HIGHER-DIMENSIONAL FROBENIUS GAPS

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In
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by

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CERTIFICATION OF APPROVAL

I certify that I have read *HIGHER-DIMENSIONAL FROBENIUS GAPS* by Jessica Delgado and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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This research expands on the well studied Frobenius problem and examines a related problem. This thesis mainly focuses on very ample polytopes of dimension three and their gaps, lattice points in the homogenization of the polytope that cannot be written as integer combinations of lattice points in the poytope. The main result is a theorem which states a universal upper bound of the gaps of very ample polytopes in dimension three does not exist. We built a program to compute the gaps of any very ample polytope. The computations of explicit examples are used to prove the main result on the nonexistence of the upper bound as well as a conjecture on the behavior of the gaps.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

1.1 Frobenius Problem

In the late nineteenth century Ferdinand Georg Frobenius gave the following problem to his students: given relatively prime positive integers $a_1, a_2, \ldots, a_n$ find the largest natural number such that is not representable as a non-negative integer linear combination of $a_1, a_2, \ldots, a_n$ [11].

Definition 1.1. Given $A = \{a_1, a_2, \ldots, a_n\} \in \mathbb{Z}_+$, an integer m is **representable** if $m = x_1a_1 + x_2a_1 + \cdots + x_na_n$ where $x_i \in \mathbb{Z}_{\geq 0}$ and **non-representable** otherwise.

For instance if $A = \{5, 7, 9\}$ then the set of non-representable positive integers is

$\{1, 2, 3, 4, 6, 8, 11, 13\}$. The largest non-representable number in this case is 13. This largest natural number in the general case became known as the **Frobenius number**, denoted as $F(a_1, a_2, \ldots, a_n)$.

This problem has also been referred to as the coin exchange problem. Suppose we abandoned our current coin system and only used 3 cent and 7 cent pieces. Although this seems foolish, this new system poses an interesting problem: there are amounts we cannot make change for. What is the largest amount that cannot be changed? Our set of unchangeable amounts are $\{1, 2, 4, 5, 8, 11\}$. Therefore if we changed our coin system to only 3 and 7 cent pieces the largest amount we could not make change for is 11 cents. In general the problem goes as follows: $a_1, a_2, \ldots, a_n$ are relatively prime coin denominations, find the largest amount that you cannot make change for.

More recently in the 1980s, there was a twist put on the problem by coming up with the McNugget number. McDonalds originally sold chicken nuggets in packages of 6, 9, and 20, and you could only order representable amounts of McNuggets. The largest number of McNuggets that you cannot order from McDonalds is precisely $F(6, 9, 20) = 43$ [14].

Over 100 years mathematicians from different areas have been researching this problem. The main area of interest is trying to produce formulas or algorithms for $F(a_1, a_2, \ldots, a_n)$. So far there has been success for producing a formula for $n = 2$; beyond this, however, there has been computational algorithms that run in polynomial time and formulas for bounds [11]. The following is the closed formula for $n = 2$ [11].

Theorem 1.1. *Given* $a_1, a_2 \in \mathbb{N}$ *such that* $\gcd(a_1, a_2) = 1$ *then* $F(a_1, a_2) = a_2a_1 - a_1 - a_2$.

Proof. Since $gcd(a_1, a_2) = 1$ then any integer p can be written as $p = xa_1 + ya_2$ with $x, y \in \mathbb{Z}$. It can be seen that p can be expressed in many different ways however the expression becomes unique when $0 \leq x < a_2$. In this case p is representable if $y \geq 0$ and not if $y < 0$. Therefore the largest non-representable integer will occur when $x = a_2 - 1$ and $y = -1$. Thus

$$F(a_1, a_2) = (a_2 - 1)a_1 + (-1)a_2 = a_2a_1 - a_1 - a_2.$$

□

It is also known how many of the integers will be representable. Sylvester came up with a theorem for the amount of representable integers when $n = 2$ [1].

Theorem 1.2. *(Sylvester's theorem) Let* a_1 *and* a_2 *be relatively prime positive integers. Exactly half of the integers between* 1 *and* $(a_1 - 1)(a_2 - 1)$ *are representable.*

To prove this theorem we will use the following lemma [1].

Lemma 1.3. *If* a_1 *and* a_2 *are relatively prime positive integers and* $n \in [1, a_1a_2 - 1]$ *is not a multiple of* a_1 *or* a_2 *then exactly one of the two integers* n *and* $a_1a_2 - n$ *is representable in terms of* a_1 *and* a_2.

Proof of Theorem 1.2. From Lemma 1.3 we know that for n between 1 and $a_1a_2 - 1$ and not divisible by a_1 or a_2 exactly one of n and $a_1a_2 - n$ is representable. There are $a_2 - 1$ numbers divisible by a_1 and $a_1 - 1$ numbers divisible by a_2. Therefore there are

$$(a_1a_2 - 1) - (a_1 - 1) - (a_2 - 1) = a_1a_2 - a_1 - a_2 + 1 = (a_1 - 1)(a_2 - 1)$$

integers between 1 and $a_1a_2 - 1$ that are not divisible by a_1 and a_2. Since exactly one of n or $a_1a_2 - n$ is representable, the number of non-representable integers is $\frac{1}{2}(a_1 - 1)(a_2 - 1)$. $\square$

In the case of $n = 3$ there has been much less success coming up with a closed-form expression, instead there are a couple of well-known algorithms such as Davison's algorithm [6] and Rødseth's algorithm [13]. There has in fact been proof that one cannot come up with an explicit formula for $n > 2$ by F. Curtis [5].

Theorem 1.4. *Let* $A = \{(a_1, a_2, a_3) \in \mathbb{N}^3 | a_1 < a_2 < a_3,\ a_1$ *and* a_2 *are prime, and* $a_1, a_2 \nmid a_3\}$ *then there is no nonzero polynomial* $H \in \mathbb{C}[X_1, X_2, X_3, Y]$ *such that*

$$H(a_1, a_2, a_3, F(a_1, a_2, a_3)) = 0 \quad \text{for all } a_1, a_2, a_3 \in A.$$

The following corollary shows that $F(a_1, a_2, a_3)$ cannot be determined by any set of closed formulas that could be reduced to a finite set of polynomials when restricted to A.

Corollary 1.5. *There is no finite set of polynomials* $\{h_1, \ldots, h_n\}$ *such that for each choice of* a_1, a_2, a_3 *there is some* i *such that* $h_i(a_1, a_2, a_3) = F(a_1, a_2, a_3)$.

Proof. The polynomial $H = \prod_{i=1}^{n}(h_i(X_1, X_2, X_3) - Y)$ would vanish on A. □

The Frobenius problem is fascinating in its own right, however this problem is the inspiration and not the focus of this paper. What do the non-representable integers look like if we lift this problem into a higher dimension? The answer to that question as well as the behavior of the higher dimensional non-representable integers will be the focus of this thesis. The rest of the introduction will focus on the background knowledge required for the main results.

1.2 Monoids and Numerical Semigroups

Definition 1.2. A **numerical semigroup** S is a subset of $\mathbb{N}$ that is closed under addition, contains 0, and whose complement in $\mathbb{N}$ is finite.

If $A = \{a_1, a_2, \ldots, a_n\}$ contains positive integers with $\gcd(a_1, a_2, \ldots, a_n) = 1$ then the set generated by A is $\{m_1a_1 + m_2a_2 + \cdots + m_na_n | m_1, \ldots, m_n \in \mathbb{N}\}$, a numerical semigroup. Every numerical semigroup is of this form [12]. A trivial example of a numerical semigroup is $\mathbb{N} \cup \{0\}$. A non-trival example is the set $\{0, 3, 5, 6, 7, 8, 9, 10, \ldots\}$. In this example the integers 1, 2 and 4 are what I will refer to as **gaps**. A non-example is the set $\{0, 1, 3, 4, 5, 6, \ldots\}$ because 1 is in the set but $1 + 1 = 2$ is not.

Definition 1.3. An integer q is a **gap** if $q \in \mathbb{N} \setminus S$.

We can now see how the Frobenius problem relates to numerical semigroups, the set of representable integers is in fact a numerical semigroup. If we go back to our first example where $A = \{5, 7, 9\}$ then our numerical semigroup S generated by A is $\{5, 7, 9, 10, 12, 14, 15, 16, 17, 18, \ldots\}$ and the set of gaps is precisely the set of non-representables $\{1, 2, 3, 4, 6, 8, 11, 13\}$. Note that if x is a gap then so are all the non-negative integers that divide it. We can also see that the largest gap equals to the Frobenius number of the set A.

Numerical semigroups live in the world of commutative monoids.

Definition 1.4. A **commutative monoid** is a set M paired with a binary operations $+$ that satisfies the following axioms:

1. For all $a, b \in M$ $a + b$ is also in M.

2. For all $a, b \in M$ $a + b = b + a$.

3. For all $a, b, c \in M$ $(a + b) + c = a + (b + c)$.

4. There exists an element e in M such that for all elements $a \in M$ $e + a = a + e = a$.

Examples of monoids are $\{0\}$, $(\mathbb{Z}_{\geq 0}, +)$, $(\mathbb{Q}, +)$ and $(\mathbb{Z}_{\geq 0}, \cdot)$ Notice that elements of a monoid do not necessarily have inverses. A monoid where every element has an inverse is a group.

1.3 Polytopes

Definition 1.5. Let V be a vector space. $A \subset V$ is an **affine subspace** if $A = U + v = \{u + v | u \in U\}$ for some linear subspace $U \subset V$ and $v \in V$. Affine subspaces can be viewed as a linear subspace with a shift.

Definition 1.6. Let V_1 and V_2 be vector spaces, $A_1 \subset V_1$ and $A_2 \subset V_2$ affine spaces. A map $f : A_1 \to A_2$ is **affine** if for some linear map $\phi : V_1 \to V_2$ and $v \in V_2$ the following diagram commutes:

$$\begin{tikzcd} V_1 \arrow[r, "\phi + v"] \arrow[d, leftrightarrow] & V_2 \arrow[d, leftrightarrow] \\ A_1 \arrow[r, "f"] & A_2 \end{tikzcd}$$

Definition 1.7. Let V be a vector space and $X \subset V$ finite. The **affine hull** $\operatorname{aff}(X)$ is the smallest affine subspace containing X. This is

$$\operatorname{aff}(X) = \operatorname{aff}(x_1, x_2, \ldots, x_n) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \middle| \lambda_i \in \mathbb{R}, \ \sum \lambda_i = 1 \right\}.$$

Definition 1.8. An affine **half-space**, denoted as either H_α^+ or H_α^- is the set $H_\alpha^+ = \{x \in V : \alpha(x) \geq b\}$ or $H_\alpha^- = \{x \in V : \alpha(x) \leq b\}$ where $b \in \mathbb{R}$ and $\alpha : V \to \mathbb{R}$ is a linear map.

The set that is defined by $\{x \in V : \alpha(x) = b\}$ is a hyperplane. An example of a hyperplane H can be seen in Figure 1.1. In this case the hyperplane is the line H.

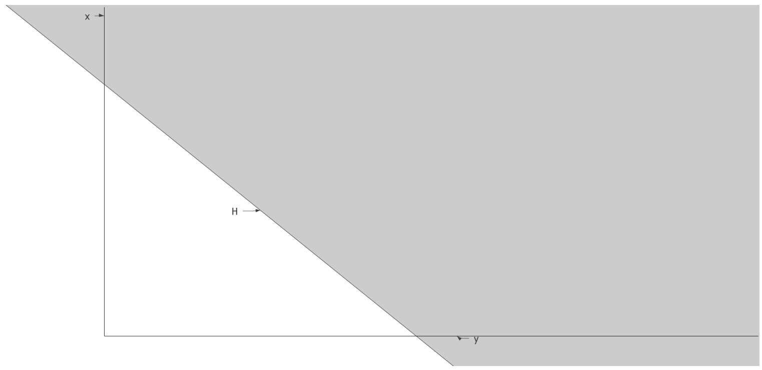


Figure 1.1: The shaded area is H_α^+.

Definition 1.9. A **polyhedron** is a finite intersection of half-spaces, $\bigcap_{i=1}^n H_{\alpha_i}^+$where $\alpha_i : V \to \mathbb{R}$ are affine maps.

Notice that a polyhedron does not necessarily have to be bounded.

Definition 1.10. A bounded polyhedron is a **polytope**.

Definition 1.11. Let V be a vector space and $X = \{x_1, x_2, \ldots, x_n\} \subset V$. The **convex hull** of X is

$$\operatorname{conv}(X) = \left\{ \sum_{i=1}^n \lambda_i x_i \middle| x_i \in X,\ \lambda_i \geq 0,\ \sum \lambda_i = 1 \right\}.$$

This is the smallest convex set containing $\{x_1, x_2, \ldots, x_n\}$.

A polytope can be described in two different ways: there is the H-description which is the intersection of half-spaces or the V-description which is the convex hull of a set of vertices [3]. For example, a simple two-dimensional triangle can be

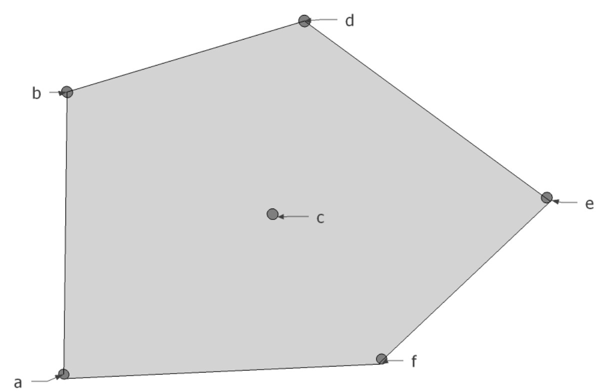


Figure 1.2: The convex hull of $\{a, b, c, d, e, f\}$.

described the following ways:

$$\operatorname{conv}((0,0),(0,1),(1,0)) \qquad \text{or} \qquad \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{c} x \geq 0 \\ y \geq 0 \\ y + x \leq 1 \end{array} \right\}.$$

Definition 1.12. A **face** of a polytope is defined as the intersection of the polytope with an affine hyperplane such that H_α^+ or H_α^- contains the polytope.

Definition 1.13. Given a polytope P, a hyperplane that has the property that H_α^+ or H_α^- contains P is a **support hyperplane**.

Faces of a polytope P include $\emptyset$ and P itself. Suppose that the dimension of P is d:

0-dimesional faces: vertices

1-dimensional faces: edges

$\vdots$

$d-2$ dimensional faces: ridges

$d-1$ dimensional faces: facets

d-dimensional face: polytope

According to [3], any face of a polytope P is a polytope itself whose faces come from the lower-dimensional faces of P.

1.4 Cones

Definition 1.14. Let V be a vector space and $\{x_1, x_2, \ldots, x_n\}$ be a collection of vectors in V. The **cone** spanned by $\{x_1, x_2, \ldots, x_n\}$, denoted by $C(x_1, x_2, \ldots, x_n)$, is the set $\{\sum_{i=1}^{n} \lambda_i x_i | \lambda_i \geq 0\} \subset V$.

Notation. $\mathbb{R}_+ X = \{\sum_{i=1}^{n} \alpha_i x_i | \alpha_i \in \mathbb{R}_+,\ x_i \in X\}$.

Theorem 1.6. *C is a cone if $\mathbb{R}_+ X = C$ for some finite non-empty set $X \subset V$ [3].*

Definition 1.15. We say that C is a **simplicial cone** if and only if

$$C = \mathbb{R}_+\{x_1, x_2, \ldots, x_n\}$$

such that $\{x_1, x_2, \ldots, x_n\}$ is linearly independent.

Definition 1.16. Let C be a cone. We say that C is **pointed** if C contains no non-zero linear subspaces.

Unless otherwise stated all cones in this paper will be pointed. A cone is also used in conjunction with polytopes. A polytope can generate a cone in the following way:

$$C(P) = \mathbb{R}_+(P, 1).$$

Here $(P, 1) = \{(x, 1)|x \in P\}$ and thus $C(P)$ refers to the cone of P by raising the polytope into the next dimension at height one and taking all the positive linear combinations. Figure 1.4 shows a cone generated by a polytope.

Notation. For $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$, let $\mathbb{Z}_+(\mathcal{A}, 1) = \{\lambda_1(a_1, 1) + \cdots + \lambda_n(a_n, 1)|\lambda_i \in \mathbb{Z}_+\}$.

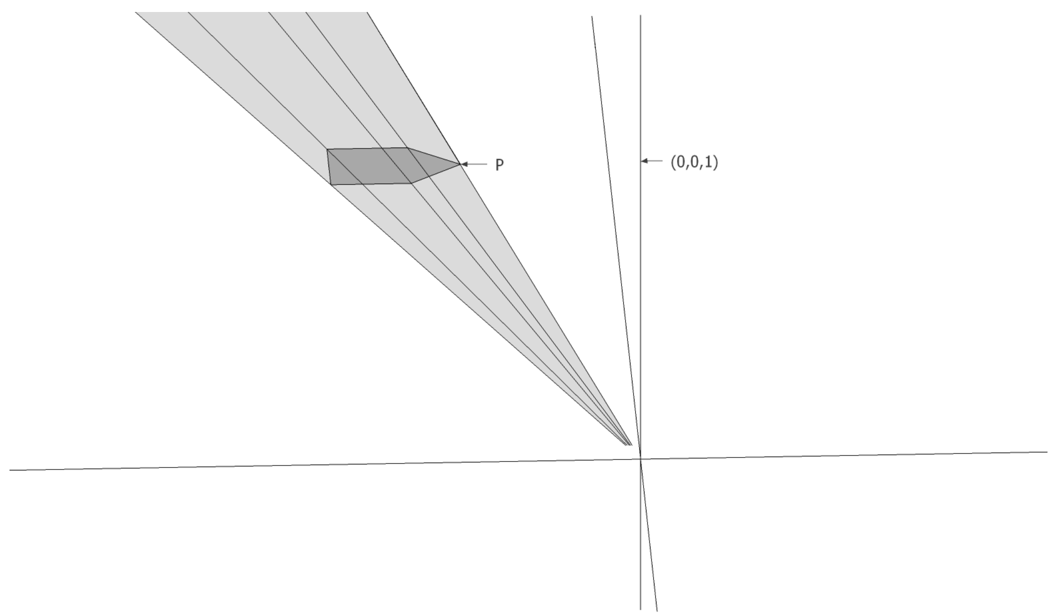


Figure 1.3: C(P).

Chapter 2

Dimension two

In this chapter we will examine the gaps of the special case of dimension two. There are a couple differences already between this and the one-dimensional Frobenius problem: the first, our gaps are now lattice points or points with integral coordinates and second, we will be using linear combinations of vectors instead of integers. This next dimension offers a new challenge: how to ensure a finite amount of gaps? This question and the question of how to locate the gaps will be answered in this chapter.

2.1 Condition for a finite number of gaps

Definition 2.1. Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{Z}_+$. A lattice point in $\mathbb{Z}^2$ is a **gap** if it is an element of $\mathbb{R}_+(\mathcal{A}, 1) \setminus \mathbb{Z}_+(\mathcal{A}, 1)$.

Theorem 2.1. *The number of gaps in* $\mathbb{R}_+(\mathcal{A}, 1) \setminus \mathbb{Z}_+(\mathcal{A}, 1)$ *are finite if and only* *if* $a_1, a_2, \ldots, a_{n-1}, a_n \in \mathbb{Z}_{\geq 0}$ *such that* $a_1 < a_2 < \cdots < a_{n-1} < a_n$ *and* $a_2 - a_1 = a_n - a_{n-1} = 1$

In order to prove Theorem 2.1 we want to show that at some height the gaps disappear and furthermore every height above will have no gaps as well. We first need the following lemma which will allow us to shift our set to the origin.

Lemma 2.2. *For* $A = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{Z}$ *and* $B = \{0, a_2 - a_1, a_3 - a_1, \ldots, a_n - a_1 | a_i \in A\}$,

$$\mathbb{R}_+(A, 1) \cong \mathbb{R}_+(B, 1)$$

$$\mathbb{R}_+(A, 1) \cap \mathbb{Z}^2 \cong \mathbb{R}_+(B, 1) \cap \mathbb{Z}^2$$

$$\mathbb{Z}_+(A, 1) \cong \mathbb{Z}_+(B, 1).$$

Here $X \cong Y$ *means* X *is isomorphic to* Y *as semigroups.*

Proof. Using $\{(a_1, 1), (a_2, 1)\}$ as a basis of $\mathbb{R}^2$ as well as $\{(0, 1), (a_2 - a_1, 1)\}$, we define a linear map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ through

$$\phi(\lambda_1(a_1, 1) + \lambda_2(a_2, 1)) = \lambda_1(0, 1) + \lambda_2(a_2 - a_1, 1).$$

In order to prove that ϕ is injective we shall show that null$\phi = \{0\}$.

$$0 = 0(0, 1) + 0(a_2 - a_1, 1)$$

$$\phi^{-1}(0(0,1)+0(a_2-a_1,1))=0(a_1,1)+0(a_2,1)=0.$$

In order to prove that ϕ is surjective let $z' \in \mathbb{R}^2$. Then

$$\phi^{-1}(z')=\phi^{-1}(\lambda_1(0,1)+\lambda_2(a_2-a_1,1))=\lambda_1(a_1,1)+\lambda_2(a_2,1),$$

which is some element of the domain $\mathbb{R}^2$, therefore our range is all of $\mathbb{R}^2$. We now know that ϕ is a linear, injective and surjective map.

In order to prove $\mathbb{R}_+(A,1)\cong\mathbb{R}_+(B,1)$, we shall use $\phi|_{\mathbb{R}_+(A,1)}:\mathbb{R}_+(A,1)\to\mathbb{R}^2$. We already know that $\phi|_{\mathbb{R}_+(A,1)}$ is linear and since $\mathbb{R}_+(A,1)$ is a subset of $\mathbb{R}^2$ it is also injective. It remains to show that the range of $\phi|_{\mathbb{R}_+(A,1)}=\mathbb{R}_+(B,1)$. Because of linearity,

$$\phi|_{\mathbb{R}_+(A,1)}\left(\sum_{j=1}^{n}\lambda_j(a_j,1)\right)=\sum_{j=1}^{n}\lambda_j\phi|_{\mathbb{R}_+(A,1)}(a_j,1).$$

We claim that $\phi|_{\mathbb{R}_+(A,1)}(a_j,1)=(a_j-a_1,1)$ giving us our desired range. Solving $(a_j,1)=\lambda(a_1,1)+\pi(a_2,1)$ for λ and π gives us the system of equations

$$\lambda a_1+\pi a_2=a_j$$

$$\lambda+\pi=1.$$

Setting $\pi = 1 - \lambda$ we obtain:

$$\lambda = \frac{a_j - a_2}{a_1 - a_2} \qquad\qquad \pi = 1 - \frac{a_j - a_2}{a_1 - a_2}$$

$$\begin{aligned}
\phi(a_j, 1) &= \frac{a_j - a_2}{a_1 - a_2}(0,1) + \left(1 - \frac{a_j - a_2}{a_1 - a_2}\right)(a_2 - a_1, 1) \\
&= \left(0, \frac{a_j - a_2}{a_1 - a_2}\right) + \left((a_2 - a_1) + (a_1 - a_2)\left(\frac{a_j - a_2}{a_1 - a_2}\right), 1 - \frac{a_j - a_2}{a_1 - a_2}\right) \\
&= \left(0, \frac{a_j - a_2}{a_1 - a_2}\right) + \left(a_2 - a_1 + a_j - a_2, 1 - \frac{a_j - a_2}{a_1 - a_2}\right) \\
&= \left(0, \frac{a_j - a_2}{a_1 - a_2}\right) + \left(a_j - a_1, 1 - \frac{a_j - a_2}{a_1 - a_2}\right) \\
&= (a_j - a_1, 1)
\end{aligned}$$

Therefore $\phi|_{\mathbb{R}_+(A,1)}(\mathbb{R}_+(A,1)) = \{\sum_{j=1}^{n} \lambda_j (a_j - a_1, 1)\} = \mathbb{R}_+(B,1)$. We now have $\phi|_{\mathbb{R}_+(A,1)} : \mathbb{R}_+(A,1) \to \mathbb{R}_+(B,1)$ as a linear bijective map. This shows that $\mathbb{R}_+(A,1) \cong \mathbb{R}_+(B,1)$.

In order to prove $\mathbb{R}_+(A,1) \cap \mathbb{Z}^2 \cong \mathbb{R}_+(B,1) \cap \mathbb{Z}^2$, we shall use $\phi|_{\mathbb{R}_+(A,1)\cap\mathbb{Z}^2} : \mathbb{R}_+(A,1) \cap \mathbb{Z}^2 \to \mathbb{R}_+(B,1)$. We already know that $\phi|_{\mathbb{R}_+(A,1)\cap\mathbb{Z}^2}$ is linear and since $\mathbb{R}_+(A,1) \cap \mathbb{Z}^2$ is a subset of $\mathbb{R}_+(A,1)$ it is also injective. It remains to show that the range of $\phi|_{\mathbb{R}_+(A,1)\cap\mathbb{Z}^2} = \mathbb{R}_+(B,1) \cap \mathbb{Z}^2$. Using linearity and our previous result,

$$\phi|_{\mathbb{R}_+(A,1)\cap\mathbb{Z}^2}\left\{\sum_{j=1}^{n} \lambda_j (a_j, 1)\right\} = \left\{\sum_{j=1}^{n} \lambda_j \phi|_{\mathbb{R}_+(A,1)\cap\mathbb{Z}^2}(a_j, 1)\right\} = \left\{\sum_{j=1}^{n} \lambda_j (a_j - a_1, 1)\right\}.$$

An element $x \in \mathbb{R}_+(B,1) \cap \mathbb{Z}^2$ has the form $(\lambda_2 + \sum_{j=3}^n \lambda_j(a_j - a_1), \sum_{j=1}^n \lambda_j)$ such that $\sum_{j=1}^n \lambda_j \in \mathbb{Z}$ and $\lambda_2 + \sum_{j=3}^n \lambda_j(a_j - a_1) \in \mathbb{Z}$. We know that $\phi|_{\mathbb{R}_+(A,1)}$: $\mathbb{R}_+(A,1) \to \mathbb{R}_+(B,1) \cap \mathbb{Z}^2$ is surjective since $\mathbb{R}_+(B,1) \cap \mathbb{Z}^2 \subset \mathbb{R}_+(B,1)$. In order to show that $\left\{\sum_{j=1}^n \lambda_j(a_j - a_1, 1)\right\} = \mathbb{R}_+(B,1) \cap \mathbb{Z}^2$ we must show that $\sum_{j=1}^n \lambda_j \in \mathbb{Z}$ and $a_j - a_1 \in \mathbb{Z}$. Since both a_j and a_1 are define to be integers $a_j - a_1 \in \mathbb{Z}$. The $\sum_{j=1}^n \lambda_j \in \mathbb{Z}$ is not affected by $\phi|_{\mathbb{R}_+(A,1)\cap\mathbb{Z}^2}$ and therefore will remain in $\mathbb{Z}$.

Now we show

$$\mathbb{Z}_+(A,1) \cong \mathbb{Z}_+(B,1).$$

There exists a bijection $\Phi : A \to B$ such that $a_i \mapsto a_i - a_1$. Therefore $\mathbb{Z}_+(A,1) \cong \mathbb{Z}_+(B,1)$. $\square$

Now that we know we can shift our set to the origin we prove there are no gaps in $C((a_1,1),(a_2,1))$ and $C((a_{n-1},1),(a_n,1))$ with the next lemma.

Lemma 2.3. *For $A = \{a_1, a_2\}$, $a_1, a_2 \in \mathbb{Z}_+$ such that $a_1 < a_2$ and $a_2 - a_1 = 1$, no gaps exist in $\mathbb{R}_+(A,1) \setminus M(A)$.*

Proof. Using Lemma 2.2, subtracting a_1 from each entry of A gives $A = \{0,1\}$ and $(A,1) = \{(0,1),(1,1)\}$. Our $\mathbb{R}_+(A,1) \cap \mathbb{Z}^2$ in this case is $\{\lambda_1(0,1)| + \lambda_2(1,1)|\lambda_i \geq 0\} \cap \mathbb{Z}^2$. The integer span of these two vectors is $\mathbb{Z}_+(A,1) = \{v_1(1,1) + v_2(0,1)|v_i \in \mathbb{Z}_{\geq 0}\}$. We can tile $\mathbb{R}_+(A,1)\cap\mathbb{Z}^2$ with parallelograms such that the first parallelogram has vertices at $(0,0)$, $(1,1)$, $(1,2)$, $(0,1)$, as shown in Figure 2.1. Every other parallelogram can be represented with the vertex set $\{(p,p),(p,p+1),(p+1,p+$

1), $(p+1, p+2)\}$ where $p = 0, 1, 2, \ldots$. If every element of $\mathbb{R}_+(A, 1) \cap \mathbb{Z}^2$ is contained

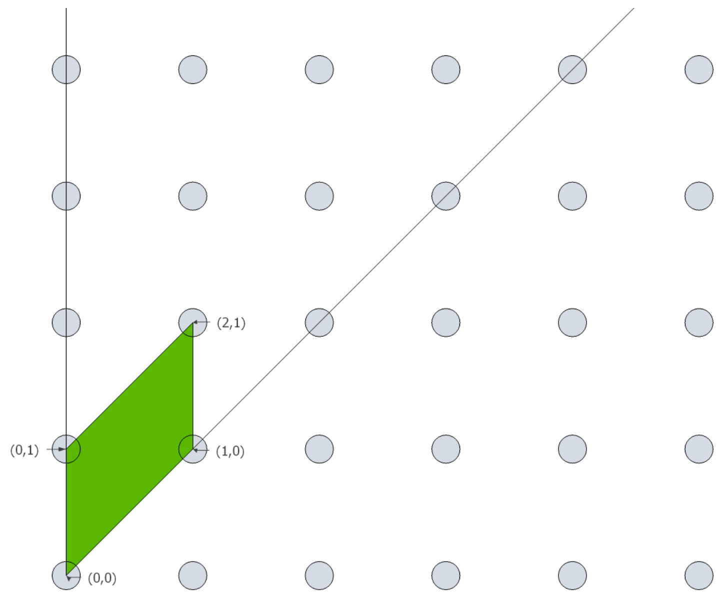


Figure 2.1: The first parallelogram with verticies $(0, 0)$, $(1, 1)$, $(1, 2)$, $(0, 1)$.

in the vertex set of a translated copy of our first parallelogram then every element of $\mathbb{R}_+(A, 1) \cap \mathbb{Z}^2$ is covered by an element from $\mathbb{Z}_+(A, 1)$. Therefore it is enough to show that there does not exist any integer lattice points in the interior of our first parallelogram. Suppose there was an integer lattice point (p, q) in the interior of our upper triangle. Then (p, q) would also be an integer vector and the vector $(1, 2) - (p, q)$ would also be an integer vector however if this were true that would imply that $(1, 2) - (p, q)$ leads to an integer lattice point in the lower triangle which cannot be true. $\square$

We will use these two lemmas to show that after some height every lattice point within $\mathbb{R}_+(A,1)$ can be written as a linear combination of points from our gap-free cones. This will ensure that from some height and above there will be no more gaps.

Proof of Theorem 2.1. Using Lemma 2.2 and the set $\{0, 1, a, a+1\}$, the minimum amount of elements that meets the criteria of Theorem 2.1, we shall prove the theorem. If we lift this set into dimension two we get $\{(0,1),(1,1),(a,1),(a+1,1)\}$. By Lemma 2.3 we know

$$\mathbb{R}_+\{(0,1),(1,1)\} \cap \mathbb{Z}^2 \setminus \mathbb{Z}_+\{(0,1),(1,1)\} = \emptyset$$

and

$$\mathbb{R}_+\{(a,1),(a+1,1)\} \cap \mathbb{Z}^2 \setminus \mathbb{Z}_+\{(a,1),(a+1,1)\} = \emptyset.$$

At an arbitrary height n, which is simply the second coordinate of a lattice point, the lattice points in $\mathbb{Z}_+(A,1)$ are

$$\{(0,n),\ldots,(n,n)\} \cup \{(a,n),\ldots,(a+n,n)\} \cup \cdots \cup \{(na,n),\ldots,(na+n,n)\}.$$

If we let $n = a$ this set becomes

$$\{(0,a),\ldots,(a,a)\} \cup \{(a,a),\ldots,(2a,a)\} \cup \cdots \cup \{(a^2,a),\ldots,(a^2+a,a)\}.$$

For height a these are all the lattice points between $(0,a)$ and (a^2,a) which are the two extremal points of the cone. At this point the cones $C((0,1),(1,1))$ and $C((a,1),(a+1,1))$ intersect and by Lemma 2.3 which implies there are no gaps within these cones everything beyond this height will be covered. $\square$

2.2 Where the gaps are located

In the previous section we proved a theorem that guaranteed a finite amount of gaps as long as A met a certain criteria. In this section we will use Theorem 2.1 and some programming to show where these gaps are located. The gaps we are locating are generated by the set $A = \{0, 1, a, a+1\}$ this way the programming can be done with respect to one parameter, a.

The first step in the process was to write a JAVA program which is located in Apendix A. The idea behind the program is to find all lattice points in $\mathbb{R}_+(A,1) \cap \mathbb{Z}^2$ up to a bound and then test each point to see if it is in $\mathbb{Z}_+(A,1)$. The program then returns all the points of $\mathbb{R}_+(A,1) \cap \mathbb{Z}^2 \setminus \mathbb{Z}_+(A,1)$, or our gaps. This data was then reformatted and input into R to generate a graphic. Figures 2.2 and 2.3 are pictures of the gaps for $a = 5$ and 20 respectively.

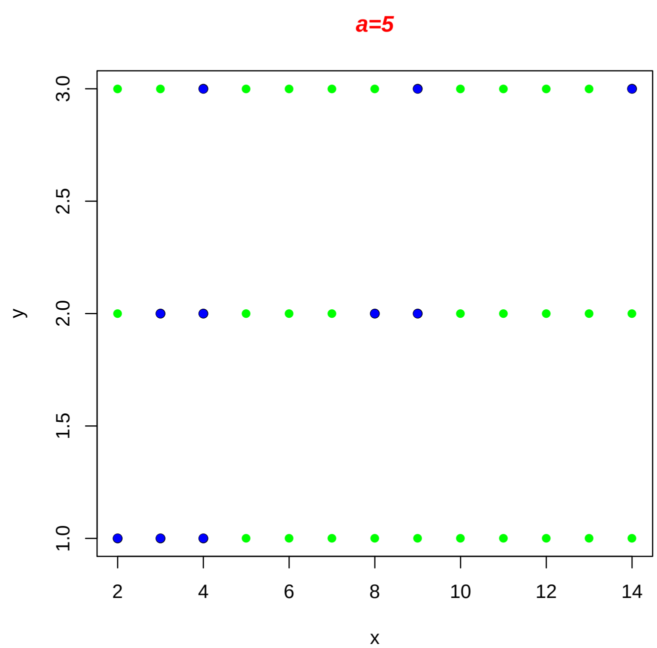


Figure 2.2: The darker dots are the gaps.

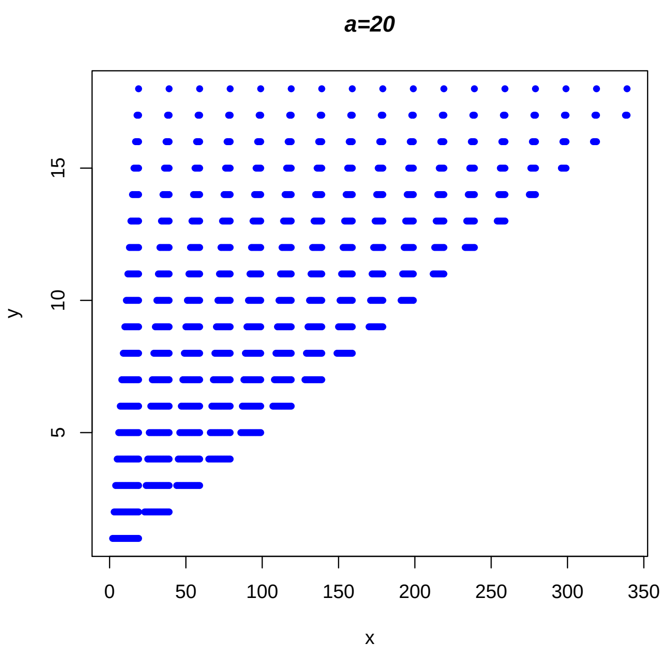


Figure 2.3: The dots are the gaps.

Chapter 3

Very Ample Polytopes

In the last chapter we showed how to guarantee a finite amount of gaps in dimension two and where these gaps were located. In this chapter, with the help of algebraic geometry, we will show the condition for a finite amount of gaps for any dimension. We will also show how to find these gaps.

3.1 What is a very ample polytope?

In this section we will define what a very ample polytope is and why we care about them. Before we do this we will need some tools from algebraic geometry.

We will be working in an affine space which is the set of all n-tuples of numbers in a field $\mathbb{K}$ and we will denote it as $\mathbb{A}^n$. For our field we will be using $\mathbb{C}$ because the complex numbers are algebraically closed, every polynomial has a root in $\mathbb{C}$. The next definitions will give some needed information about $\mathbb{C}$-algebras.

Definition 3.1. A ring A is a **$\mathbb{C}$-algebra** if it is a ring and a $\mathbb{C}$-vector space in a compatible way. Note that A contains an isomorphic copy of $\mathbb{C}$.

Definition 3.2. A $\mathbb{C}$-algebra A is **finitely generated** if there exist finitely many elements $a_1, a_2, \ldots, a_n \in A$, called a generating set, such that A is the smallest $\mathbb{C}$-subalgebra of A containing $\{a_1, a_2, \ldots, a_n\}$.

A $\mathbb{C}$-algebra A is finitely generated if and only if A is the smallest set containing $\mathbb{C}$ and $\{a_1, a_2, \ldots, a_n\}$ and every element $a \in A$ can be represented as

$$a = \sum_{i=1}^{n} c^i a_1^{d_{i_1}} \cdots a_n^{d_{i_n}}, \quad c_i \in \mathbb{C},\ d_{i_j} \geq 0.$$

Note that generating sets in general are not unique [7].

Notation. The $\mathbb{C}$-algebra generated by $a_1, a_2, \ldots, a_n$ is denoted by $\mathbb{C}[a_1, a_2, \ldots, a_n]$.

Definition 3.3. Let $x_1, x_2, \ldots, x_n$ be variables, which we regard as coordinate functions on $\mathbb{A}^n$ and $\mathbb{C}[x_1, x_2, \ldots, x_n]$ be the polynomial ring with coefficients in $\mathbb{C}$.

We may evaluate a polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ at a point $a \in \mathbb{A}^n$ to get a number $f(a) \in \mathbb{C}$. An example of this is if $f = x_1^2 - x_2^3$ then $f(0,0) = 0$. As one could notice $(0,0)$ is not the only point in $\mathbb{A}^2$ that would make $f(x_1, x_2) = 0$, in fact every time $x_2 = \sqrt[3]{x_1^2}$ this would be true. The points in which a given $f = 0$ are of importance to us.

Definition 3.4. An **affine variety** is the set of common zeros of a collection of polynomials. Given $F \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$, let $V(F) = \{a \in \mathbb{A}^n | f(a) = 0 \text{ for all } f \in F\}$.

Some trivial examples of affine varieties are $V(1) = \emptyset$ and $V(0) = \mathbb{A}^n$. Also any linear or affine subspace L of A^n is a variety. We know that L has a defining equation $Ax = b$ where A is a matrix and b is a vector so $L = V(Ax - b)$ is defined by the linear polynomials which form the rows of $Ax - b$.

Notation. Starting with a finitely generated $\mathbb{C}$-algebra A, the variety that is generated by A is the **affine spectrum** of A, denoted as $\mathrm{Spec}(A)$.

Definition 3.5. $F \in \mathbb{C}[x_1, \ldots, x_d]$ is a **homogeneous polynomial** if

$$F = \sum_i c_i x_0^{a_{i0}} \ldots x_d^{a_{id}}$$

such that for all i, $a_{i0} + \ldots + a_{id} = k$, for some fixed integer k.

Definition 3.6. The **projective space** $\mathbb{P}^d$ is the set of one-dimensional linear subspaces of $\mathbb{A}^{d+1}$, which is equivalent to $\mathbb{C}^{d+1}$. A point in $\mathbb{P}^n$ has homogeneous coordinates $(a_0, a_1, \ldots, a_n)$, where $a_i \in \mathbb{C}$ are not all zero, and another set of coordinates $(b_0, b_1, \ldots, b_n)$ give the same point in $\mathbb{P}^n$ if and only if there is a complex number $\lambda \neq 0$ such that $(b_0, b_1, \ldots, b_n) = \lambda(a_0, a_1, \ldots, a_n)$. Also if $F(x_0, x_1, \ldots, x_n)$ is a homogeneous of degree d and $(b_0, b_1, \ldots, b_n) = \lambda(a_0, a_1, \ldots, a_n)$ are two sets of homogeneous coordinates for some point $p \in \mathbb{P}^n$, then

$$F(b_0, b_1, \ldots, b_n) = \lambda^d F(a_0, a_1, \ldots, a_n).$$

By using $d + 1$ affine charts from the $(d + 1)$-dimensional affine space, a d-dimensional projective space can be covered. In order to cover a d-dimensional projective space with affine charts, start with a homogeneous polynomial $F \in \mathbb{C}[x_1, \ldots, x_d]$ and let

$$V(F) \subset \mathbb{P}^d = \{(x_0, \ldots, x_d) | F(x_0, \ldots, x_d) = 0\}.$$

Note that $V(F_1, \ldots, F_n) = \cap_i^n V(F_i)$. Denote $A_i = \mathbb{P}^n \setminus V(x_i)$ therefore

$$A_i = \{a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_d\} \cong \mathbb{A}^d.$$

Since $A_i \cong \mathbb{A}^d$ then $A_i = \mathrm{Spec}(\mathbb{C}[\frac{x_0}{x_i}, \ldots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_d}{x_i}]) \cong \mathbb{A}^d$. Therefore $\mathbb{P}^d = \bigcup_{i=0}^d \mathrm{Spec}(\mathbb{C}[\frac{x_0}{x_i}, \ldots, \frac{x_d}{x_i}])$ where $\mathrm{Spec}(\mathbb{C}[\frac{x_0}{x_i}, \ldots, \frac{x_d}{x_i}])$ are the affine charts. This construction will be used in the proof of Theorem 3.3.

Definition 3.7. $J \subset \mathbb{C}[x_1, \ldots, x_d]$ is a **homogeneous ideal** if for all $F \in J$, $F = F^{k_1} + F^{k_2} + \ldots + F^{k_n}$ where F^{k_i} are homogeneous parts of degrees $k_1, k_2, \ldots, k_n$.

Note that J is a homogeneous ideal if and only if it is generated by homogeneous polynomials.

Theorem 3.1 (Projective Nullstellensatz)**.** *Let $J \subset \mathbb{C}[x_1, \ldots, x_d]$ where $J = (F_1, \ldots, F_n)$ is a homogeneous ideal. Then the following are true [4]:*

1. $V(J) = V(\sqrt{J})$.

2. $V(J) = \emptyset \iff x_0^{a_1}, \ldots, x_d^{a_d} \in J$ *for some* $a_1, \ldots, a_d \in \mathbb{N}$.

Note: $(x_1, \ldots x_d) \in \mathbb{C}[x_1, \ldots, x_d]$ is called the **irrelevant maximal ideal**.

Definition 3.8. A **graded $\mathbb{C}$ algebra** is of the form $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \cdots$ where each A_i is a $\mathbb{C}$-vector space and $A_i A_j \subseteq A_{i+j}$.

We will now begin to discuss varieties that live inside to $\mathbb{P}^d$, denoted as Proj(A) when A is a graded $\mathbb{C}$-algebra.

Definition 3.9. A **finitely generated graded $\mathbb{C}$-algebra** , $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \ldots$ is generated in degree one with $\dim A_1 = d$, $\{a_1, \ldots, a_d\} \subset A_1$ being a $\mathbb{C}$ basis.

Suppose A is a finitely generated $\mathbb{C}$-algebra then Proj$(A) \subset \mathbb{P}^{d-1}$ is the projective variety

$$\text{Proj}(A) = V(J) \text{ where } J = ker(\mathbb{C}[x_1, \ldots, x_d] \to A) \text{ such that } x_i \mapsto a_i.$$

If $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \ldots$ is a graded finitely generated $\mathbb{C}$ algebra generated in degree one then Spec(A) is the unions of the one-dimensional linear subspaces in $\mathbb{A}^d$, $d = \dim A_1$, representing the points of Proj$(A) \subset \mathbb{P}^{d-1}$.

Just as $\mathbb{P}^d$ is covered by affine charts Proj(A) is as well. Let A be as above, then the description of the affine charts of the projective varieties in terms of ideals are

$$A_{(a_i)} = \mathbb{C} \cup \frac{A_1}{a_1} \cup \frac{A_2}{a_1^2} \cup \frac{A_3}{a_1^3} \cup \ldots \subset A_{a_i}$$

where $A_{(a_i)}$ is the localization at $a_i$and $\frac{A_i}{a_1^i} = \{\frac{a}{a_i} | a \in A_i\}$. Let $A_{(a_i)}$ be a finitely generated $\mathbb{C}$ algebra $A_{(a_i)} = \mathbb{C}[\frac{a_1}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i} \ldots \frac{a_d}{a_i}]$ and therefore

$$\text{Proj}(A) = \bigcup_{i=1}^{d} \text{Spec}\left(A_{(a_i)}\right) = \bigcup_{i=1}^{d} Spec\left(\mathbb{C}\left[\frac{a_1}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i} \ldots \frac{a_d}{a_i}\right]\right).$$

These are the affine charts of Proj(A).

Definition 3.10. An affine monoid is a subset $M \subset \mathbb{Z}^d$ for some $d \in \mathbb{Z}_{\geq 0}$ with $0 \in M$ and $m_1, m_2 \in M \implies m_1 + m_2 \in M$, and admitting a finitely generated subset $\{m_1, m_2, \ldots, m_n\} \subset M$, i.e., for all m, $m = a_1 m_1 + \cdots + a_n m_n$ for some $a_1, a_2, \ldots, a_n \in \mathbb{Z}_{\geq 0}$. The affine monoid M is positive if the cone $\mathbb{R}_+ M \subset \mathbb{R}^d$ is pointed.

Proposition 3.2. *A positive affine monoid has a smallest generating set, consisting of the indecomposible elements of M. These are elements that cannot be made from combinations of other elements of M. These elements are the Hilbert Basis,* Hilb*(M) of M.* Hilb*(M)=$\{m \in M | m \neq 0 \; m = m_1 + m_2$ for some $m_1, m_2 \in M \implies m_1 = 0$ or $m_2 = 0\}$*

Theorem 3.3. *Let $\mathcal{A} \subset \mathbb{Z}^d$ be a finite set. Recall, $\mathbb{C}[\mathcal{A}] = \mathbb{C}[M]$ where $M \subset \mathbb{Z}^{d+1}$ is the submonoid generated by $\{(a,1)|a \in \mathcal{A}\}$. Assume P=conv$(\mathcal{A}) \subset \mathbb{R}^d$. The following are equivalent*

1. $\mathrm{Proj}(\mathbb{C}[\mathcal{A}]) = \mathrm{Proj}(\mathbb{C}[C(P) \cap \mathbb{Z}^{d+1}])$

2. *For every $v \in \mathrm{vert}(P)$, $\mathrm{Hilb}(\mathbb{R}_+(P - v)) \subseteq \mathcal{A} - v$*

3. *$C(P) \cap \mathbb{Z}^{d+1} \setminus M$ is a finite set*

Proof. (1) $\iff$ (2) Let $\mathcal{A}_w$ denote $\mathcal{A} - w$ for all $w \in V$, where $V = \mathrm{vert}(P)$ and $P = \mathrm{conv}(\mathcal{A})$. Similarly we shall let $P_w = C(P - w)$ for all $w \in V$, the corner cones of P. $\mathrm{Proj}(\mathbb{C}[M])$ is covered by the affine varieties $\mathrm{Spec}(\mathbb{C}[M_w])$ for all $w \in V$, where M_w denotes $\mathbb{Z}_+(\mathcal{A}_w)$. Let N denote $C(P \times 1) \cap \mathbb{Z}^{d+1}$ and $N_w = P_w \cap \mathbb{Z}^d$ for each $w \in V$. Then likewise $\mathrm{Proj}(\mathbb{C}[N])$ is covered by $\mathrm{Spec}(\mathbb{C}[N_w])$ for all $w \in V$. $\mathrm{Proj}(\mathbb{C}[M]) = \mathrm{Proj}(\mathbb{C}[N])$ if and only if the affine varieties that cover each Proj are equal and that holds if and only if the rings $\mathbb{C}[M_w]$ and $\mathbb{C}[N_w]$ are equal. This will hold if and only if $\mathrm{Hilb}(\mathbb{R}_+(P - v)) \subseteq \mathcal{A} - v$ is an equality.

Lemma 3.4. *If $(\boldsymbol{x}, n) \in N$ and $w \in \mathcal{V}$ then there exists $m_0 \in \mathbb{Z}_+$ such that $m \geq m_0 \implies (\boldsymbol{x}, n) + m(w, 1) \in M$.*

Proof. Since $(\mathbf{x}, n) \in N$ then $(\mathbf{x}, n) = \sum_{v \in \mathcal{V}} q_v(v, 1)$ where $q_v \in \mathbb{Q}_+$. This implies that $\mathbf{x} = \sum_{v \in \mathcal{V}} q_v v$ and $\sum_{v \in \mathcal{V}} q_v = n$. Therefore

$$\mathbf{x} - nw = \sum_{v \in \mathcal{V}} q_v(v - w) \in N_w$$

Since $M_w = N_w$, $\mathbf{x} - n\mathbf{w} = \sum_{a \in \mathcal{A}} \alpha_a(\mathbf{a} - \mathbf{w})$, where $\alpha_a \in \mathbb{Z}_+$. Therefore in $\mathbb{Z}^{d+1}$,

$$(\mathbf{x}, n) = \sum_{a \in \mathcal{A}} \alpha_a(\mathbf{a}, 1) + (n - \sum_{a \in \mathcal{A}} \alpha_a)(\mathbf{w}, 1)$$

$$(\mathbf{x}, n) + m(\mathbf{w}, 1) = \sum_{a \in \mathcal{A}} \alpha_a(\mathbf{a}, 1) + (n - \sum_{a \in \mathcal{A}} \alpha_a + m)(\mathbf{w}, 1).$$

Therefore $(\mathbf{x}, n) + m(\mathbf{w}, 1) \in M$ if $m \geq \sum_{a \in \mathcal{A}} \alpha_a - n$. □

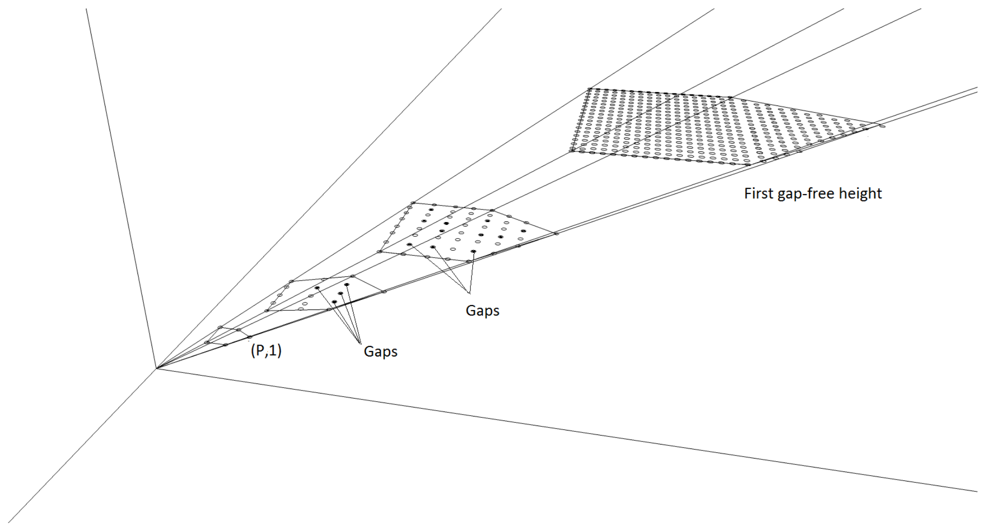
Let $B = \{\mathbf{y} \in N : \mathbf{y} = \sum_{\mathbf{v} \in \mathcal{V}} q_v v \text{ where } 0 \leq q_v < 1\}$, then B is finite. By Lemma 1 there exists an integer m_0 such that $m \geq m_0$ implies $\mathbf{b} + m(\mathbf{w}, 1) \in M$ for all $\mathbf{b} \in B$. Suppose $\mathbf{y} = \sum_{\mathbf{v} \in \mathcal{V}} q_v v \in N$ and there exists $\mathbf{w} \in \mathcal{V}$ such that $q_w \geq m_0$. Then $\mathbf{y} \in M$. □

Proof. (3) $\Longrightarrow$ (2)
Fix $\mathbf{w} \in \mathcal{V}$. Suppose $\mathbf{x} \in N_w$. Then $\mathbf{x} = \sum_{\mathbf{v} \in \mathcal{V}} q_v(\mathbf{v} - \mathbf{w})$. For m sufficiently large we have

$$(\mathbf{x} + m\mathbf{w}, m) = \sum_{\mathbf{v} \in \mathcal{V}} q_v(\mathbf{v}, 1) + (m - \sum_{\mathbf{v} \in \mathcal{V}} q_v)(\mathbf{w}, 1) \in N$$

Note that the two summands are in $C(P \times \{1\})$ so their sum is in $C(P \times \{1\}) \cap \mathbb{Z}^{d+1} = N$. The two individual summands are not necessarily in N. Since $N \setminus M$ is a finite set there exists $m \in \mathbb{Z}_+$ such that $(\mathbf{x} + m\mathbf{w}, m) \in M$. Therefore $(\mathbf{x} + m\mathbf{w}, m) \sum_{\mathbf{a} \in \mathcal{A}} \alpha_a(\mathbf{a}, 1)$ which implies $\mathbf{x} + m\mathbf{w} = \sum_{\mathbf{a} \in \mathcal{A}} \alpha_a \mathbf{a}$ and $m = \sum_{\mathbf{a} \in \mathcal{A}} \alpha_a$. Thus $\mathbf{x} = \sum_{\mathbf{a} \in \mathcal{A}} \alpha_a(\mathbf{a} - \mathbf{w}) \in M_w$. □

Figure 3.1: The gap behavior of a very ample polytope P.



Definition 3.11. Let $\mathcal{A} \subset \mathbb{Z}^d$ be a finite set and $P = \text{conv}(\mathcal{A}) \subset \mathbb{R}^d$. The polytope P is **very ample** if for every $v \in \text{vert}(P)$, $\text{Hilb}(\mathbb{R}_+(P - v)) \subseteq \mathcal{A} - v$.

Definition 3.12. A polytope $P \subset \mathbb{R}^d$ is **normal** if $\text{vert}(P) \subset \mathbb{Z}^d$ and for all $c \in \mathbb{N}$, $z \in (cP) \cap \mathbb{Z}^d$ there exists $x_1, x_2, \ldots, x_c \in P \cap \mathbb{Z}^d$ such that $z = x_1 + x_2 + \cdots + x_c$.

It is known that all normal polytopes are very ample [3]. In [10] it is proved that there are very ample polytopes of dimension $n \geq 3$ such that the $n - 2$ multiples of them are not normal. This implies that there exist very ample polytopes that generate a non-zero finite amount of gaps. The natural question to ask is where do these gaps exist. In the rest of this chapter we will explore these non-normal very ample polytopes and their gaps.

3.2 Non-Normal Very Ample Polytope of Dimension d With h Gaps

In [8] Higashitani showed that for given integers $h \geq 1$ and $d \geq 3$, there exists a non-normal very ample polytope of dimension d with exactly h gaps: let

$$u_i = \begin{cases} 0 & \text{if } i = 1, \\ e_d & \text{if } i = 2, \\ e_2 + \cdots + e_{d-1} & \text{if } i = 3, \\ h(e_2 + \cdots + e_{d-1} + e_d) & \text{if } i = 4, \\ (h-1)(e_2 + \cdots + e_{d-1}) + he_d & \text{if } i = 5, \\ h(e_2 + \cdots + e_{d-1}) + (h-1)e_d & \text{if } i = 6, \\ e_1 + (h+1)e_d & \text{if } i = 7, \\ e_1 + (h+2)e_d & \text{if } i = 8, \\ e_1 + e_2 + \cdots + e_{d-1} + (h-3)e_d & \text{if } i = 9 \\ e_1 + e_2 + \cdots + e_{d-1} + (h-2)e_d & \text{if } i = 10, \end{cases}$$

$$v_i = e_i, \qquad i = 2, \dots, d-1,$$

$$v_i' = e_i + e_d, \qquad i = 2, \dots, d-1,$$

where $0 = (0, 0, 0, \dots, 0) \in \mathbb{R}^d$ and $e_1, e_2, \dots, e_d$ are the unit vectors of $\mathbb{R}^d$. The convex hull of

$$\{u_1, \dots, u_10\} \cup \{v_i, v_i' : i = 2, \dots, d-1\}$$

is a non-normal very ample polytope in dimension d with exactly h gaps. For this polytope the set of gaps is $\{(u_j', 2) \in \mathbb{Z}^{d+1} : j = 1, \dots, h\}$, where $u_j' = e_1 + j(e_2 + \cdots + e_{d-1}) + (j + h - 1)e_d$.

Let's see an example of a polytope P in dimension six with ten gaps, $d = 6$, $h = 10$:

$$u_i = \begin{cases} 0 & \text{if } i = 1, \\ (0,0,0,0,0,1) & \text{if } i = 2, \\ (0,1,1,1,1,0) & \text{if } i = 3, \\ (0,10,10,10,10,10) & \text{if } i = 4, \\ (0,9,9,9,9,10) & \text{if } i = 5, \\ (0,10,10,10,10,9) & \text{if } i = 6, \\ (1,0,0,0,0,11) & \text{if } i = 7, \\ (1,0,0,0,0,12) & \text{if } i = 8, \\ (1,1,1,1,1,7) & \text{if } i = 9 \\ (1,1,1,1,1,8) & \text{if } i = 10, \end{cases}$$

$$v_1 = (0,1,0,0,0,0) \qquad v_1' = (0,1,0,0,0,1)$$
$$v_2 = (0,0,1,0,0,0) \qquad v_2' = (0,0,1,0,0,1)$$
$$v_3 = (0,0,0,1,0,0) \qquad v_3' = (0,0,0,1,0,1)$$
$$v_4 = (0,0,0,0,1,0) \qquad v_4' = (0,0,0,0,1,1).$$

The convex hull of all of these points forms our polytope P. Using *Normaliz* [2], a software package that computes the Hilbert basis of coned polytopes, one can input an integral polytope and the software will output the Hilbert basis of the cone generated by that polytope. Looking at the Hilbert basis of the cone, any elements that are not on height one are gaps: the Hilbert basis consists of all the indecomposable elements of the cone, therefore all elements on height one are included. If an element is in the Hilbert basis and is not on height one, it can not be reached with positive integer linear combinations of the height-one elements and by definition that makes it a gap. If the Hilbert basis is completely contained in height one the polytope is normal and there will be no gaps. According to *Normaliz* the

Hilbert basis elements not on height one for P are:

$(1,1,1,1,1,10,2)$ $(1,7,7,7,7,16,2)$ $(1,3,3,3,3,12,2)$

$(1,8,8,8,8,17,2)$ $(1,4,4,4,4,13,2)$ $(1,9,9,9,9,18,2)$

$(1,5,5,5,5,14,2)$ $(1,10,10,10,10,19,2)$ $(1,6,6,6,6,15,2)$

$(1,2,2,2,2,11,2)$

As we can see we have exactly ten gaps and, as Higashitani claimed, all the gaps are on height two and are the elements of the set $\{(u'_j, 2) \in \mathbb{Z}^7 : j = 1, \ldots, 10\}$, where

$$u'_j = e_1 + j(e_2 + \cdots + e_5) + (j+9)e_6.$$

Higashitani was able to create a non-normal very ample polytope in dimension d with h many gaps, however, there are no results about the gaps of an arbitrary non-normal very ample polytope.

3.3 Polytopal Affine Semigroups With Gaps Deep Inside

Katthän constructed a non-normal very ample polytope P such that each gap has lattice distance at least k above every facet of P, where k is a positive integer [9]. Lattice distance between two lattice points x and y is the number of lattice points contained in $\operatorname{conv}(x, y) - 1$. Katthäns's polytope is a 3-simplex that is a special case of a rectangular simplex. The construction of a rectangular simplex is as follows: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ a vector of positive integers. Let $\Delta = \Delta(\lambda) \subset \mathbb{R}^{n+1}$ with vertices

$$\begin{aligned}
&v_0 := (0, 0, \ldots, 0, 1)\\
&v_1 := (\lambda_1, 0, 0, \ldots, 0, 1)\\
&v_2 := (0, \lambda_2, 0, \ldots, 0, 1)\\
&\vdots\\
&v_n := (0, 0, \ldots, 0, \lambda_n, 1)
\end{aligned}$$

The polytope we are using is a 3-simplex and we will denote it as $P(\lambda_1, \lambda_2, \lambda_3)$. As one would guess, not every $\lambda_i \in \mathbb{Z}_+$ will give us our desired result, Katthän describes the conditions, which he refers to as a **good triple**, for the λ_i as $\lambda_1 \geq 5$ is odd, $\lambda_2 = 2\lambda_1 - 1$ and $\lambda_3 = 2\lambda_1^2 - \lambda_1 - 2$. Katthan also goes on to prove that if $P(\lambda_1, \lambda_2, \lambda_3)$ is a good triple then every gap has lattice distance at least $\lambda_1 + 2$ above the closest facet. Lets see an example of this.

Let $\lambda_1 = 7$ then our good triple will be $(7, 13, 89)$ and the vertices of the polytope are:

$$v_0 = (0, 0, 0, 1)$$

$$v_1 = (7, 0, 0, 1)$$

$$v_2 = (0, 13, 0, 1)$$

$$v_3 = (0, 0, 89, 1)$$

Using *Normaliz* we get$\{(5, 11, 39, 2), (5, 12, 32, 2), (6, 9, 40, 2), (6, 10, 33, 2),$ $(6, 11, 26, 2), (6, 12, 19, 2)\}$ as the set of gaps.

3.4 How to find gaps of very ample polytopes

In the above papers the common theme was constructing very ample polytopes such that the gaps meet certain criteria. In [9] Katthän showed a construction of a very ample polytope that has gaps deep inside and in [8] Higashitani constructed a very ample polytope in dimension d with h gaps. Neither of these papers demonstrated a method to find the gaps of a general very ample polytope, which is what we will be discussing next.

In order to compute the gaps of a very ample polytope we must first start with one. Given an arbitrary set of vertices, *Normaliz* can check each of the corner cones, also known as tangent cones, and verify whether or not the Hilbert basis of the cone is contained in the polytope.

Definition 3.13. For a lattice polytope $P \subset \mathbb{Z}^d$ and $v \in \text{vert}(P)$ a **corner cone** of P is $\mathbb{R}_+(P - v)$.

If the Hilbert Basis of every corner cone is contained within the polytope then P is very ample. As will be shown later, all the polytopes that were computed for this research were already known to be very ample.

The best way to explain how to find the gaps of a very ample polytope is by example. The set of vertices $\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,1,0),(1,1,1),(1,0,5),$ $(1,0,6)\}$ forms a very ample polytope P in dimension three. The following steps show how to find the gaps of P:

1. Input the vertices of P into *Normaliz* under the `polytope` mode.

```
8
3
0 0 0
0 0 1
0 1 0
0 1 1
1 1 0
1 1 1
```

```
1 0 5

1 0 6

polytope
```

2. Record the Hilbert basis elements of degree 1.

3. Input these elements into the *R* formatting program to return the proper format for *Python*.

4. The elements will now be in a nested list format and can be pasted into *Python* as level 1.

```
level1=[[1,0,6,1],[1,0,5,1],[1,1,1,1],[1,1,0,1],[0,1,1,1],
[0,1,0,1],[0,0,1,1],[0,0,0,1]]
```

5. Use the `addlevel` function to return all elements of $\mathbb{Z}_+(P, 1)$ on level 2, record output as replevel2.

```
addlevel(level1,level1)
```

6. Use the `scalarmult` function to return a dilation of P in *Normaliz* format.

```
scalarmult(2,level1)
```

7. Copy and paste this output into *Normaliz* to obtain the degree 1 elements.

8. Copy and paste these degree 1 elements into the *R* formatting program to return proper *Python* format.

```
level2=[[0,0,0,2],[0,0,2,2],[0,2,0,2],[0,2,2,2],[2,2,0,2],
[2,2,2,2],[2,0,10,2],[2,0,12,2],[0,0,1,2],[0,1,0,2],[0,1,1,2],
[0,1,2,2],[0,2,1,2],[1,0,5,2],[1,0,6,2],[1,0,7,2],[1,1,0,2],
[1,1,1,2],[1,1,2,2],[1,1,3,2],[1,1,4,2],[1,1,5,2],[1,1,6,2],
[1,1,7,2],[1,2,0,2],[1,2,1,2],[1,2,2,2],[2,0,11,2],[2,1,5,2],
[2,1,6,2],[2,1,7,2],[2,2,1,2]]
```

9. Copy and paste this properly formatted data for $\mathbb{R}_+(P,1)$ on level 2 into Python then use the `gaps` function to compare the lists of $\mathbb{R}_+(P,1)$ and $\mathbb{Z}_+(P,1)$ on level 2. What is returned will be the gaps on level 2.

```
gaps(level2,replevel2)
gaps on level2=[1, 1, 3, 2],[1, 1, 4, 2]
```

10. Continue this process until the `gaps` function returns zero gaps.

3.5 Bruns' Polytopes

In this section we will discuss a special variety of very ample polytopes constructed, and kindly provided, by Winfried Bruns. Bruns constructed this list of very ample polytopes by taking normal polytopes and, at random, removing elements contained

Table 3.1: Gaps of Bruns' polytopes.

Number of generators	First height of no gaps	gap vector
8	3	(0,1,0)
9	4	(0,2,3,0)
10	5	(0,3,6,3,0)
11	5	(0,4,6,6,0)
12	4	(0,2,3,0)
13	3	(0,1,0)
14	6	(0,7,12,12,10,0)
15	6	(0,7,12,12,10,0)
17	4	(0,4,3,0)
18	3	(0,1,0)
19	7	(0,14,30,42,40,30,0)
31	3	(0,1,0)

in the polytope's corner cones. He compiled a list of hundreds of very ample polytopes in dimensions three, four, five and six. Table 3.1 shows the dimension, number of vertices and the first gap-free height of some of Bruns' polytopes.

3.6 Very ample polytopes in dimension 3

Besides the polytopes in Bruns' list, the other family of very ample polytopes that we will focus on are the Bruns and Gubeladze (BG) polytopes. Let I_i, $i = 1, 2, 3, 4$ be intervals in $\mathbb{Z}$, each containing at least two integers. The BG polytope is defined as the convex hull of

$$((0,0) \times I_1) \cup ((0,1) \times I_2) \cup ((1,1) \times I_3) \cup ((1,0) \times I_4).$$

Figure 3.1 shows an example of a BG polytope. This class of polytopes can be found

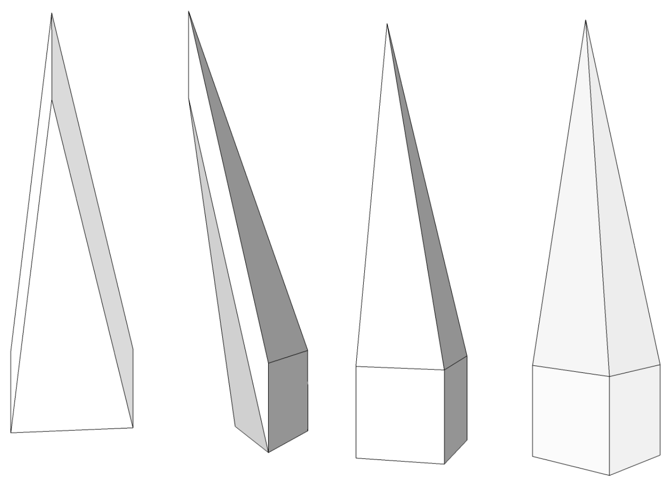


Figure 3.2: $I_1 = \{0,1\}$ $I_2 = \{0,1\}$ $I_3 = \{0,1\}$ $I_4 = \{4,5\}$

in [3].

Theorem 3.5. *The class of BG polytopes are very ample.*

Proof. Consider the subgroup Γ of the group of affine isometries of $\mathbb{R}^3$ that respect the integer lattice $\mathbb{Z}^d$, generated by the following three isometries:

- the rotation by $90°$ around the axis $(\frac{1}{2}, \frac{1}{2}, \mathbb{R})$,

- $(x, y, z) \mapsto (x, y, -z)$,

- $(x, y, z) \mapsto (x, y, z+1)$.

For every vertex $v \in P(I_1, I_2, I_3, I_4)$ there exists $g \in \Gamma$ such that $g(v) = 0$. Moreover, every element of Γ permutes the set of lines

$$(0,0,\mathbb{R}),\ (1,0,\mathbb{R}),\ (0,1,\mathbb{R}),\ (1,1,\mathbb{R}) \ \subset \ \mathbb{R}^3$$

Based on these observations, without loss of generality we can assume $I_1 = [0, b_1]$ and we only need to check that

$$C = \mathbb{R}_+ P(I_1, I_2, I_3, I_4) \cap \mathbb{Z}^3 = \mathbb{Z}_+(1, 0, a_2) + \mathbb{Z}_+(0, 1, a_3) + \mathbb{Z}_+(1, 1, a_4) + \mathbb{Z}_+ e, \quad (3.1)$$

where $C = \mathbb{R}_+ P(I_1, I_2, I_3, I_4)$ and $e = (0, 0, 1)$. There are two possibilities: either

$$C = \mathbb{R}_+ e + \mathbb{R}_+(1, 0, a_2) + \mathbb{R}_+(0, 1, a_3), \text{ or}$$
$$C = \big(\mathbb{R}_+ e + \mathbb{R}_+(1, 0, a_2) + \mathbb{R}_+(1, 1, a_4)\big) \ \cup \ \big(\mathbb{R}_+ e + \mathbb{R}_+(0, 1, a_3) + \mathbb{R}_+(1, 1, a_4)\big).$$

In the first case, (3.1) follows from the observation that $\{e, (1, 0, a_2), (0, 1, a_3)\}$ is a basis of $\mathbb{Z}^3$:

$$\det \begin{pmatrix} 1 & 0 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix} = 1\,.$$

In the second case, (3.1) follows from the observation that the two cones on the right-hand side are spanned by bases of $\mathbb{Z}^3$:

$$\det\begin{pmatrix}1 & 0 & a_2\\ 1 & 1 & a_4\\ 0 & 0 & 1\end{pmatrix} = 1\,, \qquad \det\begin{pmatrix}0 & 1 & a_3\\ 1 & 1 & a_4\\ 0 & 0 & 1\end{pmatrix} = -1\,,$$

and, therefore, every lattice point in C is a non-negative integer linear combination of either $\{(1,0,a_2),(1,1,a_4),e\}$, or $\{(0,1,a_3),(1,1,a_4),e\}$. $\square$

Table 3.2 shows the gaps of some of these polytopes for specific I_i.

Table 3.2: Gap heights of various BG polytopes.

I_1	I_2	I_3	I_4	First height of no gaps
$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{5,6\}$	4
$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{6,7\}$	5
$\{0,1\}$	$\{3,4\}$	$\{1,2\}$	$\{5,6\}$	6
$\{0,1\}$	$\{3,4\}$	$\{1,2\}$	$\{7,8\}$	8
$\{0,1\}$	$\{3,4\}$	$\{1,2\}$	$\{8,9\}$	9
$\{0,2\}$	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$	2
$\{0,5\}$	$\{7,10\}$	$\{2,5\}$	$\{3,7\}$	2
$\{0,1\}$	$\{2,3\}$	$\{1,2\}$	$\{3,4\}$	3
$\{0,1\}$	$\{5,6\}$	$\{1,2\}$	$\{7,8\}$	10
$\{0,1\}$	$\{4,5\}$	$\{6,7\}$	$\{8,9\}$	5

3.7 Is there a universal bound for the gaps?

A natural question to ask, is there a universal bound for the gaps of all very ample polytopes of a certain dimension? Using the BG polytope construction for dimension three we will show that there is not. In order to accomplish this we are going to introduce Ehrhart polynomials. Ehrhart polynomials are used to count lattice points within integral polytopes. In our case we are going to use them to count the lattice points at each height of $\mathbb{R}_+(P,1)$ where P is a BG polytope.

Definition 3.14. Let P be a polytope in dimension d and tP be an integer dilation of P. Then $L(P,t)$ is the number of lattice points in tP. Moreover $L(P,t)$ is a polynomial of the form $a_d t^d + a_{d-1}t^{d-1} + \cdots + a_0$ where $a_d, a_{d-1}, \ldots, a_0$ are rational [1]. This polynomial is known as the **Ehrhart polynomial** for a polytope P. It is known that a_d is the volume of the polytope, a_{d-1} is half of the surface area of P and $a_0 = 1$ [1].

Theorem 3.6. *A universal upper bound for the degrees of gaps of all very ample polytopes of dimension three does not exist.*

Proof. We will use the BG polytope construction for this proof. As a reminder, the construction goes as follows: Let I_i, $i = 1,2,3,4$ be intervals in $\mathbb{Z}$, each containing at least two integers. The BG polytope is defined as the convex hull of

$$((0,0)\times I_1)\cup((0,1)\times I_2)\cup((1,1)\times I_3)\cup((1,0)\times I_4).$$

For this proof we will let $I_1 = I_2 = I_3 = \{0,1\}$ and $I_4 = \{k, k+1\}$ where $k \in \mathbb{Z}_{\geq 0}$ and denote this polytope by P_k. An example of this polytope can be seen in Figure

3.1. The volume of P_k is

$$\frac{k}{3} + 1 - \frac{k}{6} = \frac{k}{6} + 1;$$

furthermore, P_k has four triangular facets and four square facets all with empty interiors, which implies the surface area is $4(\frac{1}{2}(1)) + 4(1)) = 6$. Therefore $L(P_k, t) = (\frac{k}{6} + 1)t^3 + 3t^2 + l_k t + 1$, we also know $L(P_k, 1) = 8$ which implies $l_k = 3 - \frac{k}{6}$, thus

$$L(P_k, t) = \left(\frac{k}{6} + 1\right) t^3 + 3t^2 + \left(3 - \frac{k}{6}\right) t + 1.$$

Note that for $t \geq 2$,

$$L(P_k, t) \geq \frac{k}{6} t^3 - \frac{k}{6} t \geq \frac{k}{2} \tag{3.2}$$

Assume by contradiction that there exists a bound $b \in \mathbb{Z}_+$ such that for all k

$$\gamma(P_k) \leq b \tag{3.3}$$

where $\gamma(P_k) \in \mathbb{Z}$ is the largest height containing gaps. The underlying point configuration $(P_k, 1) \in \mathbb{Z}^4$ is

$$(P_k, 1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & k & k+1 \end{bmatrix}.$$

Since P_k is very ample for $t > \gamma(P_k)$, $\dim_{\mathbb{K}}(\mathbb{K}(\mathbb{Z}_+(P_k, 1))$ at height t is equal to $L(P_k, t) \leq 8^t$. Using the 8 vertices of P_k, for each height t there is 8^t ways to sum t-many of the 8 vertices where repeats are allowed. Of course many combinations may sum to the same amount which is why $L(P_k, t) \leq 8^t$. Combining this with (3.2) and (3.3) gives for $t > b$

$$\frac{k}{2} \leq 8^t$$

which is a contradiction since k can be arbitrarily large. $\square$

Notice that the last non-zero index of each gap vector of Table 3.3 is the $(k-3)$rd tetrahedral number. So far we have not been able to explain why this is the case. Also we have not been able to explain why every diagonal is a multiple of its corresponding tetrahedral number.

Table 3.3: Gaps of BG polytope when $I_1 = I_2 = I_3 = \{0, 1\}$ and $I_4 = \{k, k+1\}$.

k	First height of no gaps	Gap vector	Gaps
4	3	(0,1,0)	1
5	4	(0,2,4,0)	6
6	5	(0,3,8,10,0)	21
7	6	(0,4,12,20,20,0)	56
8	7	(0,5,16,30,40, 35,0)	126
9	8	(0,6,20,40,60,70,56,0)	252
10	9	(0,7,24,50,80,105,112,84,0)	462
11	10	(0,8,28,60,100,140,168,168,120,0)	792
12	11	(0,9,32,70,120,175,224,252,240,165,0)	1287
13	12	(0,10,36,80,140,210,280,336,360,330,220,0)	2002

Conjecture 3.1. *For any arbitrary k let $G_{k,i}$ denote the i^{th} index of the gap vector*

associated with k. We conjecture that

$$G_{k,i} = \binom{i+1}{3}(k-i-1) \qquad i = 1, 2, \ldots, k-1$$

with $k-1$ being the first height of no gaps.

Appendix A: JAVA code to generate gaps for a given a

```
import java.util.Scanner;



public class FastFrobenius {

import java.util.Scanner;



          public static void main(String[] args) {
             Scanner scanner = new Scanner( System.in );

              System.out.println("enter a positive integer");
```

```
            double a = scanner.nextDouble();
            double bound=1/(a+1);
            double b=Math.pow(a,3);
            double[][] pq = new double[(int) (b+1)][2];


            for (int i=0; i<=b;i++){

                     pq[i][0] = (int) (i%(a*a)+1);
                     pq[i][1] = (int) (((i%(a*a))-i)/(-(a*a))+1);
                      if( (pq[i][1]/pq[i][0])<bound ){
                              pq[i][0]=0;
                              pq[i][1]=0;
                              }
}


            for (int i=0; i<=b;i++){
  double a2mod= i%(a*a);
  double w1=(int) (a2mod+1);
  double w2=(int) ((a2mod-i)/(-(a*a))+1);
```

```
                    if( (w2/w1)>=bound ){
                            pq[i][0] = w1;
                            pq[i][1] = w2;
                            }
    }

                  System.out.println("OK");



               for (int i=0; i<=b;i++){


               for (int y=0; y<=2*a;y++){
               for(int z=0;z<=2*a;z++){
                     if(  (pq[i][0]-(a*y)-(z*a)-z>=0) &&
(pq[i][0]-(a*y)-(z*a)-z<=2*a) &&
(pq[i][1]-(pq[i][0]-(a*y)-(z*a)-z)-y-z>=0) &&
(pq[i][1]-(pq[i][0]-(a*y)-(z*a)-z)-y-z<=2*a) ){
                              pq[i][0]=0;
                              pq[i][1]=0;
                      }
                      }
```

```
            }


}
        for (int r=0; r<pq.length; r++) {
          for (int c=0; c<pq[r].length; c++) {
             System.out.print((int) pq[r][0]+",");
           }
           System.out.println("");
       }

}
      }
```

Appendix B: Python code that computes gaps of very ample polytopes

```
def getRow(mat, row):
    "return row row from matrix mat"
    return(mat[row])


def cols(mat):
    "return number of cols"
    return(len(mat[0]))


def addVectors(A,B):
    "add two vectors"
    if len(A) != len(B):
        print("addVectors: different lengths")
```

```
        return()
    return([A[i]+B[i] for i in range(len(A))])

def rows(mat):
    "return number of rows"
    return(len(mat))

def addlevel(A,B):
    "Adds each vector to each vector"
    L= [0]*(rows(A)*rows(B))
    for i in range(rows(A)):
        for j in range (rows(B)):
            "print (addVectors(A[i],B[j]))"
            L[(rows(B)*i)+j]=addVectors(A[i],B[j])

    duplicates(L)
    L.remove(0)
    print (L)

def gaps (A,B):
    "A is big list B is small list"
```

```
    for i in range(rows(A)):
        if A[i] not in B:
            print (A[i])


def duplicates (mat):
    "deletes duplicate vectors"
    for i in range(rows(mat)):
        for j in range (rows(mat)):
            if i== j:
                continue
            else:
                if mat[i]==mat[j]:
                    mat[i]=0


def scalarMult(a,mat):
    "multiplies a scalar times a matrix"
    "If mat is a vector it returns a vector."
    if vectorQ(mat):
```

```
        return([a*m for m in mat])
    for row in range(rows(mat)):
        for col in range(cols(mat)):
            mat[row][col] = a*mat[row][col]
    return(mat)


def vectorQ(V):
    "mild test to see if V is a vector"
    if type(V) != type([1]):
        return(False)
    if type(V[0]) == type([1]):
        return(False)
    return(True)
```

Appendix C: R formatting program

```
myformat=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",",normaliz[i,4],"],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",",normaliz[n,4],"]]",sep="")
  return(x)
}

myformat2=function(normaliz){
  n=nrow(normaliz)
```

```
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","2","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","2","]]",sep="")
  return(x)
}

myformat3=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","3","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","3","]]",sep="")
```

```
  return(x)
}

myformat4=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","4","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","4","]]",sep="")
  return(x)
}

myformat5=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
```

```
    ,normaliz[i,3],",","5","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","5","]]",sep="")
  return(x)
}

myformat6=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","6","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","6","]]",sep="")
  return(x)
}
```

```
myformat7=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","7","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","7","]]",sep="")
  return(x)
}

myformat8=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","8","],",sep="")
  }
```

```
  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","8","]]",sep="")
  return(x)
}

myformat9=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","9","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","9","]]",sep="")
  return(x)
}

myformat10=function(normaliz){
  n=nrow(normaliz)
  x="["
```

```
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","10","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","10","]]",sep="")
  return(x)
}

myformat11=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","11","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","11","]]",sep="")
  return(x)
```

```
}

myformat12=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","12","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","12","]]",sep="")
  return(x)
}

myformat13=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","13","],",sep="")
```

```
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","13","]]",sep="")
  return(x)
}

myformat14=function(normaliz){
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","14","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","14","]]",sep="")
  return(x)
}

myformat15=function(normaliz){
```

```
  n=nrow(normaliz)
  x="["
  for(i in 1:(n-1)){
    x=paste(x,"[",normaliz[i,1],",",normaliz[i,2],","
    ,normaliz[i,3],",","15","],",sep="")
  }

  x=paste(x,"[",normaliz[n,1],",",normaliz[n,2],","
  ,normaliz[n,3],",","15","]]",sep="")
  return(x)
}
```

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