THE BETTIN–CONREY RECIPROCITY THEOREM AND INFLATED EULERIAN POLYNOMIALS

Juan Sebastián Auli

Version of July 22, 2015

THE BETTIN–CONREY RECIPROCITY THEOREM AND INFLATED EULERIAN POLYNOMIALS

Juan Sebastián Auli
San Francisco State University
2015

This thesis is composed of two independent parts. The first part is motivated by a recent paper by Bettin and Conrey, introducing a family of cotangent sums that generalize the classical notion of Dedekind sum and share with it the property of satisfying a reciprocity law. We study particular instances of these arithmetic sums for which it is possible to obtain a simpler reciprocity.

The second part focuses on the inflated $\mathbf{s}$-Eulerian polynomial $Q_n^{(\mathbf{s})}(x)$, introduced by Pensyl and Savage. We show that $Q_n^{(\mathbf{s})}(x)/\left(1 + x + \cdots + x^{s_n - 1}\right)$ is a polynomial for all positive integer sequences $\mathbf{s}$ and characterize those sequences $\mathbf{s}$ for which the sequence of nonzero coefficients of $Q_{n-1}^{(\mathbf{s})}(x)$ coincides with that of the polynomial $Q_n^{(\mathbf{s})}(x)/\left(1 + x + \cdots + x^{s_n - 1}\right)$. In particular, we show that all nondecreasing sequences satisfy this condition.

ACKNOWLEDGMENTS

Foremost, I would like to thank my advisors and mentors, Dr. Federico Ardila and Dr. Matthias Beck, for their relentless support and encouragement. I have been extremely lucky to have not one but two incredible advisors. Working with them has brought me great joy and tons of learning. Their supervision has been crucial during my time at San Francisco State University. Their excellent advice and patient guidance certainly kept me on track.

My deepest gratitude to Dr. Abdelmejid Bayad and Dr. Carla Savage. Collaborating with them has been a wonderful opportunity. Reading their work always inspires me to continue doing research.

I would also like to thank Dr. Bruce Berndt for his opportune suggestions, which led to extending the work I have being doing, together with Dr. Bayad and Dr. Beck, on generalizations of Dedekind sums.

Contents

Chapter 1

Introduction

This thesis consists of two independent parts, which we will introduce now.

1.1 The Bettin–Conrey Reciprocity Theorem

In Chapter 2, we prove a reciprocity law for a family of finite arithmetic sums introduced by Bettin and Conrey [10]. They define

$$c_a\left(\frac{h}{k}\right) := k^a \sum_{m=1}^{k-1} \cot\left(\frac{\pi m h}{k}\right) \zeta\left(-a, \frac{m}{k}\right),$$

where h and k are positive coprime integers and $\zeta(a,z)$ denotes the *Hurwitz zeta function*.

Bettin [8] points out that the sum c_0 is particularly interesting because of its relevance to the Nyman–Beurling–Báez–Duarte approach to the Riemann hypothesis (see, for example, [5]). Their work shows that the Riemann hypothesis holds if and only if

$$\lim_{N\to\infty} \left(\inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta\left(\frac{1}{2}+it\right) A_N\left(\frac{1}{2}+it\right) \right|^2 \frac{dt}{\frac{1}{4}+t^2} \right)^{\frac{1}{2}} = 0, \tag{1.1}$$

where the infimum is taken over all Dirichlet polynomials $A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$ of length N. The computation of the integral in (1.1) involves the integrals [34]

$$\begin{aligned}\nu\left(\frac{h}{k}\right) &:= \frac{1}{2\pi\sqrt{hk}} \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left(\frac{h}{k}\right)^{it} \frac{dt}{\frac{1}{4}+t^2} \\ &= \frac{\log 2\pi - \gamma}{2}\left(\frac{1}{h}+\frac{1}{k}\right) + \frac{k-h}{2hk}\log\frac{h}{k} - \frac{\pi}{2hk}\left(V\left(\frac{h}{k}\right) + V\left(\frac{k}{h}\right)\right),\end{aligned}$$

where γ denotes Euler's constant (see, for instance [26, eq. 5.2(ii)]) and

$$V\left(\frac{h}{k}\right) = \sum_{m=1}^{k-1} \left\{\frac{mh}{k}\right\} \cot\left(\frac{\pi m}{k}\right)$$

is the *Vasyunin sum* (as usual, $\{x\}$ denotes the fractional part of x). The connection to c_0 then follows from the fact that for $\overline{h}$, the inverse of h modulo k, we know that there exists an integer q such that $h\overline{h} = qk + 1$, which implies that

$$\begin{aligned} V\left(\frac{h}{k}\right) &= \sum_{m=1}^{k-1} \left\{\frac{m\overline{h}h}{k}\right\} \cot\left(\frac{\pi m\overline{h}}{k}\right) \\ &= \sum_{m=1}^{k-1} \left\{mq + \frac{m}{k}\right\} \cot\left(\frac{\pi m\overline{h}}{k}\right) \\ &= \sum_{m=1}^{k-1} \frac{m}{k} \cot\left(\frac{\pi m\overline{h}}{k}\right) \\ &= -c_0\left(\frac{\overline{h}}{k}\right). \end{aligned}$$

The last equality is a consequence of the fact that $\zeta(0, z) = \frac{1}{2} - z$ (see, for instance, [26, eq. 25.11.13]).

Bettin and Conrey [10] show that the sum c_0 satisfies a reciprocity law, which they then generalize to sums of the form c_a with $a \in \mathbb{C}$ [9],

$$c_a\left(\frac{h}{k}\right) - \left(\frac{k}{h}\right)^{1+a} c_a\left(\frac{-k}{h}\right) + \frac{k^a a\,\zeta(1-a)}{\pi h} = -i\,\zeta(-a)\,\psi_a\left(\frac{h}{k}\right). \tag{1.2}$$

The function ψ_a in (1.2) is a *period function* defined for $\Im(z) > 0$ and $a \in \mathbb{C}$ by

$$\psi_a(z) := E_{a+1}(z) - \frac{1}{z^{a+1}} E_{a+1}\left(-\frac{1}{z}\right).$$

Here, $E_{a+1}(z)$ denotes the real analytic Eisenstein series (see, for example, [37]). This period function extends to an analytic function on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ via the representation

$$\psi_a(z) = \frac{i}{\pi z}\frac{\zeta(1-a)}{\zeta(-a)} - i\frac{1}{z^{1+a}} \cot\frac{\pi a}{2} + i\frac{g_a(z)}{\zeta(-a)},$$

where

$$\begin{aligned} g_a(z) := -2 \sum_{1\leq n\leq M} (-1)^n \frac{B_{2n}}{(2n)!}\zeta(1-2n-a)(2\pi z)^{2n-1} \\ + \frac{1}{\pi i}\int_{(-\frac{1}{2}-2M)} \zeta(s)\zeta(s-a)\Gamma(s)\frac{\cos\frac{\pi a}{2}}{\sin\frac{\pi(s-a)}{2}}(2\pi z)^{-s} ds. \end{aligned}$$

Here B_n is the n-th Bernoulli polynomial (see, for example, [26, eq. 24.2.1]) and M is any integer greater or equal to $-\frac{1}{2}\min(0, \Re(a))$ [9, Thm. 1].

Bettin–Conrey sums are also related to Dedekind sums $s(h, k)$, treated extensively by Rademacher [31]. These sums are defined via the *sawtooth function*,

$$((x)) = \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases}$$

as (see, for instance, [3, p. 72])

$$s(h,k) := \sum_{m=1}^{k-1} \left(\left(\frac{mh}{k}\right)\right)\left(\left(\frac{m}{k}\right)\right) = \frac{1}{4k}\sum_{m=1}^{k-1} \cot\left(\frac{\pi hm}{k}\right)\cot\left(\frac{\pi m}{k}\right). \tag{1.3}$$

Bettin–Conrey sums generalize Dedekind sums, in the sense that

$$c_{-1}\left(\frac{h}{k}\right) = \frac{\pi}{2k}\sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right)\cot\left(\frac{\pi m}{k}\right) = 2\pi s(h,k).$$

Dedekind sums satisfy the famous reciprocity law

$$s(h,k) + s(k,h) - \frac{1}{12hk} = \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} - 3\right), \tag{1.4}$$

proved originally by Dedekind [20], which coincides with (1.2) in the case $a = -1$.

Our main goal in Chapter 2 is to provide a novel reciprocity law for Bettin–Conrey sums of the form c_{-a} with $\Re(a) > 1$. Namely, we show the following result.

Theorem 1.1. *Let a be a complex number such that $\Re(a) > 1$ and suppose h and k are positive coprime integers, then for any $0 < \epsilon < \min\left\{\frac{1}{h}, \frac{1}{k}\right\}$,*

$$h^{1-a}c_{-a}\left(\frac{h}{k}\right) + k^{1-a}c_{-a}\left(\frac{k}{h}\right) = \frac{a\,\zeta(a+1)}{\pi(hk)^a} + \int_{\epsilon+i\infty}^{\epsilon-i\infty} \frac{\cot(\pi hz)\cot(\pi kz)}{z^a}dz.$$

This reciprocity is an alternative to that given by Bettin and Conrey and involves an integrand simpler than that in (1.2).

The results presented in Chapter 2 were obtained in joint work with Abdelmejid Bayad and Matthias Beck. Our approach mimics a technique relying on Cauchy's residue theorem and introduced by Rademacher [31, p. 21] in one of the several proofs of (1.4) he gave.

Chapter 2 is organized as follows. Section 2.1 provides a historical background of Bettin–Conrey sums, expanding on the relevance of Dedekind sums and introducing various generalizations. In Section 2.2, we prove Theorem 1.1 and present several corollaries regarding the work of Bettin and Conrey on period functions. We also generalize Bettin-Conrey sums to a broader family of cotangent sums and deduce a reciprocity law for these, generalizing Theorem 1.1.

1.2 Inflated Eulerian Polynomials

In Chapter 3, we present the solution to a problem on **s**-lecture hall sequences posed by Carla Savage at the 2014 AMS Fall Western Sectional Meeting. This problem was motivated by the paper [14] of Chung and Graham on the maxdrop statistic in permutations.

Let S_n be the set of permutations of $[n] = \{1, 2, \cdots, n\}$. Given a permutation

$$\pi = \pi_1\pi_2\ldots\pi_n$$

in S_n, define $\operatorname{Des}\pi = \{i \in [n-1] \mid \pi_i > \pi_{i-1}\}$ and $\operatorname{des}\pi = |\operatorname{Des}\pi|$. The following is a consequence of [14, Thm. 4.1] and [14, Thm. 4.2].

Proposition 1.2 (Chung–Graham [14]). *Let $P_n(x)$ be defined by*

$$P_n(x) = \sum_{\pi \in S_n} x^{n \cdot \operatorname{des}(\pi) + \pi_n}.$$

Then

$$\frac{P_n(x)}{1 + x + x^2 + \cdots + x^{n-1}} = \sum_{\pi \in S_{n-1}} x^{n \cdot \operatorname{des}(\pi) + \pi_{n-1}}.$$

In particular, this means that $P_n(x)/(1 + x + x^2 + \cdots + x^{n-1})$ is a polynomial and that its sequence of nonzero coefficients coincides with that of $P_{n-1}(x)$.

We will see that Proposition 1.2 is a special case of our main result (Thm. 1.3 below) when $\mathbf{s} = (1, 2, \ldots, n)$.

Given a sequence $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ of positive integers, we define the $\mathbf{s}$-*lecture hall cone* to be

$$C_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\}.$$

The *generators* of this cone are $\{v_i = (0, \ldots, 0, s_i, \ldots, s_n) : 1 \leq i \leq n\}$. Its *fundamental half-open parallelepiped* is $\Pi_n^{(\mathbf{s})} = \{\sum_{i=1}^n \alpha_i v_i \mid 0 \leq \alpha_i < 1\}$. The lattice points $C_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ are called $\mathbf{s}$-*lecture hall sequences* [12, 32].

The generating function for the lattice points in $C_n^{(\mathbf{s})}$ can be computed from its fundamental parallelepiped and generators as

$$\sum_{\lambda \in C_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} = \frac{\sum_{\lambda \in \Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}}{\prod_{i=1}^n (1 - x_i^{s_i} \cdots x_n^{s_n})}. \tag{1.5}$$

Setting $x_1 = \ldots = x_{n-1} = 1$ and $x_n = x$ in (1.5) gives

$$\sum_{\lambda \in C_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n} = \frac{\sum_{\lambda \in \Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n}}{(1 - x^{s_n})^n}. \tag{1.6}$$

Define $Q_n^{(\mathbf{s})}(x)$ by

$$Q_n^{(\mathbf{s})}(x) = \sum_{\lambda \in \Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n}.$$

We will show in Section 3.2 that $Q_n^{(\mathbf{s})}(x)$ is the *inflated* $\mathbf{s}$-*Eulerian polynomial* associated with $C_n^{(\mathbf{s})}$, introduced in [27].

Savage observed that for particular (infinite) sequences $\mathbf{s}$ and positive integers n, the sequence of nonzero coefficients of $Q_{n-1}^{(\mathbf{s})}(x)$ coincides with that of the polynomial

$$\frac{Q_n^{(\mathbf{s})}(x)}{(1 + x + \cdots + x^{s_n - 1})}.$$

We refer to a sequence for which this occurs for a specific n as n-*contractible*. The surprising prevalence of n-contractible sequences for small n led us to consider the problem of characterizing all sequences $\mathbf{s}$ satisfying this condition for all n. We call these sequences *contractible*.

Note that the polynomiality of $Q_n^{(\mathbf{s})}(x)/\left(1+x+\cdots+x^{s_n-1}\right)$ is not evident from the definition of $Q_n^{(\mathbf{s})}(x)$.

Example 1.1. Let $\mathbf{s}$ be the Fibonacci sequence. Then the nth inflated $\mathbf{s}$-Eulerian polynomials for the first few n are as follows:

$$\begin{aligned}
Q_5^{(1,1,2,3,5)}(x) &= \mathbf{1}x^{11} + \mathbf{1}x^{10} + \mathbf{2}x^9 + \mathbf{4}x^8 + \mathbf{4}x^7 + \mathbf{4}x^6 + \mathbf{4}x^5 + \mathbf{4}x^4 \\
&\qquad + \mathbf{2}x^3 + \mathbf{2}x^2 + \mathbf{1}x + \mathbf{1}, \\
Q_4^{(1,1,2,3)}(x) &= \mathbf{1}x^4 + \mathbf{1}x^3 + \mathbf{2}x^2 + \mathbf{1}x + \mathbf{1}, \\
Q_3^{(1,1,2)}(x) &= \mathbf{1}x + \mathbf{1}, \text{ and} \\
Q_2^{(1,1)}(x) &= \mathbf{1}.
\end{aligned}$$

On the other hand, the polynomials $Q_n^{(\mathbf{s})}(x)/\left(1+x+\cdots+x^{s_n-1}\right)$ for the first few n are

$$\begin{aligned}
\frac{Q_6^{(1,1,2,3,5,8)}(x)}{1+x+\cdots+x^7} &= \mathbf{1}x^{18} + \mathbf{1}x^{16} + \mathbf{2}x^{15} + \mathbf{4}x^{13} + \mathbf{4}x^{12} + \mathbf{4}x^{10} + \mathbf{4}x^8 + \mathbf{4}x^7 \\
&\qquad + \mathbf{2}x^5 + \mathbf{2}x^4 + \mathbf{1}x^2 + \mathbf{1}, \\
\frac{Q_5^{(1,1,2,3,5)}(x)}{1+x+\cdots+x^4} &= \mathbf{1}x^7 + \mathbf{1}x^5 + \mathbf{2}x^4 + \mathbf{1}x^2 + \mathbf{1}, \\
\frac{Q_4^{(1,1,2,3)}(x)}{1+x+x^2} &= \mathbf{1}x^2 + \mathbf{1}, \text{ and} \\
\frac{Q_3^{(1,1,2)}(x)}{1+x} &= \mathbf{1}.
\end{aligned}$$

Thus, the Fibonacci sequence is *n-contractible* for $n = 3,4,5,6$.

We will prove that all nondecreasing sequences are contractible. Moreover, we characterize contractible sequences as follows.

Theorem 1.3. *A sequence $\mathbf{s}$ of positive integers is contractible if and only if either $(s_n)_{n=3}^{\infty}$ is nondecreasing, or there exists $N \geq 3$ such that $(s_n)_{n=N}^{\infty}$ is nondecreasing, $s_N = s_{N-1} - 1$ and $s_j = 1$ for $j = 1,2,\ldots,N-2$.*

The results presented in Chapter 3 were obtained in joint work with Carla Savage [4]. Our approach relies on a combinatorial interpretation of $Q_n^{(\mathbf{s})}(x)$ due to Pensyl and Savage [27], which involves $\mathbf{s}$-*inversion sequences*.

Chapter 3 is organized as follows. In Section 3.1, we provide historical background to the problem. In particular, we explain how $Q_n^{(\mathbf{s})}(x)$ is related to the $\mathbf{s}$-lecture hall sequences introduced by Bousquet-Mélou and Eriksson [12]. Section 3.2 presents Pensyl and Savage's description of $Q_n^{(\mathbf{s})}(x)$ in terms of certain statistics on $\mathbf{s}$-inversion sequences. In Section 3.3, we provide a combinatorial characterization of $Q_n^{(\mathbf{s})}(x)/\left(1+x+\cdots+x^{s_n-1}\right)$ that, in particular, proves its polynomiality. In Section 3.4, we introduce a lemma (Lemma 3.9), which is the main ingredient in the proof of Theorem 1.3. We then use this lemma to show that all nondecreasing sequences are contractible. Finally, we conclude the proof of Theorem 1.3 in Section 3.5.

Chapter 2

The Bettin–Conrey Reciprocity Theorem

2.1 Historical Background and Definitions

2.1.1 Dedekind Sums

Dedekind sums were initially defined by Richard Dedekind (1831–1916), while studying the *eta function*, an essential function in the theory of elliptic functions, defined as

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau}),$$

with the condition $\Im \tau > 0$ ensuring convergence.

To define the Dedekind sums, we need to introduce the following function.

Definition 2.1. The *sawtooth function* is defined as

$$((x)) = \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \text{ is not an integer, and} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

Note that the sawtooth function is odd, piecewise linear and periodic, with period 1 (see Figure 2.1).

Definition 2.2. The Dedekind sum $s(h, k)$ is defined for coprime positive integers h and k as

$$s(h, k) = \sum_{m=1}^{k-1} \left(\left(\frac{mh}{k} \right) \right) \left(\left(\frac{m}{k} \right) \right)$$

The periodicity of the sawtooth function implies that $s(h, k)$ is periodic in h, with period k. Since $((-x)) = ((x))$, we have $s(-h, k) = s(h, k)$.

Although Dedekind sums may seem to be a specialized subject, they appear naturally in many areas of mathematics. For instance, they are present in the convergent series for the

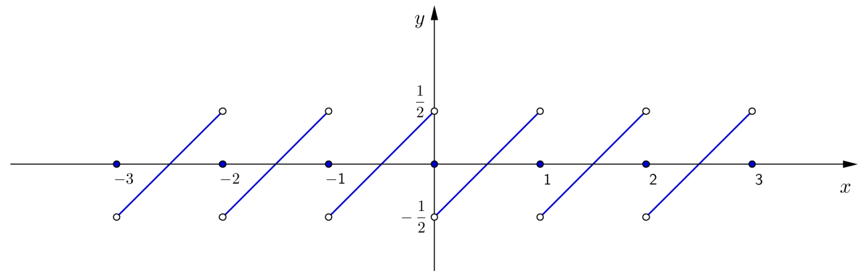

Figure 2.1: Sawtooth function.

partition function obtained by Rademacher [29],

$$p(n) = \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\sum_{\substack{0\le h<k\\(h,k)=1}} e^{\pi i\, s(h,k)-2\pi inh/k}\right)\frac{d}{dn}\left(\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right\}}{\sqrt{n-\frac{1}{24}}}\right),$$

as an improvement upon the asymptotic formula due to Hardy and Ramanujan [22],

$$p(n) \sim \frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Dedekind sums are also connected to Ehrhart polynomials. Suppose a, b and c are pairwise coprime, and let $L_{\mathcal{P}}(t)$ be the Ehrhart polynomial of the tetrahedron

$$\mathcal{P} = \left\{(x,y,z)\in\mathbb{R}^3 \,:\, x,y,z\ge 0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1\right\},$$

that is,

$$L_{\mathcal{P}}(t) = \#\left\{(k,l,m)\in\mathbb{Z}^3 \,:\, k,l,m\ge 0,\ \frac{k}{a}+\frac{l}{b}+\frac{m}{c}\le t\right\}.$$

Pommersheim [28] proved that

$$L_{\mathcal{P}}(t) = \frac{abc}{6}t^3 + \frac{ab+ac+bc+1}{4}t^2 + \left(\frac{1}{12}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}+\frac{1}{abc}\right)\right.$$
$$\left. + \frac{3}{4} + \frac{a+b+c}{4} - s(bc,a) - s(ca,b) - s(ab,c)\right)t + 1.$$

One of the most fundamental properties of Dedekind sums is the famous reciprocity law (1.4), which we recall,

$$s(h,k) + s(k,h) - \frac{1}{12hk} = \frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}-3\right).$$

This formula was first derived by Dedekind [20] as a property of the eta function, using Jacobi's theory of elliptic functions. Many alternative proofs appeared since then, several of which are due to Rademacher [31] and involve a wide variety of techniques, ranging from lattice point enumeration to computation of residues of complex functions.

The reciprocity of Dedekind sums is connected to the well-known reciprocity of the Jacobi symbol $\left(\frac{h}{k}\right)$, which is known as quadratic reciprocity:

$$\left(\frac{h}{k}\right)\left(\frac{k}{h}\right) = (-1)^{\frac{h-1}{2}\cdot\frac{k-1}{2}}. \tag{2.1}$$

Indeed, it is possible to show that (1.4) implies (2.1). This was proved by Dedekind [19], using the fact that

$$12k\, s(h,k) \equiv k + 1 - 2\left(\frac{h}{k}\right) \pmod 8.$$

The reader interested in a more comprehensive treatment of Dedekind sums is referred to [31].

2.1.2 Various Generalizations of Dedekind Sums

Dedekind sums have been studied and generalized by many mathematicians throughout the years. We now introduce some generalizations relevant to our work.

Definition 2.3. Let n be a positive integer and suppose h, k are coprime. We define the *Dedekind–Apostol sum*

$$s_n(h,k) = \sum_{m=1}^{k-1} \frac{m}{k}\, \overline{B}_n\left(\frac{mh}{k}\right) = \sum_{m=1}^{k-1} \frac{m}{k}\, B_n\left(\frac{mh}{k} - \left[\frac{mh}{k}\right]\right),$$

where $\overline{B}_n(z)$ is the n-th Bernoulli function and $B_n(z)$ is the n-th Bernoulli polynomial (see [1, eq. (2.11)] and [1, eq. (2.12)], respectively). These sums were introduced by Apostol in 1950. He proved that they satisfy a reciprocity law involving Bernoulli numbers [1]:

$$(n+1)(hk^n s_n(h,k)+kh^n s_n(k,h)) = nB_{n+1}+\sum_{m=0}^{n+1}\binom{n+1}{m}(-1)^m B_m B_{n+1-m} h^m k^{n+1-m}. \tag{2.2}$$

When $n = 1$, $s_1(h,k)$ is the Dedekind sum and (2.2) coincides with (1.4). Dedekind–Apostol sums are a particular instance of *Dedekind–Rademacher sums*, defined by

$$s(h,k;x,y) = \sum_{m=1}^{k} \left(\left(h\left(\frac{m+y}{k} + \frac{x}{h}\right)\right)\right)\left(\left(\frac{m+y}{k}\right)\right),$$

for coprime integers h, k and arbitrary reals x, y. Rademacher proved that the sum $s(h,k;x,y)$ satisfies a reciprocity law [30, Thm. 2]. If x, y are both integers, then

$$s(h,k;x,y) + s(k,h;y,x) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right),$$

which is (1.4). Also, if x, y are not both integers, then

$$s(h,k;x,y)+s(k,h;y,x)=((x))((y))+\frac{1}{2}\left\{\frac{h}{k}\rho_2(y)+\frac{1}{hk}\rho_2(hy+kx)+\frac{k}{h}\rho_2(x)\right\},$$

where $\rho_2(y)=y^2-y+\frac{1}{6}$.

Many generalizations of Dedekind sums do not involve the sawtooth function. Instead, they are given in terms of cotangents. This is a consequence of the fact that $s(h,k)$ may be written in terms of cotangents (see (1.3)).

We now define one such cotangent sum.

Definition 2.4. Let $a_0,\dots,a_d\in\mathbb{N}, m_0,\dots,m_d\in\mathbb{N}\cup\{0\}, z_0,\dots,z_d\in\mathbb{C}$. We define the *Dedekind cotangent sum* as

$$\mathfrak{c}\left(\begin{array}{c|ccc} a_0 & a_1 & \cdots & a_d \\ m_0 & m_1 & \cdots & m_d \\ z_0 & z_1 & \cdots & z_d \end{array}\right)=\frac{1}{a_0^{m_0+1}}\sum_{k \bmod a_0}\prod_{j=1}^{d}\cot^{(m_j)}\pi\left(a_j\frac{k+z_0}{a_0}-z_j\right),$$

where the sum is taken over all k mod a_0 for which the summand is not singular.

Dedekind cotangent sums were introduced by Beck. They satisfy the reciprocity formula [6, Thm. 2]:

$$\sum_{n=0}^{d}(-1)^{m_n}m_n!\sum_{\substack{l_0,\dots,\widehat{l_n},\dots,l_d\geq 0\\ l_0+\cdots+\widehat{l_n}+\cdots+l_d=m_n}}\frac{a_0^{l_0}\cdots\widehat{a_n^{l_n}}\cdots a_d^{l_d}}{l_0!\cdots\widehat{l_n!}\cdots l_d!}\mathfrak{c}_n=\begin{cases}(-1)^{\frac{d}{2}} & \text{if all } m_k=0 \text{ and } d \text{ is even},\\ 0 & \text{otherwise.}\end{cases}$$

Here $\widehat{x_n}$ means we omit the term x_n and

$$\mathfrak{c}_n=\mathfrak{c}\left(\begin{array}{c|ccccc} a_n & a_0 & \cdots & \widehat{a_n} & \cdots & a_d \\ m_n & m_0+l_0 & \cdots & \widehat{m_n+l_n} & \cdots & m_d+l_d \\ z_n & z_0 & \cdots & \widehat{z_n} & \cdots & z_d \end{array}\right).$$

2.1.3 Bettin–Conrey Sums

We now study the basic properties of Bettin–Conrey sums.

Definition 2.5. Let $a, z\in\mathbb{C}$ be such that $\Re(a)>1$ and $z\neq 0,-1,-2,\dots$ We define the *Hurwitz zeta function* by

$$\zeta(a,z)=\sum_{n=0}^{\infty}\frac{1}{(n+z)^a}.$$

Note that the *Riemann zeta function* is a special case, for $\zeta(a,1)=\zeta(a)$. The Hurwitz zeta function has a meromorphic continuation in the a-plane, its only singularity being $a=1$, a simple pole with residue 1 [26, Sec. 25.11].

Definition 2.6. Let $a \in \mathbb{C}$ and h, k be coprime integers. We define the *Bettin–Conrey sum* as

$$c_a\left(\frac{h}{k}\right) = k^a \sum_{m=1}^{k-1} \cot\left(\frac{\pi m h}{k}\right) \zeta\left(-a, \frac{m}{k}\right).$$

This definition of $c_a\left(\frac{h}{k}\right)$ is to be understood as an equality of meromorphic functions of a. In particular, this means that although $\zeta(a, z)$ has a pole at $a = 1$, the sum $c_{-1}\left(\frac{h}{k}\right)$ may be defined by taking the limit of

$$k^{-a} \sum_{m=1}^{k-1} \cot\left(\frac{\pi m h}{k}\right) \zeta\left(a, \frac{m}{k}\right)$$

as $a \longrightarrow 1$. Indeed, given that $a = 1$ is a simple pole of $\zeta(a, z)$ with residue 1, we have the expansion

$$\zeta(a, z) = \frac{1}{a-1} + \sum_{n=0}^{\infty} \gamma_n(z)(a-1)^n,$$

near 1, where $\gamma_n(z)$ denotes the generalized Stieltjes constants (see, for instance [7]). It follows that

$$\begin{aligned} c_{-1}\left(\frac{h}{k}\right) &= \frac{1}{k} \lim_{a\to 1} \sum_{m=1}^{k-1} \cot\left(\frac{\pi m h}{k}\right) \zeta\left(a, \frac{m}{k}\right) \\ &= \frac{1}{k} \lim_{a\to 1} \sum_{m=1}^{k-1} \cot\left(\frac{\pi m h}{k}\right) \left\{\frac{1}{a-1} + \sum_{n=0}^{\infty} \gamma_n\left(\frac{m}{k}\right)(a-1)^n\right\} \\ &= \frac{1}{k} \sum_{m=1}^{k-1} \cot\left(\frac{\pi m h}{k}\right) \gamma_0\left(\frac{m}{k}\right), \end{aligned}$$

where the last equality follows from the fact that $\cot\left(\frac{\pi h z}{k}\right)$ is a periodic odd function of z with period k. Using techniques of Lammel [23], and Briggs and Chowla [13], Berndt [7] proved that for $0 < z \le 1$ and $n = 0, 1, 2, \ldots$,

$$\gamma_n(z) = \gamma_n = \frac{(-1)^n}{n!} \lim_{m\to\infty} \left(\sum_{l=0}^{m} \frac{\log^n(l+z)}{l+z} - \frac{\log^{n+1}(m+z)}{n+1}\right),$$

where γ_n denotes the n-th Stieltjes constant, so we may write

$$c_{-1}\left(\frac{h}{k}\right) = \frac{1}{k} \sum_{m=1}^{k-1} \cot\left(\frac{\pi m h}{k}\right) \lim_{l\to\infty} \left(\sum_{q=0}^{l} \frac{1}{q+\frac{m}{k}} - \log\left(l + \frac{m}{k}\right)\right).$$

We now show that Bettin–Conrey sums generalize Dedekind sums. It is well known that for $0 < z \le 1$,

$$\lim_{a\to\infty} \left(\zeta(a, z) - \frac{1}{a-1}\right) = -\Psi(z),$$

where Ψ denotes the *digamma function* $\Psi(z) = \Gamma'(z)/\Gamma(z)$ (see, for example, [35, p. 271]). Therefore,

$$\begin{aligned} c_{-1}\left(\frac{h}{k}\right) &= \lim_{a\to 1} k^{-a}\sum_{m=1}^{k-1}\cot\left(\frac{\pi mh}{k}\right)\zeta\left(a,\frac{m}{k}\right) \\ &= \frac{1}{k}\sum_{m=1}^{k-1}\lim_{a\to 1}\cot\left(\frac{\pi mh}{k}\right)\left(\zeta\left(a,\frac{m}{k}\right)-\frac{1}{a-1}\right) \\ &= -\frac{1}{k}\sum_{m=1}^{k-1}\cot\left(\frac{\pi mh}{k}\right)\Psi\left(\frac{m}{k}\right), \end{aligned}$$

and it follows from the reflection formula for the digamma function [26, eq. 5.5.4],

$$\Psi(z) - \Psi(1-z) = -\pi\cot(\pi z),$$

that

$$c_{-1}\left(\frac{h}{k}\right) = \frac{\pi}{2k}\sum_{m=1}^{k-1}\cot\left(\frac{\pi mh}{k}\right)\cot\left(\frac{\pi m}{k}\right) = 2\pi s(h,k).$$

2.2 A Reciprocity Law for Bettin–Conrey Sums

2.2.1 Proof of Theorem 1.1

We now provide a proof of Theorem 1.1, which we recall.

Theorem 1.1. *Let a be a complex number such that $\Re(a) > 1$ and suppose h and k are positive coprime integers, then for any $0 < \epsilon < \min\left\{\frac{1}{h},\frac{1}{k}\right\}$,*

$$h^{1-a}c_{-a}\left(\frac{h}{k}\right) + k^{1-a}c_{-a}\left(\frac{k}{h}\right) = \frac{a\,\zeta(a+1)}{\pi(hk)^a} + \int_{\epsilon+i\infty}^{\epsilon-i\infty}\frac{\cot(\pi hz)\cot(\pi kz)}{z^a}dz.$$

In order to prove this, we need two lemmas.

Lemma 2.1. *Let m be a nonnegative integer. Then*

$$\lim_{y\to\infty}\cot^{(m)}\pi(x\pm iy) = \begin{cases} \mp i & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

Furthermore, this convergence is uniform with respect to x in a fixed bounded interval.

Proof. Since $\cot z = \frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$, we may write

$$|i + \cot\pi(x+iy)| = \frac{2}{|e^{i(2\pi x)} - e^{2\pi y}|} \le \frac{2}{||e^{i(2\pi x)}| - |e^{2\pi y}||} = \frac{2}{|1-e^{2\pi y}|}.$$

Given that the rightmost term in this inequality vanishes as $y \longrightarrow \infty$, we see that

$$\lim_{y\to\infty}\cot\pi(x+iy) = -i.$$

Similarly, the inequality

$$|-i+\cot\pi(x-iy)| = \frac{2}{|e^{i(2\pi x)}e^{2\pi y}-1|} \le \frac{2}{||e^{i(2\pi x)}e^{2\pi y}|-1|} = \frac{2}{|e^{2\pi y}-1|}$$

implies that $\lim_{y\to\infty}\cot\pi(x-iy)=i$. Since

$$|\csc\pi(x+iy)| = \frac{2e^{\pi y}}{|e^{i\pi x}-e^{-i\pi x}e^{2\pi y}|} \le \frac{2e^{\pi y}}{||e^{i\pi x}|-|e^{-i\pi x}e^{2\pi y}||} = \frac{2e^{\pi y}}{|1-e^{2\pi y}|},$$

it follows that $\lim_{y\to\infty}\csc\pi(x+iy)=0$. Similarly,

$$|\csc\pi(x-iy)| = \frac{2e^{\pi y}}{|e^{i\pi x}e^{2\pi y}-e^{-i\pi x}|} \le \frac{2e^{\pi y}}{||e^{i\pi x}e^{2\pi y}|-|e^{-i\pi x}||} = \frac{2e^{\pi y}}{|e^{2\pi y}-1|}$$

implies that $\lim_{y\to\infty}\csc\pi(x-iy)=0$. We know that $\frac{d}{dz}(\cot z)=-\csc^2 z$ and

$$\frac{d}{dz}(\csc z) = -\csc z\cot z,$$

so all the derivatives of $\cot z$ have a $\csc z$ factor, and therefore,

$$\lim_{y\to\infty}\cot^{(m)}\pi(x\pm iy) = \begin{cases}\mp i & \text{if } m=0,\\ 0 & \text{if } m>0.\end{cases}$$

Since the convergence above is independent of x, the limit is uniform with respect to x in a fixed bounded interval. □

Lemma 2.1 implies that

$$\lim_{y\to\infty}\cot^{(m)}\pi h(x\pm iy) = \lim_{y\to\infty}\cot^{(m)}\pi k(x\pm iy) = \begin{cases}\mp i & \text{if } m=0,\\ 0 & \text{if } m>0,\end{cases}$$

uniformly with respect to x in a fixed bounded interval.

The proof of the following lemma is hinted at by Apostol [2].

Lemma 2.2. *Let $\Re(a)>1$ and $R>0$, then $\zeta(a,x+iy)$ vanishes uniformly with respect to $x\in[0,R]$ as $y\longrightarrow\pm\infty$.*

Proof. We begin by showing that $\zeta(a,z)$ vanishes as $\Im(z)\longrightarrow\pm\infty$ if $\Re(z)>0$. Since $\Re(a)>1$ and $\Re(z)>0$, we have the integral representation [26, eq. 25.11.25]

$$\zeta(a,z) = \frac{1}{\Gamma(a)}\int_0^\infty \frac{t^{a-1}e^{-zt}}{1-e^{-t}}dt,$$

which may be written as

$$\zeta(a,z) = \frac{1}{\Gamma(a)}\int_0^\infty \frac{t^{a-1}e^{-t\Re(z)}}{1-e^{-t}}e^{-it\Im(z)}dt. \tag{2.3}$$

Note that for $\Re(z)$ fixed,

$$\int_0^\infty \frac{t^{a-1}e^{-t\Re(z)}}{1-e^{-t}}dt = \zeta(a,\Re(z))\,\Gamma(a)$$

and

$$\int_0^\infty \left|\frac{t^{a-1}e^{-t\,\Re(z)}}{1-e^{-t}}\right| dt = \int_0^\infty \frac{t^{\Re(a)-1}e^{-t\,\Re(z)}}{1-e^{-t}}dt = \zeta(\Re(a),\Re(z))\,\Gamma(a),$$

so the Riemann–Lebesgue lemma (see, for example, [25, Thm. 16]) implies that

$$\int_0^\infty \frac{t^{a-1}e^{-t\Re(z)}}{1-e^{-t}}e^{-it\Im(z)}dt$$

vanishes as $\Im(z) \longrightarrow \pm\infty$. By (2.3), this means that for $\Re(z)$ fixed, $\zeta(a,z)$ vanishes as $\Im(z) \longrightarrow \pm\infty$. In other words, $\zeta(a, x+iy) \longrightarrow 0$ pointwise with respect to $x > 0$ as $y \longrightarrow \pm\infty$.

Moreover, the vanishing of $\zeta(a, x+iy)$ as $y \longrightarrow \pm\infty$ is uniform with respect to $x \in (0, R]$. Indeed, denote $g(t) = \frac{t^{a-1}e^{-tR}}{1-e^{-t}}$, then (2.3) implies that

$$\int_0^\infty g(t)dt = \Gamma(a)\,\zeta(a,R)$$

and

$$\int_0^\infty |g(t)|\,dt = \Gamma(a)\,\zeta(\Re(a),R).$$

It then follows from the Riemann–Lebesgue lemma that $\lim_{|z|\to\infty}\int_0^\infty g(t)e^{-itz}dt = 0$. If $x \in (0, R]$, we may write

$$\Gamma(a)\,\zeta(a,x\pm iy) = \int_0^\infty \frac{t^{a-1}e^{-tx}}{1-e^{-t}}\,e^{\mp ity}dt = \int_0^\infty g(t)\,e^{-it(\pm y - i(x-R))}dt.$$

Since $g(t)$ does not depend on x, the speed at which $\zeta(a, x \pm iM)$ vanishes depends on R and $|y^2 + (x-R)^2|$. However, we know that $0 \le |x-R| < R$, so the speed of the vanishing depends only on R.

Finally, note that

$$\zeta(a,iy) = \sum_{n=0}^\infty \frac{1}{(iy+n)^a} = \sum_{n=0}^\infty \frac{1}{(1+iy+n)^a} + \frac{1}{(iy)^a} = \zeta(a,1+iy) + \frac{1}{(iy)^a},$$

so $\zeta(a,iy) \longrightarrow 0$ as $y \longrightarrow \pm\infty$ and the speed at which $\zeta(s,iy)$ vanishes depends on that of $\zeta(s,1+iy)$. Thus, $\zeta(s,x+iy) \longrightarrow 0$ uniformly as $y \longrightarrow \pm\infty$, as long as $x \in [0,R]$. □

We now prove the main result of this chapter. The idea is to use Cauchy's residue theorem to integrate the function

$$f(z) = \cot(\pi hz)\cot(\pi kz)\,\zeta(a,z)$$

along $C(M,\epsilon)$ as $M \longrightarrow \infty$, where $C(M,\epsilon)$ denotes the positively oriented rectangle with vertices $1+\epsilon+iM$, $\epsilon+iM$, $\epsilon-iM$ and $1+\epsilon-iM$, for $M>0$ and $0<\epsilon<\min\left\{\frac{1}{h},\frac{1}{k}\right\}$ (see Figure 2.2).

Henceforth, $a \in \mathbb{C}$ is such that $\Re(a) > 1$, (h,k) is a pair of coprime positive intergers, and $f(z)$ and $C(M,\epsilon)$ are as above, unless otherwise stated.

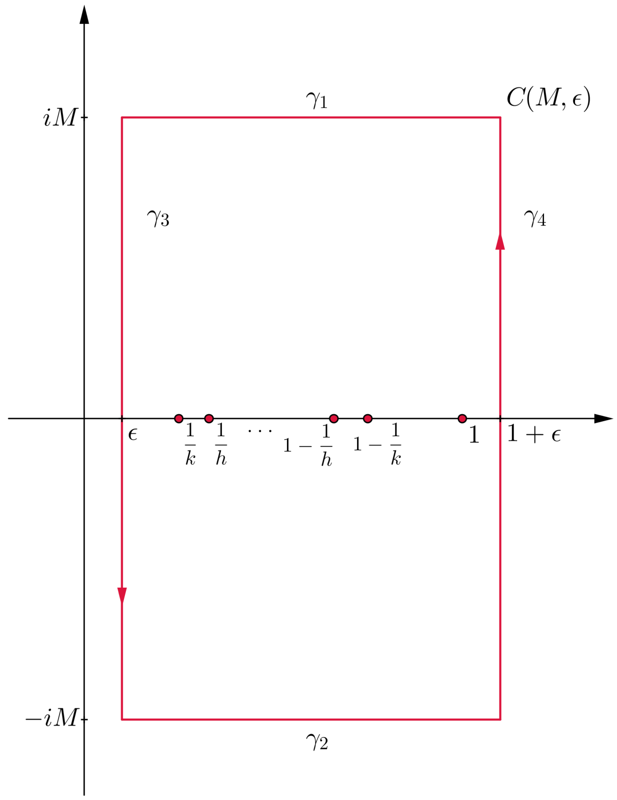

Figure 2.2: The closed contour $C(M, \epsilon)$.

Proof of Theorem 1.1. Since $\zeta(a, z)$ is analytic inside $C(M, \epsilon)$, the only poles of $f(z)$ are those of the cotangent factors. Thus, the fact that h and k are coprime implies that a complete list of the possible poles of $f(z)$ inside $C(M, \epsilon)$ is

$$E = \left\{ \frac{1}{h}, \ldots, \frac{h-1}{h}, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1 \right\}$$

and each of these poles is (at most) simple, with the exception of 1, which is (at most) double. Let us determine $\operatorname{Res}_{z=\frac{m}{h}} f(z)$ for $m \in \{1, 2, \ldots, h-1\}$. Given that $\cot(\pi k z) \cot(\pi h z)\, \zeta(a, z)$ is analytic at $z = \frac{m}{h}$, we may apply Lemma 1.3(a) in [24, p. 174] to deduce that

$$\operatorname*{Res}_{z=\frac{m}{h}} f(z) = \cot\left(\frac{\pi k m}{h}\right) \cos(\pi m)\, \zeta\left(a, \frac{m}{h}\right) \operatorname*{Res}_{z=\frac{m}{h}} \frac{1}{\sin(\pi h z)}.$$

By Lemma 1.3(b) in [24, p. 174],

$$\operatorname*{Res}_{z=\frac{m}{h}} \frac{1}{\sin(\pi h z)} = \frac{1}{\sin(\pi h z)'_{|z=\frac{m}{h}}} = \frac{1}{\pi h \cos(\pi m)}$$

and consequently,

$$\operatorname*{Res}_{z=\frac{m}{h}} f(z) = \frac{1}{\pi h} \cot\left(\frac{\pi k m}{h}\right) \zeta\left(a, \frac{m}{h}\right).$$

Of course, an analogous result is true for $\operatorname{Res}_{z=\frac{m}{k}} f(z)$ for all $m \in \{1,2,\ldots,k-1\}$, and therefore

$$\sum_{z_0\in E}\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=1} f(z) + \frac{1}{\pi h}\sum_{m=1}^{h-1}\cot\left(\frac{\pi k m}{h}\right)\zeta\left(a,\frac{m}{h}\right) + \frac{1}{\pi k}\sum_{m=1}^{k-1}\cot\left(\frac{\pi h m}{k}\right)\zeta\left(a,\frac{m}{k}\right)$$

or, equivalently,

$$h^{1-a}c_{-a}\left(\frac{h}{k}\right) + k^{1-a}c_{-a}\left(\frac{k}{h}\right) = \pi(hk)^{1-a}\left(\left(\sum_{z_0\in E}\operatorname*{Res}_{z=z_0} f(z)\right) - \operatorname*{Res}_{z=1} f(z)\right). \tag{2.4}$$

We now determine $\operatorname{Res}_{z=1} f(z)$. The Laurent series of the cotangent function about 0 is given by

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \frac{2}{945}z^5 + \cdots,$$

so, by the periodicity of $\cot z$, for $z \neq 1$ in a small neighborhood of $z = 1$,

$$\cot(\pi k z) = \left(\frac{1}{\pi k}\right)\frac{1}{z-1} - \frac{\pi k}{3}(z-1) - \frac{(\pi k)^3}{45}(z-1)^3 - \frac{2(\pi k)^5}{945}(z-1)^5 + \cdots$$

and, similarly,

$$\cot(\pi h z) = \left(\frac{1}{\pi h}\right)\frac{1}{z-1} - \frac{\pi h}{3}(z-1) - \frac{(\pi h)^3}{45}(z-1)^3 - \frac{2(\pi h)^5}{945}(z-1)^5 + \cdots.$$

Since $\zeta(a,z)$ is analytic in small neighborhood of 1, Taylor's theorem implies that

$$\zeta(a,z) = \sum_{n=0}^{\infty} b_n (z-1)^n,$$

where $b_n = \frac{\zeta^{(n)}(a,1)}{n!}$ for $n = 0,1,2,\ldots$ (derivatives relative to z). Thus, the expansion of $f(z)$ about 1 is of the form

$$\frac{b_0}{\pi^2 hk}\left(\frac{1}{z-1}\right)^2 + \left(\frac{b_1}{\pi^2 hk}\right)\frac{1}{z-1} + (\text{analytic part}).$$

Given that $a \neq 0,1$, we know that $\frac{\partial}{\partial z}\zeta(a,z) = -a\,\zeta(a+1,z)$ [26, eq. 25.11.17], so $b_1 = -a\,\zeta(a+1,1) = -a\,\zeta(a+1)$. We conclude that $\operatorname{Res}_{z=1} f(z) = -\frac{a\,\zeta(a+1)}{\pi^2 hk}$ and it then follows from (2.4) that

$$h^{1-a}c_{-a}\left(\frac{h}{k}\right) + k^{1-a}c_{-a}\left(\frac{k}{h}\right) = \frac{a\,\zeta(a+1)}{\pi(hk)^a} + \frac{\pi}{(hk)^{a-1}}\sum_{z_0\in E}\operatorname*{Res}_{z=z_0} f(z). \tag{2.5}$$

We now turn to the computation of $\sum_{z_0\in E}\operatorname{Res}_{z=z_0} f(z)$ via Cauchy's residue theorem, which together with (2.5) will provide the reciprocity we are after. Note that the function $f(z)$ is analytic on any two closed contours $C(M_1,\epsilon)$ and $C(M_2,\epsilon)$ and since the poles inside these two contours are the same, we may apply Cauchy's residue theorem to both contours and deduce that

$$\int_{C(M_1,\epsilon)} f(z)dz = \int_{C(M_2,\epsilon)} f(z)dz.$$

In particular, this implies that

$$\lim_{M\to\infty} \int_{C(M,\epsilon)} f(z)dz = 2\pi i \sum_{z_0\in E} \operatorname*{Res}_{z=z_0} f(z). \tag{2.6}$$

Let γ_1 be the path along $C(M,\epsilon)$ from $1+\epsilon+iM$ to $\epsilon+iM$. Similarly, define γ_2 from $\epsilon - iM$ to $1+\epsilon-iM$, γ_3 from $\epsilon+iM$ to $\epsilon-iM$, and γ_4 from $1+\epsilon-iM$ to $1+\epsilon+iM$ (See Figure 2.2).

Since $\Re(a) > 1$, we know that

$$\zeta(a, z+1) = \sum_{n=0}^{\infty} \frac{1}{(n+z+1)^a} = \sum_{n=1}^{\infty} \frac{1}{(n+z)^a} = \zeta(a,z) - \frac{1}{z^a},$$

so the periodicity of $\cot z$ implies that

$$\int_{\gamma_4} f(z)dz = -\int_{\gamma_3} f(z)dz + \int_{\gamma_3} \frac{\cot(\pi hz)\cot(\pi kz)}{z^a} dz.$$

Lemmas 2.1 and 2.2 imply that $f(z)$ vanishes uniformly as $M\to\infty$ (uniformity with respect to $\Re(z)\in[\epsilon, 1+\epsilon]$) so

$$\lim_{M\to\infty}\int_{\gamma_1} f(z)dz = 0 = \lim_{M\to\infty}\int_{\gamma_2} f(z)dz.$$

This means that

$$\lim_{M\to\infty}\int_{C(M,\epsilon)} f(z)dz = \lim_{M\to\infty}\left(\int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz\right)$$

and it follows from (2.5) and (2.6) that

$$h^{1-a} c_{-a}\left(\frac{h}{k}\right) + k^{1-a} c_{-a}\left(\frac{k}{h}\right) = \frac{a\,\zeta(a+1)}{\pi (hk)^a} + \int_{\epsilon+i\infty}^{\epsilon-i\infty} \frac{\cot(\pi hz)\cot(\pi kz)}{z^a} dz\,. \qquad \square$$

This proof mimics one of many proofs of the reciprocity law for Dedekind sums due to Rademacher [31, p. 21].

Note that the only way for 1 not to be a pole of $f(z)$ is if $\zeta(a)$ and $\zeta(a+1)$ are both zero. However, this never happens because of the product representation of the Riemann zeta function [26, eq. 25.2.11], given that $\Re(a), \Re(a+1) > 1$. This means that 1 is always a pole of $f(z)$. Furthermore, 1 is a double pole of $f(z)$.

Suppose $m \in \{1, 2, \ldots, k-1\}$, note that $\frac{m}{k} \in E$ may not be a pole of f. Indeed, given that $\frac{m}{k}$ is a simple pole of $\cot(\pi kz)$, if $\frac{m}{k}$ is a zero of $\cot(\pi hz)\,\zeta(a,z)$, then $f(z)$ is analytic at $\frac{m}{k}$. Of course, this does not affect our computation in (2.5) because if this is the case, then $\operatorname{Res}_{z=\frac{m}{k}} f(z) = 0$. Nevertheless, we examine this setting (when $\frac{m}{k}$ is not a pole of $f(z)$).

Case 1 ($\cot\frac{\pi hm}{k}=0$)*:* This occurs if and only if $\frac{hm}{k}$ is a multiple of $\frac{1}{2}$. Now, if k divides $2m$, then $k\le 2m$. But $2m\le 2(k-1)<2k$, so $k\le 2m<2k$ and since k divides $2m$, it follows that $k=2m$. Hence, $\cot\frac{\pi hm}{k}=0$ if and only if $\frac{m}{k}=\frac{1}{2}$.

Case 2 ($\zeta\left(a,\frac{m}{k}\right)=0$)*:* By [33, Thm. 1], $\zeta\left(a,\frac{m}{k}\right)$ does not vanish if $\Re(a)\ge 2$. Also, the identity $\zeta\left(a,\frac{1}{2}\right)=(2^a-1)\zeta(a)$ [26, eq. 25.11.11] implies that $\zeta\left(a,\frac{m}{k}\right)\neq 0$ if $\frac{m}{k}=\frac{1}{2}$, given that $\zeta(a)\neq 0$ for $\Re(a)>1$ [26, eq. 25.2.11].

Davenport and Heilbronn [18] showed that if $\frac{m}{k}\neq\frac{1}{2}$, there are infinitely many values of a such that $\Re(a)>1$ and $\zeta\left(a,\frac{m}{k}\right)=0$. Thus, if $\frac{m}{k}\neq\frac{1}{2}$, there are infinitely many values of a for which $\frac{m}{k}$ is not a pole of $f(z)$, all of which lie in the a-strip $1<\Re(a)<2$.

In any case, we see that $\frac{m}{k}$ is not a pole of $f(z)$ for infinitely many values of a such that $\Re(a)>1$. Also, we realize that $\cot\left(\frac{\pi hm}{k}\right)$ and $\zeta\left(a,\frac{m}{k}\right)$ never vanish simultaneously.

2.2.2 A Particular Case: $a=n>1$ is an Odd Integer

We now turn to the particular case in which $a=n>1$ is an odd integer and study Bettin–Conrey sums of the form c_{-n}.

Let $\Psi^{(n)}(z)$ denote the $(n+2)$-th polygamma function (see, for example, [26, Sec. 5.15]). It is well known that for n a positive integer,

$$\zeta(n+1,z)=\frac{(-1)^{n+1}\Psi^{(n)}(z)}{n!}$$

whenever $\Re(z)>0$ (see, for instance, [26, eq. 25.11.12]), so for $n>1$, we may write

$$c_{-n}\left(\frac{h}{k}\right)=\frac{(-1)^n}{k^n(n-1)!}\sum_{m=1}^{k-1}\cot\left(\frac{\pi mh}{k}\right)\Psi^{(n-1)}\left(\frac{m}{k}\right).$$

By the reflection formula for the polygamma functions [26, eq. 5.15.6],

$$\Psi^{(n)}(1-z)+(-1)^{n+1}\Psi^{(n)}(z)=(-1)^n\pi\cot^{(n)}(\pi z),$$

we know that if n is odd, then

$$\Psi^{(n-1)}\left(1-\frac{m}{k}\right)-\Psi^{(n-1)}\left(\frac{m}{k}\right)=-\pi\cot^{(n-1)}\left(\frac{\pi m}{k}\right)$$

for each $m\in\{1,2,\ldots,k-1\}$. Therefore,

$$\begin{aligned}2\sum_{m=1}^{k-1}\cot\left(\frac{\pi mh}{k}\right)\Psi^{(n-1)}\left(\frac{m}{k}\right)&=\sum_{m=1}^{k-1}\cot\left(\frac{\pi mh}{k}\right)\Psi^{(n-1)}\left(\frac{m}{k}\right)\\&\quad+\sum_{m=1}^{k-1}\cot\left(\frac{\pi(k-m)h}{k}\right)\Psi^{(n-1)}\left(1-\frac{m}{k}\right),\end{aligned}$$

which implies that

$$\begin{aligned}2\sum_{m=1}^{k-1}\cot\left(\frac{\pi mh}{k}\right)\Psi^{(n-1)}\left(\frac{m}{k}\right)&=\sum_{m=1}^{k-1}\cot\left(\frac{\pi mh}{k}\right)\left(\Psi^{(n-1)}\left(\frac{m}{k}\right)-\Psi^{(n-1)}\left(1-\frac{m}{k}\right)\right)\\&=\pi\sum_{m=1}^{k-1}\cot\left(\frac{\pi mh}{k}\right)\cot^{(n-1)}\left(\frac{\pi m}{k}\right).\end{aligned}$$

This means that for $n > 1$ odd, c_{-n} is essentially a Dedekind cotangent sum. Indeed,

$$c_{-n}\left(\frac{h}{k}\right) = -\frac{\pi}{2k^n(n-1)!}\sum_{m=1}^{k-1}\cot\left(\frac{\pi m h}{k}\right)\cot^{(n-1)}\left(\frac{\pi m}{k}\right)$$
$$= -\frac{\pi}{2(n-1)!}\,\mathfrak{c}\left(\begin{array}{c|cc} k & h & 1 \\ n-1 & 0 & n-1 \\ 0 & 0 & 0 \end{array}\right).$$

The following theorem is an instance of Theorem 1.1. It provides a reciprocity law for Bettin–Conrey sums of the form c_{-n} in terms of Bernoulli numbers.

Theorem 2.3. *Let $n > 1$ be an odd integer and suppose h and k are positive coprime integers, then*

$$h^{1-n}c_{-n}\left(\frac{h}{k}\right) + k^{1-n}c_{-n}\left(\frac{k}{h}\right) = \left(\frac{2\pi i}{hk}\right)^n \frac{1}{i(n+1)!}$$
$$\times\left(n\,B_{n+1} + \sum_{m=0}^{n+1}\binom{n+1}{m}B_m B_{n+1-m}h^m k^{n+1-m}\right),$$

where B_n denotes the n-th Bernoulli number.

Proof. We consider the closed contour $\widetilde{C}(M,\epsilon)$ defined as the positively oriented rectangle with vertices $1+iM$, iM, $-iM$ and $1-iM$, with indentations (to the right) of radius $0 < \epsilon < \min\left\{\frac{1}{h},\frac{1}{k}\right\}$ around 0 and 1 (see Figure 2.3).

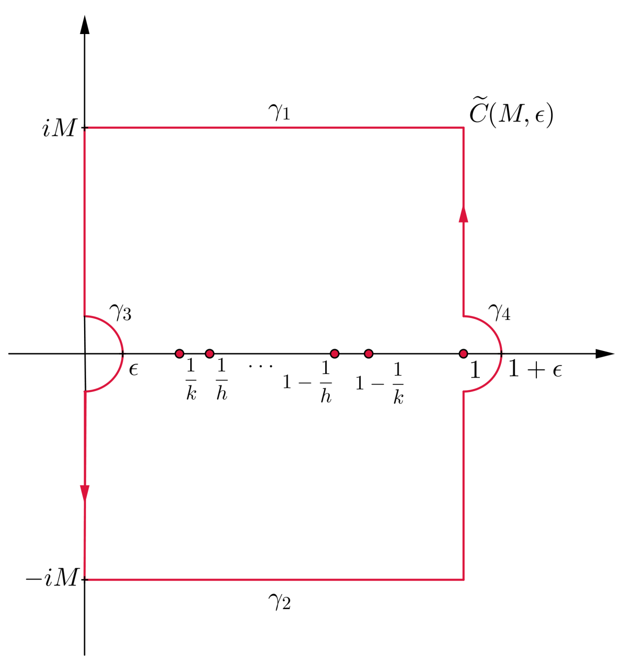

Figure 2.3: The closed contour $\widetilde{C}(M,\epsilon)$.

Since $\widetilde{C}(M,\epsilon)$ contains the same poles of $f(z)$ as the closed contour $C(M,\epsilon)$ (see Figure 2.2) used to prove Theorem 1.1, we may apply Cauchy's residue theorem, letting $M \longrightarrow \infty$ and we only need to determine $\lim_{M\to\infty} \int_{\widetilde{C}(M,\epsilon)} f(z)dz$ in order to deduce a reciprocity law for the sums c_{-n}.

As in the case of $C(M,\epsilon)$, the integrals along the horizontal paths vanish, so using the periodicity of the cotangent to add integrals along parallel paths, as we did when considering $C(M,\epsilon)$, we write

$$\lim_{M\to\infty} \int_{\widetilde{C}(M,\epsilon)} f(z)dz = \lim_{M\to\infty} \left(\int_{iM}^{i\epsilon} g(z)dz + \int_{-i\epsilon}^{-iM} g(z)dz \right) + \int_{\gamma_3} g(z)dz, \tag{2.7}$$

where γ_3 denotes the indented path around 0 and

$$g(z) = \frac{\cot(\pi kz)\cot(\pi hz)}{z^n}.$$

Given that $g(z)$ is an odd function, the vertical integrals cancel and we may apply Cauchy's residue theorem to integrate $g(z)$ along the positively oriented circle of radius ϵ and centered at 0, to deduce that

$$\lim_{M\to\infty} \int_{\widetilde{C}(M,\epsilon)} f(z)dz = -\pi i \operatorname*{Res}_{z=0} g(z).$$

This is the main reason for us to use the contour $\widetilde{C}(M,\epsilon)$ instead of $C(M,\epsilon)$. Indeed, integration along $\widetilde{C}(M,\epsilon)$ exploits the parity of the function $g(z)$, allowing us to cancel the vertical integrals in (2.7).

We know the expansion of the cotangent function to be

$$\pi z \cot(\pi z) = \sum_{m=0}^{\infty} \frac{(2\pi i)^m B_m}{m!} z^m,$$

with the convention that B_1 must be redefined to be zero. Thus, we have the expansion

$$\cot(\pi kz) = \sum_{m=0}^{\infty} \frac{(2i)(2\pi ik)^m B_{m+1}}{(m+1)!} z^m$$

and of course, an analogous result holds for h. Hence,

$$\operatorname*{Res}_{z=0} g(z) = \frac{(2i)(2\pi i)^n}{\pi hk(n+1)!} \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} h^m k^{n+1-m}, \tag{2.8}$$

and given that $\zeta(n+1) = -\frac{(2\pi i)^{n+1}}{2(n+1)!} B_{n+1}$ (see [26, eq. 25.6.2]), the Cauchy residue theorem and (2.5) yield

$$\left(\frac{2\pi i}{hk}\right)^n \frac{1}{i(n+1)!} \left(nB_{n+1} + \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} h^m k^{n+1-m} \right)$$
$$= h^{1-n} c_{-n}\left(\frac{h}{k}\right) + k^{1-n} c_{-n}\left(\frac{k}{h}\right). \tag{2.9}$$

Finally, note that the convention $B_1 := 0$ is irrelevant in (2.9), since B_1 in this sum is always multiplied by a Bernoulli number with odd index larger than 1. □

Note that this result is essentially the same as (2.2), the reciprocity deduced by Apostol for Dedekind–Apostol sums [1]. This is a consequence of the fact that for $n > 1$ an odd integer, $c_{-n}\left(\frac{h}{k}\right)$ is a multiple of the Dedekind–Apostol sum $s_n(h,k)$. Indeed, for such n [2, Thm. 1]

$$s_n(h,k) = in!(2\pi i)^{-n} c_{-n}\left(\frac{h}{k}\right).$$

It is worth mentioning that although the Dedekind–Apostol sum $s_n(h,k)$ is trivial for n even [1, eq. (4.13)], in the sense that $s_n(h,k)$ is independent of h, the Bettin–Conrey sum $c_{-n}\left(\frac{h}{k}\right)$ is not, in general.

The following corollary is an immediate consequence of Theorems 1.1 and 2.3.

Corollary 2.4. *Let $n > 1$ be an odd integer and suppose h and k are positive coprime integers, then for any $0 < \epsilon < \min\left\{\frac{1}{h}, \frac{1}{k}\right\}$,*

$$\int_{\epsilon+i\infty}^{\epsilon-i\infty} \frac{\cot(\pi hz)\cot(\pi kz)}{z^n} dz = \frac{2(2\pi i)^n}{hk(n+1)!} \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} h^m k^{n+1-m}.$$

2.2.3 Some Consequences Regarding the Functions ψ_a and g_a

We now use Theorem 1.1 to derive analytical properties of the functions ψ_{-a} and g_{-a} studied by Bettin and Conrey.

Recall the function ψ_a is defined for $\Im(z) > 0$ and $a \in \mathbb{C}$ by

$$\psi_a(z) = E_{a+1}(z) - \frac{1}{z^{a+1}} E_{a+1}\left(-\frac{1}{z}\right)$$

and it extends to an analytic function on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ via the representation

$$\psi_a(z) = \frac{i}{\pi z}\frac{\zeta(1-a)}{\zeta(-a)} - i\frac{1}{z^{1+a}}\cot\frac{\pi a}{2} + i\frac{g_a(z)}{\zeta(-a)}.$$

The function g_a is given by

$$g_a(z) = -2\sum_{1\leq n\leq M}(-1)^n \frac{B_{2n}}{(2n)!}\zeta(1-2n-a)(2\pi z)^{2n-1} + \frac{1}{\pi i}\int_{(-\frac{1}{2}-2M)} \zeta(s)\,\zeta(s-a)\,\Gamma(s)\frac{\cos\frac{\pi a}{2}}{\sin\frac{\pi(s-a)}{2}}(2\pi z)^{-s} ds,$$

where M is any integer greater or equal to $-\frac{1}{2}\min(0, \Re(a))$.

Proposition 2.5. *Let $\Re(a) > 1$ and $h, k \in \mathbb{Z}_{\geq 1}$ such that $(h,k) = 1$. Then for all $0 < \epsilon < \min\left\{1, \frac{k}{h}\right\}$,*

$$\psi_{-a}\left(\frac{h}{k}\right) = \frac{1}{2\zeta(a)} \lim_{M\to\infty} \int_{\epsilon+ikM}^{\epsilon-ikM} \frac{\cot\left(\frac{\pi hz}{k}\right)\cot(\pi z)}{z^a} dz,$$

and

$$-g_{-a}\left(\frac{h}{k}\right)=-\frac{k\,\zeta(1+a)}{\pi h}+\zeta(a)\left(\frac{k}{h}\right)^{1-a}\cot\left(\frac{\pi a}{2}\right)+\frac{i}{2}\lim_{M\to\infty}\int_{\epsilon+ikM}^{\epsilon-ikM}\frac{\cot\left(\frac{\pi hz}{k}\right)\cot(\pi z)}{z^a}dz.$$

Moreover, let r *be a positive real number,* $\left(\frac{h_n}{k_n}\right)_{n\in\mathbb{N}}$ *a sequence of rational numbers* $\frac{h_n}{k_n}\longrightarrow r^-$ *such that* $(h_n,k_n)=1$ *and* $h_n,k_n\in\mathbb{Z}_{\geq 1}$ *for each* n*. Then, for all* $0<\delta<\min\left\{1,\frac{1}{r}\right\}$*,*

$$\psi_{-a}(r)=\frac{1}{2\zeta(a)}\lim_{n\to\infty}\lim_{m\to\infty}\int_{\delta+imk_n}^{\delta-imk_n}\frac{\cot\left(\frac{\pi h_n z}{k_n}\right)\cot(\pi z)}{z^a}dz$$

and

$$-g_{-a}(r)=\frac{\zeta(1+a)}{\pi r}+\frac{\zeta(a)}{r^{1-a}}\cot\left(\frac{\pi a}{2}\right)+\frac{i}{2}\lim_{n\to\infty}\lim_{m\to\infty}\int_{\delta+imk_n}^{\delta-imk_n}\frac{\cot\left(\frac{\pi h_n z}{k_n}\right)\cot(\pi z)}{z^a}dz.$$

Proof. By Theorem 1.1, we know that for $0<\epsilon<\min\left\{\frac{1}{h},\frac{1}{k}\right\}$,

$$h^{1-a}c_{-a}\left(\frac{h}{k}\right)+k^{1-a}c_{-a}\left(\frac{k}{h}\right)=\frac{a\,\zeta(a+1)}{\pi(hk)^a}+\frac{1}{2i(hk)^{a-1}}\int_{\epsilon+i\infty}^{\epsilon-i\infty}\frac{\cot(\pi hz)\cot(\pi kz)}{z^a}dz$$

and given that $c_{-a}\left(\frac{-k}{h}\right)=-c_{-a}\left(\frac{k}{h}\right)$, we may write

$$c_{-a}\left(\frac{h}{k}\right)-\left(\frac{k}{h}\right)^{1-a}c_{-a}\left(\frac{-k}{h}\right)=\frac{a\,\zeta(a+1)}{\pi k^a h}+\frac{1}{2ik^{a-1}}\int_{\epsilon+i\infty}^{\epsilon-i\infty}\frac{\cot(\pi hz)\cot(\pi kz)}{z^a}dz.$$

Making a change of variables, we deduce that for $0<\epsilon<\min\left\{1,\frac{k}{h}\right\}$,

$$c_{-a}\left(\frac{h}{k}\right)-\left(\frac{k}{h}\right)^{1-a}c_{-a}\left(\frac{-k}{h}\right)=\frac{a\,\zeta(a+1)}{\pi k^a h}+\frac{1}{2i}\lim_{M\to\infty}\int_{\epsilon+ikM}^{\epsilon-ikM}\frac{\cot\left(\frac{\pi hz}{k}\right)\cot(\pi z)}{z^a}dz.$$

Comparing this reciprocity with (1.2), we deduce that

$$\psi_{-a}\left(\frac{h}{k}\right)=\frac{1}{2\,\zeta(a)}\lim_{M\to\infty}\int_{\epsilon+ikM}^{\epsilon-ikM}\frac{\cot\left(\frac{\pi hz}{k}\right)\cot(\pi z)}{z^a}dz.$$

Therefore [9, Thm. 1],

$$\psi_{-a}(z)=\frac{i}{\pi z}\frac{\zeta(1+a)}{\zeta(a)}+i\frac{1}{z^{1-a}}\cot\frac{\pi a}{2}+i\frac{g_{-a}(z)}{\zeta(a)}$$

implies that

$$-g_{-a}\left(\frac{h}{k}\right)=\frac{k\,\zeta(1+a)}{\pi h}+\zeta(a)\left(\frac{k}{h}\right)^{1-a}\cot\left(\frac{\pi a}{2}\right)+\frac{i}{2}\lim_{M\to\infty}\int_{\epsilon+ikM}^{\epsilon-ikM}\frac{\cot\left(\frac{\pi hz}{k}\right)\cot(\pi z)}{z^a}dz.$$

Now, let r be a positive real number and suppose $\left(\frac{h_n}{k_n}\right)_{n\in\mathbb{N}}$ is a sequence of rational numbers $\frac{h_n}{k_n} \longrightarrow r^-$ such that $(h_n, k_n) = 1$ and $h_n, k_n \in \mathbb{Z}_{\geq 1}$ for all n. Then, for $0 < \delta < \min\left\{1, \frac{1}{r}\right\}$,

$$\psi_{-a}\left(\frac{h_n}{k_n}\right) = \frac{1}{2\,\zeta(a)} \lim_{M\to\infty} \int_{\delta+ik_nM}^{\delta-ik_nM} \frac{\cot\left(\frac{\pi h_n z}{k_n}\right)\cot(\pi z)}{z^a} dz, \tag{2.10}$$

because $\frac{1}{r} \leq \frac{k_n}{h_n}$ for each n. Hence, if we denote

$$\alpha(n,m) := \int_{\delta+imk_n}^{\delta-imk_n} \frac{\cot\left(\frac{\pi h_n z}{k_n}\right)\cot(\pi z)}{z^a} dz,$$

it follows from (2.10) that $\lim_{m\to\infty}\alpha(n,m)$ exists for each n. Furthermore, for each n, we know that $\lim_{m\to\infty}\alpha(n,m) = 2\,\zeta(a)\,\psi_{-a}\left(\frac{h_n}{k_n}\right)$.

Since ψ_{-a} is an analytic function on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$,

$$\begin{aligned}\psi_{-a}(r) = \psi_{-a}\left(\lim_{n\to\infty}\frac{h_n}{k_n}\right) = \lim_{n\to\infty}\psi_{-a}\left(\frac{h_n}{k_n}\right) &= \frac{1}{2\,\zeta(a)}\lim_{n\to\infty}\lim_{m\to\infty}\alpha(n,m)\\ &= \frac{1}{2\,\zeta(a)}\lim_{n\to\infty}\lim_{m\to\infty}\int_{\delta+imk_n}^{\delta-imk_n} \frac{\cot\left(\frac{\pi h_n z}{k_n}\right)\cot(\pi z)}{z^a} dz. \end{aligned}\tag{2.11}$$

Given that $\psi_{-a}(z) = \frac{i}{\pi z}\frac{\zeta(1+a)}{\zeta(a)} + i\frac{1}{z^{1-a}}\cot\frac{\pi a}{2} + i\frac{g_{-a}(z)}{\zeta(a)}$,

$$-g_{-a}(z) = i\,\zeta(a)\,\psi_{-a}(z) + \frac{\zeta(1+a)}{\pi z} + \frac{\zeta(a)}{z^{1-a}}\cot\left(\frac{\pi a}{2}\right).$$

Thus, (2.11) implies that

$$-g_{-a}(r) = \frac{\zeta(1+a)}{\pi r} + \frac{\zeta(a)}{r^{1-a}}\cot\left(\frac{\pi a}{2}\right) + \frac{i}{2}\lim_{n\to\infty}\lim_{m\to\infty}\int_{\delta+imk_n}^{\delta-imk_n} \frac{\cot\left(\frac{\pi h_n z}{k_n}\right)\cot(\pi z)}{z^a} dz,$$

which concludes the proof. □

If n is an odd integer we have a stronger result, describing ψ_{-n} and g_{-n} on the domain $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ in terms of Bernoulli numbers.

Proposition 2.6. *Let $n > 1$ be an odd integer, then*

$$\psi_{-n}(z) = \frac{(2\pi i)^n}{\zeta(n)(n+1)!}\sum_{m=0}^{n+1}\binom{n+1}{m}B_m B_{n+1-m} z^{m-1}$$

and

$$g_{-n}(z) = -\frac{i(2\pi i)^n}{(n+1)!}\sum_{m=0}^{n}\binom{n+1}{m+1}B_{m+1}B_{n-m} z^m$$

for all $z \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$.

Proof. Given that $c_{-n}\left(\frac{-k}{h}\right) = -c_{-n}\left(\frac{k}{h}\right)$, it follows from (1.2) and Theorem 2.3 that

$$\psi_{-n}\left(\frac{h}{k}\right) = \frac{(2\pi i)^n}{\zeta(n)(n+1)!}\sum_{m=0}^{n+1}\binom{n+1}{m}B_m B_{n+1-m}\left(\frac{h}{k}\right)^{m-1}. \tag{2.12}$$

Define the function

$$\phi_{-n}(z) = \frac{(2\pi i)^n}{\zeta(n)(n+1)!}\sum_{m=0}^{n+1}\binom{n+1}{m}B_m B_{n+1-m}z^{m-1}.$$

This function is analytic on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ and by [9, Thm.1], so is ψ_{-n}. Let

$$S_n = \left\{z \in \mathbb{C}\backslash\mathbb{R}_{\leq 0} \mid \psi_{-n}(z) = \phi_{-n}(z)\right\}.$$

Since all positive rationals can be written in reduced form, it follows from (2.12) that $\mathbb{Q}_{>0} \subseteq S_n$. Thus, S_n is not a discrete set and given that both ψ_{-n} and ϕ_{-n} are analytic on the connected open set $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$, Theorem 1.2(*ii*) in [24, p. 90] implies that $\psi_{-n} = \phi_{-n}$ on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$. That is,

$$\psi_{-n}(z) = \frac{(2\pi i)^n}{\zeta(n)(n+1)!}\sum_{m=0}^{n+1}\binom{n+1}{m}B_m B_{n+1-m}z^{m-1}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$. Now, we know that

$$\psi_{-n}(z) = \frac{i}{\pi z}\frac{\zeta(1+n)}{\zeta(n)} - iz^{n-1}\cot\left(\frac{-\pi n}{2}\right) + i\frac{g_{-n}(z)}{\zeta(n)},$$

and since n is odd, $\cot\left(\frac{-\pi n}{2}\right) = 0$ and $\zeta(n+1) = -\frac{(2\pi i)^{n+1}}{2(n+1)!}B_{n+1}$ [26, eq. 25.6.2], so

$$\begin{aligned}
g_{-n}(z) &= \frac{i(2\pi i)^n B_{n+1}}{(n+1)!}\left(\frac{1}{z}\right) - i\,\zeta(n)\,\psi_{-n}(z) \\
&= \frac{-i(2\pi i)^n}{(n+1)!}\sum_{m=1}^{n+1}\binom{n+1}{m}B_m B_{n+1-m}z^{m-1} \\
&= \frac{-i(2\pi i)^n}{(n+1)!}\sum_{m=0}^{n}\binom{n+1}{m+1}B_{m+1}B_{n-m}z^{m}. \qquad \square
\end{aligned}$$

Bettin and Conrey [9, Thm. 3] prove the following result.

Theorem 2.7. *Let $\Re(\tau) > 0$ for $|z| < |\tau|$, let $g_a(\tau + z) := \sum_{m=0}^{\infty}\frac{g_a^{(m)}(\tau)}{m!}z^m$ be the Taylor series of $g_a(z)$ about τ. Then*

$$\begin{aligned}
\frac{g_a^{(m)}(1)}{m!} = &-\sum_{\substack{2n-1+k=m \\ n,k\geq 1}}(-1)^{n+m}B_{2n}\zeta(1-2n-a)\frac{\Gamma(2n+a+k)}{\Gamma(2n+a)k!(2n)!}2(2\pi)^{2n-1} \\
&+(-1)^m\cot\frac{\pi a}{2}\zeta(-a)\frac{\Gamma(1+a+m)}{\Gamma(1+a)m!} \\
&+(-1)^m\left(\frac{\Gamma(1+a+m)}{\Gamma(a)(m+1)!} - 1\right)\frac{\zeta(1-a)}{\pi},
\end{aligned}$$

and in particular, if $a \in \mathbb{Z}_{\le 0}$ *and* $(a,0) \neq (0,0)$*, then* $\pi g_a^{(m)}(1)$ *is a rational polynomial in* π^2*. Moreover,*

$$\frac{g_a^{(m)}(\tau)}{m!} = \cos\left(\frac{\pi a}{2}\right) \frac{2^{\frac{7}{4}-\frac{a}{2}}}{\pi^{\frac{3}{4}-\frac{a}{2}}} \cdot \frac{e^{-2\sqrt{\pi\tau m}}}{m^{\frac{1}{4}-\frac{a}{2}}\tau^{m+\frac{3}{4}+\frac{a}{2}}} \times \left(\cos\left(2\sqrt{\pi\tau m} - \frac{1}{8}\pi(2a-1) + (\tau+m)\pi\right) + O_{\tau,a}\left(\frac{1}{\sqrt{m}}\right)\right),$$

as $m \longrightarrow \infty$*.*

Clearly, Proposition 2.6 is a particular case of Theorem 2.7. Also, the proofs of Theorem 2.7 and (1.2) are independent [9], so Proposition 2.6 provides an alternative proof of Theorem 2.7 in the case $a = -n$, with $n > 1$ an odd integer. In fact, Proposition 2.6 is much stronger (in this particular case), because it completely determines g_{-n} and it shows g_{-n} is a polynomial.

In particular, it becomes obvious that if $a \in \mathbb{Z}_{\le 1}$ is odd and $(a,m) \neq (0,0)$, then $\pi g_a^{(m)}(1)$ is a rational polynomial in π^2.

2.2.4 Generalizations of Bettin–Conrey Sums

The notion of Bettin–Conrey sum may be generalized to a broader family of cotangent sums satisfying a reciprocity law analogous to that of Theorem 1.1.

Definition 2.7. Let $k_0, k_1, \ldots, k_n$ be positive integers such that $(k_0, k_j) = 1$ for $j = 1, 2, \ldots, n$. Suppose $m_0, m_1, \ldots, m_n$ are nonnegative integers and $a \neq -1$ is a complex number, then we define the *generalized Bettin–Conrey sum*

$$c_a\left(\begin{array}{c|ccc} k_0 & k_1 & \cdots & k_n \\ m_0 & m_1 & \cdots & m_n \end{array}\right) = k_0^{a-m_0} \sum_{l=1}^{k_0-1} \zeta^{(m_0)}\left(-a, \frac{l}{k_0}\right) \prod_{j=1}^{n} \cot^{(m_j)}\left(\frac{\pi k_j l}{k_0}\right),$$

Here $\zeta^{(m_0)}(a,z)$ denotes the m_0-th derivative of the Hurwitz zeta function with respect to z.

This notation mimics that of Dedekind cotangent sums. Note that

$$c_s\left(\frac{h}{k}\right) = c_s\left(\begin{array}{c|c} k & h \\ 0 & 0 \end{array}\right).$$

We will prove a reciprocity law for generalized Bettin–Conrey sums of the form c_{-a} whenever $\Re(a) > 1$.

Definition 2.8. Suppose m is a nonnegative integer, then we define the sequence $\{B(m)\}_{j=0}^{\infty}$ by

$$B(m)_j := \begin{cases} 0 & \text{if } j \in \{1, \ldots, \max\{1, m\}\}, \\ B_j & \text{otherwise.} \end{cases}$$

The following result generalizes Theorem 1.1.

Theorem 2.8. *Let $d \geq 2$ and suppose that $k_1, \dots, k_d$ is a list of pairwise coprime positive integers and $m_0, m_1, \dots, m_d$ are nonnegative integers. If a is such that $\Re(a) > 1$ and $0 < \epsilon < \min_{1\leq j\leq d}\left\{\frac{1}{k_j}\right\}$, then*

$$\sum_{j=1}^{d}(-1)^{m_j}\left(\prod_{\substack{t=1\\t\neq j}}^{d} k_t^{1-(a+m_0)}\right)\sum_{\substack{l_1+\cdots+l_d=m_j\\l_1,\dots,l_d\geq 0}}\frac{1}{\pi^{l_j+1}}\left(\prod_{t=1}^{d}\frac{1}{l_t!}\left(\frac{k_t}{k_j}\right)^{l_t}\right)c_{-a}(j)$$
$$= (-1)^{m_0}\prod_{j=1}^{d}k_j^{1-(a+m_0)}\Bigg\{\frac{a^{(m_0)}}{2\pi i}\int_{\epsilon+i\infty}^{\epsilon-i\infty}\frac{\prod_{j=1}^{d}\cot^{(m_j)}(\pi k_j z)}{z^{a+m_0}}dz + (2i)^{d+\sum_{j=1}^{d}m_j}$$
$$\times\sum_{m=0}^{m_1+\cdots+m_d+d-1} b_m \sum_{\substack{l_1+\cdots+l_d=-(m+1)\\l_j\geq -(m_j+1)}}\prod_{j=1}^{d}\frac{(l_j+1)^{(m_j)}\,k_j^{l_j}B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!}\Bigg\},$$

where $x^{(n)} = \prod_{l=0}^{n-1}(x+l)$ is the rising factorial,

$$c_{-a}(j) = c_{-a}\left(\begin{array}{c|ccccc} k_j & k_1 & \cdots & \widehat{k_j} & \cdots & k_d \\ m_0+l_j & m_1+l_1 & \cdots & \widehat{m_j+l_j} & \cdots & m_d+l_d \end{array}\right)$$

and for $m \in \mathbb{Z}_{\geq 0}$

$$b_m := \frac{\zeta(a+m+m_0)\,a^{(m+m_0)}}{(-2\pi i)^{m+1}m!}.$$

The proof of this theorem is analogous to that of Theorem 1.1. Henceforth, a is such that $\Re(a) > 1$, $k_1, \dots, k_d$ is a list of pairwise coprime positive integers and $0 < \epsilon < \min_{1\leq j\leq d}\left\{\frac{1}{k_j}\right\}$. In addition, $m_0, m_1, \dots, m_d$ is a list of nonnegative integers,

$$f(z) = \zeta^{(m_0)}(a, z)\prod_{j=1}^{d}\cot^{(m_j)}(\pi k_j z)$$

and, as before, $C(M, \epsilon)$ denotes the positively oriented rectangle with vertices $1+\epsilon+iM$, $\epsilon+iM$, $\epsilon-iM$ and $1+\epsilon-iM$, where $M > 0$ (see Figure 2.2).

Note that if we compute the residues of $f(z) = \zeta(a, z)\prod_{j=1}^{d}\cot^{(m_j)}(\pi k_j z)$, that is, if we consider the case $m_0 = 0$, then we obtain generalized Bettin–Conrey sums of the form

$$c_{-a}\left(\begin{array}{c|ccccc} k_j & k_1 & \cdots & \widehat{k_j} & \cdots & k_d \\ l_j & m_1+l_1 & \cdots & \widehat{m_j+l_j} & \cdots & m_d+l_d \end{array}\right),$$

with no powers of k_j as factors. This fact is the main reason for the term $k_0^{a-m_0}$ in our definition of the generalized Bettin–Conrey sum

$$c_a\left(\begin{array}{c|ccc} k_0 & k_1 & \cdots & k_n \\ m_0 & m_1 & \cdots & m_n \end{array}\right) = k_0^{a-m_0}\sum_{l=1}^{k_0-1}\zeta^{(m_0)}\left(-a, \frac{l}{k_0}\right)\prod_{j=1}^{n}\cot^{(m_j)}\left(\frac{\pi k_j l}{k_0}\right).$$

Proof of Theorem 2.8. For each j, we know that $\cot(\pi k_j z)$ is analytic on and inside $C(M,\epsilon)$, with the exception of the poles $\frac{1}{k_j},\ldots,\frac{k_j-1}{k_j}$. This means that except for the aforementioned poles, $\cot^{(m_j)}(\pi k_j z)$ is analytic on and inside $C(M,\epsilon)$, so the analyticity of $\zeta^{(m_0)}(a,z)$ on and inside $C(M,\epsilon)$ implies that a complete list of (possible) poles of f is

$$E=\left\{\frac{1}{k_1},\ldots,\frac{k_1-1}{k_1},\ldots\frac{1}{k_d},\ldots,\frac{k_d-1}{k_d},1\right\}.$$

Let $j\in\{1,2,\ldots d\}$ and $q\in\{1,2,\ldots,k_j-1\}$, then the Laurent series of $\cot(\pi k_j z)$ about $\frac{q}{k_j}$ is of the form $\left(\frac{1}{\pi k_j}\right)\frac{1}{z-\frac{q}{k_j}}+$ (analytic part), so near $\frac{q}{k_j}$,

$$\cot^{(m_j)}(\pi k_j z)=\frac{(-1)^{m_j}m_j!}{(\pi k_j)^{m_j+1}}\left(z-\frac{q}{k_j}\right)^{-(m_j+1)}+\text{analytic part.}$$

Since $(k_j,k_t)=1$ for $t\neq j$, it follows from Taylor's theorem that for $t\neq j$ the expansion

$$\cot^{(m_t)}(\pi k_t z)=\sum_{l_t=0}^{\infty}\frac{(\pi k_t)^{l_t}}{l_t!}\cot^{(m_t+l_t)}\left(\frac{\pi k_t q}{k_j}\right)\left(z-\frac{q}{k_j}\right)^{l_t}$$

is valid near $\frac{q}{k_j}$. Taylor's theorem also yields that

$$\zeta^{(m_0)}(a,z)=\sum_{l_j=0}^{\infty}\frac{\zeta^{(m_0+l_j)}\left(a,\frac{q}{k_j}\right)}{l_j!}\left(z-\frac{q}{k_j}\right)^{l_j}$$

near $\frac{q}{k_j}$. Hence, we may write $\operatorname{Res}_{z=\frac{q}{k_j}}f(z)$ as

$$\frac{(-1)^{m_j}m_j!}{(\pi k_j)^{m_j+1}}\sum_{\substack{l_1+\cdots l_d=m_j\\ l_1,\ldots,l_d\geq 0}}\frac{\zeta^{(m_0+l_j)}\left(a,\frac{q}{k_j}\right)}{l_j!}\prod_{\substack{t=1\\ t\neq j}}\frac{(\pi k_t)^{l_t}}{l_t!}\cot^{(m_t+l_t)}\left(\frac{\pi k_t q}{k_j}\right).$$

Therefore, $\sum_{q=1}^{k_j-1}\operatorname{Res}_{z=\frac{q}{k_j}}f(z)$ is given by

$$\begin{aligned}&(-1)^{m_j}k_j^{a-1}m_j!\left(\prod_{t=1}^{d}\frac{1}{l_t!}\left(\frac{k_t}{k_j}\right)^{l_t}\right)\\&\qquad\times\sum_{\substack{l_1+\cdots+l_d=m_j\\ l_1,\ldots,l_d\geq 0}}\frac{1}{\pi^{l_j+1}k_j^{(a+l_j)}}\sum_{q=1}^{k_j-1}\zeta^{(m_0+l_j)}\left(a,\frac{q}{k_j}\right)\prod_{\substack{t=1\\ t\neq j}}^{d}\cot^{(m_t+l_t)}\left(\frac{\pi k_t q}{k_j}\right)\\&\qquad\qquad=(-1)^{m_j}k_j^{m_0+a-1}m_j!\sum_{\substack{l_1+\cdots+l_d=m_j\\ l_1,\ldots,l_d\geq 0}}\frac{1}{\pi^{l_j+1}}\left(\prod_{t=1}^{d}\frac{1}{l_t!}\left(\frac{k_t}{k_j}\right)^{l_t}\right)c_{-a}(j).\end{aligned}$$

Given that this holds for all j, we conclude that

$$\sum_{z_0 \in E} \operatorname*{Res}_{z=z_0} f(z) - \operatorname*{Res}_{z=1} f(z) = \sum_{j=1}^{d} (-1)^{m_j} k_j^{m_0+a-1} m_j! \\ \times \sum_{\substack{l_1+\cdots l_d=m_j \\ l_1,\ldots,l_d \geq 0}} \frac{1}{\pi^{l_j+1}} \left(\prod_{t=1}^{d} \frac{1}{l_t!} \left(\frac{k_t}{k_j} \right)^{l_t} \right) c_{-a}(j). \quad (2.13)$$

We now compute $\operatorname{Res}_{z=1} f(z)$. For each $j \in \{1, 2, \ldots, d\}$ we know that the Laurent series of $\cot^{(m_j)}(\pi k_j z)$ about 1 is of the form $\sum_{l_j=-(m_j+1)}^{\infty} a_{l_j} (z-1)^{l_j}$, where

$$a_{l_j} = \frac{(2i)^{l_j+m_j+1} (\pi k_j)^{l_j} (l_j+1)^{(m_j)} B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!}.$$

Also, by Taylor's theorem, we have an expansion about 1 of the form

$$\zeta^{(m_0)}(a,z) = \sum_{l_{d+1}=0}^{\infty} a_{l_{d+1}} (z-1)^{l_{d+1}},$$

where

$$a_{l_{d+1}} = \frac{\zeta^{(m_0+l_{d+1})}(a,1)}{l_{d+1}!} = \frac{(-1)^{m_0+l_{d+1}} \zeta(a+m_0+l_{d+1})\, a^{(m_0+l_{d+1})}}{l_{d+1}!}.$$

Hence

$$\operatorname*{Res}_{z=1} f(z) = \sum_{l_{d+1}=0}^{m_1+\cdots+m_d+d-1} \sum_{\substack{l_1+\cdots+l_{d+1}=-1 \\ l_j \geq -(m_j+1)}} \prod_{j=1}^{d+1} a_{l_j}.$$

Denoting $b_m = \frac{(-1)^{m_0}}{m!(-2\pi i)^{m+1}} \zeta(s+m+m_0) a^{(m+m_0)}$, we may write

$$\operatorname*{Res}_{z=1} f(z) = -(2i)^{d+\sum_{j=1}^{d} m_j} \sum_{m=0}^{m_1+\cdots+m_d+d-1} b_m \sum_{\substack{l_1+\cdots+l_d=-(m+1) \\ l_j \geq -(m_j+1)}} \prod_{j=1}^{d} \frac{(l_j+1)^{(m_j)} k_j^{l_j} B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!}.$$

Since $\frac{\partial}{\partial z} \zeta(a,z) = -a\, \zeta(a+1,z)$, we know that

$$\begin{aligned} \zeta^{(m_0)}(a,z+1) &= (-1)^{m_0} \zeta(a+m_0, z+1) a^{(m_0)} \\ &= (-1)^{m_0} a^{(m_0)} \sum_{n=0}^{\infty} \frac{1}{(n+z+1)^{a+m_0}} \\ &= (-1)^{m_0} a^{(m_0)} \left(\zeta(a,z) - \frac{1}{z^{a+m_0}} \right) \\ &= \zeta^{(m_0)}(a,z) - \frac{(-1)^{m_0} a^{(m_0)}}{z^{a+m_0}}. \end{aligned}$$

This means that

$$\int_{1+\epsilon-iM}^{1+\epsilon+iM} f(z)dz + \int_{\epsilon+iM}^{\epsilon-iM} f(z)dz = (-1)^{m_0} a^{(m_0)} \int_{\epsilon+iM}^{\epsilon-iM} \frac{\prod_{j=1}^d \cot^{(m_j)}(\pi k_j z)}{z^{a+m_0}} dz.$$

As in the proof of Theorem 1.1, it follows from Lemmas 2.1 and 2.2 that the integrals along the horizontal segments of $C(M,\epsilon)$ vanish, so the Cauchy residue theorem implies that

$$\sum_{z_0 \in E} \operatorname*{Res}_{z=z_0} f(z) = \frac{(-1)^{m_0} a^{(m_0)}}{2\pi i} \int_{\epsilon+i\infty}^{\epsilon-i\infty} \frac{\prod_{j=1}^d \cot^{(m_j)}(\pi k_j z)}{z^{a+m_0}} dz.$$

The result then follows from (2.13) and the computation of $\operatorname{Res}_{z=1} f(z)$. $\square$

An analogue Theorem 2.3 is valid for generalized Bettin–Conrey sums.

Theorem 2.9. *Let $d \geq 2$ and suppose that $k_1, \dots, k_d$ is a list of pairwise coprime positive integers and $m_0, m_1, \dots, m_d$ are nonnegative integers. If $n > 1$ is an integer and $n + d + \sum_{j=1}^d m_j$ is odd, then*

$$\sum_{j=1}^d (-1)^{m_j} \left(\prod_{\substack{t=1 \\ t \neq j}}^d k_t^{1-(n+m_0)} \right) \sum_{\substack{l_1+\cdots+l_d=m_j \\ l_1,\dots,l_d \geq 0}} \frac{1}{\pi^{l_j+1}} \left(\prod_{t=1}^d \frac{1}{l_t!} \left(\frac{k_t}{k_j} \right)^{l_t} \right) c_{-n}(j)$$

$$= (-1)^{m_0} (2i)^{d+\sum_{j=1}^d m_j} \prod_{j=1}^d k_j^{1-(n+m_0)}$$

$$\times \sum_{m \in \mathbb{Z}} b_m \sum_{\substack{l_1+\cdots+l_d=-(m+1) \\ l_j \geq -(m_j+1)}} \prod_{j=1}^d \frac{(l_j+1)^{(m_j)} k_j^{l_j} B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!},$$

where

$$c_{-n}(j) = c_{-n} \left(\begin{array}{c|ccccc} k_j & k_1 & \cdots & \widehat{k_j} & \cdots & k_d \\ m_0 + l_j & m_1 + l_1 & \cdots & \underbrace{m_j + l_j} & \cdots & m_d + l_d \end{array} \right)$$

and

$$b_m := \begin{cases} \frac{\zeta(n+m+m_0)\, n^{(m+m_0)}}{(-2\pi i)^{m+1} m!} & \text{if } m \in \left\{0, 1, \dots, d-1+\sum_{j=1}^d m_j\right\}, \\ -\frac{(2\pi i)^{m_0+n-1} n^{(m_0)}}{2} & \text{if } m = -(n+m_0), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As in the proof of Theorem 2.3, the contour $\widetilde{C}(M,\epsilon)$ is defined as the positively oriented rectangle with vertices $1+iM$, iM, $-iM$ and $1-iM$, with indentations (to the right) of radius $0 < \epsilon < \min_{1 \leq j \leq d} \left\{ \frac{1}{k_j} \right\}$ around 0 and 1 (see Figure 2.3). Since this closed contour contains the same poles of f as $C(M,\epsilon)$, we may apply Cauchy's residue theorem letting $M \longrightarrow \infty$ and we only need to determine $\lim_{M\to\infty} \int_{\widetilde{C}(M,\epsilon)} f(z)dz$ in order to deduce a reciprocity law for the generalized Bettin–Conrey sums of the form c_{-n}.

Given that $m_0 + n + d + \sum_{j=1}^{d} m_j$ is odd, the function

$$g(z) := \frac{(-1)^{m_0} a^{(m_0)} \prod_{j=1}^{d} \cot^{(m_j)}(\pi k_j z)}{z^{n+m_0}}$$

is odd. Indeed,

$$g(z) = \frac{(-1)^{m_0+d+\sum_{j=1}^{d} m_j} a^{(m_0)}}{(-1)^{n+m_0} z^{n+m_0}} \prod_{j=1}^{d} \cot^{(m_j)}(\pi k_j z) = \frac{(-1)^{d+\sum_{j=1}^{d} m_j}}{(-1)^{n+m_0}} g(z) = -g(z).$$

Let $\gamma(M, \epsilon)$ be the indentation around zero along $C(M, \epsilon)$, then

$$\begin{aligned} \lim_{M\to\infty} \int_{\widetilde{C}(M,\epsilon)} f(z)dz &= \lim_{M\to\infty} \left(\int_{iM}^{\epsilon i} g(z)dz + \int_{\epsilon i}^{iM} g(z)dz \right) + \int_{\gamma(M,\epsilon)} g(z)dz \\ &= \int_{\gamma(M,\epsilon)} g(z)dz. \end{aligned}$$

Given that g is odd, the Cauchy residue theorem implies that

$$\int_{\gamma(M,\epsilon)} g(z)dz = -\pi i \operatorname*{Res}_{z=0} g(z)$$

and it follows that

$$\sum_{z_0\in E} \operatorname*{Res}_{z=z_0} f(z) = -\frac{1}{2} \operatorname*{Res}_{z=0} g(z).$$

For each $j \in \{1, 2, \ldots, d\}$ we have an expansion of the form

$$\cot^{(m_j)}(\pi k_j z) \sum_{l_j=-(m_j+1)}^{\infty} \frac{(2i)^{l_j+m_j+1} (\pi k_j)^{l_j} (l_j+1)^{(m_j)} B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!} z^{l_j},$$

so $\operatorname{Res}_{z=0} g(z)$ is given by

$$(-1)^{m_0} a^{(m_0)} \sum_{\substack{l_1+\cdots l_d = n+m_0-1 \\ l_j \geq -(m_j+1)}} (2\pi i)^{n+m_0-1} (2i)^{d+\sum_{j=1}^{d} m_j} \prod_{j=1}^{d} \frac{k_j^{l_j} (l_j+1)^{(m_j)} B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!},$$

which concludes our proof. □

From Theorems 2.8 and 2.9, we deduce a computation of the integral

$$\int_{\epsilon+i\infty}^{\epsilon-i\infty} \frac{\prod_{j=1}^{d} \cot^{(m_j)}(\pi k_j z)}{z^{n+m_0}} dz$$

in terms of the sequences $B(m_j)_{l_j}$, whenever $n \in \mathbb{Z}_{>1}$ and $n + d + \sum_{j=1}^{d} m_j$ is odd, which generalizes Corollary 2.4.

Corollary 2.10. *Let $d \geq 2$ and suppose that $k_1, \ldots, k_d$ is a list of pairwise coprime positive integers and $m_0, m_1, \ldots, m_d$ are nonnegative integers. If $n > 1$ is an integer and*

$$n + d + \sum_{j=1}^{d} m_j$$

is odd, then for all $0 < \epsilon < \min_{1 \leq j \leq d} \left\{ \frac{1}{k_j} \right\}$,

$$\int_{\epsilon+i\infty}^{\epsilon-i\infty} \frac{\prod_{j=1}^{d} \cot^{(m_j)}(\pi k_j z)}{z^{n+m_0}} dz = \frac{(2\pi i)^{n+m_0} (2i)^{d-1+\sum_{j=1}^{d} m_j}}{i} \times \sum_{\substack{l_1+\cdots+l_d=n+m_0-1 \\ l_j \geq -(m_j+1)}} \prod_{j=1}^{d} \frac{(l_j+1)^{(m_j)} k_j^{l_j} B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!}.$$

The phrasing of Theorems 2.8, 2.9 and Corollary 2.10 is particularly aesthetically pleasing in the case $m_0 = 0$.

Corollary 2.11. *Let $d \geq 2$ and $0 < \epsilon < \min_{1 \leq j \leq d} \left\{ \frac{1}{k_j} \right\}$. Suppose that $k_1, \ldots, k_d$ is a list of pairwise coprime positive integers and $m_1, \ldots, m_d$ are nonnegative integers. For $a \in \mathbb{C}$, denote*

$$c_{-a}(j) = c_{-a} \left(\begin{array}{c|ccccc} k_j & k_1 & \cdots & \widehat{k_j} & \cdots & k_d \\ l_j & m_1 + l_1 & \cdots & \widehat{m_j + l_j} & \cdots & m_d + l_d \end{array} \right).$$

(i) If a is a complex number such that $\Re(a) > 1$, then

$$\sum_{j=1}^{d} (-1)^{m_j} \left(\prod_{\substack{t=1 \\ t \neq j}}^{d} k_t^{1-a} \right) \sum_{\substack{l_1+\cdots+l_d=m_j \\ l_1,\ldots,l_d \geq 0}} \frac{1}{\pi^{l_j+1}} \left(\prod_{t=1}^{d} \frac{1}{l_t!} \left(\frac{k_t}{k_j} \right)^{l_t} \right) c_{-a}(j)$$
$$= \prod_{j=1}^{d} k_j^{1-a} \Bigg\{ \frac{1}{2\pi i} \int_{\epsilon+i\infty}^{\epsilon-i\infty} \frac{\prod_{j=1}^{d} \cot^{(m_j)}(\pi k_j z)}{z^a} dz + (2i)^{d+\sum_{j=1}^{d} m_j} \times \sum_{m=0}^{m_1+\cdots+m_d+d-1} b_m \sum_{\substack{l_1+\cdots+l_d=-(m+1) \\ l_j \geq -(m_j+1)}} \prod_{j=1}^{d} \frac{(l_j+1)^{(m_j)} k_j^{l_j} B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!} \Bigg\},$$

where $b_m = \left(\frac{-1}{2\pi i}\right)^{m+1} \binom{a+m-1}{m} \zeta(a+m)$ for $m \in \mathbb{Z}_{\geq 0}$.

(ii) If $n \in \mathbb{Z}_{>1}$ and $n + d + \sum_{j=1}^{d} m_j$ is odd, then

$$\int_{\epsilon+i\infty}^{\epsilon-i\infty} \frac{\prod_{j=1}^{d} \cot^{(m_j)}(\pi k_j z)}{z^n} dz = \frac{(2\pi i)^n (2i)^{d-1+\sum_{j=1}^{d} m_j}}{i} \times \sum_{\substack{l_1+\cdots+l_d=n-1 \\ l_j \geq -(m_j+1)}} \prod_{j=1}^{d} \frac{(l_j+1)^{(m_j)} k_j^{l_j} B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!}.$$

In addition,

$$\left(\prod_{j=1}^{d} k_j^{1-n}\right)(2i)^{d+\sum_{j=1}^{d} m_j}\sum_{m\in\mathbb{Z}} b_m \sum_{\substack{l_1+\cdots+l_d=-(m+1)\\ l_j\geq -(m_j+1)}} \prod_{j=1}^{d} \frac{(l_j+1)^{(m_j)} k_j^{l_j} B(m_j)_{l_j+m_j+1}}{(l_j+m_j+1)!}$$
$$= \sum_{j=1}^{d}(-1)^{m_j}\left(\prod_{\substack{t=1\\ t\neq j}}^{d} k_t^{1-n}\right)\sum_{\substack{l_1+\cdots+l_d=m_j\\ l_1,\ldots,l_d\geq 0}} \frac{1}{\pi^{l_j+1}}\left(\prod_{t=1}^{d}\frac{1}{l_t!}\left(\frac{k_t}{k_j}\right)^{l_t}\right) c_{-n}(j),$$

where

$$b_m := \begin{cases} \left(\frac{-1}{2\pi i}\right)^{m+1}\binom{n+m-1}{m}\zeta(n+m) & \text{if } m\in\left\{0,1,\ldots,d-1+\sum_{j=1}^{d} m_j\right\},\\ -(2\pi i)^{n-1}/2 & \text{if } m=-n, \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

The consideration of the case $m_0 = 0$ also leads to the definition of *higher-dimensional Bettin–Conrey sums*,

$$c_a(k_0;k_1,\ldots,k_n) := c_a\left(\begin{array}{c|ccc} k_0 & k_1 & \cdots & k_n\\ 0 & 0 & \cdots & 0\end{array}\right) = k_0^a\sum_{m=1}^{k_0-1}\zeta\left(-a,\frac{m}{k_0}\right)\prod_{l=1}^{n}\cot\left(\frac{\pi k_l m}{k_0}\right),$$

for $a \neq -1$ complex and $k_0, k_1, \ldots, k_n$ a list of positive numbers such that $(k_0, k_j) = 1$ for each $j \neq 0$.

Of course, higher-dimensional Bettin–Conrey sums satisfy Corollary 2.11. In particular, if $0 < \epsilon < \min_{1\leq l\leq d}\left\{\frac{1}{k_l}\right\}$, $\Re(a) > 1$ and $k_1, \ldots, k_d$ is a list of pairwise coprime positive integers, then

$$\sum_{j=1}^{d}\left(\prod_{\substack{l=1\\ l\neq j}}^{d} k_l^{1-a}\right) c_{-a}(k_j;k_1,\ldots,\widehat{k_j},\ldots,k_d)$$
$$= K_a(k_1,\ldots,k_d) + \frac{1}{2i\prod_{l=1}^{d} k_l^{a-1}}\int_{\epsilon+i\infty}^{\epsilon-i\infty}\frac{\prod_{l=1}^{d}\cot(\pi k_l z)}{z^a}dz,$$

where

$$K_a(k_1,\ldots,k_d) = \frac{(2i)^d\pi}{(d-1)!\prod_{l=1}^{d} k_l^a}\sum_{m=0}^{d-1}\sum_{\substack{m_1+\cdots+m_d+m=d-1\\ m_1,\ldots,m_d\geq 0}}\left(\frac{-1}{2\pi i}\right)^{m+1}$$
$$\times\binom{a+m-1}{m}\binom{d-1}{m_1,\ldots,m_d}\left(\prod_{l=1}^{d} k_l^{m_l} B_{m_l}\right)\zeta(a+m),$$

with the convention $B_1 := 0$. Note that unlike in the two-dimensional case, the convention $B_1 := 0$ cannot be dropped. Indeed, consider the example $d = 3$ and the factor $k_1k_2k_3B_1^3$ in the product $\prod_{l=1}^{d} k_l^{m_l} B_{m_l}$.

Chapter 3

Inflated Eulerian Polynomials

3.1 Lecture Hall Sequences and $Q_n^{(\mathbf{s})}(x)$

Lecture hall partitions were introduced in 1997 by Bousquet-Mélou and Eriksson [11]. They defined a *lecture hall partition* into n parts as a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying the inequalities

$$0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \cdots \le \frac{\lambda_n}{n}. \tag{3.1}$$

Note that this condition ensures that $\lambda_i \le \lambda_{i+1}$ for $1 \le i \le n-1$. Furthermore, if we order the parts $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of a lecture hall partition from left to right in a diagram (see Figure 3.1 for an example), we obtain a figure resembling a lecture hall. Indeed, (3.1) is a sufficient condition to allow students (A, B, C and D in the picture) in each row (part) to see the professor (O in the picture). This fact led to the name of these restricted partitions.

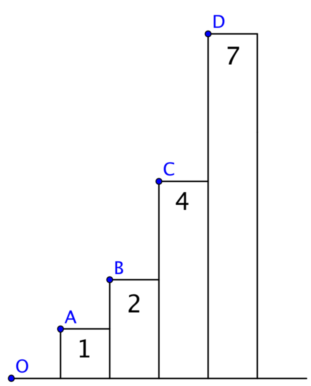

Figure 3.1: The lecture hall partition $14 = 1 + 2 + 4 + 7$.

Example 3.1. The partition $14 = 1 + 2 + 4 + 7$ is a lecture hall partition because

$$\frac{1}{1} \le \frac{2}{2} \le \frac{4}{3} \le \frac{7}{4}.$$

Note that a partition into 4 parts such that $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 4$ must satisfy $\frac{4}{3} \le \frac{\lambda_4}{4}$ in order to be a lecture hall partition, so necessarily, $\lambda_4 \ge 6$. Thus, $12 = 1+2+4+5$ is not a lecture hall partition.

The most remarkable result about lecture hall partitions is the Lecture Hall Theorem, due to Bousquet-Mélou and Eriksson [11]. This theorem relates lecture hall partitions to partitions into bounded odd parts.

Theorem 3.1 (Bousquet-Mélou and Eriksson)**.** *For n fixed, the generating function for the number $LH(N,n)$ of lecture hall partitions of N into n parts is given by*

$$\sum_{N=0}^{\infty} LH(N,n)q^N = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}.$$

That is, it coincides with the generating function of integer partitions into odd parts less than $2n$.

The reader interested in a bijective proof of this result is referred to [21, 36].

Since lecture hall partitions have distinct parts, we may think of this result as a finite version of Euler's theorem asserting that the number of partitions of N into distinct parts coincides with the number of partitions of N into odd parts.

A natural generalization of lecture hall partitions is to consider sequences

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

of positive integers such that

$$0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n},$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ is a sequence of positive integers. These $\mathbf{s}$-*lecture hall sequences* were introduced by Bousquet-Mélou and Eriksson for nondecreasing $\mathbf{s}$ in [12] and by Savage and others for arbitrary positive integer sequences in [15–17, 32].

Definition 3.1. Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ be a sequence of positive integers. We define the $\mathbf{s}$-*lecture hall cone* to be

$$C_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n} \right\}.$$

This cone has *generators* $\{v_i = (0, \ldots, 0, s_i, \ldots, s_n) : 1 \le i \le n\}$ and its *fundamental half-open parallelepiped* is given by

$$\Pi_n^{(\mathbf{s})} = \left\{ \sum_{i=1}^{n} \alpha_i v_i \mid 0 \le \alpha_i < 1 \right\}.$$

Example 3.2. Let $\mathbf{r} = (3,2)$, then there are six $\mathbf{r}$-lecture hall sequences in the fundamental half-open parallelepiped of $\mathbf{r}$. Indeed, Figure 3.4 shows that

$$\Pi_2^{(3,2)} \cap \mathbb{Z}^2 = \{(0,0), (0,1), (1,1), (1,2), (2,2), (2,3)\}.$$

In general, for a positive sequence $\mathbf{s}$ it is true that $\left|\Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n\right| = s_1 s_2 \cdots s_n$.

Note that each **s**-lecture hall sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an element of $C_n^{(\mathbf{s})}$ when considered as an ordered n-dimensional vector. For instance, $(1, 2)$ and $(2, 1)$ are points of the $(5, 2)$-lecture hall cone and, by definition, they are distinct $(5, 2)$-lecture hall sequences of 3 (see Figure 3.2). However, if **s** is nondecreasing, then the points in $C_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ are actually partitions on the traditional sense, in that the sequence λ is nondecreasing, so no permutation of its coordinates appears more than once. For this reason, some authors [12, 15–17, 32] refer to **s**-lecture hall sequences as **s**-lecture hall partitions, although in general they are compositions.

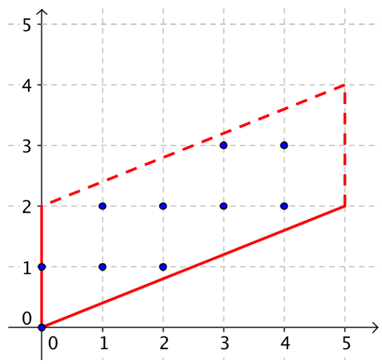

Figure 3.2: The fundamental half-open parallelepiped of $C_2^{(5,2)}$.

Definition 3.2. Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ be a sequence of positive integers. We define the *n-th inflated* **s**-*Eulerian polynomial* by

$$Q_n^{(\mathbf{s})}(x) = \sum_{\lambda \in \Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n}.$$

Example 3.3. The sequence $\mathbf{s} = (5, 3)$ has **s**-lecture hall cone

$$C_2^{(5,3)} = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 : 0 \leq \frac{\lambda_1}{5} \leq \frac{\lambda_2}{3} \right\},$$

with generators $[5, 3]$ and $[0, 3]$. There are $s_1 \cdot s_2 = 15$ **s**-lecture hall sequences in the fundamental half-open parallelepiped (see Figure 3.3), namely, the elements of $\Pi_2^{(5,3)} \cap \mathbb{Z}^2$ are $(0, 0)$, $(0, 1)$, $(1, 1)$, $(0, 2)$, $(1, 2)$, $(2, 2)$, $(3, 2)$, $(1, 3)$, $(2, 3)$, $(3, 3)$, $(4, 3)$, $(2, 4)$, $(3, 4)$, $(4, 4)$ and $(4, 5)$. Therefore, the inflated **s**-Eulerian polynomial is given by

$$Q_2^{(5,3)}(x) = \sum_{(\lambda_1, \lambda_2) \in \Pi_2^{(5,3)} \cap \mathbb{Z}^2} x^{\lambda_2} = 1 + 2x + 4x^2 + 4x^3 + 3x^4 + x^5.$$

The following example illustrates that the inflated **s**-Eulerian polynomial depends not only on the positive integers in a sequence, but also on the order in which they appear.

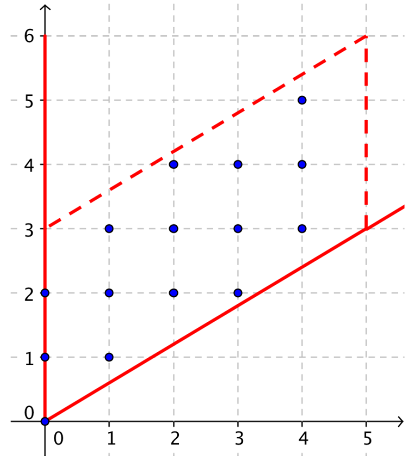

Figure 3.3: The cone $C_2^{(5,3)}$ and its fundamental half-open parallelepiped.

Example 3.4. Let $\mathbf{s} = (2,3)$ and $\mathbf{r} = (3,2)$. These sequences have lecture hall cones

$$C_2^{(\mathbf{s})}=\left\{(\lambda_1,\lambda_2) \in \mathbb{R}^2 : 0 \leq \frac{\lambda_1}{2} \leq \frac{\lambda_2}{3}\right\} \quad \text{and} \quad C_2^{(\mathbf{r})}=\left\{(\lambda_1,\lambda_2) \in \mathbb{R}^2 : 0 \leq \frac{\lambda_1}{3} \leq \frac{\lambda_2}{2}\right\},$$

respectively (see Figure 3.4). Since $s_1 \cdot s_2 = 6 = r_1 \cdot r_2$, there are six $\mathbf{s}$-lecture hall (resp. $\mathbf{r}$-lecture hall) sequences in the fundamental parallelepiped $\Pi_2^{(\mathbf{s})}$ (resp. $\Pi_2^{(\mathbf{r})}$), which implies that $Q_2^{(\mathbf{s})}(1) = 6 = Q_2^{(\mathbf{r})}(1)$. However, the inflated Eulerian polynomials of these sequences do not coincide, for $Q_2^{(\mathbf{s})}(x) = 1 + x + 2x^2 + x^3 + x^4$ and $Q_2^{(\mathbf{r})}(x) = 1 + 2x + 2x^2 + x^3$.

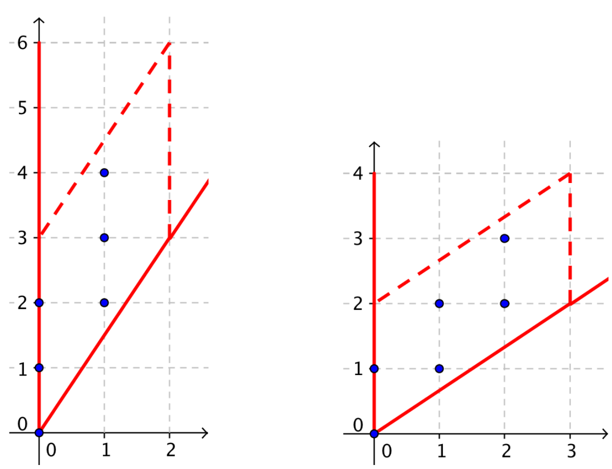

Figure 3.4: Comparison between $\Pi_2^{(2,3)}$ and $\Pi_2^{(3,2)}$.

3.2 An Alternative Description of $Q_n^{(\mathbf{s})}(x)$

The definition of the inflated $\mathbf{s}$-Eulerian polynomial is intuitive. However, we now introduce an alternative description of $Q_n^{(\mathbf{s})}(x)$, due to Pensyl and Savage [27], that is more convenient for our purposes.

Definition 3.3. Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ be a finite sequence of positive integers. We define the $\mathbf{s}$-*inversion sequences* as

$$I_n^{(\mathbf{s})} = \{\mathbf{e} = (e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

Given $\mathbf{e} \in I_n^{(\mathbf{s})}$, we say $1 \leq i < n$ is an *ascent* of $\mathbf{e}$ if $\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}}$. We say 0 is an ascent of $\mathbf{e}$ if $e_1 > 0$.

It is customary to denote the set of ascents of $\mathbf{e} \in I_n^{(\mathbf{s})}$ by $\mathrm{Asc}(\mathbf{e})$ and its cardinality by $\mathrm{asc}(\mathbf{e})$. We denote the collection $\left\{\mathrm{Asc}(\mathbf{e}) \mid \mathbf{e} \in I_n^{(\mathbf{s})}\right\}$ by $\mathrm{Asc}_n^{(\mathbf{s})}$.

Example 3.5. Consider the sequence $\mathbf{s} = (1, 1, 2, 2)$. By definition, the $\mathbf{s}$-inversion sequences are given by $I_4^{(1,1,2,2)} = \{(0,0,0,0), (0,0,0,1), (0,0,1,0), (0,0,1,1)\}$. Note that $3 \notin \mathrm{Asc}(0,0,1,1)$, because $\frac{1}{2} \not< \frac{1}{2}$. However, $\frac{0}{1} < \frac{1}{2}$ implies that 2 is an element of $\mathrm{Asc}(0,0,1,1)$.

We now state Pensyl and Savage's result. It leads to a description of $Q_n^{(\mathbf{s})}(x)$ in terms of the ascents of the $\mathbf{s}$-inversion sequences $I_n^{(\mathbf{s})}$.

Theorem 3.2 (Pensyl–Savage [27])**.** *Let* $\mathbf{s}$ *be a sequence of positive integers. Then*

$$\sum_{\lambda \in C_n^{(\mathbf{s})} \cap \mathbb{Z}^n} x^{\lambda_n} = \frac{\sum_{\mathbf{e} \in I_n^{(\mathbf{s})}} x^{\mathrm{asc}\,(\mathbf{e}) - e_n}}{(1 - x^{s_n})^n}. \tag{3.2}$$

Combining Theorem 3.2 with (1.6) we obtain the following corollary.

Corollary 3.3. *Let* $\mathbf{s}$ *be a sequence of positive integers, then*

$$Q_n^{(\mathbf{s})}(x) = \sum_{\mathbf{e} \in I_n^{(\mathbf{s})}} x^{s_n \mathrm{asc}(\mathbf{e}) - e_n}.$$

Example 3.6. Consider the sequence $\mathbf{s} = (1, 2, 3)$. The $\mathbf{s}$-inversion sequences

$$I_3^{(1,2,3)} = \{(0,0,0), (0,0,1), (0,0,2), (0,1,0), (0,1,1), (0,1,2)\}$$

have ascent sets $\mathrm{Asc}_3^{(1,2,3)} = \{\emptyset, \{2\}, \{2\}, \{1\}, \{1\}, \{1,2\}\}$. By Corollary 3.3,

$$\begin{aligned} Q_3^{(1,2,3)}(x) &= x^{3(0)-0} + x^{3(1)-1} + x^{3(1)-2} + x^{3(1)-0} + x^{3(1)-1} + x^{3(2)-2} \\ &= x^0 + x^2 + x + x^3 + x^2 + x^4 \\ &= x^4 + x^3 + 2x^2 + x + 1. \end{aligned}$$

3.3 Contractible Sequences

In this section, we introduce the notion of contractibility for a sequence $\mathbf{s}$ of positive integers. We begin by showing that for any sequence $\mathbf{s}$ and any n, the expression

$$\frac{Q_n^{(\mathbf{s})}(x)}{(1 + x + \cdots + x^{s_n-1})}$$

is a polynomial with integer coefficients. Furthermore, we provide a convenient combinatorial description of this polynomial.

Given a sequence $\mathbf{e} = (e_1, e_2, \ldots, e_{n-1}) \in I_{n-1}$ and $0 \leq k < s_n$, we denote

$$(e_1, e_2, \ldots, e_{n-1}, k) \in I_n$$

by $(\mathbf{e}, k)$.

Theorem 3.4. *Let* $\mathbf{s}$ *be a sequence of positive integers, then*

$$\frac{Q_n^{(\mathbf{s})}(x)}{1 + x + \cdots + x^{s_n-1}} = \sum_{\mathbf{e} \in I_{n-1}^{(\mathbf{s})}} x^{s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor}.$$

Proof. Let $\mathbf{s}$ be a sequence of positive integers and $\mathbf{e} \in I_{n-1}^{(\mathbf{s})}$. Let a be the smallest positive integer such that $\frac{e_{n-1}}{s_{n-1}} < \frac{a}{s_n}$, then $a - 1 \leq \frac{s_n e_{n-1}}{s_{n-1}} < a$, so $\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = a - 1$. Note that $\frac{e_{n-1}}{s_{n-1}} < 1$ implies that $a \leq s_n$. Write

$$\begin{aligned}
\left(\sum_{k=0}^{s_n-1} x^k\right) x^{s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor} &= \left(\sum_{k=0}^{s_n-1} x^k\right) x^{s_n \operatorname{asc}(\mathbf{e}) - a + 1} \\
&= \left(\sum_{k=0}^{a-1} x^{s_n \operatorname{asc}(\mathbf{e}) - (a-k-1)}\right) + \left(\sum_{k=a}^{s_n-1} x^{s_n \operatorname{asc}(\mathbf{e}) - (a-k-1)}\right). \qquad (3.3)
\end{aligned}$$

If $a = s_n$, the rightmost sum is empty. Since $\frac{e_{n-1}}{s_{n-1}} \geq \frac{k}{s_n}$ for $0 \leq k \leq a - 1$,

$$\begin{aligned}
\sum_{k=0}^{a-1} x^{s_n \operatorname{asc}(\mathbf{e}) - (a-k-1)} &= x^{s_n \operatorname{asc}(\mathbf{e}) - 0} + x^{s_n \operatorname{asc}(\mathbf{e}) - 1} + \cdots + x^{s_n \operatorname{asc}(\mathbf{e}) - (a-1)} \\
&= x^{s_n \operatorname{asc}(\mathbf{e},0) - 0} + x^{s_n \operatorname{asc}(\mathbf{e},1) - 1} + \cdots + x^{s_n \operatorname{asc}(\mathbf{e},a-1) - (a-1)}.
\end{aligned}$$

Similarly, the fact that $\frac{e_{n-1}}{s_{n-1}} < \frac{k}{s_n}$ for $a \leq k \leq s_n - 1$ implies that

$$\begin{aligned}
\sum_{k=a}^{s_n-1} x^{s_n \operatorname{asc}(\mathbf{e}) - (a-k-1)} &= x^{s_n \operatorname{asc}(\mathbf{e}) + 1} + x^{s_n \operatorname{asc}(\mathbf{e}) + 2} + \cdots + x^{s_n \operatorname{asc}(\mathbf{e}) + s_n - a} \\
&= x^{s_n(1 + \operatorname{asc}(\mathbf{e})) - (s_n - 1)} + x^{s_n(1 + \operatorname{asc}(\mathbf{e})) - (s_n - 2)} + \cdots + x^{s_n(1 + \operatorname{asc}(\mathbf{e})) - a} \\
&= x^{s_n \operatorname{asc}(\mathbf{e}, s_n - 1) - (s_n - 1)} + x^{s_n \operatorname{asc}(\mathbf{e}, s_n - 2) - (s_n - 2)} + \cdots + x^{s_n \operatorname{asc}(\mathbf{e}, a) - a}.
\end{aligned}$$

By (3.3), we deduce that for each $\mathbf{e} \in I_{n-1}^{(\mathbf{s})}$,

$$\left(\sum_{k=0}^{s_n-1} x^k\right) x^{s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor} = \sum_{k=0}^{s_n-1} x^{s_n \operatorname{asc}(\mathbf{e},k)-k}, \tag{3.4}$$

so we may write

$$\left(1 + x + \cdots + x^{s_n-1}\right) \sum_{\mathbf{e} \in I_{n-1}^{(\mathbf{s})}} x^{s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor} = \sum_{\mathbf{e} \in I_n^{(\mathbf{s})}} x^{s_n \operatorname{asc}(\mathbf{e}) - e_n} = Q_n^{(\mathbf{s})}(x).$$

Note that the rightmost equality follows from Corollary 3.3. $\square$

The following corollary is an immediate consequence of Theorem 3.4.

Corollary 3.5 (Chung–Graham [14]). *Let $Q_n^{(1,2,\ldots,n)}(x)$ be the inflated $[n]$-Eulerian polynomial, then*

$$\frac{Q_n^{(1,2,\ldots,n)}(x)}{1 + x + \cdots + x^{n-1}} = \sum_{\mathbf{e} \in I_{n-1}} x^{n \cdot \operatorname{asc}(\mathbf{e}) - e_{n-1}}.$$

To see that Corollary 3.5 is equivalent to Proposition 1.2, consider the mapping

$$\phi : S_n \to I_n$$

defined by $\phi(\pi) = (e_1, \ldots, e_n)$, where $e_i = |\{j > 0 \mid j < i \text{ and } \pi_j > \pi_i\}|$. Then $\operatorname{Des} \pi = \operatorname{Asc}(\phi(\pi))$ and $e_n = n - \pi_n$.

It follows from Corollaries 3.3 and 3.5 that the sequence of nonzero coefficients of

$$\frac{Q_n^{(1,2,\ldots,n)}(x)}{(1 + x + \cdots + x^{n-1})}$$

and $Q_{n-1}^{(1,2,\ldots,n-1)}(x)$ coincide for all n. Computational experiments show that this property is common among positive sequences $\mathbf{s}$, at least for small n. This fact motivates the next definition.

Definition 3.4. If $n \geq 3$, we say a positive sequence $\mathbf{s}$ is *n-contractible* if the sequence of nonzero coefficients of $Q_{n-1}^{(\mathbf{s})}(x)$ and $Q_n^{(\mathbf{s})}(x) / \left(1 + x + \cdots + x^{s_n-1}\right)$ coincide. If a sequence $\mathbf{s}$ is n-contractible for $n \geq 3$, then $\mathbf{s}$ is *contractible*.

Example 3.7. Let $\mathbf{s}$ be the Fibonacci sequence. Using Corollary 3.3, it is possible to compute $Q_{n-1}^{(\mathbf{s})}(x)$ and $Q_n^{(\mathbf{s})}(x) / \left(1 + x + \cdots + x^{s_n-1}\right)$ for the first few n (see Example 1.1), and verify that the Fibonacci sequence is n-contractible for $n = 3, 4, 5, 6$.

The following corollary shows that all positive constant sequences are contractible. Furthermore, the polynomials $Q_{n-1}^{(\mathbf{s})}(x)$ and $Q_n^{(\mathbf{s})}(x) / \left(1 + x + \cdots + x^{s_n-1}\right)$ coincide if $\mathbf{s}$ is constant.

Corollary 3.6. *Let* $\mathbf{s}$ *be a positive constant sequence, then for all* $n \geq 2$

$$Q_{n-1}^{(\mathbf{s})}(x) = \frac{Q_n^{(\mathbf{s})}(x)}{1 + x + \cdots + x^{s_n - 1}}.$$

Proof. Say $s_n = k$ for all n, then Theorems 3.2 and 3.4 imply that

$$\frac{Q_n^{(\mathbf{s})}(x)}{1 + x + \cdots + x^{s_n-1}} = \sum_{\mathbf{e} \in I_{n-1}^{(\mathbf{s})}} x^{s_n \mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor} = \sum_{\mathbf{e} \in I_{n-1}^{(\mathbf{s})}} x^{k \cdot \mathrm{asc}(\mathbf{e}) - e_{n-1}} = Q_{n-1}^{(\mathbf{s})}(x)\,. \qquad \square$$

3.4 The Case of Nondecreasing Sequences

In this section, we prove that all nondecreasing sequences are contractible. As we mentioned before, this case is of particular interest because for such $\mathbf{s}$, the set $C_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ consists of the $\mathbf{s}$-lecture hall sequences with n parts, all of which are partitions.

Lemma 3.7. *Let* $n \geq 3$. *An* n*-contractible sequence* $\mathbf{s}$ *is one such that for all* $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$,

$$s_{n-1}\mathrm{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1} \textit{ if and only if } s_n\mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n\mathrm{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor. \quad (3.5)$$

Proof. By Corollary 3.3 and Theorem 3.4, we know that for any sequence $\mathbf{s}$ of positive integers

$$Q_{n-1}^{(\mathbf{s})}(x) = \sum_{\mathbf{e} \in I_{n-1}^{(\mathbf{s})}} x^{s_{n-1}\mathrm{asc}(\mathbf{e}) - e_{n-1}} \text{ and,}$$

$$\frac{Q_n^{(\mathbf{s})}(x)}{1 + x + \cdots + x^{s_n-1}} = \sum_{\mathbf{e} \in I_{n-1}^{(\mathbf{s})}} x^{s_n \mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor}. \qquad \square$$

We now show that the forward direction of (3.5) holds for arbitrary positive sequences $\mathbf{s}$ and any $n \geq 3$.

Lemma 3.8. *Let* $\mathbf{s}$ *be a sequence of positive integers and* $n \geq 3$. *Suppose* $\mathbf{e}$ *and* $\overline{\mathbf{e}}$ *are elements of* $I_{n-1}^{(\mathbf{s})}$ *such that*

$$s_{n-1}\mathrm{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1},$$

then

$$s_n\mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n\mathrm{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor.$$

Proof. Let $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$ and suppose that $s_{n-1}\mathrm{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}$. Without loss of generality, say $e_{n-1} \geq \overline{e}_{n-1}$ and write

$$s_{n-1}\mathrm{asc}(\mathbf{e}) = s_{n-1}\mathrm{asc}(\overline{\mathbf{e}}) + (e_{n-1} - \overline{e}_{n-1}).$$

This means that $e_{n-1} - \overline{e}_{n-1} \equiv 0 \bmod s_{n-1}$, but

$$0 \leq e_{n-1} - \overline{e}_{n-1} < s_{n-1} - \overline{e}_{n-1} \leq s_{n-1},$$

so it must be the case that $e_{n-1} = \overline{e}_{n-1}$ and this forces $\mathrm{asc}(\mathbf{e}) = \mathrm{asc}(\overline{\mathbf{e}})$. Hence,

$$s_n \mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n \mathrm{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor. \qquad \square$$

By Lemmas 3.7 and 3.8, we have the following characterization of n-contractible sequences.

Lemma 3.9. *Let* $\mathbf{s}$ *be a sequence of positive integers and* $n \geq 3$*. Then* $\mathbf{s}$ *is* n*-contractible if and only if whenever* $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$ *satisfy*

$$s_n \mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n \mathrm{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor,$$

then

$$s_{n-1} \mathrm{asc}(\mathbf{e}) - e_{n-1} = s_{n-1} \mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}.$$

We now show that $s_n \geq s_{n-1}$ implies that $\mathbf{s}$ is n-contractible, and hence nondecreasing sequences are contractible.

Proposition 3.10. *Let* $\mathbf{s}$ *be a sequence of positive integers and* $n \geq 3$*. If* $s_n \geq s_{n-1}$*, then* $\mathbf{s}$ *is* n*-contractible. Therefore, nondecreasing sequences are contractible.*

Proof. Let $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$ be such that

$$s_n \mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n \mathrm{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor. \tag{3.6}$$

Since $0 \leq e_{n-s}, \overline{e}_{n-1} < s_{n-1}$, we know that the inequalities

$$0 \leq \frac{s_n e_{n-1}}{s_{n-1}}, \frac{s_n \overline{e}_{n-1}}{s_{n-1}} < s_n$$

hold. Thus

$$0 \leq \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor, \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor \leq s_n - 1.$$

By (3.6), we know that $\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor$ and $\left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor$ are congruent modulo s_n, so we deduce that

$$\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor,$$

and therefore $\mathrm{asc}(\mathbf{e}) = \mathrm{asc}(\overline{\mathbf{e}})$. Assume that

$$s_{n-1} \mathrm{asc}(\mathbf{e}) - e_{n-1} \neq s_{n-1} \mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1},$$

then it must be that $e_{n-1} \neq \overline{e}_{n-1}$. Say $e_{n-1} > \overline{e}_{n-1}$, then

$$\left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor \leq \left\lfloor \frac{s_n(\overline{e}_{n-1}+1)}{s_{n-1}} \right\rfloor \leq \cdots \leq \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor.$$

Thus,

$$\left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} + \frac{s_n}{s_{n-1}} \right\rfloor \geq \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} + 1 \right\rfloor = \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor + 1,$$

which is a contradiction. We conclude that $s_{n-1}\mathrm{asc}(\mathbf{e}) - e_{n-1} = s_{n-1}\mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}$. The result then follows from Lemma 3.9. □

Remark. Note that in the preceding proof the condition $s_n \geq s_{n-1}$ is not used to conclude that

$$\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor.$$

Therefore, for any $\mathbf{s}$, if $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$ satisfy

$$s_n\mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n\mathrm{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor,$$

then $\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n\overline{e}_{n-1}}{s_{n-1}} \right\rfloor$ and $\mathrm{asc}(\mathbf{e}) = \mathrm{asc}(\overline{\mathbf{e}})$.

3.5 A Characterization of Contractible Sequences

Although the criterion for contractibility provided by Proposition 3.10 requires relatively weak conditions on $\mathbf{s}$, there do exist noncontractible sequences. Indeed, the next example exhibits an infinite family of noncontractible sequences.

Example 3.8. Consider the finite sequence $(1, 7, 2)$. Corollary 3.3 and Theorem 3.4 imply that

$$Q_2^{(1,7)}(x) = \mathbf{1}x^6 + \mathbf{1}x^5 + \mathbf{1}x^4 + \mathbf{1}x^3 + \mathbf{1}x^2 + \mathbf{1}x + \mathbf{1}, \text{ and}$$
$$\frac{Q_3^{(1,7,2)}(x)}{1+x} = \mathbf{3}x^2 + \mathbf{3}x + \mathbf{1}.$$

Therefore, any $\mathbf{s}$ such that $(s_1, s_2, s_3) = (1, 7, 2)$ is not contractible.

By Proposition 3.10, we know that all nondecreasing sequences are contractible. Furthermore, this proposition implies that if $\mathbf{s}$ is a sequence of positive integers such that $(s_n)_{n=3}^{\infty}$ is nondecreasing, then $\mathbf{s}$ is contractible. The next example shows that this sufficient criterion is not necessary.

Example 3.9. Let $\mathbf{s}$ be the sequence defined by

$$s_n = \begin{cases} 1 & \text{if } n \in \{1, 2\}, \\ 3 & \text{if } n = 3 \text{ and}, \\ 2 & \text{if } n \geq 4. \end{cases}$$

Using Corollary 3.3, we find that

$$Q_3^{(\mathbf{s})}(x) = \mathbf{1}x^2 + \mathbf{1}x + \mathbf{1}, \text{ and}$$
$$Q_2^{(\mathbf{s})}(x) = \mathbf{1}.$$

On the other hand, Theorem 3.4 implies that

$$\frac{Q_4^{(\mathbf{s})}(x)}{1+x} = \mathbf{1}x^2 + \mathbf{1}x + \mathbf{1} \text{ and}$$
$$\frac{Q_3^{(\mathbf{s})}(x)}{1+x+x^2} = \mathbf{1}.$$

Suppose that $n > 4$, then $s_n \geq s_{n-1}$ and it follows from Proposition 3.10 that $\mathbf{s}$ is n-contractible. We see that although $s_4 < s_3$, the sequence $\mathbf{s}$ is contractible.

We now conclude the proof of Theorem 1.3, which we recall.

Theorem 2.2. *A sequence* $\mathbf{s}$ *of positive integers is contractible if and only if either* $(s_n)_{n=3}^{\infty}$ *is nondecreasing, or there exists* $N \geq 3$ *such that* $(s_n)_{n=N}^{\infty}$ *is nondecreasing,* $s_N = s_{N-1} - 1$ *and* $s_j = 1$ *for* $j = 1, 2, \ldots, N-2$.

Our strategy is to exploit the characterization of n-contractible sequences provided by Lemma 3.9. In order to do this, we need the following lemma.

Lemma 3.11. *Let* $\mathbf{s}$ *be a sequence of positive integers. Suppose* $n \geq 3$ *is such that* $s_n \leq s_{n-1} - 1$. *If there exists* $1 \leq j \leq n-2$ *such that* $s_j > 1$, *then* $\mathbf{s}$ *is not* n*-contractible.*

Proof. Note that $\mathbf{e} = (0, 0, \ldots, 0, 1)$ and $\overline{\mathbf{e}} = (0, \ldots, 0, 1, 0, \ldots 0)$ (1 in the jth entry) are elements of $I_{n-1}^{(\mathbf{s})}$ such that $\operatorname{asc}(\mathbf{e}) = 1 = \operatorname{asc}(\overline{\mathbf{e}})$ and $e_{n-1} = 1 \neq 0 = \overline{e}_{n-1}$. Given that $\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = 0 = \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor$,

$$s_{n-1}\operatorname{asc}(\mathbf{e}) - e_{n-1} \neq s_{n-1}\operatorname{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}$$

and

$$s_n\operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n\operatorname{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor.$$

The result then follows from Lemma 3.9. □

By Lemma 3.11, it remains to consider sequences of the form

$$(1, 1, \ldots, 1, s_{N-1}, s_N, \ldots),$$

where $N \geq 3$, $s_N \leq s_{N-1} - 1$ and $(s_n)_{n=N}^{\infty}$ is nondecreasing. Indeed, in order to conclude the proof of Theorem 1.3, it suffices to show that if $n \geq 3$ and $\mathbf{s}$ is a sequence such that $s_1 = s_2 = \cdots = s_{n-2} = 1$ and $s_n \leq s_{n-1} - 1$, then $\mathbf{s}$ is n-contractible if and only if $s_n = s_{n-1} - 1$.

Proof of Theorem 1.3. Let $n \geq 3$ and suppose $\mathbf{s} = (s_1, s_2, \ldots)$ is a sequence such that

$$s_1 = s_2 = \cdots = s_{n-2} = 1$$

and $s_n \leq s_{n-1} - 1$. Assume $s_n = s_{n-1} - 1$ and let $\mathbf{e}, \overline{\mathbf{e}} \in I_{n-1}^{(\mathbf{s})}$ be such that

$$s_n \mathrm{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n \mathrm{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor.$$

By the remark following Proposition 3.10, we know that $\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor$ and $\mathrm{asc}(\mathbf{e}) = \mathrm{asc}(\overline{\mathbf{e}})$. Write

$$\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \left\lfloor \frac{(s_{n-1} - 1) e_{n-1}}{s_{n-1}} \right\rfloor = e_{n-1} + \left\lfloor \frac{-e_{n-1}}{s_{n-1}} \right\rfloor.$$

This means that

$$\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = \begin{cases} 0 & \text{if } e_{n-1} = 0, \\ e_{n-1} - 1 & \text{otherwise.} \end{cases}$$

Hence, if $e_{n-1} \neq \overline{e}_{n-1}$, then (without loss of generality) $e_{n-1} = 0$ and $\overline{e}_{n-1} = 1$; that is, $\mathbf{e} = (0, 0, \ldots, 0)$ and $\overline{\mathbf{e}} = (0, \ldots, 0, 1)$, but then $\mathrm{asc}(\mathbf{e}) = 0 \neq 1 = \mathrm{asc}(\overline{\mathbf{e}})$, a contradiction. We deduce that $e_{n-1} = \overline{e}_{n-1}$ and consequently,

$$s_{n-1} \mathrm{asc}(\mathbf{e}) - e_{n-1} = s_{n-1} \mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}.$$

Using Lemma 3.9, we conclude that $\mathbf{s}$ is n-contractible if $s_n = s_{n-1} - 1$.

Now, suppose $s_n < s_{n-1} - 1$. Say $s_n = s_{n-1} - l$ for $2 \leq l < s_{n-1}$. If $0 \leq m < s_{n-1}$, then

$$\left\lfloor \frac{s_n m}{s_{n-1}} \right\rfloor = \left\lfloor \frac{(s_{n-1} - l) m}{s_{n-1}} \right\rfloor = m + \left\lfloor \frac{-lm}{s_{n-1}} \right\rfloor. \tag{3.7}$$

Note that $\left\lfloor \frac{-l(s_{n-1}-1)}{s_{n-1}} \right\rfloor = -l + \left\lfloor \frac{l}{s_{n-1}} \right\rfloor = -l \leq -2$. Thus, we may choose k to be the smallest integer in $\{2, 3, \ldots, s_{n-1} - 1\}$ such that $\left\lfloor \frac{-lk}{s_{n-1}} \right\rfloor \leq -2$. Since $k - 1 \geq 1$, it must be that $\left\lfloor \frac{-l(k-1)}{s_{n-1}} \right\rfloor = -1$. By (3.7), we may write

$$\left\lfloor \frac{s_n (k-1)}{s_{n-1}} \right\rfloor = (k-1) + (-1) = k - 2.$$

Now,

$$\left\lfloor \frac{-lk}{s_{n-1}} \right\rfloor = \left\lfloor \frac{-l(k-1)}{s_{n-1}} - \frac{l}{s_{n-1}} \right\rfloor \geq \left\lfloor \frac{-l(k-1)}{s_{n-1}} \right\rfloor + \left\lfloor \frac{-l}{s_{n-1}} \right\rfloor = (-1) + (-1) = -2.$$

This means that $\left\lfloor \frac{-lk}{s_{n-1}} \right\rfloor = -2$ and it follows from (3.7) that $\left\lfloor \frac{s_n k}{s_{n-1}} \right\rfloor = k - 2$. Let $\mathbf{e}$ and $\overline{\mathbf{e}}$ in $I_{n-1}^{(\mathbf{s})}$ be given by $\mathbf{e} = (0, \ldots, 0, k)$ and $\overline{\mathbf{e}} = (0, \ldots, 0, k-1)$, then $\mathrm{asc}(\mathbf{e}) = 1 = \mathrm{asc}(\overline{\mathbf{e}})$ and $\left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = k - 2 = \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor$, so

$$s_{n-1} \mathrm{asc}(\mathbf{e}) - e_{n-1} \neq s_{n-1} \mathrm{asc}(\overline{\mathbf{e}}) - \overline{e}_{n-1}$$

and

$$s_n \operatorname{asc}(\mathbf{e}) - \left\lfloor \frac{s_n e_{n-1}}{s_{n-1}} \right\rfloor = s_n \operatorname{asc}(\overline{\mathbf{e}}) - \left\lfloor \frac{s_n \overline{e}_{n-1}}{s_{n-1}} \right\rfloor .$$

By Lemma 3.9, we deduce that $\mathbf{s}$ is not $n-$contractible if $s_n < s_{n-1} - 1$. □

Bibliography

[1] Tom M. Apostol, *Generalized Dedekind sums and transformation formulae of certain Lambert series*, Duke Math. J. **17** (1950), 147–157.

[2] ———, *Theorems on generalized Dedekind sums*, Pacific J. Math. **2** (1952), 1–9.

[3] ———, *Modular functions and Dirichlet series in number theory*, second ed., Graduate Texts in Mathematics, vol. 41, Springer-Verlag, New York, 1990.

[4] Juan S. Auli and Carla D. Savage, *Coefficients of the inflated Eulerian polynomial*, arXiv preprint arXiv:1504.01089 (2015).

[5] Bhaskar Bagchi, *On Nyman, Beurling and Baez-Duarte's Hilbert space reformulation of the Riemann hypothesis*, Proc. Indian Acad. Sci. Math. Sci. **116** (2006), no. 2, 137–146.

[6] Matthias Beck, *Dedekind cotangent sums*, Acta Arith. **109** (2003), no. 2, 109–130.

[7] Bruce C. Berndt, *On the Hurwitz zeta-function*, Rocky Mountain J. Math. **2** (1972), no. 1, 151–157.

[8] Sandro Bettin, *A generalization of Rademacher's reciprocity law*, Acta Arith. **159** (2013), no. 4, 363–374.

[9] Sandro Bettin and Brian Conrey, *Period functions and cotangent sums*, Algebra Number Theory **7** (2013), no. 1, 215–242.

[10] ———, *A reciprocity formula for a cotangent sum*, Int. Math. Res. Not. IMRN (2013), no. 24, 5709–5726.

[11] Mireille Bousquet-Mélou and Kimmo Eriksson, *Lecture hall partitions*, The Ramanujan Journal **1** (1997), no. 1, 101–111.

[12] ———, *Lecture hall partitions II*, The Ramanujan Journal **1** (1997), no. 2, 165–185.

[13] William E. Briggs and Sarvadaman Chowla, *The power series coefficients of* $\zeta(s)$, Amer. Math. Monthly **62** (1955), 323–325.

[14] Fan Chung and Ron Graham, *Inversion-descent polynomials for restricted permutations*, J. Comb. Theory Ser. A **120** (2013), no. 2, 366–378.

[15] Sylvie Corteel, Sunyoung Lee, and Carla D. Savage, *Enumeration of sequences constrained by the ratio of consecutive parts*, Sém. Lothar. Combin. **54A** (2005/07), Art. B54Aa, 12.

[16] Sylvie Corteel and Carla D. Savage, *Anti-lecture hall compositions*, Discrete Math. **263** (2003), no. 1-3, 275–280.

[17] Sylvie Corteel, Carla D. Savage, and Andrew V. Sills, *Lecture hall sequences, q-series, and asymmetric partition identities*, Partitions, q-series, and modular forms, Dev. Math., vol. 23, Springer, New York, 2012, pp. 53–68.

[18] Harold Davenport and Hans Heilbronn, *On the zeros of certain Dirichlet series*, Journal of the London Mathematical Society **s1-11** (1936), no. 4, 307–312.

[19] Richard Dedekind, *Schreiben an Herrn Borchardt über die Theorie der elliptischen Modul-Functionen.*, Journal für die reine und angewandte Mathematik **83** (1877), 265–292 (ger).

[20] ———, *Erläuterungen zu zwei Fragmenten von Riemann*, Dedekinds Gesammelte Math. Werke, I (R. Fricke et al., Eds.) (1930), 159–173.

[21] Niklas Eriksen, *A simple bijection between lecture hall partitions and partitions into odd integers*, FPSAC 2002 Proceedings (Melbourne, Australia), 2002.

[22] Godfrey H. Hardy and Srinivasa Ramanujan, *Asymptotic formulae in combinatory analysis*, Proceedings of the London Mathematical Society **2** (1918), no. 1, 75–115.

[23] Ernst Lammel, *Ein Beweis, dass die Riemannsche Zetafunktion $\zeta(s)$ in $|s-1| \leqq 1$ keine Nullstelle besitzt*, Univ. Nac. Tucumán Rev. Ser. A **16** (1966), 209–217.

[24] Serge Lang, *Complex analysis*, fourth ed., Graduate Texts in Mathematics, vol. 103, Springer-Verlag, New York, 1999.

[25] Michael J. Lighthill, *An introduction to Fourier analysis and generalised functions*, Cambridge University Press, 1958.

[26] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark (eds.), *NIST handbook of mathematical functions*, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

[27] Thomas W. Pensyl and Carla D. Savage, *Rational lecture hall polytopes and inflated Eulerian polynomials*, The Ramanujan Journal **31** (2013), no. 1-2, 97–114.

[28] James E. Pommersheim, *Toric varieties, lattice points and Dedekind sums*, Mathematische Annalen **295** (1993), no. 1, 1–24.

[29] Hans Rademacher, *On the partition function p(n)*, Proceedings of the London Mathematical Society **2** (1938), no. 1, 241–254.

[30] ———, *Some remarks on certain generalized Dedekind sums*, Acta Arithmetica **1** (1964), no. 9, 97–105.

[31] Hans Rademacher and Emil Grosswald, *Dedekind sums*, The Mathematical Association of America, Washington, D.C., 1972, The Carus Mathematical Monographs, No. 16.

[32] Carla D. Savage and Michael J. Schuster, *Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences*, J. Combin. Theory Ser. A **119** (2012), no. 4, 850–870.

[33] Robert Spira, *Zeros of Hurwitz zeta functions*, Math. Comp. **30** (1976), no. 136, 863–866.

[34] Vasiliy I. Vasyunin, *On a biorthogonal system associated with the Riemann hypothesis*, Algebra i Analiz **7** (1995), no. 3, 118–135.

[35] Edmund T. Whittaker and George N. Watson, *A course of modern analysis*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996, Reprint of the fourth (1927) edition.

[36] Ae Ja Yee, *On the combinatorics of lecture hall partitions*, The Ramanujan Journal **5** (2001), no. 3, 247–262.

[37] Don Zagier, *Eisenstein series and the Riemann zeta-function*, Automorphic forms, representation theory and arithmetic, Springer, 1981, pp. 275–301.