Using Polytopes to Derive Growth Series for Classical Root Lattices

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by

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San Francisco, California

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CERTIFICATION OF APPROVAL

I certify that I have read *Using Polytopes to Derive Growth Series for Classical Root Lattices* by Kimberly Holmes Seashore and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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Using Polytopes to Derive Growth Series for Classical Root Lattices

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Given a lattice $\mathcal{L}$ finitely generated as a monoid by a set $\mathcal{M}$, the growth series of $\mathcal{L}$ is a generating function which encodes the number of elements with word length k in $\mathcal{L}$. The growth series is $G(x) = \frac{h(x)}{(1-x)^d}$ where d is the rank of the lattice and $h(x)$ is the coordinator polynomial with degree $\leq d$. This thesis investigates the growth series for the classical root lattices A_{n-1} and C_n, reproving formulae given by Conway, Sloane, Baake, and Grimm. Our approach is an application the theory presented by Beck and Hoşten using a regular unimodular triangulation of the polytope $\mathcal{P} = \text{conv}(\mathcal{M})$. We also apply Ehrhart theory to derive the coordinator polynomial for C_n.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee   Date

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Chapter 1

Introduction

1.1 Lattices and Growth Series

A **lattice** $\mathcal{L}$ is a discrete subgroup of $\mathbb{R}^n$ for some $n \in \mathbb{Z}_{>0}$. The **rank** of a lattice is the dimension of the subspace spanned by the lattice. We say that a lattice $\mathcal{L}$ is **generated as a monoid** by a finite collection of vectors $\mathcal{M} = \{a_1, \ldots, a_r\} \subset \mathbb{R}^n$ if each point $u \in \mathcal{L}$ is a non-negative integer combination of the vectors in $\mathcal{M}$. For convenience, we often write the vectors from $\mathcal{M}$ as columns of a matrix $M \in \mathbb{R}^{n \times r}$ and to make the connection between $\mathcal{L}$ and M more transparent, we refer to the lattices as $\mathcal{L}_M$. The **word length** of u with respect to $\mathcal{M}$, denoted $\mathrm{w}(u)$ is the $\min(\sum c_i)$ where $u = \sum c_i a_i$. The **growth function** $\mathrm{S}(k)$ is defined to be the number of points $u \in \mathcal{L}$ with word length $\mathrm{w}(u) = k$ with respect to $\mathcal{M}$. The growth function of a lattice generated as a monoid allows us to determine how rapidly the lattice expands. We define the growth series to be the generating function $G(x) := \sum_{k\geq 0} S(k)x^k$. From [4], $G(x) = \frac{h(x)}{(1-x)^d}$ where $h(x)$ is a polynomial of degree less than or equal to the rank d of $\mathcal{L}_M$. We call $h(x)$ the **coordinator polynomial** of the growth series. Explicit formulae for the growth series for the root lattices A_{n-1}, C_n and D_n, which we will define below, are given by Baake and Grimm in [1]; the formulae for A_{n-1} and D_n are proven in Conway and Sloane [5].

In this paper we examine the growth series for the classical root lattices in terms of their standard set of generators. We rederive formulae for A_{n-1} and for C_n and conjecture a method for determining the formula for the growth series for D_n. The approach presented here is a natural extension of the proofs related to the growth series of cyclotomic lattices presented in [2]. Using regular triangulations of the polytopes formed by the convex hull of the generating vectors in $\mathcal{M}$, we both determine the growth series and extend the given set of lattice generators. Along the way, we will show specific triangulations of these contact polytopes which facilitate these calculations and determine that the coordinator polynomials arising from these constructions must be palindromic. Finally we provide an argument for the application of this technique to proving the growth series for D_n.

The majority of the material presented here is an explanation and synthesis of theorems and techniques proved elsewhere. The results proven in this thesis are summarized in the following theorems. The terms in these theorems are fully defined in subsequent chapters.

Theorem 1.1. *Given the lattice A_{n-1}, generated as a monoid by the vector configuration $\mathcal{M}_{A_d} = \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i, j \leq n\}$, the coordinator polynomial of the growth series of A_{n-1} is the h-polynomial of the boundary complex of any regular, unimodular* *triangulation of* $\mathcal{P}_{A_d} = \text{conv}(\mathcal{M}_{A_{n-1}})$.

Corollary 1.2. *The coordinator polynomial for the growth series of* A_{n-1} *generated as a monoid by* $M_{n-1} = \{\mathbf{e}_i - \mathbf{e}_j, : 0 \leq i, j \leq n \text{ with } i \neq j\}$ *is*

$$h(x) = \sum_{k=0}^{n} -1 \begin{pmatrix} d \\ k \end{pmatrix}^2 x^k.$$

Theorem 1.3. *The coordinator polynomial for the lattice* C_n *generated as a monoid by the standard generators* $\mathcal{M}_{C_n} = \{\pm \mathbf{e}_i \pm \mathbf{e}_j : 0 \leq i, j \leq n\}$ *is given by*

$$h(x) = \sum_{k=0}^{n} \begin{pmatrix} 2n \\ 2k \end{pmatrix} x^k.$$

While two different sets of generators $\mathcal{M}_1$ and $\mathcal{M}_2$ may generate the same lattice, the rate at which the lattice grows relative to these generators might be quite different. The growth functions for $\mathcal{L}_{M_1}$ and $\mathcal{L}_{M_2}$ allow us to quantify this difference.

Example 1.1. Let $M_1 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 1 \end{pmatrix}$. Then $\mathcal{L}_{M_1} = \mathcal{L}_{M_2} = \mathbb{Z}^2$. Figure 1.1 shows the lattices generated by these matrices with the generators shown in bold and points with word length less than 2 indicated on the lattice. Let $\mathrm{S}_1(k)$ and $\mathrm{S}_2(k)$ denote the growth sequence for $\mathcal{L}_{M_1}$ and $\mathcal{L}_{M_2}$ respetively. In every lattice, the origin is the only point with word length 0, so $S_1(0) = S_2(0) = 1$. We can easily compute that for $n \geq 1$, $\mathrm{S}_1(k) = 3k$ while $\mathrm{S}_2(k) = 6k$. That is, $\mathcal{L}_{M_2}$ is growing twice as fast as $\mathcal{L}_{M_1}$.

In many cases it is easier to work with the growth function embedded in a series.

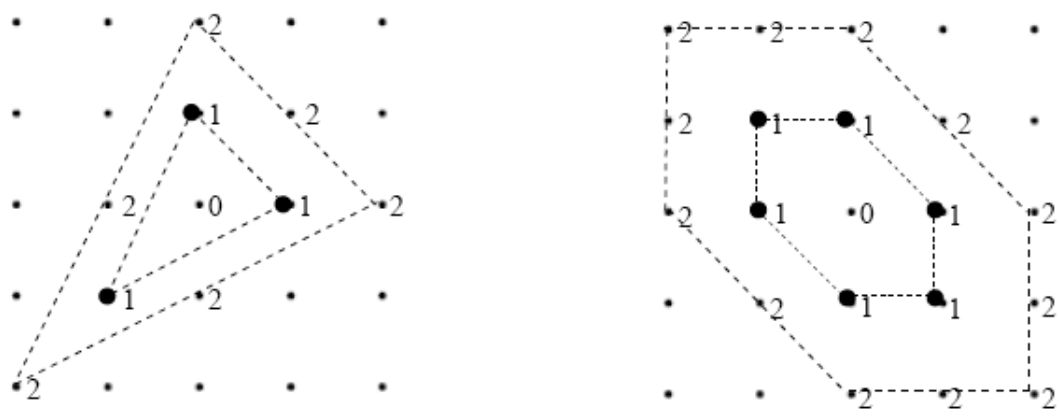

Figure 1.1: $\mathcal{L}_{M_1}$ and $\mathcal{L}_{M_2}$ with generators.

Rather than looking for an explicit formula for $\mathrm{S}(k)$ in terms of k, we define the generating function

$$G(x) := \sum_{k\geq 0} \mathrm{S}(k)x^k.$$

As cited above, $G(x) = \frac{h(x)}{(1-x)^r}$ where $h(x)$ is called the **coordinator polynomial of $\mathcal{L}$ with respect to $\mathcal{M}$**. The degree of $h(x)$ is $\leq r$.

1.2 Polytope Basics

The theory that we use to determine the growth series for the root lattices depends on some basic knowledge of polytopes and cones. The definitions and theorems summarized here are from [3],[15], and [13], where the proofs and motivation are explained more fully.

A *polytope* $\mathcal{P} \subseteq \mathbb{R}^n$ can be defined either as the convex hull of a finite set of points or as the bounded intersection of a finite collection of halfspaces.

Definition 1.1. A $\mathcal{V}-$**polytope** $\mathcal{P}_{\mathcal{M}}$ is the **convex hull** of a finite collection of points in $\mathcal{M} = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subset \mathbb{R}^n$. Formally,

$$\begin{aligned} P_{\mathcal{M}} &:= \left\{ \sum_{i=1}^{r} \lambda_i \mathbf{v}_i \;:\; \lambda_i \geq 0 \,\text{with} \sum_{i=1}^{r} \lambda_i = 1 \right\}. \\ &:= \operatorname{conv}(\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}). \end{aligned}$$

A **hyperplane** $H \subset \mathbb{R}^n$ is a subset of the form

$$H = \{\mathbf{x} \in \mathbb{R}^n : a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = b\} \text{ where } a_1, a_2, \ldots, a_n, b \in \mathbb{R}.$$

A hyperplane H separates $\mathbb{R}^n$ in two **halfspaces** H^+ and H^- given by the points satisfying the following inequalities:

$$\begin{aligned} H^+ &= \{\mathbf{x} \in \mathbb{R}^n : a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n \geq b\} \\ H^- &= \{\mathbf{x} \in \mathbb{R}^n : a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n \leq b\}. \end{aligned}$$

Any halfspace H^+ can be written as H^- by multiplying the both sides of the defining inequality by -1. Thus we can write any finite collection of m halfspaces in $\mathbb{R}^n$ as $\mathcal{H} = \{H_1^-, \ldots, H_m^-\}$ where $H_i^- = \{\mathbf{x} \in \mathbb{R}^n : a_{i1}\mathbf{x}_1 + a_{i2}\mathbf{x}_2 + \cdots + a_{in}\mathbf{x}_n \leq b_i\}$. We collect the coefficients from these inequalities in the $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$$

and let $\mathbf{b} = (b_1, \ldots, b_m)$.

Definition 1.2. An $\mathcal{H}-$**polytope** $\mathcal{P}_{\mathcal{H}}$ is the bounded intersection of a finite collection of halfspaces $\mathcal{H}$. Formally,

$$\mathcal{P}_{\mathcal{H}} := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\},$$

where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

The following theorem establishes that definitions 1.1 and 1.2 describe the same set of objects.

Theorem 1.1. *[15] Every $\mathcal{V}$-polytope can be described by a finite set of inequalities as an $\mathcal{H}$-polytope. Every $\mathcal{H}$-polytope is the convex hull of a minimal finite collections of points called its vertices and is thus a $\mathcal{V}$-polytope.*

The proof of this theorem takes considerable work and is explained in fully [15]. As a result of this theorem, it makes sense to use the term **polytope** denoted $\mathcal{P}$ to describe either a $\mathcal{V}$-polytope or an $\mathcal{H}$-polytope. We scale the polytope $\mathcal{P}$ by a factor k to form the $\mathbf{k^{th}}$ **dilate** of a polytope $\mathcal{P}$ given by

$$k\mathcal{P} = \{(kx_1, kx_1, \ldots, kx_n) : (x_1, x_2, \ldots, x_n) \in \mathcal{P}\}.$$

The dimension of a polytope $\mathcal{P}$ is one of the most fundamental invariants of $\mathcal{P}$. To precisely define the dimension of a polytope, we first consider the notion of an **affine space**.

Definition 1.3. An **affine space** is any set of the form $\mathcal{U} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$. In other words, a non-empty affine space is simply a translation of a vector space

$\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$. The **dimension** of an affine space $\mathcal{U}$ is the same as the dimension of the translated vector space.

$$\dim(\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}) = \dim(\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}).$$

The **affine hull** of a set of points $\mathcal{M} = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is the set of all affine combinations of the points in $\mathcal{M}$ denoted

$$\operatorname{aff}(\mathcal{M}) := \left\{ \sum_{i=1}^{r} \lambda_i \mathbf{v}_i \;:\; \lambda_i \in \mathbb{R} \text{ and} \sum_{i=1}^{r} \lambda_i = 1 \right\}.$$

Thus every affine hull is an affine space.

Definition 1.4. The **dimension** d of a polytope $\mathcal{P}$ is the dimension of the affine hull of $\mathcal{P}$. We denote this $\dim(\mathcal{P}) = d$ and call $\mathcal{P}$ a d-polytope.

Given a polytope $\mathcal{P} \in \mathbb{R}^n$, $\dim(\mathcal{P}) \leq n$. If $\dim(\mathcal{P}) = n$ we say that $\mathcal{P}$ is **full dimensional**. A **supporting hyperplane** of $\mathcal{P}$ is any hyperplane $H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{c}\mathbf{x} = c_0\}$ such that $\mathcal{P}$ is completely contained in either H^- or H^+.

Definition 1.5. A **face** of $\mathcal{P}$ is a set of the form $\mathcal{F} = \mathcal{P} \cap H$ where H is a supporting hyperplane of $\mathcal{P}$. If we let $\mathcal{H}$ be the degenerate hyperplane $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{0}\mathbf{x} = \mathbf{0}\}$, then $\mathcal{P}$ is a face of itself. If we let $\mathcal{H}$ be any hyperplane that does not intersect $\mathcal{P}$ at all, then $\emptyset$ is a face of $\mathcal{P}$.

If we let $\mathcal{P} = \{x \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ and $H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{cx} = c_0\}$ then

$$\mathcal{F} = \mathcal{P} \cap H = \left\{ x \in \mathbb{R}^n : \begin{pmatrix} A \\ \mathbf{c} \\ -\mathbf{c} \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} \mathbf{b} \\ c_0 \\ -c_0 \end{pmatrix} \right\}.$$

It follows that every face of a polytope is itself a polytope. A face $\mathcal{F}$ of $\mathcal{P}$ is considered a **proper face** if $\dim(\mathcal{F}) < \dim(\mathcal{P})$. The 0-dimensional faces of $\mathcal{P}$ are called **vertices**, the 1-dimensional faces are **edges** and the $(d-1)$-dimensional faces are **facets** of $\mathcal{P}$. The set of vertices of $\mathcal{P}$ is called the **vertex set** of $\mathcal{P}$ denoted $\text{vert}(\mathcal{P})$. The **boundary** of $\mathcal{P}$, denoted $\partial\mathcal{P}$, is the union of all proper faces of $\mathcal{P}$.

Definition 1.6. Given a d-polytope $\mathcal{P} \in R^n$ where $n > d$,and a point $v_0 \notin \text{aff}(\mathcal{P})$, we call the polytope $Q = \text{conv}(\{v_0\} \cup \mathcal{P})$ the **cone over $\mathcal{P}$ from** v_0.

Definition 1.7. A d-polytope $\mathcal{P}$ is a **d-simplex** if $\mathcal{P}$ has exactly $d+1$ vertices. In this case we denote $\mathcal{P}$ by $\triangle_d$.

When listing several simplices, we use δ_i to denote the i^{th} simplex regardless of the dimension.

Definition 1.8. A **simplicial complex** $\mathcal{C}$ is a finite collection of simplices in $\mathbb{R}^n$ such that

1. the empty polytope $\emptyset$ is in $\mathcal{C}$
2. if $\delta \in \mathcal{C}$, then every face of δ is in $\mathcal{C}$
3. if $\delta_1 \in \mathcal{C}$ and $\delta_2 \in \mathcal{C}$, then $\delta_1 \cap \delta_2$ is a face of both δ_1 and δ_2.

Each $\delta \in \mathcal{C}$ is called **face** of $\mathcal{C}$. A simplicial complex is **pure** if all maximal faces of $\mathcal{C}$ have the same dimension.

Chapter 2

The Root Lattice A_{n-1}

We now take a closer look at the lattice A_{n-1} and examples of the growth series for specific values of n. In this chapter, we also define the contact polytope of A_{n-1} and examine the structure of this contact polytope.

2.1 Monoid Generators and Growth Series of A_{n-1}

The lattice A_{n-1} is a subgroup of $\mathbb{Z}^n$ given by $A_{n-1} = \{x \in \mathbb{Z}^n \mid \sum_{k=1}^{n} x_k = 0\}$. The rank of this lattice is $d = n - 1$. To simplify the subscript, we will refer to this lattice as A_d.

Proposition 2.1. *The lattice A_d is generated as a monoid by the set of vectors $\mathcal{M}_{A_d} = \{\mathbf{e}_i - \mathbf{e}_j, \mid 0 \leq i, j \leq d+1 \text{ with } i \neq j\}$*

Proof. Let $\mathcal{L}_{\mathcal{M}_{A_d}}$ be the lattice generated by $\mathcal{M}_{A_d}$. Choose $\mathbf{u} \in \mathcal{L}_{\mathcal{M}_{A_d}}$, so $\mathbf{u} = \sum_{i,j} c_{ij}(\mathbf{e}_i - \mathbf{e}_j) = \sum_{i,j} c_{ij}\mathbf{e}_i - c_{ij}\mathbf{e}_j$. Then

$$\sum_{k=1}^{n} u_k = \sum_i \left(\sum_j c_{ij} \right) - \sum_j \left(\sum_i c_{ij} \right) = \mathbf{0}.$$

Thus $\mathbf{u} \in A_d$ and $\mathcal{L}_{\mathcal{M}_{A_d}} \subseteq A_d$.

Now choose $\mathbf{u} = (u_1, u_2, \ldots, u_k) \in A_d$. Then $\forall k,\ u_k \in \mathbb{Z}$. Also

$$\sum_{k=1}^{n} u_k = 0 \text{ or } u_n = -\sum_{k=1}^{n-1} u_k.$$

We now construct the coefficients c_{ij} such that $\mathbf{u} = \sum_{i,j} c_{ij}(\mathbf{e}_i - \mathbf{e}_j)$. If $(\sum_{k=1}^{i} u_k) \geq 0$,let $c_{i(i+1)} = \sum_{k=1}^{i} u_k$. If $(\sum_{k=1}^{i} u_k) < 0$, let $c_{(i+1)i} = -(\sum_{k=1}^{i} u_k)$. Otherwise let $c_{ij} = 0$.

$$\begin{aligned}
\sum_{i,j} c_{ij}(\mathbf{e}_i - \mathbf{e}_j) &= u_1(\mathbf{e}_1 - \mathbf{e}_2) + [(u_2 + u_1)(\mathbf{e}_2 - \mathbf{e}_3)] + \cdots + \left(\sum_{k=1}^{n-1} u_k \right) (e_{n-1} - e_n) \\
&= u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \cdots + u_{n-1}\mathbf{e}_{n-1} + \left(-\sum_{k=1}^{n-1} u_k \right) e_n \\
&= \mathbf{u}.
\end{aligned}$$

Thus $\mathbf{u} \in \mathcal{L}_{M_{A_d}}$ and $A_d = \mathcal{L}_{\mathcal{M}_{A_d}}$. □

We represent the $d(d+1)$ vectors in $\mathcal{M}_{A_d}$, one for each possible combination of i and j, as the columns in a $(d+1) \times d(d+1)$ matrix M_{A_d}.

Example 2.1. A_2 is a rank 2 lattice in $\mathbb{R}^3$ generated as a monoid by the vectors

$$M_2 = \begin{pmatrix} 1 & 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 & 1 \end{pmatrix}.$$

While the rank of A_2 is 2, this lattice, shown in Figure 2.1,is not a subset of $\mathbb{Z}^2$. Instead, A_2 lies in the hyperplane $H = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ in $\mathbb{Z}^3$.

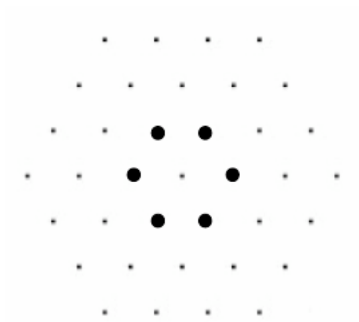

Figure 2.1: The hexagonal lattice A_2.

Recall that for the lattice A_d, the word length $\mathrm{w}(p)$ of a point $p \in A_d$ is the minimum number of generators from $\mathcal{M}_{A_d}$ used to obtain p. The growth function $S_d(k)$ is the number of points $p \in A_d$ with word length $\mathrm{w}(p) = k$ with respect to $\mathcal{M}_{A_d}$. Since the origin is always considered to have word length 0, $S_d(0) = 1$. Each generating vector in M_{A_d} has word length 1, so $S_d(1) = d(d+1)$. Figure 2.2 shows the points in A_2 with word length ≤ 3 as sets of radiating hexagons. The hexagon $\mathcal{P}_2 = \mathrm{conv}(\mathcal{M}_2)$ is called the **contact polytope** of A_2. The radiating hexagons are dilations of $\mathcal{P}_2$ around the origin. All of the points $p \in A_2$ such that $\mathrm{w}(p) = k$ will lie on $\partial(k\mathcal{P}_{A_2})$; that is, the number of points on the boundary of the k^{th} hexagon equals $S_2(k)$. By observation, we see that $S_2(k) = 6k$ for $k \geq 1$.

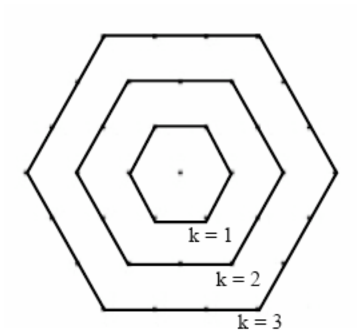

Figure 2.2: Dilations of the contact polytope $\mathcal{P}_{A_2}$ in A_2.

2.2 The Contact Polytope of A_d

Definition 2.1. Let $\mathcal{L}$ be a lattice generated as a monoid by the columns of $M \in \mathbb{R}^{n \times r}$. The **contact polytope** of $\mathcal{L}$ is $\mathcal{P}_M = \text{conv}(M)$, the convex hull of the column vectors in M. That is,

$$\mathcal{P}_M = \left\{ p \in \mathbb{R}^n : p = \sum_{i=1}^{r} \lambda_i a_i,\ \lambda_i \geq 0 \text{ and } \sum \lambda_i = 1. \right\}.$$

where a_i is the i-th column of M.

For the lattice A_d we denote the contact polytope $\mathcal{P}_{A_d}$. Since A_d is a lattice of rank d in $\mathbb{R}^{d+1}$, $\mathcal{P}_{A_d} \subset \mathbb{R}^{d+1}$ is d-dimensional.

In the example above, $\mathcal{P}_{A_2}$ is a regular hexagon containing only the lattice points in $\mathcal{M}_{A_d} \cup \mathbf{0}$. But what can we say about $\mathcal{P}_{A_d}$ for $d > 2$? To get an idea about these, we examine a larger example.

Example 2.2. Consider $\mathcal{P}_{A_4} \subset \mathbb{R}^5$. The generators of this polytope are given by

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

$\mathcal{P}_{A_4} = \mathrm{conv}(M_4)$. Consider the hyperplane $H_{12} = \{\mathbf{x} \in \mathbb{R}^5 : x_1 - x_2 = 2\}$. Then $\mathcal{P}_4 \subseteq H_{12}^-$ and

$$\mathcal{P}_{A_4} \cap H_{12} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So $\mathbf{e}_1 - \mathbf{e}_2$ is a vertex of P_M.

For any choice of i and j, $\mathcal{P}_{A_d}$ intersects the supporting hyperplane $H_{ij} = \{x \in \mathbb{R}^{d+1} : x_i - x_j = 2\}$ in the point $\mathbf{e}_i - \mathbf{e}_j$, so the generalization of Example 2.2 is that the columns in M_{A_d} are exactly the vertices of $\mathcal{P}_{A_d}$. We call these vertices $\mathbf{v}_{ij} = \mathbf{e}_i - \mathbf{e}_j$ and reorganize them in the $(d+1) \times (d+1)$ matrix $\mathcal{V}_d$,whose entries are $\mathbf{v}_{ij}$ for $i \neq j$ and 0 if $i = j$. In this context we write $\mathcal{P}_{A_d} = \mathrm{conv}(\mathcal{V}_4)$ to mean that $\mathcal{P}_{A_d}$ is convex hull of the **entries** in $\mathcal{V}_d$. While this notation may initially seem a bit awkward, we will shortly see that it is quite powerful.

Example 2.3. The contact polytope P_{A_4} from Example 2.2 can be written as $\mathrm{conv}(\mathcal{V}_4)$ where

$$\mathcal{V}_4 = \begin{pmatrix} 0 & \mathbf{v}_{12} & \mathbf{v}_{13} & \mathbf{v}_{14} & \mathbf{v}_{15} \\ \mathbf{v}_{21} & 0 & \mathbf{v}_{23} & \mathbf{v}_{24} & \mathbf{v}_{25} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & 0 & \mathbf{v}_{34} & \mathbf{v}_{35} \\ \mathbf{v}_{41} & \mathbf{v}_{42} & \mathbf{v}_{43} & 0 & \mathbf{v}_{45} \\ \mathbf{v}_{51} & \mathbf{v}_{52} & \mathbf{v}_{53} & \mathbf{v}_{54} & 0 \end{pmatrix}.$$

Whether we are referring to the vertices v_{ij} or the components of a vector $\mathbf{x} \in \mathbb{R}^n$, we often want to identify a subset of the indices. Let $[d+1] := \{1, 2, 3, \ldots, d+1\}$ and $S \subsetneq [d+1]$ is a proper subset of $\{1, 2, 3, \ldots, d+1\}$.

Proposition 2.2. *$\mathcal{P}_d$ is contained the hyperplane $H_0 = \{\mathbf{x} \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i = 0\}$ and for every $S \subsetneq [d+1],\ S \neq \{\emptyset\}$*

$$H_S := \left\{ \mathbf{x} \in \mathbb{R}^{d+1} :\ \sum_{i \in S} x_i = 1 \right\}$$

is a facet defining hyperplane. Also, each facet of $\mathcal{P}_{A_d}$ is equal to $\mathcal{P}_{A_d} \cap H_S$ for some $S \subsetneq [d+1]$.

Proof. Since $\mathcal{P}_{A_d} = \operatorname{conv}(\mathcal{V})$,each vector $\mathbf{v}_{ij} \in \mathcal{H}_0$, then $\mathcal{P}_{A_d} \subseteq H_0$. Fix $S \subset [d+1]$ with $|S| > 0$. For H_S to define a facet of $\mathcal{P}_{A_d}$, $\dim(\mathcal{P}_{A_d} \cap H_S) = d-1$. That is, there must exist d affinely independent vertices of $\mathcal{P}_{A_d}$ that also lie on H_S. Fix $s \in S$. Let $T = [d+1] - S$. Then $\forall t \in T$, $(\mathbf{v}_{st})_s = 1$ and $(\mathbf{v}_{st})_t = -1$ and $(\mathbf{v}_{st})_i = 0$ when $i \neq s,\ i \neq t$. Since $\sum_{i \in S} (v_{st})_i = 1$, then $\mathbf{v}_{st} \in H$. For a fixed $s \in S$, the set of vectors $\{v_{sj} : j \in T\}$ is affinely independent with $|T| = n-k$ vectors. Now fix a $t \in T$ and consider $\{\mathbf{v}_{it} : i \in S,\, i \neq s\}$. These form an affinely independent set with $k-1$ vectors. So $\{\mathbf{v}_{sj} | j \in T\} \cup \{\mathbf{v}_{it} | i \in S, i \neq s\}$ is the desired set of $(d+1-k)+(k-1) = d$ affinely independent vectors. So $\mathcal{P}_{A_d} \cap H_S$ is a facet of $\mathcal{P}_{A_d}$.

Let $\mathcal{F}$ be a facet of $\mathcal{P}_{A_d}$. Then there exists

$$H^- = \{\mathbf{x} \in \mathbb{R}^n : c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \leq b\}$$

such that $\mathcal{P}_{A_d} \subset H^-$ and $\mathcal{F} = (\mathcal{P}_{A_d} \cap H^-)$. Since the $\mathcal{P}_{A_d}$ contains the origin, we can assume that $b = 1$. Without loss of generality, let $\mathbf{v}_{12} \in \mathcal{F}$. Since every point in $\mathcal{P}_{A_d}$

satisfies the equation $\sum x_i = 0$, we can add enough copies of this equation to make $c_1 = 1$. Let $T = j : \mathbf{v}_{1j} \in \mathcal{F}$. For each $j \in T$, $1 - c_j = 1$ so $c_j = 0$, in particular $c_2 = 0$. Now consider $\mathbf{v}_{km}$ where $k, m \notin T$ and $k \neq 1$. Then $1 - c_m < 1$ so $c_m > 0$. If $\mathbf{v}_{km} \in \mathcal{F}$ then $c_k - c_m = 1$,so $c_k > 1$. But then $c_k - c_2 > 1$ means that $\mathbf{v}_k 2 \notin H^-$, a contradiction to the assumption $\mathcal{P}_{A_d} \subset H^-$. Therefore $\forall k, m \notin T$, $\mathbf{v}_k m \notin \mathcal{F}$. Finally, since $\mathcal{F}$ is a facet, there must be d affinely independent vertices that satisfy $x_1 + c_2 x_2 + \cdots + c_n x_n \leq 1$ with equality. Thus for each $k \notin T$, at least one vector v_{kj} with $j \in T$ must lie on $\mathcal{F}$. So $c_k = 1$. Let $S = [d+1] - T$. Then $H = \{\mathbf{x} \in \mathbb{R}^{d+1} : \sum x_i = 1 \text{ for} i \in S\}$.

□

Corollary 2.3. *The integer points in $\mathcal{P}_{A_d}$ are exactly $\mathcal{M}_{A_d} \cup \mathbf{0}$.*

Proof. By the definition of $\mathcal{P}_{A_d}$, the points in $\mathcal{M}_{A_d}$ are contained in $\mathcal{P}_{A_d}$. Since $\mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{e}_2 - \mathbf{e}_1$ are both in $\mathcal{M}_{A_d}$, then $\mathbf{0} = \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2) + \frac{1}{2}(\mathbf{e}_2 - \mathbf{e}_1) \in \mathcal{P}_{A_d}$. Let $\mathbf{x}$ be an integer point in $\mathcal{P}_{A_d}$, so $\mathbf{x} = \sum \lambda_{ij} \mathbf{v}_{ij}$ with $\sum \lambda_{ij} =$ and $\forall k \ |x_k| \leq 1$. If $\mathbf{x} \neq 0$, then there exists i such that $x_i = 1$ or $x_i = -1$. Without loss of generality, assume that $x_1 = 1$. Then

$$\sum_j \lambda_{1j} = 1$$

and $\lambda_{ij} = 0$ if $i \neq 1$. Choose s such that $\lambda_{1s} > 0$. Then $x_s = -\lambda_{1s}$. Since x_s is an integer, then $\lambda_{1s} = 1$. Thus, $\mathbf{x} = \mathbf{e}_1 - \mathbf{e}_s$. □

Given any $S \subset [n]$, the vertices of each of the facets of $\mathcal{P}_{A_d}$ are the columns of M_d where $x_s = 1$ for exactly one $s \in S$ and $x_i = 0$ for all $i \in S$, $i \neq s$. We denote this facet by $\mathcal{F}_S$, and the corresponding set of vertices $\mathcal{V}_S$. The vertices of $\mathcal{F}_S$ are

$$\mathcal{V}_S = \{v_{ij} : i \in S, \text{ and } j \in T = [n] - S\}.$$

There is a one-to-one correspondence between the vertices of a facet $\mathcal{F}_S$ and the entries in $k \times (n-k)$ submatrices of $\mathcal{V}_d$ with rows given by the indices in S and columns from the indices in T. We will call these submatrices $\mathcal{V}_S$.

Example 2.4. The vertices of $\mathcal{F}_{\{1,4\}}$, the facet given by the intersection of $\mathcal{P}_{A_4}$ and the hyperplane $x_1 + x_4 = 1$, appear in the 2×2 matrix formed by rows 1 and 4 and columns 2, 3, and 5 of the matrix $\mathcal{V}_4$:

$$F_{\{1,4\}} = \text{conv}(V_{\{1,4\}}) = \text{conv}\begin{pmatrix} \mathbf{v}_{12} & \mathbf{v}_{13} & \mathbf{v}_{15} \\ \mathbf{v}_{42} & \mathbf{v}_{43} & \mathbf{v}_{45} \end{pmatrix}. \tag{2.1}$$

Definition 2.2. [15] The **product of two polytopes** $\mathcal{P}$ and $\mathcal{Q}$ is given by

$$\mathcal{P} \times \mathcal{Q} := \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} : \mathbf{x} \in P, \mathbf{y} \in Q \right\}.$$

$\mathcal{P} \times \mathcal{Q}$ is a polytope of dimension $\dim(\mathcal{P}) + \dim(\mathcal{Q})$ whose faces are the products of faces of $\mathcal{P}$ and $\mathcal{Q}$.

Continuing with the example of $\mathcal{P}_{A_4}$, the vertices $\mathbf{v}_{12}, \mathbf{v}_{13}$, and $\mathbf{v}_{15}$ are affinely independent and the convex hull of these points is a triangular face, $\triangle_2$, of $\mathcal{F}_{\{1,4\}}$. The vertices $\mathbf{v}_{12}$ and $\mathbf{v}_{42}$ are also affinely independent with convex hull a segment, $\triangle_1$, forming an edge of $\mathcal{F}_{14}$. In this case, $\mathcal{F}_{14} = \text{conv}(\mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{15}) \times \text{conv}(\mathbf{v}_{12}, \mathbf{v}_{42}) = \triangle_2 \times \triangle_1$. This example is generalized in the following proposition.

Proposition 2.4. *Every facet of the contact polytope of A_d is a product of simplices*

$$\triangle_{k-1} \times \triangle_{(n-k-1)} \text{ for } 1 \leq k \leq n-1.$$

Proof. Let $\mathcal{F}_S$ be a facet of $\mathcal{P}_{A_d}$ with $S = \{i_1, i_2, \ldots, i_k\} \subseteq [n]$ and $T = [n] - S$. Without loss of generality, we can assume that $S = \{1, 2, \ldots, k\}$ and $T = \{k+1, \ldots, n\}$. Consider only the vertices in the first row of the submatrix of vertices of this facet. All of the these vertices will be of the form

$$\mathbf{v}_{ij} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where exactly one of the first k components is 1 and exactly one of the last $n-k$ components is -1. If I_k is the $k \times k$ identity matrix, then let $\mathcal{P} = \text{conv}(I_k) = \triangle_{(k-1)}$ and $\mathcal{Q} = \text{conv}(-I_{(n-k)}) = \triangle_{(n-k-1)}$. Then the vertices v_{ij} of F_S are exactly the vertices of $\mathcal{P} \times \mathcal{Q}$ thus $\mathcal{F}_S = \triangle_{k-1} \times \triangle_{(n-k-1)}$. $\square$

Summarizing, we know the following about the combinatorial structure of $\mathcal{P}_{A_d}$: it is a d-polytope with $d(d+1)$ vertices, all of the vectors in $\mathcal{M}_{A_d}$. Each vertex $\mathbf{v}_{ij}$ shares an edge with the vertices of the form $\mathbf{v}_{kj}$ or $\mathbf{v}_{il}$. There are $2(d-1)$ such vertices, so $\mathbf{v}_{ij}$ lies on $2(d-1)$ edges. To find the total number of edges we multiply $d(d+1)$ vertices by $2(d-1)$ edges per vertex and divide by two, since every edge contains two vertices. $\mathcal{P}_{A_d}$ has $(d-1)d(d+1)$ edges and the Pythagorean theorem gives the length of each edges as $\sqrt{2}$. $\mathcal{P}_{A_d}$ has $\binom{d+1}{k}$ facets of the form $\triangle_{(k-1)} \times \triangle_{(n-k-1)}$ for each $1 \leq k \leq n-1$ for a total of $2^k - 2$ facets. The integer points in $\mathcal{P}_{A_d}$ are $\mathcal{M}_{A_d} \cup \mathbf{0}$.

2.3 Triangulations and Unimodularity of $\mathcal{P}_{A_d}$

In the next chapters, we will explore how embedding the monoid structure of a lattice in an algebra will enable us to use more tools to determine the growth series of this lattice. A critical step is a triangulation of the polytope $\mathcal{P}_{A_d}$. In Chapter 4, we will use a specific triangulation of $\mathcal{P}_{A_d}$ to derive a formula for the coordinator polynomial of A_d. In this section, we develop the notion of triangulating a polytope and examining when this triangulation has a special property called unimodularity.

Definition 2.3. [3] A **triangulation** $\mathcal{T}$ of a convex d-polytope $\mathcal{P}$ is a pure simplicial complex where $\mathcal{P}$ is the union of the simplices in $\mathcal{T}$.

The fact that every convex polytope $\mathcal{P}$ can be triangulated using only the vertices of $\mathcal{P}$ is proven in the appendix of [3]. For a contact polytope $\mathcal{P}_{\mathcal{L}}$ of a lattice $\mathcal{L}$, we are primarily interested in triangulations which use only the points in $\mathcal{P}_{\mathcal{L}} \cap \mathcal{L}$.

Example 2.5. Consider the hexagon P_{A_2} with vertices $\{v_{12}, v_{13}, v_{21}, v_{23}, v_{31}, v_{32}\}$. $\mathcal{T} = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ is a triangulation of P_{A_2} as shown in Figure 2.3 with
$\delta_1 = \mathrm{conv}(\{v_{12}, v_{13}, v_{23}\})$, $\delta_2 = \mathrm{conv}(\{v_{12}, v_{23}, v_{21}\})$,
$\delta_3 = \mathrm{conv}(\{v_{12}, v_{21}, v_{31}\})$, $\delta_4 = \mathrm{conv}(\{v_{12}, v_{31}, v_{32}\})$.

Definition 2.4. A simplex with vertices $v_0, v_1, \ldots, v_d \in \mathbb{Z}^n$ is **unimodular** in a given lattice $\mathcal{L}$ if for any ordering of the vertices, $\{v_1 - v_0, v_2 - v_0, \ldots, v_d - v_0\}$ generates $\mathcal{L}$. In the case where $\mathcal{L}$ is isomorphic to $\mathbb{Z}^d$, this characterization is equivalent to

$$|\det(v_1 - v_0, v_2 - v_0, \ldots, v_d - v_0)| = 1$$

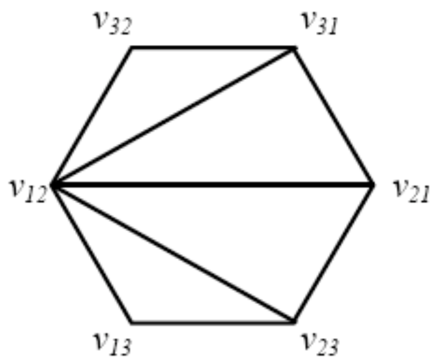

Figure 2.3: A triangulation $\mathcal{T}$ of $\mathcal{P}_{A_2}$

.

In other words, given a d-simplex in with vertices $\mathbb{Z}^d$, this simplex is unimodular if when you translate the simplex so that any one of the vertices is at the origin, then the set of vectors from the origin to the other translated vertices will span $\mathbb{Z}^d$.

Definition 2.5. A **unimodular triangulation** of a polytope P is a triangulation into unimodular simplices with vertices in $P \cap \mathbb{Z}^d$.

Definition 2.6. A matrix M is called **totally unimodular** if every square submatrix of M has determinant 0, +1, or −1.

Proposition 2.5. *The matrix M_{A_d} is totally unimodular.*

For the proof of proposition 2.5 we use an alternate characterization of totally unimodular matrices proven by Schrijver in [9].

Theorem 2.6. *[9, Theorem 19.3] If M is an $n \times m$ matrix with entries $0, +1$ and -1, then M is totally unimodular if each collection of columns of M can be split into two* *parts B_1 and B_2 so that the sum of the columns in B_1 minus the sum of the columns B_2 is a vector with entries only $0, +1$ and -1.*

We abuse the notation slightly to create the following shorthand for this condition: $S_{12} = \sum B_1 - \sum B_2 \in (-1, 0, +1)^n$.

Proof of Proposition 2.5. For $d = 1$, $M_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is totally unimodular by simply applying Definition 2.6. We proceed by induction on the dimension d. Assume that for $d < k$, M_{A_d} is totally unimodular. Let $B = \mathbf{b}_1, \dots, \mathbf{b}_r$ be a collection of columns of M_{A_k} so $\mathbf{b}_k = \mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in [n]$. We use the following algorithm to transfer the vectors from B into the sets B_1 and B_2 one at a time so that the property $S_{12} \in (-1, 0, +1)^{k+1}$ is preserved at every step. We begin with B_1 and B_2 empty.

(1) For every pair of vectors $\mathbf{b}_s$ and $\mathbf{b}_t$ such that $\mathbf{b}_s + \mathbf{b}_t = \mathbf{0}$, place both $\mathbf{b}_s$ and $\mathbf{b}_t$ in B_1. Go to (2)

(2) Let $c = 1$. Choose any of the remaining vectors $\mathbf{e}_{i_1} - \mathbf{e}_{j_1}$ in B and add it to B_1. Go to (3)

(3) At this step there are three cases:

(3a) If all of the vectors from B have been distributed to either B_1 or B_2, go to (5);

(3b) If there are no remaining vectors in B of the form $\mathbf{e}_{j_c} - \mathbf{e}_{j_{c+1}}$, go to (4);

(3c) If there remains vectors in B of the form $\mathbf{e}_{j_c} - \mathbf{e}_{j_{c+1}}$, add this vector to B_1. There are now two new cases.

i. If $j_{c+1} = i_1$, then $\sum B_1 - \sum B_2 = \mathbf{0}$, so return to (2).

ii. If $j_{c+1} \neq i_1$, let $c := c + 1$ and go to (3).

(4) Again, there are two cases:

(4a) If there are no remaining vectors with a 1 or -1 in the j_c position, go to (5);

(4b) If there is still a vector in B are of the form $\mathbf{e}_{j_{c+1}} - \mathbf{e}_{j_c}$, add this vector to B_2, let $c := c + 1$ and return to (3).

(5) Either

(5a) All of the vectors from B have been separated into B_1 and B_2; or

(5b) All of the remaining vectors have entry 0 in the j_c row.

In both cases, either $S_{12} = \mathbf{0}$ or $S_{12} = e_{i_1} - e_{j_c}$. In case (5a), we are done. In case (5b), let the matrix B' be the remaining vectors in B with the j_k row eliminated. The vectors in B' are a subset of the columns of M_{k-1}. By the induction hypothesis, M_{k-1} is totally unimodular, and thus the vectors in B' can be divided into two sets B'_3 and B'_4 such that $S'_{34} = \sum B'_3 - \sum B'_4 \in (-1, 0, +1)^{k-1}$. Thus, we separate the remaining vectors in B into the same two groups, B_3 and B_4 such that $S_{34} = \sum B_3 - \sum B_4 \in (-1, 0, +1)^k$ and the j_c component of this sum is 0. If the i_1 component of this sum is -1 or 0, then we simply add all of the columns from B_3 to B_1 and the columns from B_4 to B_2 and we are done. If the i_1 component of S_{34} is 1, then we switch and add the columns from B_4 to B_1 and the columns from B_3 to B_2 and again we are done. Thus, we conclude that M_k is totally unimodular, and by induction, M_{A_d} is totally unimodular for all $d \in \mathbb{Z}_{>0}$. $\square$

Example 2.6. Let $B \subset \mathcal{M}_{A_4}$ be given by

$$
\begin{aligned}
B &= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix} \\
&= \{\mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{41}, \mathbf{v}_{32}, \mathbf{v}_{25}, \mathbf{v}_{21}, \mathbf{v}_{35}, \mathbf{v}_{45}\}.
\end{aligned}
$$

The results for each step in the algorithm above are as follows:

- (1) $\mathbf{v}_{12} + \mathbf{v}_{21} = 0$; place both in B_1,and go to (2).

$$
\begin{array}{lll}
B_1 = \{\mathbf{v}_{12}, \mathbf{v}_{21}\} & B_2 = \{\} & B = \{\mathbf{v}_{13}, \mathbf{v}_{41}, \mathbf{v}_{32}, \mathbf{v}_{25}, \mathbf{v}_{35}, \mathbf{v}_{45}\} \\
\sum B_1 = \mathbf{0} & \sum B_2 = \mathbf{0} & \sum S_{12} = \mathbf{0}
\end{array}
$$

- (2) Place $\mathbf{v}_{13}$ in B_1, $c = 1$, $i_1 = 1$ and $j_1 = 3$. Go to (3)

$$
\begin{array}{lll}
B_1 = \{\mathbf{v}_{12}, \mathbf{v}_{21}, \mathbf{v}_{13}\} & B_2 = \{\} & B = \{\mathbf{v}_{41}, \mathbf{v}_{32}, \mathbf{v}_{25}, \mathbf{v}_{35}, \mathbf{v}_{45}\} \\
\sum B_1 = \mathbf{v}_{13} & \sum B_2 = \mathbf{0} & \sum S_{12} = \mathbf{v}_{13}
\end{array}
$$

- (3c) Place $\mathbf{v}_{35}$ in B_1, $c = 2$ and $j_2 = 5$. Go to (3)

$$
\begin{array}{lll}
B_1 = \{\mathbf{v}_{12}, \mathbf{v}_{21}, \mathbf{v}_{13}, \mathbf{v}_{35}\} & B_2 = \{\} & B = \{\mathbf{v}_{41}, \mathbf{v}_{32}, \mathbf{v}_{25}, \mathbf{v}_{45}\} \\
\sum B_1 = \mathbf{v}_{15} & \sum B_2 = \mathbf{0} & \sum S_{12} = \mathbf{v}_{15}
\end{array}
$$

- (3b to 4b) Place $\mathbf{v}_{25}$ in B_2, $c = 3$ and $j_3 = 2$. Go to (3)

$$B_1 = \{\mathbf{v}_{12}, \mathbf{v}_{21}, \mathbf{v}_{13}, \mathbf{v}_{35}\} \quad B_2 = \{\mathbf{v}_{25}\} \quad B = \{\mathbf{v}_{41}, \mathbf{v}_{32}, \mathbf{v}_{45}\}$$
$$\sum B_1 = \mathbf{v}_{15} \qquad \sum B_2 = \mathbf{0} \qquad \sum S_{12} = \mathbf{v}_{15}$$

- (3b to 4b) Place $\mathbf{v}_{32}$ in B_2, $c = 4$ and $j_4 = 3$. Go to (3)

$$B_1 = \{\mathbf{v}_{12}, \mathbf{v}_{21}, \mathbf{v}_{13}, \mathbf{v}_{35}\} \quad B_2 = \{\mathbf{v}_{25}, \mathbf{v}_{32}\} \quad B = \{\mathbf{v}_{41}, \mathbf{v}_{45}\}$$
$$\sum B_1 = \mathbf{v}_{15} \qquad \sum B_2 = \mathbf{v}_{35} \qquad \sum S_{12} = \mathbf{v}_{13}$$

- (3b to 4b)$B' = \{\mathbf{v}_{31}, \mathbf{v}_{34}\} \in M_{A_3}$. Let $B'_3 = \{\mathbf{v}_{31}\}$ and $B'_4 = \{\mathbf{v}_{34}\}$,so $S'_{34} = \mathbf{v}_{41}$. Then $B_3 = \{\mathbf{v}_{41}\}$ and $B'_4 = \{\mathbf{v}_{45}\}$ and $S_{34} = \mathbf{v}_{51}$ in $\mathbb{R}^5$ and the final result is

$$B_1 = \{\mathbf{v}_{12}, \mathbf{v}_{21}, \mathbf{v}_{13}, \mathbf{v}_{35}, \mathbf{v}_{41}\} \quad B_2\{\mathbf{v}_{25}, \mathbf{v}_{32}, \mathbf{v}_{45}\}$$
$$\sum B_1 = \mathbf{v}_{45} \qquad \sum B_2 = \mathbf{v}_{34} \qquad \sum S_{12} = \mathbf{v}_{35}$$

So the entries of $\sum B_1 + \sum B_2$ are only $-1, 0$ and 1.

We now consider the simplicial complex formed by the boundary of $\mathcal{P}_{A_d}$. Let $\mathcal{T}$ be a triangulation of $\partial(\mathcal{P}_{A_d})$ using only the vertices of $\mathcal{P}_{A_d}$. A maximal simplex of $\mathcal{T}$ is a $(d-1)$-simplex with d vertices $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ which are a subset of columns of M_{A_d}. Since $\mathbf{0} \in \mathcal{P}_{A_d}$, we can construct a triangulation T' of $\mathcal{P}_{A_d}$ by coning over each of the maximal simplices in $\mathcal{T}$ from the origin. Then the vertices of any d-simplex in T' are $\{0, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$. Since m is totally unimodular, then $|\det(v_1 - v_0, v_2 - v_0, \ldots, v_d - v_0)| = 1$. From this we conclude the following proposition.

Proposition 2.7. *Any triangulation of $\mathcal{P}_{A_d}$ constructed by coning from the origin over a triangulation of the boundary using no new vertices is unimodular.*

2.3.1 The rational cone over $\mathcal{P}_{A_d}$

In many cases, it is easier and more natural to work with a cone rather than a polytope. The cones that we are defining in this section differ from those defined in Definition 1.6 in that these are unbounded polyhedra. We construct the rational cone over a polytope $\mathcal{P}$ as follows. If $V_{\mathcal{P}} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r)$ are the vertices of $\mathcal{P}$ in $\mathbb{R}^n$ then let

$$V'_{\mathcal{P}} = \begin{pmatrix} v_1 & a_2 & \ldots & a_r \\ 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Define the **rational polyhedral cone** generated by M' as

$$\mathcal{K}_{V'_{\mathcal{P}}} = \text{cone}(V') = \left\{ \sum_{i=1}^{r+1} c_i \begin{pmatrix} a_i \\ 1 \end{pmatrix} : c_i \in \mathbb{R}_{\geq 0} \right\}.$$

Let us take a closer look at case we are concerned with here: when $\mathcal{P}$ is the contact polytope of a lattice $\mathcal{L}_{\mathcal{M}}$. Then $\{\mathcal{M} \cup \{\mathbf{0}\}\}$ is the set of *lattice points* with word length ≤ 1 contained in $\mathcal{P}$. In this case we let $M' = \begin{bmatrix} M & \mathbf{0} \\ \mathbf{1} & 1 \end{bmatrix}$. Since $\mathcal{P} = \text{conv}(\mathcal{M})$, then $\mathcal{K}_{V'_{\mathcal{P}}} = \mathcal{K}_{M'}$. This construction can be interpreted geometrically as follows. We embed the contact polytope $\mathcal{P}$ in $\mathbb{R}^{n+1}$ in the hyperplane $x_{n+1} = 1$ and draw a ray from the origin through each of the vertices of $\mathcal{P}$, and find the convex hull of these rays. $\mathcal{K}_{V'_{\mathcal{P}}} \cap \{x : x_{n+1} = k\}$ is an isomorphic copy of $k\mathcal{P}$.

We would like to say, therefore, that the integer points in $\mathcal{K}_{M'} \cap \{x : x_{n+1} = k\}$ are in bijection with the set of points in the $\mathcal{L}_M$ with word length less than or equal

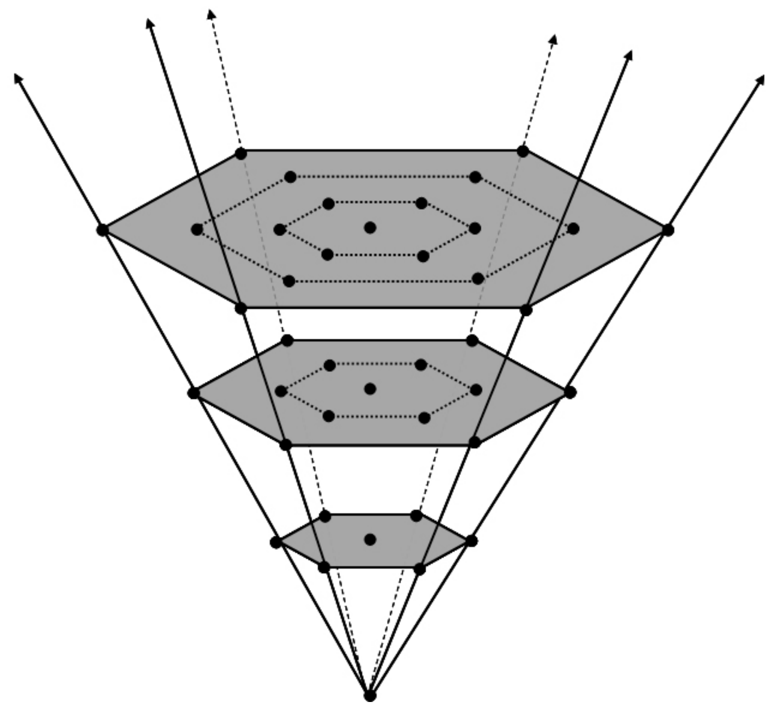

Figure 2.4: Dilations of $\mathcal{P}_{A_2}$ as cross-sections of $\mathcal{K}'_M$.

to k, however this is not always the case. This equality does hold when $\mathcal{P}_M$ has a unimodular triangulation [2, Lemma 11]. Theorem 2.7 guarantees that the polytope $\mathcal{P}_{A_d}$ has a unimodular triangulation, thus the integer points $\mathbf{x} \in \mathcal{K}'_M$ are an exact copy of the points in A_d with word length $\leq k$.

Chapter 3

Translating to Commutative Algebra

In this chapter, we consider a different problem, that of determining the number of basis vectors of a given degree in a monoid algebra. The theory outlined in this chapter is explained in depth in [7] and [13] and is the backbone of the proofs of coordinator polynomials of cyclotomic lattices in [2].

3.1 From Lattices to Monoid Algebras

Thus far, we have considered the points in a lattice $\mathcal{L}_M \in \mathbb{R}^n$ as vectors generated by non-negative integer combinations of $M = (a_1, a_2, \ldots, a_r)$. Alternately, we can choose to consider these points as exponents of monomials in a K-algebra as follows. Let $K[\mathbf{x}] = K[x_1, \ldots, x_r, x_{r+1}]$ be the ring of polynomials with coefficients in an arbitrary field K. A **monomial** of $K[\mathbf{x}]$ is a product of powers of variables, $x^u = x_1^{u_1} x_2^{u_2} \ldots x_{r+1}^{u_{r+1}}$

where $u = (u_1, u_2, \ldots, u_{r+1})$ is the **exponent vector**. Let $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ and the **Laurent polynomial ring** $T = K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ where the monomials can have exponent vectors with negative coordinates (except in the last position.) The variable s is a distinguished variable which we will use to keep track of the degree of monomials in a subring of T. As in Section 2.3.1, let

$$M' = \begin{pmatrix} a_1 & a_2 & \ldots & a_r & \mathbf{0} \\ 1 & 1 & \ldots & 1 & 1 \end{pmatrix}.$$

We now define the ring homomorphism $\psi : K[\mathbf{x}] \to T$ by $\psi(\mathbf{x}) = M'(\mathbf{t}, s)$ which gives the monomial mapping

$$\psi(x_i) = t^{a_i} s = t_1^{a_{i1}} t_2^{a_{i2}} \ldots t_n^{a_{in}} s.$$

The image of the map ψ is the monoid algebra $K[M'] = K[t^{a_1}s, t^{a_2}s, \ldots, t^{a_r}s, s]$. Let $I_{M'}$ be the kernel of ψ. By the first isomorphism theorem for rings, $I_{M'}$ is an ideal of $K[\mathbf{x}]$ called the **toric ideal**, $K[M']$ is a sub-ring of T and $K[\mathbf{x}]/I_{M'} \cong K[M']$. Elements in $K[M']$ are monomials of the form $\mathbf{t}^{\mathbf{v}} s^r$ where

$$\mathbf{v} = \sum_{u_i \in M \cup \{\mathbf{0}\}} c_i \mathbf{u}_i \text{ with } c_i \in \mathbb{Z}_{\geq 0} \text{ and } \sum c_i = r.$$

We define a grading on the monomials in $K[M']$ by letting the **degree** of the monomial $\mathbf{t}^{\mathbf{v}} s^r$ be r. $K[M']_k$ is a K vector space generated by the set of monomials in $K[M']$ with $\deg(\mathbf{t}^{\mathbf{v}} s^k) = k$. The dimension of $K[M']_k$ as a vector space, written $\dim_K(K[M']_k)$ is equal to the number of monic monomials with degree k in $K[M']$. Let us step back and look at a geometric interpretation of this construction in relationship to the lattice $\mathcal{L}_M$. By construction, we see that the set of vectors $\{\mathbf{v} : t^{\mathbf{v}} s^k \in K[M']\}$ are in bijection with the points in $\mathcal{L}_M$ with word length less than or equal to k. Thus counting the number of basis vectors of $K[M']_k$ is the same as finding the number of lattice points in $\mathcal{L}_M$ with word length less than or equal to k; that is,

$$\dim_K(K[M']_k) = \sum_{i=0}^{k} \mathrm{S}(i).$$

Example 3.1.

$$M_2' = \begin{pmatrix} 1 & 1 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then $K[M_2'] = K[\frac{t_1}{t_2}s, \frac{t_1}{t_3}s, \ldots, \frac{t_3}{t_1}s, s]$. The monomials

$$\{t_1^{v_1}t_2^{v_2}t_3^{v_3}s^k \text{ where } \sum_{i=1}^{3} v_i = 0 \text{ and } |v_i| \leq k\}$$

form a basis for $K[M_2']_k$.

3.2 The Hilbert Series

Definition 3.1. Let V be a finitely generated graded K-algebra, with grading by degree. Then the function $H_V : \mathbb{N} \to \mathbb{N}$ given by

$$H_V(k) := \dim_K(V_k)$$

is called the **Hilbert function** of V.

We apply this definition to the algebra $K[M']$ and simplify the notation by letting

$$H(k) = \dim_K(K[M']_k).$$

As with the growth function, we embed the Hilbert function in a generation function.

Definition 3.2. The **Hilbert series** of $K[M']$ is the generating function

$$H(K[M'];x) := \sum_{k\geq 0} H(k)x^k = \sum_{k\geq 0} dim_k(K[M']_k)x^k.$$

The following theorem is a standard result from algebraic geometry.

Theorem 3.1. *[6] If $H(K[M'];x)$ is the Hilbert series of the finitely generated graded standard K-algebra $K[M']$, then*

$$H(K[M'];x) \;=\; \frac{h(x)}{(1-x)^{d+1}},$$

where $h(x)$ is a polynomial of degree at most d.

Here we call $h(x)$ the **Hilbert polynomial of** $K[M']$.

Proposition 3.2. *The Hilbert polynomial of $K[M']$ is precisely the coordinator polynomial of the growth series of $\mathcal{L}_M$.*

Proof. Since the number of elements in $\mathcal{L}_M$ with word length less than or equal to k is $\dim_K(K[M']_k)$ then $\mathrm{S}(k) = \dim_K(K[M']_k) - \dim_K(K[M']_{k-1})$. The growth series of

$\mathcal{L}_M$ is given by

$$\begin{aligned} G(x) &= \sum_{k\geq 0} \mathrm{S}(x)\, x^2 \\ &= \sum_{k\geq 0} (\dim_K(K[M']_k) - \dim_K(K[M']_{k-1}))\, x^k \\ &= H(K[M'];x) - x(H(K[M'];x)) = (1-x)H(K[M'];x) \\ &= \frac{h(x)}{(1-x)^d}. \end{aligned}$$

□

3.3 Initial Ideals and Regular Triangulations

In section 3.1, we established that $\ker\psi$ is the toric ideal $I_{M'}$ of the polynomial ring $K[\mathbf{x}] = K[x_1, \ldots, x_{r+1}]$. We now take a closer look at this ideal with the goal of building a connection between the *initial ideal* of $I_{M'}$ and regular triangulations of the polytope P_M.

Definition 3.3. A **term order** $\prec$ is a well ordering of all the monomials in $K[\mathbf{x}]$ with the following properties:

1. there exists a minimum element $x^0 = 1$;
2. for any monomial $\mathbf{x}^{\mathbf{w}}$, if $\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{v}}$ then $\mathbf{x}^{\mathbf{w}}\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{w}}\mathbf{x}^{\mathbf{v}}$.

We can think of the term order $\prec$ either as applying to the monomials in $K[\mathbf{x}]$ or as applying to the exponent vectors in N^{n+1}. Since the exponent vectors that we are considering are elements in the lattice $\mathcal{L}$, this term order induces an ordering on the points in the lattice, and in particular on the lattice points in contact polytope $P_{M'}$.

Definition 3.4. Given a nonzero polynomial $f \in K[\mathbf{x}]$ and a term order $\prec$, we let $\mathsf{in}_\prec(f)$, the **initial term** of f, be the largest monomial of f with respect to $\prec$. If I is an ideal of $K[\mathbf{x}]$, the **initial ideal** of I with respect to the term order $\prec$ is the monomial ideal generated by all the initial terms of polynomials in I denoted by

$$\mathsf{in}_\prec(I) := \langle \mathsf{in}_\prec(f) \ : \ f \in I \rangle.$$

Theorem 3.3. *[6] Let I be a homogeneous ideal in $K[\mathbf{x}]$ and $\prec$ any term order. Then for all k, $(K[\mathbf{x}]/I)_k$ and $(K[\mathbf{x}]/\mathsf{in}_\prec(I))_k$ are isomorphic K-vector spaces, and therefore $H(K[\mathbf{x}]/I; x) = H(K[\mathbf{x}]/\mathsf{in}_\prec(I); x)$.*

The implication of this proposition for our problem is as follows:

$$H(K[M']; x) H(K[\mathbf{x}]/\mathsf{in}_\prec(I_{M'}); x) = \frac{h(x)}{(1-x)^{d+1}}$$

for *any* term order $\prec$. The next task is to identify a term order that enable us to determine $h(x)$. For this we introduce the notion of a weight vector.

Definition 3.5. For $\omega \in R^{n+1}$, $\prec_\omega$ is the **term order induced by** ω given by $x^\alpha \prec_\omega x^\beta$ if $\omega \cdot \alpha \leq \omega \cdot \beta$. For a polynomial $f \in I$, $\mathsf{in}_\omega(f))$ is the sum of the terms $c_\alpha x^\alpha$ of f where $\omega \cdot \alpha$ is maximal. So we define the **initial ideal with respect to** ω by

$$\mathsf{in}_\omega(I) := \langle \mathsf{in}_\omega(f) \ : \ f \in I \rangle.$$

We can also use weight vector ω to induce a triangulation of $\mathcal{P}_M$ by first triangulating the cone $\mathcal{K}_{M'}$ into simplicial cones and then finding the intersection of these cones with the hyperplane $H_1 = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 1\}$. Geometrically, given a generic weight vector ω and set of vectors M, we construct $\mathcal{T}_\omega$, the **regular triangulation** of $\mathcal{P}_M$ induced by ω, as follows:

a. Lift each vector $\mathbf{u_i} \in M \cup \{\mathbf{0}\} \subset \mathbb{R}^n$ to $(\mathbf{u_i}, 1) \in M' \subset \mathbb{R}^{n+1}$.

b. Lift the points in M' into $\mathbb{R}^{n+2}$ again using the weight vector ω to construct $\widehat{M'} = \{(\mathbf{a_1}, 1, \omega_1), \ldots, (\mathbf{a_r}, 1, \omega_r), (\mathbf{0}, 1, \omega_{r+1})\}$.

c. Form the cone $\mathcal{K}_{\widehat{M'}}$, then project the lower hull of this cone back onto the first $(n+1)$ coordinates. This projected image is a triangulation of M' written $\mathcal{T}_\omega(\mathbf{M'})$.

d. The intersection of $\mathcal{T}_\omega(M')$ with H_1 yields $\mathcal{T}_\omega$, a triangulation of $\mathcal{P}_M$ in $\mathbb{R}^r$.

In section 2.3 we defined a unimodular triangulation of a polytope. The following theorem establishes the connection between the initial ideal $\mathsf{in}_\omega(I_{M'})$ and the regular triangulation $\mathcal{T}_\omega$. A monomial ideal such as $\mathsf{in}_\omega(I_{M'})$ is **squarefree** if the minimal generators are monomials not divisible by x_i^2 for any i.

Theorem 3.4. *[7, Theorem 6.2] The initial ideal* $\mathsf{in}_\omega(I_{M'})$ *is squarefree if and only if the regular triangulation* $\mathcal{T}_\omega$ *is unimodular.*

From Theorem 2.5, we know that M_{A_d} is totally unimodular, thus every triangulation of $\mathcal{P}_{M'}$ is unimodular. In particular, any regular triangulation of $\mathcal{P}_{A_d}$ is unimodular, so every initial ideal $\mathsf{in}_\omega(I_{M'})$ will be squarefree. To complete the connection between the triangulation of $\mathcal{P}_M$ and the toric ideal I'_M we need a few more definitions. In Section 1.2, we defined the terms faces, facets and simplicial complex for polytopes. Here we generalize these definitions to work in that context of abstract algebra.

Definition 3.6. [13, Chapter 13] An **abstract simplicial complex** Γ on the set $[n]$ is a collection of subsets of $[n]$ such that if $F \in \Gamma$ and $G \subset F$, then $G \in \Gamma$ as well.

The elements $F \in \Gamma$ are called the **faces** of Γ. The **dimension** of a face $F \in \Gamma$ is $|F| - 1$ and maximal dimensional faces are called **facets**. If all of the facets of Γ have the same dimension, then we say that Γ is **pure**. Any set $H \subseteq [n]$ such that $H \notin \Gamma$ is called a **non-face** of Γ. A **minimal non-face** of Γ is a non-face H such that for all $G \subsetneq H$, $G \in \Gamma$. Notice that when the set $[n]$ is the subscripts of the vertices of a polytope, these definitions agree with those in Section 1.2.

We now return to the discussion of the columns of M' as a set of $r+1$ vectors in $\mathbb{R}^{n+1}$. Any triangulation $\mathcal{T}_\omega$ of M' is a simplicial complex on [r+1]. Then T_ω is also a triangulation of $\mathcal{P}_M$ which uses all of the lattice points in $\mathcal{P}_M$.

Definition 3.7. The **Stanley-Reisner ideal** of a simplicial complex Γ on $[n]$ is the squarefree monomial ideal

$$I_\Gamma := \langle x_{i_1} x_{i_2} \dots x_{i_n} : \{i_1, i_2, \dots, i_n\} \text{is a minimal non-face of } \Gamma \rangle.$$

In this context, the **face ring** of Γ is given by

$$K[\Gamma] := K[x_1, \dots, x_n] \backslash I_\Gamma.$$

Example 3.2. The simplicial complex

$$\mathcal{C} = \{\{x_1, x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_3, x_5\}\}$$

is shown in Figure 3.1. If we consider x_1, x_2, x_3, x_4, x_5 as indeterminants and form the ring $K[x_1, x_2, x_3, x_4, x_5]$, then the Stanley-Reisner ideal $I_\mathcal{C}$ is generated by the monomials $\{x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_4x_5\}$, corresponding to the minimal non-faces of $\mathcal{C}$.

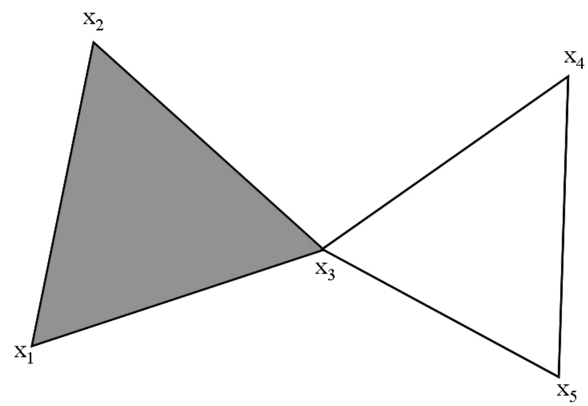

Figure 3.1: The simplicial complex $\mathcal{C}$.

The following theorem connects the squarefree ideal $\mathsf{in}_\omega(I_{M'})$ with the Stanley-Reisner ideal of a simplicial complex.

Theorem 3.5. *[11, Theorem 8.3] Let I_M' be the toric ideal of M' and let $\omega \in \mathbb{R}^{r+1}$ such that $\mathsf{in}_\prec(I_{M'}) = \mathsf{in}_\omega(I_{M'})$ is squarefree. Then $\mathsf{in}_\omega(I_{M'})$ is the Stanley-Reisner ideal of the simplicial complex Γ induced by $\mathcal{T}_\omega$ of M'.*

3.4 From h-vectors to Coordinator Polynomials

The **f-vector** of a simplicial complex Γ is given by $f(\Gamma) = (f_o, f_1, \ldots, f_{n-1})$ where $f_{-1} = 1$, $f_0 = r+1$, and $f_i =$ the number of i-dimensional faces of Γ. The **f-polynomial** is given by

$$f_\Gamma(x) = \sum_{i=0}^{n} f_{i-1} x^{d-i}.$$

We define the **h-polynomial** in terms of the f-polynomial by

$$h_\Gamma(x) = f_\Gamma(x-1) = \sum_{i=0}^{n} f_{i-1}(x-1)^{d-i}.$$

The following theorem from Stanley makes the final link in the connection between the coordinator polynomial of $\mathcal{L}_M$ and the h-polynomial of a simplicial complex.

Theorem 3.6. *[10, Theorem II.1.4] For a simplicial complex Γ with vertex set $\{x_1, x_2, \ldots, x_r\}$ and Stanley-Reisner ideal I_Γ, the Hilbert series of $K[\Gamma]$ is given by*

$$H(K[\Gamma]; x) = \frac{h_\Gamma(x)}{(1-x)^{d+1}}$$

where $h_\Gamma(x)$ is the h-polynomial of the complex Γ.

We summarize the results from this chapter as follows: By Theorem 3.1, the numerator of the Hilbert series $H(K[M']; x)$ is equal to the coordinator polynomial of the growth series for $\mathcal{L}_M$. From Theorem 3.3, $H(K[\mathbf{x}]/I; x) = H(K[\mathbf{x}]/\mathsf{in}_\prec(I); x)$ for any term order $\prec$. Next, we consider the triangulation $\Gamma(P_M)$ induced by a weight vector ω (called regular triangulations.) Theorem 3.4 tells us that if $\Gamma(P_M)$ is unimodular, then $\mathsf{in}_\omega(I)$ is squarefree and by Theorem 3.5 $\mathsf{in}_\omega(I)$ is the Stanley-Reisner ideal I_Γ of the boundary complex of $\Gamma(P_M)$. Finally, with Theorem 3.6, we get the full connection:

$$\begin{aligned} G(x) &= (1-x)H(K[M']; x) = H(K[\mathbf{x}]/I; x) = H(K[\mathbf{x}]/\mathsf{in}_w \prec (I); x) \\ &= H(K[\mathbf{x}]/I_\Gamma); x) = \frac{h_\Gamma(x)}{(1-x)^{d+1}}. \end{aligned}$$

This chain of reasoning depends on the unimodularity of the regular triangulation induced by ω.

Now consider the lattice A_n. By choosing a weight vector such that the origin is the lowest point in the lifting, we can induce a regular triangulation of P_{A_n} equal to a union of cones from the origin over a triangulation of the boundary of P_{A_n}. By Theorem 2.7, every such triangulation will be unimodular. Thus, we have completed the proof of Theorem 1.

Theorem 1.1. *Given the lattice A_{n-1}, generated as a monoid by the vector configuration $\mathcal{M}_{A_{n-1}}$, the coordinator polynomial of the growth series of A_{n-1} is the h-polynomial of the boundary complex of any regular, unimodular triangulation of* $\mathcal{P}_{\mathcal{M}} = \text{conv}(\mathcal{M}_{A_{n-1}})$.

The **Dehn-Sommerville Relations** [15, Theorem 8.21] give that the coefficients of the h-polynomial of the boundary complex of a simplicial d-polytope satisfy the relationship

$$h_k = h_{d-k}.$$

Thus, the h-polynomial is palindromic. Hence, we conclude that the coordinator polynomial for the growth series for A_d is palindromic. In the next chapter, we describe a specific regular triangulation of P_{A_d} and use the face numbers resulting from this triangulation to derive the explicit formula for the Hilbert polynomial for $K[M'_{A_d}]$ and thus the coordinator polynomial for the lattice A_d.

The formulae for the coordinator polynomial of the root lattices C_n and D_n given in [1] are also palindromic, leading to the conjecture that the growth series for these lattices can also be derived using regular, unimodular triangulations of the contact polytopes. Conversely, since the coordinator polynomial for B_n is *not* palindromic, we know that it is not the h-vector of a simplicial polytope, hence this method for deriving the coordinator polynomial will not apply. In Chapter 5, we will define C_n, derive the growth series using Ehrhart theory, and also demonstrate the specific unimodular triangulation that can be used to form the Hilbert series proving that the coordinator polynomial is palindromic.

Chapter 4

The Growth Series of A_d

We now describe a specific regular, unimodular triangulation of the boundary of $\mathcal{P}_{A_d}$ and apply the theorems from Chapter 3 to derive an explicit formula for the coordinator polynomial of A_d. To simplify notation in this chapter, we describe a polytope (or face of a polytope) using only its vertex set and leave the "conv" implied.

Let $\mathcal{M}$ be a totally ordered set of points $\mathbf{a}_i \in \mathbb{R}^d$, where $\mathbf{a}_1 \prec \mathbf{a}_2 \prec \cdots \prec \mathbf{a}_r$. We define a **face** $\mathcal{F}$ of this point configuration to be an ordered subset of points given by the intersection of $\mathcal{M}$ and a face $\mathcal{F} \in \mathcal{P}_M$.

Definition 4.1. [11, Proposition 8.6] The **reverse lexicographic triangulation** (also called the **pulling triangulation**) $\Gamma_{revlex}(\mathcal{M})$ is defined recursively as follows:

- If $\mathcal{M}$ is affinely independent, then $\Gamma_{revlex}(\mathcal{M}) = \{\mathcal{M}\}$.

- Otherwise,

$$\Gamma_{revlex}(\mathcal{M}) = \bigcup_{\mathcal{F}} \{\{\mathbf{a}_1\} \cup \mathcal{G} : \mathcal{G} \in \Gamma_{revlex}(\mathcal{F})\}$$

  where the first union is taken over all facets $\mathcal{F}$ of $\mathcal{M}$ not containing $\mathbf{a}_1$.

We use the following example to illustrate this definition.

Example 4.1. Let $\mathcal{M} = \{\mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{15}, \mathbf{v}_{42}, \mathbf{v}_{43}, \mathbf{v}_{45}\} \subset \mathcal{M}_{A_d}$ with term order $\mathbf{v}_{12} \prec \mathbf{v}_{13} \prec \mathbf{v}_{15} \prec \mathbf{v}_{42} \prec \mathbf{v}_{43} \prec \mathbf{v}_{45}$. $\mathcal{P}_{\mathcal{M}}$ is the facet $\mathcal{F}_{14}$ of $\mathcal{P}_{A_4}$, shown on the left in Figure 4.1. Since $\mathcal{M}$ is not affinely independent, we look at each facet of $\mathcal{P}_{\mathcal{M}}$ that does not contain $\mathbf{v}_{12}$. There are two such facets:

$$\begin{aligned}\mathcal{F}_1 &= \{\mathbf{v}_{42}, \mathbf{v}_{43}, \mathbf{v}_{45}\}\\ \mathcal{F}_2 &= \{\mathbf{v}_{13}, \mathbf{v}_{15}, \mathbf{v}_{43}, \mathbf{v}_{45}\}.\end{aligned}$$

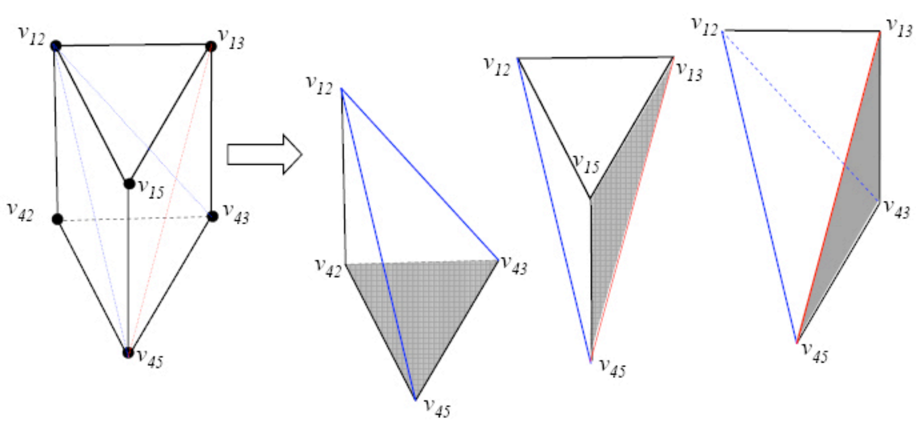

Figure 4.1: Reverse lexicographic triangulation of $\mathcal{F}_{14}$.

$\mathcal{F}_1$ is affinely independent, so $\Gamma_{revlex}(\mathcal{F}_1) = \{\mathcal{F}_1\}$. $\mathcal{F}_2$ is not affinely independent, so we apply the second part of the definition to $\mathcal{F}_2$. The facets of $\mathcal{F}_2$ that do not contain $\mathbf{v}_{13}$ are $\mathcal{F}_3 = \{\mathbf{v}_{15}, \mathbf{v}_{45}\}$ and $\mathcal{F}_4 = \{\mathbf{v}_{43}, \mathbf{v}_{45}\}$ both of which are affinely independent. Assembling the pieces we get

$$\begin{aligned}
\Gamma_{revlex}(\mathcal{F}_2) &= \{\{\{\mathbf{v}_{13}\} \cup \mathcal{F}3\}, \{\{\mathbf{v}_{13}\} \cup \mathcal{F}4\}\} \\
&= \{\{\mathbf{v}_{13}, \mathbf{v}_{15}, \mathbf{v}_{45}\}, \{\mathbf{v}_{13}, \mathbf{v}_{43}, \mathbf{v}_{45}\}\} \\
\Gamma_{revlex}(\mathcal{M}) &= \{\{\{\mathbf{v}_{12}\} \cup \mathcal{F}_1\}, \{\{\mathbf{v}_{12}\} \cup \{\mathbf{v}_{13}, \mathbf{v}_{15}, \mathbf{v}_{45}\}\}, \{\{\mathbf{v}_{12}\} \cup \{\mathbf{v}_{13}, \mathbf{v}_{43}, \mathbf{v}_{45}\}\} \\
&= \{\{\mathbf{v}_{12}, \mathbf{v}_{42}, \mathbf{v}_{43}, \mathbf{v}_{45}\}, \{\mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{15}, \mathbf{v}_{45}\}, \{\mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{43}, \mathbf{v}_{45}\}\}.
\end{aligned}$$

The tetrahedra on the right of Figure 4.1 show the facets of $\Gamma_{revlex}(\mathcal{M})$.

By [11, Chapter 8] we know that every reverse lexicographic triangulation can be induced by some weight vector ω and is therefore regular. In [11], Sturmfels describes a specific reverse lexicographic triangulation, the **staircase triangulation**, on a polytope that is a product of simplices. In Chapter 2 we proved that every facet of $\mathcal{P}_{A_d}$ is the product of simplices. By ordering the integer points in $\mathcal{P}_{A_d}$ appropriately, we can use the staircase triangulation on each of the facets to create a reverse lexicographic triangulation for $\mathcal{P}_{A_d}$. Recall that the integer points in $\mathcal{P}_{A_d}$ are $\mathbf{0} \cup \mathcal{M}_{A_d}$ where $\mathbf{v}_{ij} \in \mathcal{M}_{A_d}$ is given by $\mathbf{v}_{ij} = \mathbf{e}_i - \mathbf{e}_j$. Let $\mathbf{v}_0 = \mathbf{0}$ and define a partial ordering on these lattice points by:

$$\mathbf{v}_0 \prec \mathbf{v}_{ij} \text{ and } \mathbf{v}_{ij} \prec \mathbf{v}_{km} \text{ if } i \leq k \text{ and } j \leq m.$$

Letting v_0 be the minimal element in this ordering allows us triangulate each of the facets of $\mathcal{P}_{A_d}$ and then cone over them from the origin to construct the triangulation $\Gamma(\mathcal{P}_{A_d})$. We will see that these cones match up; that is, their intersections are faces of both cones.

The staircase triangulation on a polytope $\mathcal{P} = \triangle_{s-1} \times \triangle_{t-1}$ is obtained as follows: Label each of the vertices of $\mathbf{x}_{ij} \in \mathcal{P}$ so that for any fix k, $\text{conv}(\{\mathbf{x}_{kj}\}) = \triangle_{s-1}$. Similarly for a given m, $\text{conv}(\{\mathbf{x}_{im}\}) = \triangle_{t-1}$. Let $X = [\mathbf{x}_{ij}] \in \mathbb{R}^{s\times t}$ be the matrix whose entries or the vertices of $\mathcal{P}$. Order the vertices by $\mathbf{x}_{ij} \prec \mathbf{x}_{kl}$ whenever $i \leq k$ and $j \leq l$. The facets of the staircase triangulation are the convex hull of sets of vertices that form maximal chains in this poset. These sets of $s+t-1$ vertices will lie on each of the maximal increasing paths from x_{11} to x_{st}. In the matrix X, these paths appear as staircases, hence the name staircase triangulation.

Example 4.2. If the matrix

$$X = \begin{pmatrix} \mathbf{x}_{11} & \mathbf{x}_{12} & \mathbf{x}_{13} & \mathbf{x}_{14} \\ \mathbf{x}_{21} & \mathbf{x}_{22} & \mathbf{x}_{23} & \mathbf{x}_{24} \\ \mathbf{x}_{31} & \mathbf{x}_{32} & \mathbf{x}_{33} & \mathbf{x}_{34} \end{pmatrix}$$

represents the vertices of a polytope $\mathcal{P} = \text{conv}(X) = \triangle_2 \times \triangle_3$. The set $\{\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{22}, \mathbf{x}_{23}, \mathbf{x}_{33}, \mathbf{x}_{34}\}$ is the vertices of one facet $\mathcal{Q}$ of the staircase triangulation $\Gamma(\mathcal{P})$. Placing these vertices in the original matrix X, it is obvious that they form a staircase from the top left corner to the bottom right.

$$\mathcal{Q} = \begin{bmatrix} \mathbf{x}_{11} & \mathbf{x}_{12} & - & - \\ - & \mathbf{x}_{22} & \mathbf{x}_{23} & - \\ - & - & \mathbf{x}_{33} & \mathbf{x}_{34} \end{bmatrix}$$

There are 10 facets in $\Gamma(\mathcal{P})$ corresponding to the 10 increasing paths from $\mathbf{x}_{11}$ to $\mathbf{x}_{34}$.

4.1 Triangulating $\mathcal{P}_{A_d}$

We now apply the staircase triangulation to each facet $\mathcal{F}_S$ of $\mathcal{P}_{A_d}$. We let $\Gamma(\mathcal{F}_S)$ denote the triangulation of a facet $\mathcal{F}_S$ and $\Gamma(\partial\mathcal{P}_{A_d})$ represent the simplicial complex formed by $\bigcup_{S\subset[n]}\Gamma(\mathcal{F}_S)$. It will follow from Proposition **??** that by ordering the vertices in $\mathcal{V}_d$ as above, the triangulations of the facets will patch together nicely so that $\Gamma(\mathcal{F}_S)$ and $\Gamma(\mathcal{F}_T)$ induce the same triangulation on $\mathcal{F}_S \cap F_T$.

Example 4.3. $\mathcal{F}_{14}$ and $\mathcal{F}_{124}$, shown in Figure 4.2 are facets of $\mathcal{P}_4$ given by:

$$\mathcal{F}_{14} = \text{conv}\begin{pmatrix} \mathbf{v}_{12} & \mathbf{v}_{13} & \mathbf{v}_{15} \\ \mathbf{v}_{42} & \mathbf{v}_{43} & \mathbf{v}_{45} \end{pmatrix}$$

$$\mathcal{F}_{124} = \text{conv}\begin{pmatrix} \mathbf{v}_{13} & \mathbf{v}_{15} \\ \mathbf{v}_{23} & \mathbf{v}_{25} \\ \mathbf{v}_{43} & \mathbf{v}_{45} \end{pmatrix}$$

These facets intersect in the square $\mathcal{F}_{14} \cap \mathcal{F}_{124} = \text{conv}\begin{pmatrix} \mathbf{v}_{13} & \mathbf{v}_{15} \\ \mathbf{v}_{43} & \mathbf{v}_{45} \end{pmatrix}$ As we saw in

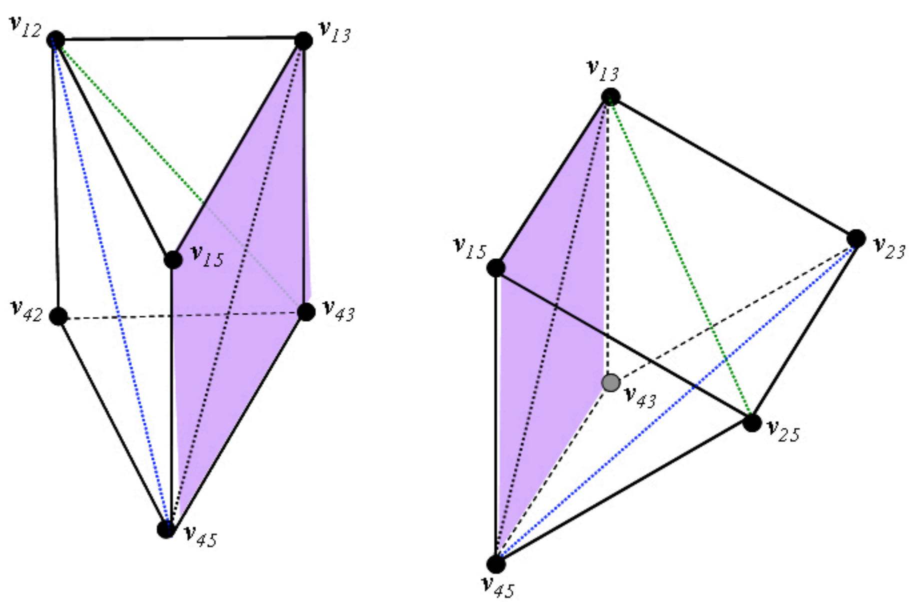

Figure 4.2: The intersection of $\Gamma(\mathcal{F}_{14})$ and $\Gamma(\mathcal{F}_{124})$ is $\Gamma(\mathcal{F}_{14} \cap \mathcal{F}_{124})$.

Example 4.1, $\Gamma(\mathcal{F}_{14})$ is the simplicial complex with facets:

$$\begin{aligned}\delta_1 &= \{\mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{15}, \mathbf{v}_{45}\}\\ \delta_2 &= \{\mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{43}, \mathbf{v}_{45}\}\\ \delta_3 &= \{\mathbf{v}_{12}, \mathbf{v}_{42}, \mathbf{v}_{43}, \mathbf{v}_{45}\}\end{aligned}$$

Using the same term order, $\Gamma(\mathcal{F}_{124})$ is the simplicial complex with facets:

$$\begin{aligned}\delta_4 &= \{\mathbf{v}_{13}, \mathbf{v}_{15}, \mathbf{v}_{25}, \mathbf{v}_{45}\},\\ \delta_5 &= \{\mathbf{v}_{13}, \mathbf{v}_{23}, \mathbf{v}_{25}, \mathbf{v}_{45}\},\\ \delta_6 &= \{\mathbf{v}_{13}, \mathbf{v}_{23}, \mathbf{v}_{43}, \mathbf{v}_{45}\}.\end{aligned}$$

In both cases, $\Gamma(\mathcal{F}_{14} \cap \mathcal{F}_{124}) = \{\{\mathbf{v}_{13}, \mathbf{v}_{15}, \mathbf{v}_{45}\}, \{\mathbf{v}_{13}, \mathbf{v}_{43}, \mathbf{v}_{45}\}$.

The total number of facets in $\Gamma(\mathcal{P}_{A_d})$ can be found by triangulating each of the facets $\mathcal{F}_S \in \mathcal{P}_{A_d}$, then summing across all facets $\mathcal{F}_S$. For $S \subseteq [n]$ with $|S| = k$, there are $\binom{n}{k}$ distinct facets F_S corresponding to $k \times (n-k)$ submatrices of $V_I \subseteq \mathcal{V}_d$. Each of these matrices contains $\binom{n-2}{k-1}$ staircases, thus in the staircase triangulation, these facets are divided into $\binom{n-2}{k-1}$ simplices. Summing over all $k \leq n$ yields

$$\sum_{k=1}^{n-1} \binom{n}{k} \binom{n-2}{k-1}$$

facets in $\Gamma(P_{A_d})$.

While this approach works for determining the number of facets of $\Gamma(\mathcal{P}_{A_d})$, we need a more global approach to count the lower dimensional faces of the triangulation. Simply counting the number of m-faces in each facet will result in serious over-counting. Rather, we will count the number of m-faces that belong to $\Gamma(\mathcal{P}_{A_d})$ by determining the locations of the vertices of these faces in the matrix $\mathcal{V}_d$.

4.2 The f-vector of $\Gamma(\partial\mathcal{P}_{A_d})$

If $\mathcal{F}$ is an m-face of $\Gamma(\mathcal{P}_{A_d})$, $\mathcal{F}$ will be the convex hull of $m+1$ affinely independent vertices and $\mathcal{F}$ must be contained in a facet of $\Gamma(\mathcal{P}_{A_d})$. So there exists an $S \subset [d+1]$ such that the vertices of $\mathcal{F}$ are all contained in a staircase (an increasing path) of $\mathcal{F}_S$ as defined in Section 2.2.

Proposition 4.1. *The m-dimensional faces of the triangulation $\Gamma(\mathcal{P}_{A_d})$ are the polytopes of the form $\mathcal{F} = \mathrm{conv}(\{v_{i_r j_r} \in \mathcal{V}_d$ for $1 \leq r \leq m+1\})$, that meet the following conditions:*

1. $i_s \neq j_t$, $1 \leq s, t \leq m+1$, *and*

2. $1 \leq i_1 \leq i_2 \leq \ldots \leq i_{m+1} \leq d+1$ *and* $1 \leq j_1 \leq j_2 \leq \ldots \leq j_{m+1} \leq d+1$.

Proof. Let $\mathcal{F}$ be an m-face of $\Gamma(\mathcal{P}_{A_d})$. Then $\mathcal{F}$ is an m-simplex with $m+1$ vertices given by $\{v_{i_r j_r} \in \mathcal{V}_d\, 1 \leq r \leq m+1\}$. Since $\mathcal{F}$ is a face of $\Gamma(\mathcal{P}_{A_d})$, the vertices of $\mathcal{F}$ must all be contained in a common facet of $\mathcal{P}_{A_d}$. From Section 2.2, the vertices of a facet of $\mathcal{P}_{A_d}$ are given by v_{ij} where $i \in S$ and $j \in T = [n] - S$ for some $S \subset [n]$. This is precisely the same condition as is given in 1. Further, $\mathcal{F}$ must be contained in a facet of $\Gamma(\mathcal{P}_{A_d})$, so $v_{i_r j_r} \prec v_{i_{r+1} j_{r+1}}$ for all $1 \leq r \leq m$. By the definition of $\prec$, we have that $i_r \leq i_{r+1}$ and $j_r \leq j_{r+1}$ which give condition 2.
Assume that $\mathcal{F}$ satisfies conditions *1* and *2*. Let $S = \{i : \exists j$ such that $v_i j \in \mathcal{F}\}$ and $T = \{j : \exists i$ such that $v_i j \in \mathcal{F}\}$. By condition *1*, $S \cap T = \emptyset$, so $T \subset [d+1] - S$. Thus $\{v_{i_r j_r} \in \mathcal{V}_d$ for $1 \leq r \leq m+1\} \subset \mathrm{vert}(\mathcal{F}_S)$ where $\mathcal{F}_S$ is the facet of $\mathcal{P}_{A_d}$ defined by S. By condition *2*,$\{v_{i_r j_r}\}$ lie on an increasing path in the matrix of vertices of $\mathcal{F}_S$. Thus, $\mathcal{F}$ is a face of the triangulation $\Gamma(\mathcal{P}_{A_d})$. $\square$

The problem of counting the m-faces of $\Gamma(P_{A_d})$ has now been reduced to counting the possible choice for indices $i_r j_r$ of the vertices $v_{i_r j_r}$ that meet the conditions in Proposition 4.1. From condition 2, the possible choices for $\{i_1, \ldots, i_{m+1}\}$ must satisfy $1 \leq i_1 \leq i_2 \leq \ldots \leq i_{m+1} \leq n$. We can replace this weakly increasing condition with the following strictly increasing condition:

$$1 \leq i_1 < i_2 + 1 < i_3 + 2 < \ldots < i_{m+1} + m \leq d + 1 + m.$$

There are $\binom{d+1+m}{m+1}$ possible choices for the set of indices $\{i_1, \ldots, i_{m+1}\}$. Similarly, we change condition 2 for the j's to the strictly increasing condition

$$1 \leq j_1 < j_2 + 1 < i_3 + 2 \ldots < j_{m+1} + m \leq d + 1 + m.$$

But condition 1 gives that the choices for j must be distinct from the choices for the i, thus there are $\binom{(d+1+m)-(m+1)}{m+1} = \binom{d}{m+1}$ possible choices for the indices $\{j_1, \ldots, j_{m+1}\}$. The resulting number of possibilities for the vertices $v_{i_r j_r} \in \mathcal{V}_d$ for $1 \leq r \leq m+1$ is $\binom{d+1+m}{m+1}\binom{d}{m+1}$. We summarize this result as a proposition.

Proposition 4.2. *Let $\Gamma(P_{A_d})$ be the simplicial complex resulting from the reverse lexicographic triangulation of the boundary of P_{A_d} with the ordering $\mathbf{0} < \mathbf{v}_{ij}\ \forall i, j$ and $\mathbf{v}_{ij} \prec \mathbf{v}_{kl}$ if $i \leq k$ and $j \leq l$. The f-vector of $\Gamma(P_{A_d})$ is $f(\Gamma) = \{f_0, \ldots, f_m, \ldots, f_{d-1}\}$ where*

$$f_m = \binom{d+1+m}{m+1}\binom{d}{m+1} = \frac{(d+1+m)!}{(m+1)!\,(m+1)!\,(d-m-1)!}.$$

Then the h-polynomial of $\Gamma(P_{A_d})$ is

$$h_\Gamma(x) = f_\Gamma(x-1) = \sum_{m=-1}^{d-1} f_m (x-1)^{d+1-i},$$

where $f_{-1} = 1$ and $f_m = \frac{(d+1+m)!}{(m+1)!(m+1)!(d-(m+1))!}$. From Theorem 3.4, the coordinator polynomial of the growth series of the lattice A_d with respect to the generators $\mathcal{M} = \{e_i - e_j : 0 \leq i, j \leq d+1\}$ is the precisely $h_\Gamma(x)$. So the growth series of A_d is

$$G(x) = \frac{h(x)}{(1-x)^d} = \frac{\sum_{m=-1}^{d-1} \frac{(d+1+m)!}{(m+1)!\,(m+1)!\,(d-(m+1))!}(x-1)^{d-(m+1)}}{(1-x)^d}. \qquad (4.1)$$

4.3 Simplifying $h(x)$ with a WZ Proof

While we have reached our goal of finding an explicit expression for the coordinator polynomial $h_d(x)$, this formula is still quite complicated. In the paper *Low-Dimensional Lattice VII: Coordination Sequence*[5], Conway and Sloane prove the coordinator polynomial for A_d is given by

$$h(x) = \sum_{k=0}^{d} \binom{d}{k}^2 x^k. \tag{4.2}$$

We now show that these formula for $h(x)$ given in equations (4.1) and (4.2) are equivalent. First we simplify equation (4.1) by the change of variable $m = r - 1$ and rewriting the fraction as a product of binomials.

$$\begin{aligned} h(x) &= \sum_{m=-1}^{d-1} \frac{(d+1+m)!}{(m+1)!(m+1)!(d-(m+1))!}(x-1)^{d-(m+1)} \\ &= \sum_{r=0}^{d} \frac{(d+r)!}{(r)!(r)!(d-r)!}(x-1)^{d-r} \\ &= \sum_{r=0}^{d} \binom{2r}{r}\binom{d+r}{d-r}(x-1)^{d-r} \\ &= (x-1)^d + \binom{2}{1}\binom{d+1}{d-1}(x-1)^{d-1} + \\ \ldots &+ \binom{2r}{r}\binom{d+r}{d-r}(x-1)^{d-r} + \binom{2d}{d}\binom{2d}{0}. \end{aligned}$$

Next, we expand the binomials $(x-1)^{d-r}$ to obtain h_k, the coefficient of x^k in $h(x)$.

$$\begin{aligned} h_k &= (-1)^{d-k}\binom{d}{k} + (-1)^{(d-1)-k}\binom{2}{1}\binom{d+1}{d-1}\binom{d-1}{k} + \\ &\quad +(-1)^{(d-r)-k}\binom{2r}{r}\binom{d+r}{d-r}\binom{d-r}{k} + \binom{2(d-k)}{d-k}\binom{d-k}{k} \\ &= \sum_{r=0}^{d-k}(-1)^{(d-r)-k}\binom{2r}{r}\binom{d+r}{d-r}\binom{d-r}{k}. \end{aligned}$$

To show that Equations (4.1) and (4.2) are the same, it suffices to prove the following identity.

$$\sum_{r=0}^{d-k}(-1)^{(d-r)-k}\binom{2r}{r}\binom{d+r}{d-r}\binom{d-r}{k} = \binom{d}{k}^2. \tag{4.3}$$

We now employ a technique for proving combinatorial identities known as the **WZ (Wilf, Zeilberger)** method. For more details and a proof of this method, we refer the reader to [14]. We are grateful to Akalu Tefera [12] for the WZ proof we present here. Rewrite the summand from Equation 4.3 as

$$s(d,k,r) = (-1)^{(d-k+r)}\binom{2r}{r}\binom{d+r}{k+2r}\binom{k+2r}{k}$$

Divide both sides of the identity by $\binom{d}{k}^2$ and set $f(d,k,r) = \frac{s(d,k,r)}{\binom{d}{k}^2}$. Let $F(d,k) = \sum_{r=0}^{d-k} f(d,k,r)$. Proving the identity in 4.3 is equivalent to showing $F(d,k) = 1$. It is sufficient to show

$$F(d+1,k) - F(d,k) = 0 \text{ and } F(0,0) = 1.$$

This second condition is trivial since

$$F(0,0) = \frac{(-1)^0 \binom{0}{0}\binom{0}{0}\binom{0}{0}}{\binom{0}{0}^2} = 1.$$

Courtesy of the WZ method and Akalu Tefera, we introduce the function $g(d,k,r)$ given by

$$\begin{aligned} g(d,k,r) &= f(d,k,r)\frac{(2d+2-k)r^2}{(d+1-r-k)(d+1)^2} \\ &= \frac{(-1)^{(d-k+r)}(2d+2-k)r^2\binom{2r}{r}\binom{d+r}{k+2r-1}\binom{k+2r-1}{k}}{(d+1)^2}. \end{aligned}$$

The magic of the function $g(d,k,r)$ is that it satisfies the WZ-equation

$$f(d+1,k,r) - f(d,k,r) = g(d,k,r+1) - g(d,k,r).$$

Letting $\binom{m}{n} = 0$ when $b > a$, gives that

$$f(d,k,d+1-k) = (-1)\binom{2(d+1-k)}{d+1-k}\binom{2d+1-k}{2d+2-k}\binom{2d+2}{k} = 0.$$

Finally

$$\begin{aligned} \sum_{r=0}^{d+1-k}[(f(d+1,k,r) - f(d,k,r)] &= \sum_{r=0}^{d+1-k}[g(d,k,r+1) - g(d,k,r)] \\ F(d+1,k) - F(d,k) &= g(d,k,d+1-k) - g(d,k,0) \\ &= 0. \end{aligned}$$

This completes the proof that the formula that we obtained for the coordinator polynomial of the growth series of A_n is equivalent to the formula presented by in [5].

Chapter 5

The Root Lattice C_n

We now investigate the structure and the growth series of another of the root lattices, C_n. In this chapter, we demonstrate a regular unimodular triangulation of $\mathcal{P}_{C_n}$ and use the Hilbert series of the facets of $\mathcal{P}_{C_n}$ to derive a formula for the coordinator polynomial for C_n with respect to the natural generators. We also apply Ehrhart theory to derive the coordinator polynomial for C_n.

5.1 The Lattice C_n and Contact Polytope $\mathcal{P}_{C_n}$

The lattice $C_n \in \mathbb{R}^n$ is defined by $C_n := \{\mathbf{x} \in \mathbb{Z}^n : \sum_i x_i \text{ is even}\}$.

Proposition 5.1. *The lattice C_n is a rank n lattice generated as a monoid by $\mathcal{M} = \{\pm 2\mathbf{e}_i,\ \pm\mathbf{e}_i \pm \mathbf{e}_j$ for $i < j\}$.*

To prove this proposition, we denote the vectors in $\mathcal{M}$ by

$$\{\mathbf{v}_{i,j} \ \text{ for } -n \leq i \leq j \leq n,\ i,j \neq 0,\ i \neq -j\}$$

as follows: $\mathbf{e}_i + \mathbf{e}_j = \mathbf{v}_{i,j}$, $(-\mathbf{e}_i) + \mathbf{e}_j = \mathbf{v}_{-i,j}$,$(-\mathbf{e}_i) + (-\mathbf{e}_j) = \mathbf{v}_{-i,-j}$, $\mathbf{e}_i + (-\mathbf{e}_j) = \mathbf{v}_{i,-j}$, $2\mathbf{e}_i = \mathbf{v}_{i,i}$ and $-2\mathbf{e}_i = \mathbf{v}_{-i,-i}$.

Proof of Proposition 5.1. For all $\mathbf{v}_{i,j} \in \mathcal{M}$,$\sum_{k=1}^{n} v_{i,j_k}$ is $-2, 0$ or 2. Choose $\mathbf{u} \in \mathcal{L}_{\mathcal{M}}$. Then $\mathbf{u} = \sum_{i,j} c_{ij}\mathbf{v}_{i,j}$. Thus

$$\sum_k u_k = \sum_k \left(\sum_{i,j} c_{ij} v_{i,j_k} \right) = \sum_{i,j} c_{ij} \left(\sum_k v_{i,j_k} \right) = \left(\sum_k c_k a \right),$$

where a is $-2, 0$ or 2. Thus $\mathbf{u} \in C_n$.

Choose $\mathbf{u} \in C_n$. Construct the coefficients c_{ij} such that $\mathbf{u} = \sum_{i,j} c_{ij}\mathbf{v}_{i,j}$. For each u_i, if $u_i \geq 0$ then $c_{i,i} = \lfloor \frac{1}{2}(u_i) \rfloor$ and $w_i = u_i - 2\lfloor \frac{1}{2}(u_i) \rfloor$. If $u_i < 0$ then $c_{-i,-i} = \lfloor \frac{1}{2}(u_i) \rfloor$ and $w_i = u_i - 2\lfloor \frac{1}{2}(u_i) \rfloor$. Consider the vector $\mathbf{w} = (w_1, \ldots, w_n)$. Each w_k is $1, 0$ or -1, and since $\sum_k u_k$ is even, then we also know that $\sum_k w_k$ is even. Hence the total number of non-zero coordinates of w must be even and we can pair them up. For each pair w_i and w_j of non-zero coordinates of w, let $c_{w_i,w_j} = 1$. All other coefficients $c_{ij} = 0$, so

$$\sum_{i,j} c_{ij}\mathbf{v}_{i,j} = \sum_{i,i} c_{ii}\mathbf{v}_{i,i} + \sum_{i \neq j} c_{ij}\mathbf{v}_{i,j} = (\mathbf{u} - \mathbf{w}) + (\mathbf{w}) = \mathbf{u}.$$

□

Now that we have established that the vectors in $\mathcal{M}$ generate C_n, we will refer to these generators as $\mathcal{M}_{C_n}$. Since $\mathcal{M}$ consists of $2n^2$ vectors, we let the matrix $M_{C_n} \in \mathbb{R}^{n \times 2n^2}$ be the matrix whose columns are the generators in $\mathcal{M}_{C_n}$.

Example 5.1. The lattice C_2 is a rank 2 lattice generated by

$$\begin{aligned} M_{C_2} &= \begin{bmatrix} 2 & 1 & 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & -1 & -2 & -1 \end{bmatrix} \\ &= [\mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \mathbf{v}_{2,2}, \mathbf{v}_{-1,1}, \mathbf{v}_{-1,-1}, \mathbf{v}_{-1,-2}, \mathbf{v}_{-2,-2}, \mathbf{v}_{1,-1}]. \end{aligned}$$

The bold points in Figure 5.1 show a portion of C_2 as a subset of the lattice $\mathbb{Z}^2$. Since $(1,0)$ is not contained in C_2,then C_2 is a proper subset of $\mathbb{Z}^2$.

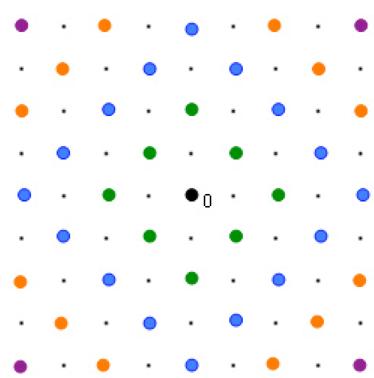

Figure 5.1: C_2 as a subset of $\mathbb{Z}^2$.

Definition 5.1. The **cross-polytope** $\diamond_n$ in $\mathbb{R}^d$ is given by the hyperplane and vertex descriptions

$$\diamond_n := \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_i |x_i| \leq 1 \right\} = \text{conv}(\{\mathbf{e}_1, -\mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_n\}).$$

As in Section 2.2 we define the contact polytope $P_{C_n} = \mathrm{conv}(\mathcal{M}_{C_n})$. It turns out that P_{C_n} is simply a dilation of the cross-polytope, $\mathcal{P}_{C_n} = 2\Diamond_n$.

Example 5.2. The 3-dimensional cross-polytope $\Diamond_3$ is a regular octahedron. Figure 5.2 shows the contact polytope P_{C_3} along with the generators $\mathcal{M}_{C_3}$ in blue. The six gray integer points **inside** of $\mathcal{P}_{C_3}$ are not contained in the lattice C_3. These points are the vertices of $\Diamond_3$. Here we can see that $\mathcal{P}_{C_3} = 2\Diamond_3$.

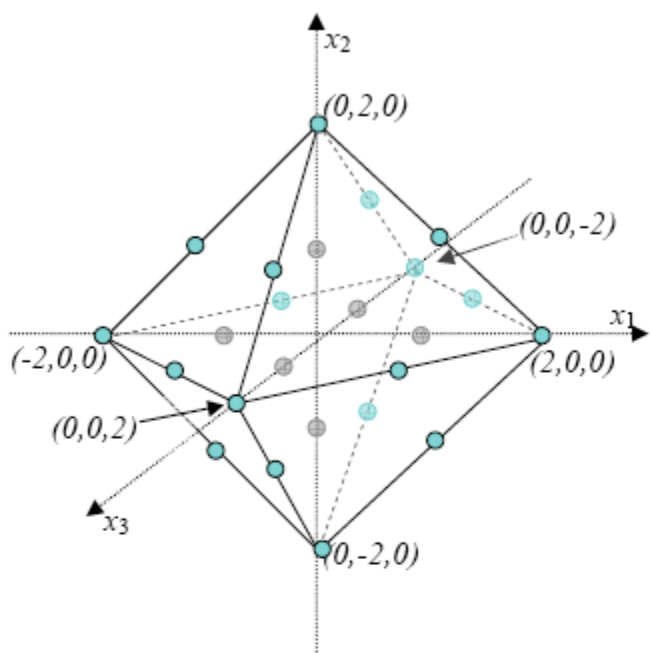

Figure 5.2: The contact polytope P_{C_3} in $\mathbb{R}^3$.

Since the hyperplane $H = \{x \in \mathbb{R}^n : x_k = 2\}$ intersects $\mathcal{P}_{C_n}$ in exactly the point $\mathbf{v}_{k,k}$, we generalize that the vertices of $\mathcal{P}_{C_n}$ are $\mathbf{v}_{k,k}$ where $-n \leq k \leq n$, $k \neq 0$. For any $\mathbf{v}_{i,j} \in \mathcal{M}_{C_n}$ with $i \neq j$, $\mathbf{v}_{i,j} = \frac{1}{2}\mathbf{v}_{i,i} + \frac{1}{2}\mathbf{v}_{j,j}$, thus $\mathbf{v}_{i,j}$ is contained in an edge of $\mathcal{P}_{C_n}$.

Therefore, all of the points in $\mathcal{M}_{C_n}$ are in $\partial\mathcal{P}_{C_n}$ and these are the only integer points contained in $\partial\mathcal{P}_{C_n}$. The only other integer points contained in $\mathcal{P}_{C_n}$ are the vertices of $\diamond_n$ and the origin.

5.2 The Zig-Zag Triangulation of $\mathcal{P}_{C_n}$

In order to apply the theorems from Chapter 3 to the lattice C_n, we must have a regular, unimodular triangulation of $\mathcal{P}_{C_n}$. As with A_d, we can do this by constructing a reverse lexicographic triangulation $\Gamma(P_{C_n})$ using only the points in $\{\mathcal{M}_{C_n} \cup \mathbf{0}\}$. If we let $\mathbf{v}_0 = \mathbf{0}$ be the least element in the ordering, then the triangulation $\Gamma(P_{C_n})$ comes from triangulating each of the facets and then coning over these facets from the origin. Since each facet of P_{C_n} is standard simplex dilated by a factor of two, we will choose to order the vertices in $\mathcal{M}_{C_n}$ in a manner which yields a nice triangulation on these simplices.

The **zig-zag triangulation** is a reverse lexicographic triangulation on $2\triangle_{k-1}$ where the lattice points are labeled $x_{i,j}$ with $i \leq j$ as follows: each of the vertices is $\mathbf{x}_{i,i}$ for $i \in [k]$ and the point in the middle of the edge joining $\mathbf{x}_{i,i}$ with $\mathbf{x}_{j,j}$ is labeled $\mathbf{x}_{i,j}$ (where $i < j$). Define the term order $\prec_z$ by

$$\mathbf{x}_{i_1,j_2} \prec_z \mathbf{x}_{i_1,j_2} \text{ if } i_1 \leq i_2 \text{ and } j_1 \geq j_2.$$

The vertices of the facets of the zig-zag triangulation are given by the maximal chains in this poset.

Example 5.3. Figure 5.3 shows the polytope $\mathcal{P} = 2\triangle_2$ with integer points $V = \{\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \mathbf{x}_{1,3}, \mathbf{x}_{2,2}, \mathbf{x}_{2,3}, \mathbf{x}_{3,3}\}$. Order the integer points in $\mathcal{P}$ using the term order $\prec_z$

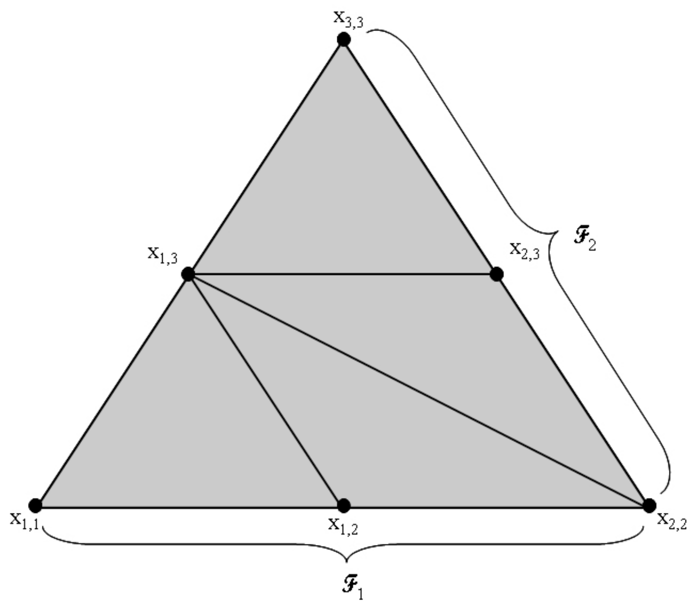

Figure 5.3: The zig-zag triangulation of $2\triangle_2$.

described above. Then $\mathbf{x}_{1,3}$ is the smallest element in this set. $\mathcal{F}_1 = \{\mathbf{x}_{1,1},\ \mathbf{x}_{1,2},\ \mathbf{x}_{2,2}\}$ and $\mathcal{F}_2 = \{\mathbf{x}_{2,2},\ \mathbf{x}_{2,3},\ \mathbf{x}_{3,3}\}$ are the facets of $\mathcal{P}$ not containing $\mathbf{x}_{1,3}$. The minimal elements of $\mathcal{F}_1$ and $\mathcal{F}_2$ are $\mathbf{x}_{1,2}$ and $\mathbf{x}_{2,3}$ respectively. Again we look at the facets of $\mathcal{F}_1$ and $\mathcal{F}_2$ not containing these minimal elements. Since the $\mathcal{F}_1$ and $\mathcal{F}_2$ are vertices, which are trivially affinely independent sets, we are ready to work our way back up by coning over faces of the triangulation from the minimal elements.

$$\begin{aligned}
\Gamma(\mathcal{F}_1) &= \{\{\mathbf{x}_{1,2}, \mathbf{x}_{2,2}\}, \{\mathbf{x}_{1,2}, \mathbf{x}_{1,1}\}\} \\
\Gamma(\mathcal{F}_1) &= \{\{\mathbf{x}_{2,3}, \mathbf{x}_{3,3}\}, \{\mathbf{x}_{2,3}, \mathbf{x}_{2,2}\}\} \\
\Gamma(\mathcal{P}_{C_n}) &= \{\{\mathbf{x}_{1,3},\ \mathbf{x}_{1,2},\ \mathbf{x}_{2,2}\}, \{\mathbf{x}_{1,3},\ \mathbf{x}_{1,2},\ \mathbf{x}_{1,1}\},\ \{\mathbf{x}_{1,3},\ \mathbf{x}_{2,3},\ \mathbf{x}_{3,3}\}, \{\mathbf{x}_{1,3},\ \mathbf{x}_{2,3},\ \mathbf{x}_{2,2}\}\}
\end{aligned}$$

The polytope $\mathcal{P} = 2\triangle_k$ has k vertices $\mathbf{x}_{i,i}$ for $i \in [k+1]$ and $\binom{k}{2} = \frac{k(k-1)}{2}$ points $\mathbf{x}_{i,j}$ with $i < j$, one in the middle of each edge. When these integer points are ordered by $\prec_z$, the point $\mathbf{x}_{1,k}$ is the minimal point of all chains in the poset. Of the k facets in $2\triangle_k$, only 2 will *not* contain $\mathbf{x}_{1,k}$. Since all of the facets of a $k-$simplex are $(k-1)-$simplices, each of these two will have a minimal point contained in all but two facets. This process will repeat until the facets that are examined are 0-simplices, that is vertices of $\mathcal{P}$. When we build up the triangulation by coning over the triangulated facets, there are 2^k facets of the triangulation $\Gamma(\mathcal{P})$.

The triangulation $\Gamma(\mathcal{P}_{C_n})$ is formed by ordering all of the vertices in $\mathbf{v}_{i,j} \in \mathcal{M}_{C_n}$ by $\prec_z$, forming the triangulation $\Gamma(\partial\mathcal{P}_{C_n})$ of the boundary of $\mathcal{P}_{C_n}$, then coning over each of facets of $\Gamma(\partial\mathcal{P}_{C_n})$ from the origin. Since each of the n-simplices in $\Gamma(\mathcal{P}_{C_n})$ contain the origin and $n-1$ affinely independent points from $\mathcal{M}_{C_n}$, these simplices will be unimodular in the lattice generated by $\mathcal{M}_{Cn}$. From the theorems in Chapter 3, the coordinator polynomial for C_n must be the h-polynomial of a simplicial complex. The Dehn-Sommerville Relations imply that the coordinator polynomial must also be palindromic.

5.2.1 A Patchwork of Hilbert Series

While the zigzag triangulation is quite simple to explain and apply, it does not provide an easy method for counting the face numbers. Here we give a different derivation of the formula for the coordinator polynomial using the Hilbert series of the facets.

We have already shown that for $0 < k < n$ every k-face of $\mathcal{P}_{C_n}$ is $2\triangle_k$. The Hilbert series for $2\triangle_k$ is given in [8, Corollary 2.6] as

$$H(K[2\triangle_k], x) = \frac{h_k(x)}{(1-x)^k} = \frac{\sum_{i=0}^{k} \binom{k}{2i} x^i}{(1-x)^k}.$$

Note. Here we are following the convention $\binom{n}{m} = 0$ if $m > n$.

The Hilbert series for the boundary complex of $\mathcal{P}_{C_n}$ can be constructed by a series of inclusions and exclusions. The product

$$f_{n-1}\frac{h_{n-1}(x)}{(1-x)^{n-1}}$$

is the sum of the Hilbert series for each of the facets; however, this product double counts the series on each of the $(n-2)$-faces. So we must subtract those. Then add back in the sum of the series for the $(n-3)$-faces and so on. The resulting growth series for C_n is

$$G(x) = \sum_{j=0}^{n}(-1)^{n-j}\left(f_{j-1}\right)\frac{h_{j-1}}{(1-x)^{j-1}}$$

where $f\_1 = 1$ and f_{j-1} is the number of $j-1$-faces of P_{C_n}. By the duality of the n-dimensional cross-polytope and the n-dimensional hypercube, we can replace f_{j-1} with g_{n-j} the number of $n-j$-faces of the n-dimensional hypercube. The number of $n-j$-faces of an n-dimensional hypercube is given by $g_{n-j} = \binom{n}{n} - j\,2^j$, thus

$$G(x) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{n-j}2^j\,\frac{\sum_{i=0}^{j-1}\binom{j-1}{2i}x^i}{(1-x)^{j-1}}.$$

Example 5.4. For the lattice C_3, we first compute the Hilbert series for the faces of $\mathcal{P}_{C_n}$: $h_2 = 1+3x$, $h_1 = 1+x$ and $h_0 = 1$. The f-vector for $\mathcal{P}_{C_n}$ is $(6, 12, 8)$ so the growth series for C_n is

$$\begin{aligned} G(x) &= -1 + \frac{6(1)}{(1-x)} - \frac{12(1+x)}{(1-x)^2} + \frac{8(1+3x)}{(1-x)^3} \\ &= \frac{1+15x+15x^2+x^3}{(1-x)^3}. \end{aligned}$$

While this formula is correct and makes good use of the Hilbert series, it is quite cumbersome to use in practice. In the next section we use the Ehrhart series of $\mathcal{P}_{C_n}$ to re-derive the concise formula for the coordinator polynomial of C_n given in [1].

5.3 Finding the Growth Series from the Ehrhart Polynomial

The growth function $\mathrm{S}_{C_n}(k)$ will count the number of point in the lattice C_n with word length k relative to the generators $\mathcal{M}_{C_n}$.

Proposition 5.2. *For* $\mathbf{u} \in C_n$, $\mathrm{w}(\mathbf{u}) = k$ *if and only if* $\mathbf{u}$ *is contained in the boundary the* k^{th} *dilate of* $\mathcal{P}_{C_n}$.

Proof. Choose $\mathbf{u} \in \mathcal{C}_n$. Since $\sum u_i$ is even, then u must be an even combination of the generators of $\diamond_n$. The $\mathrm{w}(\mathbf{u})$ with in the lattice generated by the vertices of $\diamond_n$ is $2m$ for some integer m, thus $u \in \partial(2m\diamond_n) = \partial(m(2\diamond_n)) = \partial(m\mathcal{P}_{C_n})$. So every $\mathbf{u} \in \mathcal{C}_n$ lies on the boundary of a dilate of $\mathcal{P}_{C_n}$.

Assume $\mathrm{w}(\mathbf{u}) = k$ with respect to $\mathcal{M}_{C_n}$. $\exists \{c_{ij}\}$ such that

$$\mathbf{u} = \sum_{i,j} c_{ij}\mathbf{v}_{i,j} \text{ where } \sum_{i,j} c_{ij} = k.$$

$$\mathbf{u} = \sum_{i,j} \frac{c_{ij}}{k}(k\mathbf{v}_{i,j}) \text{ where } \sum_{i,j} \frac{c_{ij}}{k} = 1.$$

Thus, $\mathbf{u} \in k(\mathcal{P}_{C_n})$. If $\mathbf{u}$ is not on the boundary of $k\mathcal{P}_{C_n}$, then $\mathbf{u} \in k'\mathcal{P}_{C_n}$ where $k' < k$ so $\mathbf{u}$ can be represented as a sum of fewer than k vectors, which contradicts $\mathrm{w}(\mathbf{u}) = k$. So $\mathbf{u} \in \partial(k\mathcal{P}_{C_n})$. Conversely, if $\mathbf{u} \in \partial(k\mathcal{P}_{C_n})$ then $\mathbf{u} \in \partial(2k\diamondsuit_n)$. If $\mathrm{w}(\mathbf{u}) = k'$, then by writing each of the vectors $\mathbf{v}_{i,j}$ as a sum of two vertices of $\diamondsuit_n$ we can see that $\mathbf{u}$ must lie on $\partial(2k'\diamondsuit_n)$. Hence $2k' = 2k$ and $\mathrm{w}(\mathbf{u}) = k$. $\square$

Since $k\mathcal{P}_{C_n} = 2k\diamond_n$, Proposition 5.2 implies that the growth function $S_{C_n}(k)$ actually counts the number of integer points on the boundary of $2k\diamond_n$. It turns out that the question of counting integer points contained in any dilation of $k\diamond_n$ is quite straight forward. To count only the points on the boundary, we can simply count all of the integer points in $k\diamond_n$ and then subtract the points in the interior of $k\diamond_n$. Fortunately, the points in the interior of $k\diamond_n$ are just the integer points contained in $(k-1)\diamond_n$. To show just how this works, we need a few definitions about counting integer points in polytopes.

Definition 5.2. [3] Given a finite set S, the function $\#(S)$ counts the number of elements in S. For a polytope $\mathcal{P} \in \mathbb{R}^n$, the **lattice-point enumerator** $L_{\mathcal{P}}(t)$ is a function that counts the number of lattice points contained in the t^{th} dilate of $\mathcal{P}$. Formally, $L_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z})$. The **Ehrhart series** of $\mathcal{P}$ is defined to be the generating function

$$\mathrm{Ehr}_{\mathcal{P}}(x) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) x^t.$$

The cross polytope $\diamond_n$ has a particularly nice feature which will help with determining a formula for $\mathrm{Ehr}_{\diamond_n}(x)$. For any $n > 0, \diamond_n$ is a **bipyramid over** $\diamond_{(n-1)}$. Thus, the number of lattice points in $t\diamond_n$ can be found as a sum of lattice points in dilations of $L_{\diamond_{(n-1)}}$ as follows.

$$\begin{aligned} L_{\diamond_n}(t) &= 2 + 2L_{\diamond_{n-1}}(1) + 2L_{\diamond_{n-1}}(2) + \cdots + 2L_{\diamond_{n-1}}(t-1) + L_{\diamond_{n-1}}(t) \\ &= 2 + \sum_{j=1}^{t-1} 2L_{\diamond_{n-1}}(j) + L_{\diamond_{n-1}}(t). \end{aligned} \tag{5.1}$$

This identity allows us to compute the Ehrhart series of $\diamond_n$ recursively resulting in the following formula.

Theorem 5.3. *The Ehrhart series of the n-dimensional cross polytope is*

$$\mathrm{Ehr}_{\diamond_n}(x) = \sum_{t\geq 0} L_{\diamond_n} x^t = \frac{(1+x)^n}{(1-x)^{n+1}}.$$

Proof. $\mathrm{Ehr}_{\diamond_n}(x) = \sum_{t\geq 0} L_{\diamond_n} x^t = 1 + \sum_{t\geq 1} L_{\diamond_n} x^t$. Substituting equation 5.1 into the definition of the Ehrhart series for $\diamond_n$ gives

$$\begin{aligned}
\mathrm{Ehr}_{\diamond_n}(x) &= 1 + \sum_{t\geq 1} L_{\diamond_n}(t)x^t = 1 + \sum_{t\geq 1}\left(2 + \sum_{j=1}^{t-1} 2L_{\diamond_{n-1}}(j) + L_{\diamond_{n-1}}(t)\right)x^t \\
&= -1 + \sum_{t\geq 0} 2x^t + \sum_{t\geq 0}\sum_{j=1}^{t-1} 2L_{\diamond_{n-1}}(j)x^t + \sum_{t\geq 0} L_{\diamond_{n-1}}(t)x^t \\
&= \frac{2}{1-x} + 2\sum_{j\geq 1}\left(L_{\diamond_{n-1}}(j)\sum_{t\geq j} x^t\right) - \left(1 + \sum_{t\geq 1} L_{\diamond_{n-1}}(t)x^t\right) \\
&= \frac{2}{1-x} + 2\sum_{j\geq 1} L_{\diamond_{n-1}}(j)\frac{x^j}{1-x} - \mathrm{Ehr}_{\diamond_{n-1}}(x) \\
&= \frac{2(1+\sum L_{\diamond_{n-1}} x^j)}{1-x} - \frac{(1-x)\,\mathrm{Ehr}_{\diamond_{n-1}}(x)}{1-x} \\
&= \frac{2\,\mathrm{Ehr}_{\diamond_{n-1}}(x) - (1-x)\,\mathrm{Ehr}_{\diamond_{n-1}}(x)}{1-x} \\
&= \frac{1+x}{1-x}\,\mathrm{Ehr}_{\diamond_{n-1}}(x).
\end{aligned}$$

A bit of induction will yield the desired result. Since $\diamond_0$ is simply the origin,$L_{\diamond_0} = 1$ and $\mathrm{Ehr}_{\diamond_0} = 1 + \sum_{t\geq 1} x^t = \frac{1}{1-x} = \frac{(1+x)^0}{(1-x)^1}$. Assume that for $k < n$, $\mathrm{Ehr}_{\diamond_k}(x) = \frac{(1+x)^k}{(1-x)^{k+1}}$.

Then, by the recursive relationship above,

$$\begin{aligned}\mathrm{Ehr}_{\diamond_n}(x) &= \frac{(1+x)}{(1-x)}\,\mathrm{Ehr}_{\diamond_{n-1}}(x) \\ &= \frac{(1+x)}{(1-x)}\frac{(1+x)^{n-1}}{(1-x)^n} \\ &= \frac{(1+x)^n}{(1-x)^{n+1}}.\end{aligned}$$

□

We now make the connection back to the growth series for C_n. By Proposition 5.2, the points $u \in C_n$ with word length k are exactly the integer points contained in the boundary of $(2k)\Diamond_n$. So $\mathrm{S}_{c_n}(k) = L_{\diamond_n}(2k) - L_{\diamond_n}(2k-1)$. This gives the growth series of C_n as

$$G(x) = \sum_{k\geq 0} \mathrm{S}(k)x^k = 1 + \sum_{k\geq 1} \left(L_{\diamond_n}(2k) - L_{\diamond_n}(2k-1)\right) x^k$$

The change of variable $y = \sqrt{x}$, gives

$$G(y^2) = \sum_{k\geq 0} L_{\diamond_n}(2k)y^{2k} - y\sum_{k\geq 1} L\Diamond_n(2k-1)y^{2k-1}.$$

The coefficients in the series $\sum_{k\geq 0} L_{\diamond_n}(2k)y^{2k}$ are simply the coefficients of the *even* terms of $\mathrm{Ehr}_{\diamond_n}(y)$ and the coefficients of $\sum_{k\geq 1} L\Diamond_n(2k-1)y^{2k-1}$ are the coefficients of the odd terms of $\mathrm{Ehr}_{\diamond_n}(y)$. We now use a little trick from [14] to pick out the desired coefficients.

$$\begin{aligned}\sum_{k\geq 0} L_{\Diamond_n}(2k)y^{(2k)} &= \frac{1}{2}\left(\mathrm{Ehr}_{\Diamond_n}(y) + \mathrm{Ehr}_{\Diamond_n}(-y)\right)\\ \sum_{k\geq 1} L_{\Diamond_n}(2k-1)y^{(2k-1)} &= \frac{1}{2}\left(\mathrm{Ehr}_{\Diamond_n}(y) - \mathrm{Ehr}_{\Diamond_n}(-y)\right).\end{aligned}$$

Replacing these expressions in the growth series $G(y^2)$ gives

$$\begin{aligned}G(y^2) &= \sum_{k\geq 0} L_{\Diamond_n}(2k)y^{2k} - y\sum_{k\geq 1} L\Diamond_n(2k-1)y^{2k-1}\\ &= \frac{1}{2}\left[\mathrm{Ehr}_{\Diamond_n}(y) + \mathrm{Ehr}_{\Diamond_n}(-y) - y(\mathrm{Ehr}_{\Diamond_n}(y) - \mathrm{Ehr}_{\Diamond_n}(-y))\right]\\ &= \frac{1}{2}\left[(1-y)\,\mathrm{Ehr}_{\Diamond_n}(y) + (1+y)\,\mathrm{Ehr}_{\Diamond_n}(-y)\right]\\ &= \frac{1}{2}\left[(1-y)\frac{(1+y)^n}{(1-y)^{n+1}} + (1+y)\frac{(1-y)^n}{(1+y)^{n+1}}\right]\\ &= \frac{1}{2}\left[\frac{(1+y)^n}{(1-y)^n} + \frac{(1-y)^n}{(1+y)^n}\right]\\ &= \frac{1}{2}\left[\frac{(1+y)^{2n} + (1-y)^{2n}}{(1+y)^n(1-y)^n}\right]\\ &= \frac{\frac{1}{2}[(1+y)^{2n} + (1-y)^{2n}]}{(1-y^2)^n}.\end{aligned} \tag{5.2}$$

Replacing y with $\sqrt{x}$ gives the following formula for the growth series of C_n.

$$G_{C_n}(x) = \frac{\frac{1}{2}[(1+\sqrt{x})^{2n} + (1-\sqrt{x})^{2n}]}{(1-x)^n}.$$

We simplify the coordinator polynomial a bit further by expanding and combining like terms:

$$(1+\sqrt{x})^{2n} = 1 + \binom{2n}{1}\sqrt{x} + \binom{2n}{2}(\sqrt{x})^2 + \binom{2n}{3}(\sqrt{x})^3 + \cdots + \binom{2n}{2n}(\sqrt{x})^{2n}$$

$$(1-\sqrt{x})^{2n} = 1 - \binom{2n}{1}\sqrt{x} + \binom{2n}{2}(\sqrt{x})^2 - \binom{2n}{3}(\sqrt{x})^3 + \cdots + \binom{2n}{2n}(\sqrt{x})^{2n}.$$

$$\begin{aligned} h_{C_n}(x) &= \frac{1}{2}[(1+\sqrt{x})^{2n} + (1-\sqrt{x})^{2n}] \\ &= 2 + 2\binom{2n}{2}x + \binom{2n}{4}x^2 + \cdots + \binom{2n}{2n}x^n \\ &= \sum_{k=0}^{n}\binom{2n}{2k}(x)^k. \end{aligned}$$

We summarize this section with the final theorem:

Theorem 1.3. *The coordinator polynomial for the lattice C_n generated as a monoid by the standard generators $\mathcal{M}_{C_n} = \{\pm\mathbf{e}_i \pm \mathbf{e}_j : 0 \leq i, j \leq n\}$ is given by*

$$h(x) = \sum_{k=0}^{n} \left(\begin{array}{c} 2n \\ 2k \end{array} \right) x^k.$$

Chapter 6

Conclusion

In this thesis, we have derived the growth series using Hilbert series and triangulation of the polytopes formed by the natural monoid generator, or roots, of these lattices. In the case of C_n, while we were able to produce a regular unimodular triangulation to satisfy Theorem 3.4, we did not use that triangulation to produce the formula for the coordinator polynomial. We are still interested in finding a method for counting the faces of the zig-zag triangulation of $\mathcal{P}_{C_n}$ or finding another triangulation which is more easily counted.

The coordinator series for the root lattice D_n is given in [1] by

$$h(x) = \sum_{k=0}^{n} \left[\binom{2n}{2k} - \frac{2k(n-k)}{n-1} \binom{n}{k} \right] x^k.$$

Since this formula is palindromic, the *Dehn-Sommerville Relations* suggest that it is the h-polynomial for a simplicial complex. We would like to extend the results from this paper to include a specific triangulation of D_n that would enable us to compute the growth series.

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