathbf{a}^*_{31}, \mathbf{a}^*_{32}\}, \{\mathbf{a}^*_{12}, \mathbf{a}^*_{21}, \mathbf{a}^*_{31}, \mathbf{a}^*_{23}\}\},$$

$$\Delta_2 = \{\{\mathbf{a}^*_{12}, \mathbf{a}^*_{21}, \mathbf{a}^*_{31}, \mathbf{a}^*_{32}\}, \{\mathbf{a}^*_{21}, \mathbf{a}^*_{13}, \mathbf{a}^*_{31}, \mathbf{a}^*_{32}\}, \{\mathbf{a}^*_{12}, \mathbf{a}^*_{31}, \mathbf{a}^*_{23}, \mathbf{a}^*_{32}\}\},$$

$$\Delta_3 = \{\{\mathbf{a}^*_{12}, \mathbf{a}^*_{13}, \mathbf{a}^*_{31}, \mathbf{a}^*_{32}\}, \{\mathbf{a}^*_{12}, \mathbf{a}^*_{21}, \mathbf{a}^*_{13}, \mathbf{a}^*_{32}\}, \{\mathbf{a}^*_{12}, \mathbf{a}^*_{31}, \mathbf{a}^*_{23}, \mathbf{a}^*_{32}\}\},$$

$$\Delta_4 = \{\{\mathbf{a}^*_{12}, \mathbf{a}^*_{13}, \mathbf{a}^*_{23}, \mathbf{a}^*_{32}\}, \{\mathbf{a}^*_{12}, \mathbf{a}^*_{13}, \mathbf{a}^*_{31}, \mathbf{a}^*_{23}\}, \{\mathbf{a}^*_{12}, \mathbf{a}^*_{21}, \mathbf{a}^*_{13}, \mathbf{a}^*_{32}\}\},$$

$$\Delta_5 = \{\{\mathbf{a}^*_{12}, \mathbf{a}^*_{21}, \mathbf{a}^*_{13}, \mathbf{a}^*_{23}\}, \{\mathbf{a}^*_{21}, \mathbf{a}^*_{13}, \mathbf{a}^*_{23}, \mathbf{a}^*_{32}\}, \{\mathbf{a}^*_{12}, \mathbf{a}^*_{13}, \mathbf{a}^*_{31}, \mathbf{a}^*_{23}\}\},$$

$$\Delta_6 = \{\{\mathbf{a}^*_{21}, \mathbf{a}^*_{13}, \mathbf{a}^*_{31}, \mathbf{a}^*_{23}\}, \{\mathbf{a}^*_{12}, \mathbf{a}^*_{21}, \mathbf{a}^*_{31}, \mathbf{a}^*_{23}\}, \{\mathbf{a}^*_{21}, \mathbf{a}^*_{13}, \mathbf{a}^*_{23}, \mathbf{a}^*_{32}\}\}.$$

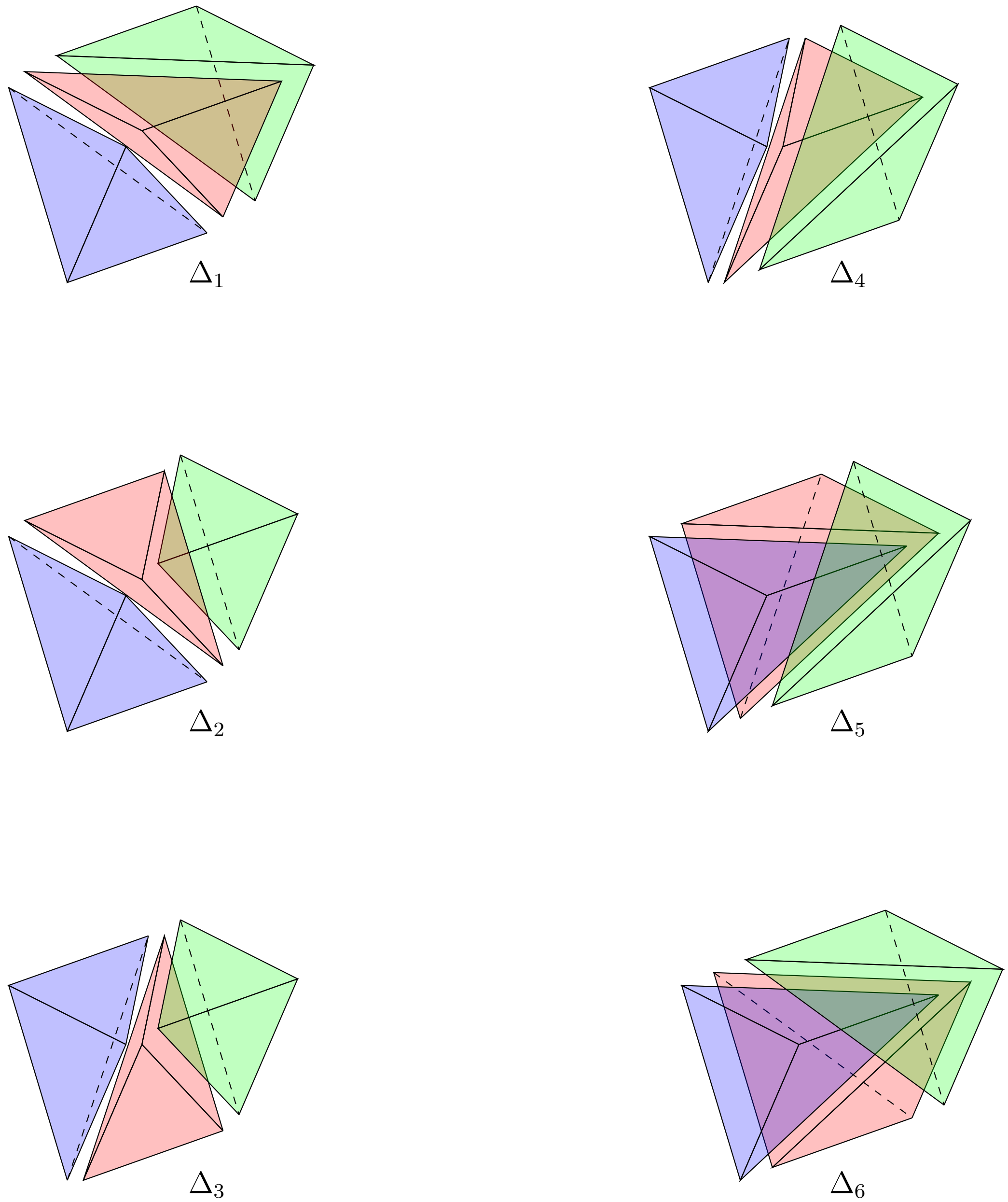


Figure 4.7: The six unimodular triangulations of $P_{A_2^*}$ using only the existing vertices.

We see that each triangulation contains exactly three simplices, and visually it appears as though each simplex is of the same volume. We verify this by computing the volume of each simplex using `polymake` [4]. The volume of $P_{A_2^*}$ is $\frac{1}{2}$, and the volume of each simplex in each triangulation is $\frac{1}{6} = \frac{1}{3!}$, as expected.

Remark. It should be noted that the type-A dual polytope $P_{A_{n-1}^*}$ is a *Lawrence polytope*, and many of the results we have regarding the polytope, specifically Theorem 4.9, can be proven using this fact. For more details on Lawrence polytopes, see [3], for example.

Seemingly, a consequence of Corollary 4.9 should be a count of the number of simplices in a given triangulation of $P_{A^*_{n-1}}$. By (4.2) we simply need to know the volume of the polytope, and presto! However, finding an explicit formula for the volume of a polytope can be tricky, and may offer little combinatorial reasoning. A bijection would be ideal, but from what object?

We use `polymake` [4] to assist us in low dimensions. We find that in every triangulation of $P_{A^*_2}$ there are three simplices, of $P_{A^*_3}$ there are 16 simplices, and of $P_{A^*_4}$ there are 125 simplices.

The pattern we see is that every triangulation of $P_{A^*_{n-1}}$ consists of exactly n^{n-2} simplices. This is a promising observation because of the known result that there are exactly n^{n-2} spanning trees of the complete graph K_n, which has vertex set $V = [n]$ and edge set $E = \{ij : 1 \leq i < j \leq n, i \neq j\}$.

Although the connection we have been utilizing is between our vector configurations and the complete digraph $\overleftrightarrow{K_n}$, the (undirected, single-edged) complete graph K_n proves to be exactly the object with which we find a bijection. But first we need additional tools.

4.5 The Chamber Complex

Given the root configuration A_{n-1}, define the cone spanned by the sub-configuration $A \subseteq A_{n-1}$ as

$$\mathrm{cone}(A) = \left\{ \sum_{\mathbf{a}_{ij} \in A} \lambda_{ij} \mathbf{a}_{ij} : \lambda_{ij} \geq 0 \right\}.$$

When the vectors in A are contained completely in one open half-space, then $\mathrm{cone}(A)$ will form a pointed cone, and when we additionally have $|A| = n - 1$ and the vectors in A are linearly independent, then $\mathrm{cone}(A)$ is called a **simplicial cone** and is necessarily full-dimensional in H_0, the $(n-1)$-dimensional subspace containing A_{n-1}. Note: From here forward our use of the word "cone" will always refer to pointed cones, unless otherwise specified.

The **chamber complex** of A_{n-1}, denoted $\mathrm{Ch}(A_{n-1})$, is the common refinement of all simplicial cones induced by taking subsets A of A_{n-1}. This decomposes H_0 into polyhedral cells, where the maximal, i.e., full dimensional, cells are called **chambers**. See Figure 4.8.

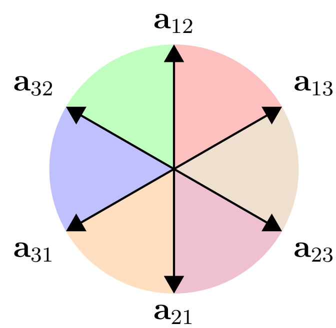


Figure 4.8: The chamber complex $\mathrm{Ch}(A_2)$.

Our main use of the chamber complex of the root configuration is its connection with triangulations of the Gale dual. The corollary below is specialized from [3], where it was stated in general.

Corollary 4.10. *The face lattice of the chamber complex of A_{n-1} is reverse-isomorphic to the refinement poset of all regular polyhedral subdivisions of A^*_{n-1}.*

While we will not go into more detail on the implications and meaning of the corollary in this paper, we will interpret it just enough to make use of it: the chambers in $\mathrm{Ch}(A_{n-1})$ are in bijection with regular triangulations of A^*_{n-1} as a point configuration and so, as all points in A^*_{n-1} are vertices of $P_{A^*_{n-1}}$, with triangulations of $P_{A^*_{n-1}}$.

The correspondence is made as follows: Given a chamber cone(A) in $\mathrm{Ch}(A_{n-1})$ for some A (so $|A| = n-1$), we will look at a sub-configuration s_i such that cone$(A) \subseteq$ cone(s_i), still with $|s_i| = n-1$. For each such s_i, let

$$s_i^* = \{\mathbf{a}_{ij}^* : \mathbf{a}_{ij} \notin s_i\}.$$

Then s_i^* will form one of the simplices in a triangulation of $P_{A^*_{n-1}}$. The resulting triangulation corresponds to the original chamber.

Alternatively, given a triangulation Δ of A^*_{n-1}, take the vertices of a simplex s_i^* in the triangulation. Take the complement of the $\mathbf{a}_{ij}^* \in s_i^*$ and convert to their associated root counterparts $\mathbf{a}_{ij}$, giving a set s_i. Then s_i will generate a simplicial cone, and the intersection over all simplicial cones induced by simplices in Δ will again be a simplicial cone, and a chamber in $\mathrm{Ch}(A_{n-1})$.

Example 4.3. Figure 4.9 shows the root configuration A_2, whose chamber complex consists of the six full-dimensional simplicial cones generated by adjacent vectors. Notice that each of the six chambers is labeled by a triangulation of $P_{A_2^*}$ (see Figure 4.7).

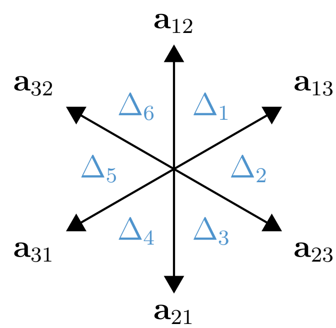


Figure 4.9: The labeling of the chambers in $\mathrm{Ch}(A_2)$ with triangulations of $P_{A_2^*}$.

Consider the chamber in $\mathrm{Ch}(A_2)$ labeled by Δ_1. The three simplicial cones containing this chamber are generated by the following vectors:

$$\begin{aligned} s_1 &= \{\mathbf{a}_{12}, \mathbf{a}_{13}\}, \\ s_2 &= \{\mathbf{a}_{12}, \mathbf{a}_{23}\}, \text{ and} \\ s_3 &= \{\mathbf{a}_{13}, \mathbf{a}_{32}\}. \end{aligned}$$

The complements in A_2 of the generating vectors are

$$\begin{aligned} A_2 \setminus s_1 &= \{\mathbf{a}_{21}, \mathbf{a}_{31}, \mathbf{a}_{23}, \mathbf{a}_{32}\}, \\ A_2 \setminus s_2 &= \{\mathbf{a}_{21}, \mathbf{a}_{13}, \mathbf{a}_{31}, \mathbf{a}_{32}\}, \text{ and} \\ A_2 \setminus s_3 &= \{\mathbf{a}_{12}, \mathbf{a}_{21}, \mathbf{a}_{31}, \mathbf{a}_{23}\}. \end{aligned}$$

Thus, the three simplices in A_2^* that form Δ_1 are

$$\begin{aligned} s_1^* &= \{\mathbf{a}_{21}^*, \mathbf{a}_{31}^*, \mathbf{a}_{23}^*, \mathbf{a}_{32}^*\}, \\ s_2^* &= \{\mathbf{a}_{21}^*, \mathbf{a}_{13}^*, \mathbf{a}_{31}^*, \mathbf{a}_{32}^*\}, \text{ and} \\ s_3^* &= \{\mathbf{a}_{12}^*, \mathbf{a}_{21}^*, \mathbf{a}_{31}^*, \mathbf{a}_{23}^*\}. \end{aligned}$$

Similarly, we can first look at triangulation Δ_1 and see that the vertices of each simplex s_i^* give rise to cone $s_i = A_2^* \setminus s_i^*$ (without asterisks). The intersection over the three cones generated by s_i for $i = 1, 2, 3$ is precisely the chamber labeled by Δ_1.

Thus, we can count simplices in a triangulation of A_{n-1}^* by first finding the associated chamber in the chamber complex of A_{n-1} and then counting how many simplicial cones generated by sub-configurations of A_{n-1} contain that chamber.

Example 4.4. Let's look at another example, this time for $n = 4$.

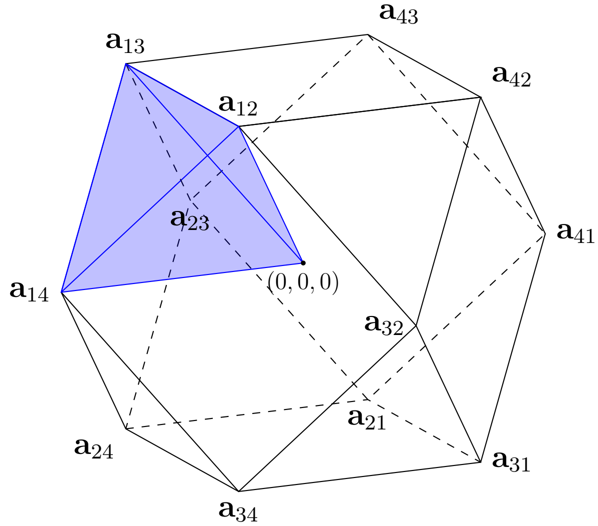


Figure 4.10: P_{A_3}, with chamber cone($\{\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}\}$) for Ch($A_3$).

We've highlighted one of the 32 chambers in Ch(A_3) which is generated by $\{\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}\}$. This corresponds to a triangulation Δ_1 of A_3^*, and we hope to find that there are 16 simplicial cones with generators in A_3 that contain cone($\{\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}\}$). We do, and in attempting to enumerate the results, we find a systematic approach to identifying said simplicial cones.

Proposition 8 in [1] tells us that all edges in $P_{A_{n-1}}$ are of the form $\mathbf{a}_{ij}\mathbf{a}_{ik}$ and $\mathbf{a}_{ik}\mathbf{a}_{jk}$ for i, j, k distinct. What this means for cone($\{\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}\}$) is that the generator $\mathbf{a}_{12}$ is adjacent to $\mathbf{a}_{32}$ and to $\mathbf{a}_{42}$, both of which are not in the cone. If we "flip" at that single point, from $\mathbf{a}_{12}$ to adjacent vector $\mathbf{a}_{32}$, then we obtain cone($\{\mathbf{a}_{32}, \mathbf{a}_{13}, \mathbf{a}_{14}\}$).

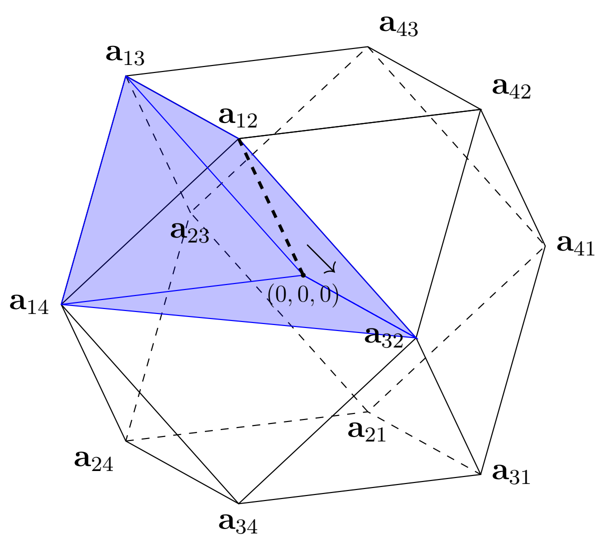


Figure 4.11: Cone obtained by flipping from generator $\mathbf{a}_{12}$ to $\mathbf{a}_{32}$.

This is in fact a simplicial cone, as the three vectors are linearly independent, and it contains the original chamber cone($\{\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}\}$), as $\mathbf{a}_{13} + \mathbf{a}_{32} = \mathbf{a}_{12}$. Thus, $A_3^* \setminus \{\mathbf{a}_{32}^*, \mathbf{a}_{13}^*, \mathbf{a}_{14}^*\}$ forms a simplex in Δ_1. Similarly, we could flip $\mathbf{a}_{12}$ to the other adjacent vector, $\mathbf{a}_{42}$, and obtain another simplicial cone, and thus another simplex in Δ_1.

We see that we obtain exactly 16 simplicial cones containing cone($\{\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}\}$) by this process of flipping at one of the three vectors or at two of the three vectors. (Note: We cannot flip at all three vectors, because the result would not form a simplicial cone.)

This method is successful and intuitive in dimensions 2 and 3, but is difficult to generalize and to prove for every n. We readily observe that, in general, for any chamber of the form cone($\{\mathbf{a}_{ij_1}, \mathbf{a}_{ij_2}, \ldots, \mathbf{a}_{ij_{n-1}}\}$), flipping at a single vertex involves fixing the second index and permuting through the available $n-2$ possibilities for the first index. However, determining which flips will form simplicial cones, and which will not, when we are flipping k generic vectors is laborious and slow.

Instead, we convert this concept of flipping at a generator into another language, namely, our language pertaining to the complete graph.

Lemma 4.11. *Simplicial cones generated by vectors in A_{n-1} are in bijection with spanning trees of $\overleftrightarrow{K_n}$.*

Proof. Let A be a set of $n-1$ vectors in A_{n-1}, and let T_A be the corresponding arcs in $\overleftrightarrow{K_n}$. Then

$$\begin{aligned}
\text{cone}(A) \text{ is simplicial} &\iff A \text{ is linearly independent} \\
&\iff \text{no dependence relation exists among vectors in } A \\
&\iff T_A \text{ contains no cycles} \\
&\iff T_A \text{ is a tree.}
\end{aligned}$$

And as T_A contains exactly $n-1$ arcs, T_A is spanning. □

Example 4.5 (Continuation of Example 4.4)**.** The original chamber cone($\{\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}\}$) in Ch($A_2$) corresponds to the spanning tree $\{\vec{12}, \vec{13}, \vec{14}\}$, as in Figure 4.12.

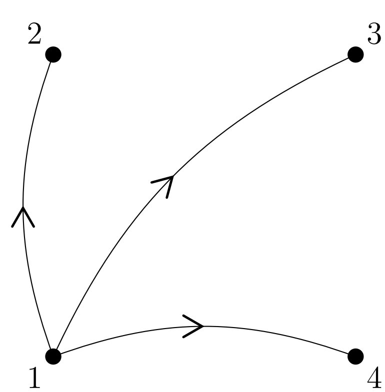


Figure 4.12: Spanning tree of $\overleftrightarrow{K_4}$.

We flipped from generator $\mathbf{a}_{12}$ to adjacent vector $\mathbf{a}_{32}$, and so we flip from arc $\vec{12}$ to arc $\vec{32}$.

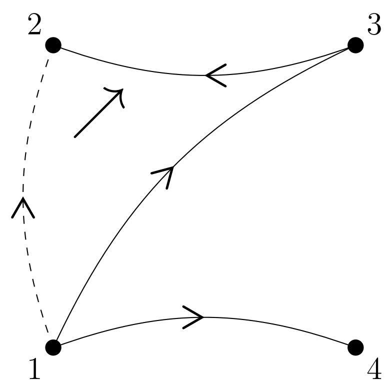


Figure 4.13: A flip at vertex 2.

Our previous action of flipping a vector to a neighboring vector, by fixing the second index of the vector and switching the first index, manifests itself as a flip at a vertex in the graph; similar to fixing the 2 in the index of $\mathbf{a}_{12}$ and flipping the 1 to a 3, we fix vertex 2 in the graph and flip the arc that connects it to vertex 1 to make it connect to vertex 3 instead, retaining the orientation of the arc so it flows into vertex 2 (i.e., 2 remains the second index).

At this point, we seem to have more questions than answers. Does this action of flipping at a vertex in the graph guarantee us a spanning tree again? Also, n^{n-2} counts spanning trees of K_n, not $\overleftrightarrow{K_n}$. How do we rectify this, when we appear to be making a connection between particular simplicial cones in A_{n-1} and spanning trees of $\overleftrightarrow{K_n}$?

As it turns out, we do not need to answer or prove the first question (although it is true), because of how we choose to deal with the second question. Observe that, necessarily, a spanning tree of $\overleftrightarrow{K_n}$ will never contain both arcs $\vec{ij}$ and $\vec{ji}$, and so any spanning tree of $\overleftrightarrow{K_n}$ *is* a spanning tree of K_n once we combine arcs $\vec{ij}$ and $\vec{ji}$ into edge ij, with $i < j$.

We cannot lose the orientation in the graph if we wish to make a map from trees to simplicial cones, however; recall that $\mathbf{a}_{ij}$ and $\mathbf{a}_{ji}$ are different vectors in the root configuration. In fact, the orientation on the arcs of the graph is only necessary to tell us which vector we are dealing with, $\mathbf{a}_{ij}$ or $\mathbf{a}_{ji}$. And that proves to be the key to creating our bijection. But first, a necessary result.

Lemma 4.12. *The simplicial cone* $\Sigma = \text{cone}(\{\mathbf{a}_{12}, \mathbf{a}_{13}, \ldots, \mathbf{a}_{1n}\})$ *is a chamber in* $\text{Ch}(A_{n-1})$.

Proof. We see that $\Sigma = \text{cone}(\{\mathbf{a}_{12}, \mathbf{a}_{13}, \ldots, \mathbf{a}_{1n}\})$ is a simplicial cone in H_0, as there are $n-1$ generators and they are linearly independent. To show it is a chamber in $\text{Ch}(A_{n-1})$, we also need to show that there is no other simplicial cone generated by vectors in A_{n-1} that it intersects. More precisely, if Σ is not a chamber, then there would exist another chamber Ω such that the interiors of Σ and Ω intersect. For this to happen, a hyperplane spanned by some generators of Ω would need to intersect the interior of Σ, effectively splitting Σ into two pieces. We will show this cannot happen.

Suppose such a simplicial cone Ω exists. As Ω has $n-1$ linearly independent generators, there exists a linearly independent $(n-2)$-subset of those generators, which, together with the origin, form a hyperplane N in H_0, splitting Σ into two pieces.

We claim that such a hyperplane has the equation $\sum_{i \in S} x_i = 0$ for some set $S \subset [n]$. This is true because if we choose any linearly independent $n-2$ vectors in A_{n-1}, they will form a set F of $n-2$ arcs in $\overleftrightarrow{K_n}$. Since the vectors are linearly independent, F will have no cycles, and since there are only $n-2$ arcs, these will form a forest with two components: S, the vertices spanned by the $n-2$ arcs, and $[n] \setminus S$.

Thus, the $n-2$ vectors corresponding to the arcs in the tree satisfy the equation $\sum_{i \in S} x_i = 0$ where S is the set of vertices spanned by the tree. As the origin always satisfies this equation for all S, this must be the equation of the hyperplane.

So, given the $(n-2)$-subset of generators of Ω which cuts through Σ's interior, we know that the hyperplane N has the equation $\sum_{i \in S} x_i = 0$ for some set $S \subset [n]$. We also know that there exist some generators $\mathbf{a}_{1i}$ and $\mathbf{a}_{1j}$ that lie on opposite sides of N. Suppose $\mathbf{a}_{1i} \in N^<$ and $\mathbf{a}_{1j} \in N^>$. For $\mathbf{a}_{1j}$ to live in $N^>$, it must be the case that $1 \in S$, but for $\mathbf{a}_{1i}$ to live in $N^<$ it must be the case that $1 \notin S$, a contradiction. Thus, no such hyperplane exists, and $\Sigma = \text{cone}(\{\mathbf{a}_{12}, \mathbf{a}_{13}, \ldots, \mathbf{a}_{1n}\})$ is a chamber in $\text{Ch}(A_{n-1})$. □

Theorem 4.13. *Every triangulation of $P_{A^*_{n-1}}$ contains exactly n^{n-2} simplices.*

Proof. Consider the chamber $\Sigma = \text{cone}(\{\mathbf{a}_{12}, \mathbf{a}_{13}, \ldots, \mathbf{a}_{1n}\})$ in $\text{Ch}(A_{n-1})$. By Corollary 4.10, there exists some triangulation Δ of $P_{A^*_{n-1}}$ corresponding to Σ. The correspondence given in Corollary 4.10 tells us that each simplicial cone containing Σ gives rise to a simplex in Δ and vice versa.

Therefore, we will show a bijection between simplicial cones containing Σ and spanning trees of K_n.

Let T be a spanning tree of K_n, so $T = \{i_1j_1, i_2j_2, \ldots, i_{n-1}j_{n-1}\}$ with $i_k < j_k$. From vertex 1 to any other vertex there exists a unique walk, otherwise T would contain a cycle and would not be a tree. Thus, we orient T away from vertex 1, i.e., orient each edge in T according to how that edge is traversed in a walk from 1 to some vertex.

The oriented tree $\vec{T}$ is now $\vec{T} = \{\overrightarrow{k_1l_1}, \ldots, \overrightarrow{k_{n-1}l_{n-1}}\}$. Notice that at least one k_i is equal to 1, otherwise we would not have a tree, and that all of the l_i are distinct, otherwise we would have had more than one walk from vertex 1 to some vertex l_i.

Take the arcs in $\vec{T}$ and set them as indices for vectors in A_{n-1}, producing $S_{\vec{T}} = \{\mathbf{a}_{k_1l_1}, \ldots, \mathbf{a}_{k_{n-1}l_{n-1}}\}$. Our claim is that $S_{\vec{T}}$ is a simplicial cone containing Σ.

We see that $S_{\vec{T}}$ is a simplicial cone by Lemma 4.11, because $\vec{T}$ is a spanning tree of $\overleftrightarrow{K_n}$.

All of the generators of Σ are of the form $\mathbf{a}_{1j}$ for every $j \in [n] \setminus \{1\}$. We see that $S_{\vec{T}}$ contains Σ because of the way it is constructed. We constructed $\vec{T}$ by orienting edges away from vertex 1, so given any vertex j there exists a combination of arcs in $\vec{T}$ that induce a direct walk from 1 to j, i.e., there is a combination of vectors in $S_{\vec{T}}$ that, when added together, give us $\mathbf{a}_{1j}$ for every j.

Clearly, this map is well defined. It is onto because given any simplicial cone containing Σ, simply "unorient" the indices of the vectors so that they are in increasing order. These unoriented pairs will form a spanning tree in K_n.

We see that this is a one-to-one map as well: Lemma 4.11 tells us that spanning trees of $\overleftrightarrow{K_n}$ are in bijection with simplicial cones whose generators are in A_{n-1}. Since $\vec{T}$ is a spanning tree of $\overleftrightarrow{K_n}$, we know that $S_{\vec{T}}$ is a simplicial cone in H_0, and therefore the vectors in $S_{\vec{T}}$ forms a basis for a subspace isomorphic to $\mathbb{R}^{n-1}$. Given a tree T, there are 2^{n-1} possible orientations of $\vec{T}$, and thus 2^{n-1} possible simplicial cones which, when unoriented, could give rise to T. However, these 2^{n-1} cones form the 2^{n-1} orthants in the subspace they span, and so their interiors are disjoint. Thus, no two simplicial cones containing Σ could correspond to the same tree.

We have proven our bijection for a particular chamber in $\mathrm{Ch}(A_{n-1})$, and so for only a particular triangulation of $P_{A^*_{n-1}}$, but recall from Corollary 4.9 that every triangulation of $P_{A^*_{n-1}}$ contains the same number of simplices. Thus, this generalizes for every triangulation and proves our theorem. $\square$

Chapter 5

Open Problems and Future Work

Theorem 4.13 makes it possible to explicitly transcribe each unimodular triangulation of $P_{A_{n-1}^*}$. We simply take a spanning tree of K_n, orient the edges away from 1, use these arcs as indices for vectors in the root configuration A_{n-1}, take the complement vectors, superscript an asterisk, and voila! This will denote a simplex in the triangulation corresponding to the chamber $\text{cone}(\{\mathbf{a}_{12}, \ldots, \mathbf{a}_{1,n-1}\})$ in $\text{Ch}(A_{n-1})$.

Additionally, we arbitrarily chose to fix 1 as the fixed first index of the root vectors generating Σ and as the vertex from which to orient the spanning trees. Due to the symmetry of the type-A root polytope, $\text{Ch}(A_{n-1})$ also has symmetry, and in particular we could have chosen any of the numbers in $\{1, \ldots, n\}$ to act as the 1 in Theorem 4.13.

We can also look into explicitly transcribing the triangulations which correspond to chambers in $\text{Ch}(A_{n-1})$ that are *not* of the form $\text{cone}(\{\mathbf{a}_{ij_1}, \mathbf{a}_{ij_2}, \ldots, \mathbf{a}_{ij_{n-1}}\})$ for a fixed i. While we do not prove it here, the symmetry of the root configuration and the complete digraph offers the observation that describing triangulations which correspond to chambers of the form $\text{cone}(\{\mathbf{a}_{j_1 i}, \mathbf{a}_{j_2 i}, \ldots, \mathbf{a}_{j_{n-1} i}\})$ for a fixed i will follow our same logic but involve the spanning tree of K_n rooted at i and oriented *towards* i. But, this is still only a small fraction of the chambers in $\text{Ch}(A_{n-1})$.

Future work in this can also be looking at other root systems, such as types C and D, both of which are also studied in [1]. Can we apply our methods and knowledge that we've learned here to these other root polytopes and their Gale duals?

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