d-Fold Partition Diamonds through Posets

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Abstract

In this thesis we consider the generalization of plane partitions known as d-fold partition diamonds first introduced by George Andrews, Peter Paule, and Axel Riese counted by $r_d(n)$. We also consider the Schmidt type d-fold partition diamond counting function $s_d(n)$. We look at these structures from the lens of posets and use poset geometry to compute their generating functions. Through this we connect d-fold partition diamonds to Euler-Mahonian polynomials. This work yields an improvement over work by Dockery et al. as we obtain that Euler-Mahonian connection along with an abstracted form that allows for the efficient computing of identities for the generating functions of analogous partition diamond constructions. One in particular is a further generalization of the Dockery et al. d-fold partition diamonds which we call multifold partition diamonds.

Acknowledgments

I want to thank my advisor Dr. Beck for every bit of guidance and support he has provided me throughout this thesis. I very much enjoyed working with him. I thank the faculty of the math department at SFSU for facilitating my mathematical growth and making me more confident as a mathematician. I thank my mother for giving everything in order to allow me to pursue something I love.

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Introduction

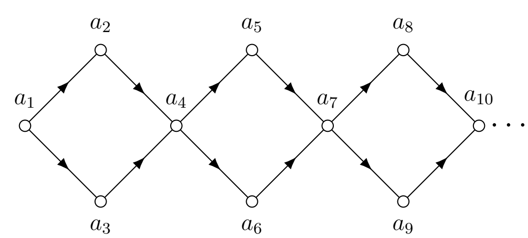

Figure 1.1: A plane partiton diamond.

In 2001, plane partition diamonds were defined by George Andrews, Peter Paule, and Axel Riese in [APR01] as integer partitions whose parts are given by the a_i's in Figure 1.1 where each directed edge represents $\geq$. In 2024, Dalen Dockery, Marie Jameson, James A. Sellers, and Samuel Wilson in [Doc+24] sought to generalize this partition construction and defined d-fold partition diamonds; partitions whose parts are given by the a_i's in Figure 1.2. Notice that Andrews, Paule, and Riese's plane partition diamond is the d-fold partition diamond for $d = 2$. Both papers computed the generating function for the counting function of these d-fold partition diamonds by means of MacMahon's partition analysis and his Ω-operator.

In this paper, we seek to look at these partitions through the lens of partially ordered sets. We will reprove and extend the results of Dockery, Jameson, Sellers, and Wilson through machinery related to posets and geometry. This ends up being versatile in computing the generating functions for analogous and further generalized constructions of these partitions. For these purposes, we define d-fold partition diamonds inversely to that of the the cited papers. Refer to Definition 2.1 and Definition 2.2 for formal definitions of a partition of n and a poset respectively.

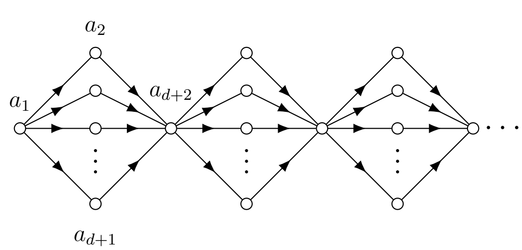

Figure 1.2: A d-fold partition diamond.

Definition 1.1. *We define a d-**fold partition diamond** of n to be a partition of n whose parts are given by the non-negative integers a_i in Figure 1.2 where each directed edge indicates a $\leq$. Let the corresponding counting function be given by $r_d(n)$.*

This construction allows for a more natural interpretation of Figure 1.2 as a poset with a minimum element. Before we tackle the infinite number of diamonds given in Figure 1.2, we seek to understand a finite variation.

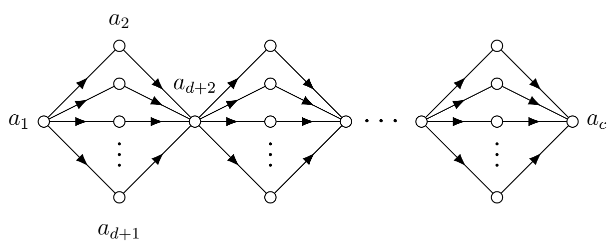

Figure 1.3: A d-fold partition diamond of length M.

Definition 1.2. *We define a* d***-fold partition diamond of*** n ***of length*** M *to be a partition whose parts are given by the non-negative integers* a_i *in Figure 1.3 for* $i = 1, 2, \ldots, c$ *where* $c = M(d+1)+1$ *and each directed edge corresponds to a* $\leq$*. Let the corresponding counting function be given by* $r_{d,M}(n)$*.*

Notice that in the finite case, this construction is nearly identical to that of Dockery, Jameson, Sellers, and Wilson. We also formally define the corresponding poset.

Definition 1.3. *We define a* d***-fold partition diamond poset of length*** M *to be the poset whose elements are the non-negative integers* a_i *in Figure 1.3 and whose partial order is given by the same figure where each directed edge indicates a* $\leq$*.*

When discussing this poset, we will be thinking of it as a finite poset, so we sometimes omit stating its length unless necessary. We now state the main results of the paper.

Theorem 1.1. *Let $\Pi = [c]$ be a naturally labeled d-fold partition diamond poset and let*

$$z_i = \begin{cases} a & \text{if } i \not\equiv 1 \text{ mod } (d+1), \\ b & \text{if } i \equiv 1 \text{ mod } (d+1). \end{cases}$$

Then,

$$\sigma_{K_\Pi}(\boldsymbol{z}) = \frac{\prod_{n=1}^{M} E_d(a^{(n-1)d} b^n, a)}{(1 - a^{Md} b^{M+1}) \prod_{n=1}^{M} \prod_{j=0}^{d} (1 - a^{d-j} a^{(n-1)d} b^n)}$$

where $\sigma_{K_\Pi}(\boldsymbol{z})$ is the integer-point transform of the order cone K_Π (see Definitions 2.5 and 2.4 respectively) and E_d is the Euler-Mahonian polynomial (see Definition 3.4).

Specializing $a = b = q$ in Theorem 1.1 gives the following result.

Corollary 1.1. *The generating function for the number of d-fold partition diamonds is given by*

$$\sum_{n=0}^{\infty} r_d(n) q^n = \prod_{n=1}^{\infty} \frac{E_d(q^{1+(n-1)(d+1)}, q)}{1 - q^n}.$$

Dockery, Jameson, Sellers, and Wilson give a similar result in [Doc+24] using a polynomial F_d, which they define via a recursion, with the exact same inputs as our polynomial E_d. As of now, the current literature for the Euler-Mahonian polynomial does not show a matching recursion. However, the similarities between our results suggest $F_d = E_d$ and their recursion is a recursion for the Euler-Mahonian polynomial. Corollary 1.1 is an improvement over the result in [Doc+24] in that our numerator has a closed form with a known structure behind it. In addition, it is a corollary to a more abstract theorem that shows versatility and ease in computing these identities. An analogous result comes in the form of the Schmidt type

d-fold partition diamond where we only sum over the connecting nodes of the the d-fold partition diamond. This result is given by specializing $a = 1$ and $b = q$ in Theorem 1.1.

Definition 1.4. *A* ***Schmidt type*** d***-fold partition diamond*** *of n is a partition of n whose parts are given by the connecting nodes of a d-fold partition diamond. Let its counting function be $s_d(n)$.*

Dockery et al. give the generating function of $s_d(n)$. In this paper we give an alternative proof using our poset machinery.

Corollary 1.2. *The generating function for the number of Schmidt type d-fold partition diamonds is given by*

$$\sum_{n=0}^{\infty} s_d(n)q^n = \prod_{n=1}^{\infty} \frac{A_d(q^n)}{(1-q^n)^{d+1}}$$

where A_d is the Eulerian polynomial (see Definition 3.4).

We also introduce a generalized construction where each diamond has a different number of folds.

Definition 1.5. *Let $\{d_i\}_{i=1}^{M}$ be some sequence of M positive integers. We define a* ***multifold partition diamond of*** n ***of length*** M *corresponding to this sequence to be a partition whose parts are given by a similar structure of Figure 1.3 and Definition 1.2, but the i^{th} diamond has d_i folds. Let the corresponding counting function be $r_{\{d_i\},M}(n)$. We similarly define the* ***multifold partition diamond poset of length*** M *to Definition 1.3.*

Theorem 1.2. *Let* $\{d_i\}_{i=1}^{M}$ *be some sequence of* M *positive integers and let* $\omega_k = \sum_{i=k+1}^{M} d_i$*. Let* Ξ *be a naturally labeled multifold partition diamond poset of length* M *and let*

$$z_i = \begin{cases} a & \text{if } i \neq 1 + k + \sum_{j=1}^{k} d_j \quad \text{for all } k, \\ b & \text{if } i = 1 + k + \sum_{j=1}^{k} d_j \quad \text{for some } k. \end{cases}$$

Then,

$$\sigma_{K_\Xi}(\boldsymbol{z}) = \frac{\prod_{k=1}^{M} E_{d_k}(a^{\omega_k} b^{1+M-k}, a)}{(1 - a^{\omega_0} b^{M+1}) \prod_{k=1}^{M} \prod_{j=0}^{d_k} (1 - a^{d_k - j} a^{\omega_k} b^{1+M-k})}.$$

Chapter 2

Background

Almost all definitions, results, and corresponding proofs in this section are provided in various chapters of [BS18]. We reference the said results accordingly.

Definition 2.1. *A* ***partition*** *of* $n \in \mathbb{Z}_{\geq 0}$ *is a multiset of positive integers whose sum is* n*. The elements of this multiset are known as the* ***parts*** *of the partition.*

Definition 2.2. *A* ***partially ordered set****, often times referred to as* ***poset****, is a pair* $(P, \leq)$ *where* P *is a set and* $\leq$ *is a relation on* P *such that for all* $x, y, z \in P$*,*

$$x \leq x,$$

$$\text{if } x \leq y \text{ and } y \leq x \text{ then } x = y,$$

$$\text{if } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

2.1 Order Cones and Integer-Point Transforms

We will first be looking at posets from a geometric point of view. The theory in this section will only be dealing with finite posets $(\Pi, \leq_\Pi)$. As a result, we may assume $\Pi = [c] := \{1, 2, \ldots, c\}$ where $\leq_\Pi$ is some partial order on the elements of $[c]$. This allows us to label the elements of Π in a convenient way.

Definition 2.3. *We say* Π *is* ***naturally labelled*** *if* $i \leq_\Pi j$ *implies* $i \leq j$ *for all* $i, j \in [d]$.

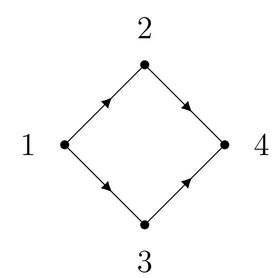

Figure 2.1: Naturally labeled 2-fold partition diamond poset of length 1.

In Figure 2.1 we provide a running example of a 2-fold partition diamond. Notice that it is labelled naturally. Now consider the set of functions from Π to $\mathbb{R}$, $\mathbb{R}^\Pi$. Notice that if $\Pi = [c]$, then $\mathbb{R}^\Pi \cong \mathbb{R}^c$. We will be considering the subset of this set of functions that contains all order preserving functions from Π to $\mathbb{R}_{\geq 0}$.

Definition 2.4. *We define the* ***order cone of*** Π *to be the set of all order preserving functions in* $\mathbb{R}^\Pi_{\geq 0}$,

$$K_\Pi := \left\{ \phi : \Pi \to \mathbb{R}_{\geq 0} \;\middle|\; \phi(a) \leq \phi(b) \qquad \textit{for all } a \leq_\Pi b \right\}.$$

Now consider the order cone for a d-fold partition diamond poset of length M. Notice that every d-fold partition diamond of n corresponds to an integer point in this order cone. The order cone corresponding to Figure 2.1 are the set of points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_1 \le x_2$, $x_1 \le x_3$, $x_2 \le x_4$, and $x_3 \le x_4$ where each point corresponds to a 2-fold partition diamond of some n. As a result, enumerating d-fold partition diamonds is equivalent to enumerating integer points in the corresponding order cone. This motivates the study of multivariate generating functions.

Definition 2.5. *The **integer-point transform** of a set $S \subset \mathbb{R}^c$ is defined as*

$$\sigma_S(z_1, \ldots, z_c) := \sum_{\boldsymbol{m} \in S \cap \mathbb{Z}^c} z_1^{m_1} z_2^{m_2} \cdots z_c^{m_c}.$$

For shorter notation, we will write

$$\sigma_S(\mathbf{z}) := \sum_{\boldsymbol{m} \in S \cap \mathbb{Z}^c} \mathbf{z}^{\boldsymbol{m}}$$

where $\mathbf{z}^{\boldsymbol{m}} = z_1^{m_1} z_2^{m_2} \cdots z_c^{m_c}$.

Notice that if we specialize $z_1 = \cdots = z_c = q$ and let S be the order cone corresponding to d-fold partition diamonds, we have exactly the generating function for $r_{d,M}(n)$. Eventually, we will find the integer point transform of K_Π and specialize it in this way. A simple property of integer point transforms to keep note of is the following: for $C = \uplus_{i=1}^n S_i$,

$$\sigma_C(\mathbf{z}) = \sum_{i=1}^{n} \sigma_{S_i}(\mathbf{z}).$$

Decomposing our order cone into smaller disjoint cones will prove to be quite useful. Our goal is to do this by avoiding inclusion-exclusion. We will be studying half-open decompositions of cones and what their integer point transforms look like to accomplish just that.

2.2 Half Open Decompositions

Definition 2.6. *We say P is a* ***polyhedron*** *if*

$$P = \{\, \boldsymbol{x} \in \mathbb{R}^c \,|\, A\boldsymbol{x} \leq \boldsymbol{b} \,\}$$

for some $A \in \mathbb{R}^{m\times c}$ and $\boldsymbol{b} \in \mathbb{R}^m$. The plural of polyhedron is ***polyhedra****. For some $\boldsymbol{a} \in \mathbb{R}^c \setminus \{\boldsymbol{0}\}$ and $b \in \mathbb{R}$, we call the sets*

$$H := \{\, \boldsymbol{x} \in \mathbb{R}^c \;\mid\; \langle \boldsymbol{a}, \boldsymbol{x} \rangle = b \,\}$$

$$H^{\leq} := \{\, \boldsymbol{x} \in \mathbb{R}^c \;\mid\; \langle \boldsymbol{a}, \boldsymbol{x} \rangle \leq b \,\}$$

a ***hyperplane*** *and a* ***halfspace*** *respectively. We say a hyperplane H is* ***admissible*** *for a polyhedron $P \subset \mathbb{R}^c$ if $P \subset H^{\leq}$. A set of the form $F = P \cap H$ is known as a* ***face*** *of P if H is admissible. The* ***dimension of a polyhedron*** *P in $\mathbb{R}^c$ is the dimension of the smallest affine subspace of $\mathbb{R}^c$ it is contained in. We call the faces of a full dimensional polyhedron $P \subset \mathbb{R}^c$ with dimension $c-1$ the* ***facets*** *of P.*

Definition 2.7. *A* ***dissection*** *of a polyhedron $P \subset \mathbb{R}^c$ is a collection of polyhedra $P_1, \ldots, P_m$ of the same dimension such that*

$$P = P_1 \cup \cdots \cup P_m \quad \text{and} \quad P_i^{\circ} \neq P_j^{\circ} \quad \text{for} \quad i \neq j.$$

Later we will go over how we can construct a dissection for our order cones, but for now let us assume we have a general polyhedron and a dissection of said polyhedron.

Definition 2.8. *Let $P \subset \mathbb{R}^c$ be a polyhedron and $q \in \mathbb{R}^c$. The* ***tangent cone*** *of P at q is defined as*

$$T_q(P) := \{\, q + u \,|\, q + \varepsilon u \in P \quad \text{for all } \varepsilon > 0 \text{ sufficiently small} \,\}\,.$$

The ***tangent cone*** *of P at face $F \subset P$ is defined as*

$$T_F(P) := T_q(P) \quad \text{for all } q \in F^\circ.$$

We say F is ***visible*** *from $p \in \mathbb{R}^c$ if $p \notin T_F$.*

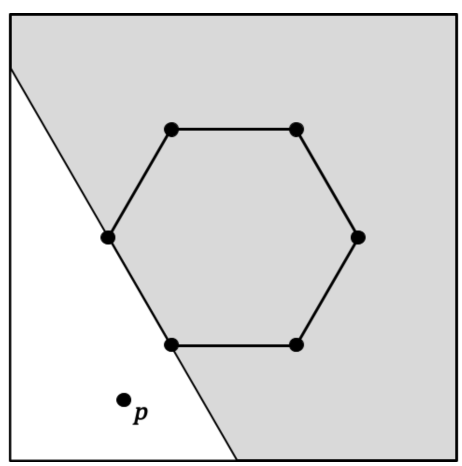

Figure 2.2: Edge of a hexagon visible from p

Note that $T_F(P)$ is well defined as every point in F° gives rise to the same tangent cone.

Definition 2.9. *Let $P \subset \mathbb{R}^c$ be a polyhedron with facets $F_1, \ldots, F_m$.*

1. *We say a point* $q \in \mathbb{R}^c$ *is* ***generic relative to*** P *if* q *is not contained in any facet defining hyperplane of* P.

2. *Let* $P \subset \mathbb{R}^d$ *and* q *be generic relative to* P. *We define the* ***half-open polyhedra***

$$\mathbb{H}_q(P) := P \setminus \bigcup_{j \in I} F_j$$

$$\mathbb{H}^q(P) := P \setminus \bigcup_{j \notin I} F_j$$

where $I := \{ j \mid F_j \text{ is visible from } q \}$.

Intuitively, in $\mathbb{H}_q(P)$, we are taking out the facets of P that are visible from q, i.e., q is not in their tangent cones. In $\mathbb{H}^q(P)$ we are taking out those that are not visible from q. This definition provides us with the following lemma with proofs provided in chapter 5 of [BS18].

Lemma 2.1. *Let* $P \subset \mathbb{R}^c$ *be a full dimensional polyhedron with dissection* $P = P_1 \cup P_2 \cdots \cup P_m$. *If* $q \in \mathbb{R}^c$ *is generic relative to each* P_i, *then*

$$\mathbb{H}_q(P) = \mathbb{H}_q(P_1) \uplus \cdots \uplus \mathbb{H}_q(P_m)$$

and

$$\mathbb{H}^q(P) = \mathbb{H}^q(P_1) \uplus \cdots \uplus \mathbb{H}^q(P_m).$$

Notice that if $q \in P^\circ$, then $\mathbb{H}_q(P) = P$ and $\mathbb{H}^q(P) = P^\circ$, giving the following.

Corollary 2.1. *Let* $P \subset \mathbb{R}^c$ *be a full dimensional polyhedron with dissection* $P = P_1 \cup P_2 \cdots \cup P_m$. *If* $q \in P^\circ$ *is generic relative to each* P_i, *then*

$$P = \mathbb{H}_q(P_1) \uplus \cdots \uplus \mathbb{H}_q(P_m)$$

and

$$P^\circ = \mathbb{H}^q(P_1) \uplus \cdots \uplus \mathbb{H}^q(P_m).$$

This is known as a half open decomposition of P. We will later formalize this in the language of order cones.

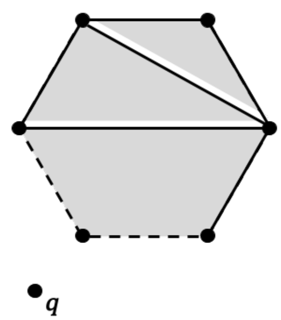

Figure 2.3: Half-open dissection $\mathbb{H}_q(P)$ of a hexagon P according to Lemma 2.1

2.3 Unimodular Cones

Definition 2.10.

a) *We define a* ***cone*** *generated by* $S = \{ s_1, \ldots, s_m \}$ *such that* $s_i \in \mathbb{R}^c$ *as*

$$cone(S) = \mathbb{R}_{\geq 0}s_1 + \cdots + \mathbb{R}_{\geq 0}s_m.$$

b) *We say a cone* $C = cone(\{ v_1, \ldots, v_m \})$ *is* ***simplicial*** *if its set of generators are linearly independent.*

c) *We say a cone $C \subset \mathbb{R}^c$ is* ***unimodular*** *if its set of primitive generators form a lattice basis for $\mathbb{Z}^c$.*

Notice that cones are polyhedra and that unimodular implies simplicial. We will show later that we can decompose an order cone K_Π into unimodular cones, so those are the cones we will be considering in this paper. The below lemma gives insight on the computation of the integer point transform of a half open unimodular cone. See chapter 4 of [BS18] for more insight.

Lemma 2.2. *Let $C \subset \mathbb{R}^c_{\geq 0}$ be a unimodular cone and $q \in \mathbb{R}^c$ be generic relative to C with*

$$\mathbb{H}_q(C) = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_{m-1} + \mathbb{R}_{>0}v_m + \cdots + \mathbb{R}_{>0}v_c.$$

Then

$$\sigma_{\mathbb{H}_q(C)}(\boldsymbol{z}) = \frac{\mathbf{z}^{v_m+\cdots+v_c}}{(1-\mathbf{z}^{v_1})\cdots(1-\mathbf{z}^{v_c})}\ .$$

Proof. Consider the fundamental parallepiped of $\mathbb{H}_q(C)$, which we define as

$$\square_{\mathbb{H}_q(C)} := [0,1)\,v_1 + \cdots + [0,1)\,v_{m-1} + (0,1]\,v_m + \cdots + (0,1]\,v_c.$$

If we let $(j_1, \ldots, j_c) \in \mathbb{Z}^c_{\geq 0}$, then

$$j_1v_1+\cdots+j_cv_c+\square_{\mathbb{H}_q(C)} = (j_1+[0,1))v_1+\cdots+(j_{m-1}+[0,1))v_{m-1}+(j_m+(0,1])v_m+\cdots+(j_c+(0,1])v_c$$

is simply a translation of $\square_{\mathbb{H}_q(C)}$. All these translations are disjoint as their open sides overlap with their closed sides following from $\square_{\mathbb{H}_q(C)}$'s half-open nature. As a result,

$$\mathbb{H}_q(C) = \biguplus_{j_1,\ldots,j_c \in \mathbb{Z}_{\geq 0}} j_1v_1 + \cdots + j_cv_c + \square_{\mathbb{H}_q(C)}.$$

Applying the integer point transform,

$$\begin{aligned}\sigma_{\mathbb{H}_q(C)}(\mathbf{z}) &= \left(\sum_{j_1 \in \mathbb{Z}_{\geq 0}} \mathbf{z}^{j_1 v_1}\right) \cdots \left(\sum_{j_c \in \mathbb{Z}_{\geq 0}} \mathbf{z}^{j_c v_c}\right) \sigma_{\square_{\mathbb{H}_q(C)}}(\mathbf{z}) \\ &= \frac{\sigma_{\square_{\mathbb{H}_q(C)}}(\mathbf{z})}{(1-\mathbf{z}^{v_1})\cdots(1-\mathbf{z}^{v_c})} \\ &= \frac{\mathbf{z}^{v_m+\cdots+v_c}}{(1-\mathbf{z}^{v_1})\cdots(1-\mathbf{z}^{v_c})}\ .\end{aligned}$$

Since the only integer point of $\square_{\mathbb{H}_q(C)}$ is given by 0 for the coefficients of $v_1, \ldots, v_{m-1}$ and 1 for the coefficients of $v_m, \ldots, v_c$, we are given the last equality above. □

This gives a formula for computing the integer point transforms of half open unimodular cones. Now we will show how to obtain a dissection of K_Π in order to apply these results for d-fold partition diamonds. For this, we consider poset refinements and linear extensions.

2.4 Order Cone Subdivisions

For this section, the ground set for our poset will be fixed, so we will write $K_{\leq_\Pi}$ for the order cone of $(\Pi, \leq_\Pi)$. We first would like to understand the relationship between pairs of order cones based on their partial orders.

Definition 2.11. *Given two partial orders $\leq_1$ and $\leq_2$ on the same set Π, we say $\leq_2$ refines $\leq_1$ if for all $a, b \in \Pi$,*

$$a \leq_1 b \quad \textit{implies} \quad a \leq_2 b.$$

This means $\leq_2$ may have more relations than $\leq_1$, but has all the relations of $\leq_1$. As a result, more relations for $\leq_2$ will give rise to an overall less amount of order preserving functions for that given order while still giving rise to the same order preserving functions that $\leq_1$ does. A rigorous proof of the following result is provided in chapter 6 of [BS18].

Proposition 2.1. *Let $\leq_1$ and $\leq_2$ be partial orders on Π, then*

$$\leq_2 \textit{ refines } \leq_1 \textit{ if and only if } K_{\leq_2} \subseteq K_{\leq_1}.$$

Notice that poset refinement is a partial order on posets with the same ground set.

Definition 2.12. *Given a poset $(\Pi, \leq_\Pi)$, we define the poset ordered by refinement,*

$$N(\Pi, \leq_\Pi) := \{ \leq' \mid \leq' \textit{ refines } \leq_\Pi \}.$$

Notice that $\leq_\Pi$ is the maximum element of $N(\Pi, \leq_\Pi)$ and that $\leq'$ is a minimal element of $N(\Pi, \leq)$ if and only if $(\Pi, \leq')$ is a total order. Assuming $\Pi = [c]$, any total order of Π is given by a permutation $\tau \in \mathfrak{S}_c$. We will denote this total order $\leq_\tau$ where

$$i <_\tau j \quad \text{implies} \quad \tau^{-1}(i) < \tau^{-1}(j)$$

for all $i, j \in [c]$. Since $\leq_\tau$ refines $\leq_\Pi$,

$$i <_\Pi j \quad \text{implies} \quad \tau^{-1}(i) < \tau^{-1}(j)$$

for all $i, j \in [c]$.

Definition 2.13. *A **linear extension** is a strictly order preserving bijection*

$$l : (\Pi, \leq_\Pi) \to ([c], \leq).$$

We denote the set of all linear extensions of Π *as* $Lin(\Pi)$*.*

Assuming $\Pi = [c]$, we have that l is simply a permutation of $[c]$ for which

$$i <_\Pi j \quad \text{implies} \quad l(i) < l(j)$$

for all $i, j \in [c]$. As a result, $\leq_\tau$ refines $\leq_\Pi$ if and only if $\tau^{-1} \in \mathrm{Lin}(\Pi)$.

Definition 2.14. *We define the **Jordan-Hölder** set of* Π *as*

$$\begin{aligned} JH(\Pi) &:= \left\{ \tau \in \mathfrak{S}_c \mid \tau^{-1} \in Lin(\Pi) \right\} \\ &= \left\{ \tau \in \mathfrak{S}_c \mid \ \leq_\tau \textit{ refines } \leq_\Pi \right\}. \end{aligned}$$

The permutation $\tau^{-1} \in \mathrm{Lin}(\Pi)$ gives the position, $\tau^{-1}(i)$, of an element $i \in \Pi$ in a linear order that respects the partial order $\leq_\Pi$. In addition, $\mathrm{JH}(\Pi)$ is precisely the set of permutations corresponding to the orders that are minimal elements of $N(\Pi, \leq_\Pi)$. Since every $\leq' \in N(\Pi, \leq_\Pi)$ refines $\leq_\Pi$, we have that $K_{\leq'} \subset K_\Pi$ which gives numerous cones to subdivide K_Π with.

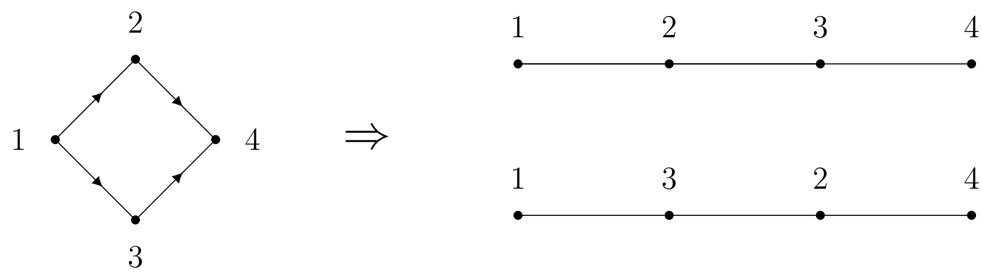

Figure 2.4: Linear extensions for the 2-fold partition diamond poset of length 1.

The linear extensions in the Jordan-Hölder set of the poset given in Figure 2.4 are $\tau_1 = (1, 2, 3, 4)$ and $\tau_2 = (1, 3, 2, 4)$.

Definition 2.15.

a) *We say poset is an* ***anti-chain*** *if none of its elements are related*

b) *We say poset is a* ***chain*** *if all of its element are related to each other. We say a chain C of maximal length in some poset N is a maximal chain.*

c) *Let $\hat{0}$ be the minimal element of N. A* ***crosscut*** *in a poset N is an anti-chain $\{ c_1, \ldots, c_m \} \subseteq N - \{ \hat{0} \}$ such that for every maximal chain $C \subseteq N$, there is a unique $c_i \in C$.*

Theorem 2.1. *Let $(\Pi, \leq_\Pi)$ be a poset and $N = N(\Pi, \leq_\Pi)$ its poset refinements. Let $\leq_1, \leq_2, \ldots, \leq_s \in N$ be a collection of refinements of Π. Then*

$$K_\Pi = K_{\leq_1} \cup \cdots \cup K_{\leq_s}$$

is a dissection of K_Π *if and only if* $\leq_1, \ldots, \leq_s$ *is a crosscut in* N *such that every minimal element is covered uniquely.*

The most convenient crosscut of refinements that covers the minimal elements is the set of minimal elements itself, i.e., JH(Π) gives a dissection of K_Π. Moreover, these cones give a unimodular subdivision of K_Π. Since total orders correspond to permutations we will write $K_{\leq_\tau}$ as K_τ. For explicit proofs of these results, see chapter 6 of [BS18].

Corollary 2.2. *Let* Π *be a finite poset. Then*

$$K_\Pi = \bigcup_{\tau \in JH(\Pi)} K_\tau$$

is a dissection of K_Π *into unimodular cones.*

In Figure 2.4, K_{τ_2} is the set of points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_1 \leq x_3 \leq x_2 \leq x_4$. Notice that for $\tau \in \mathfrak{S}_c$, any $\phi \in K_\tau^\circ$ satisfies

$$\phi(\tau(1)) < \cdots < \phi(\tau(c)).$$

Such a function ϕ is generic relative to K_π for all $\pi \in \mathfrak{S}_c$. Now the facets of K_π are the set of points $\psi \in K_\pi$ such that $\psi(\pi(i)) = \psi(\pi(i+1))$ for some $i \in \Pi$. The corresponding tangent cone for this facet is the set of points $f \in \mathbb{R}^\Pi_{\geq 0}$ such that $f(\pi(i)) \leq f(\pi(i+1))$. As a result, the corresponding facet is visible from a point $\phi \in K_\tau^\circ$ if $\phi(\pi(i)) > \phi(\pi(i+1))$. This occurs when $\pi(i)$ is in a higher position than $\pi(i+1)$ in a linear order given by τ, so when

$(\tau^{-1} \circ \pi)(i) > (\tau^{-1} \circ \pi)(i+1)$. As a result, we take out all the facets with this property to get a half-open order cone.

Definition 2.16. *Given a permutation ρ, an index $1 \le i \le c-1$ is a* ***descent*** *if $\rho(i) > \rho(i+1)$ and an* ***ascent*** *if $\rho(i) < \rho(i+1)$. All the descents and ascents of ρ are collected in the sets*

$$Des(\rho) := \left\{\, i \in [c-1] \,\middle|\, \rho(i) > \rho(i+1) \,\right\}$$

and

$$Asc(\rho) := \left\{\, i \in [c-1] \,\middle|\, \rho(i) < \rho(i+1) \,\right\}$$

respectively. In addition, we call $des(\rho) := |Des(\rho)|$ the ***descent number*** *of ρ and $asc(\rho) := |Asc(\rho)|$ the* ***ascent number*** *of ρ.*

We can now formalize a half open order cone as given in chapter 6 of [BS18].

Lemma 2.3. *Let $\tau, \pi \in \mathfrak{S}_c$ be permutations and let $\phi \in K_\tau^\circ$. Then*

$$\mathbb{H}_\phi K_\pi = \left\{ \boldsymbol{x} \in \mathbb{R}^c \;\middle|\; \begin{array}{l} 0 \le x_{\pi(1)} \le x_{\pi(2)} \le \cdots \le x_{\pi(c)} \\ x_{\pi(i)} < x_{\pi(i+1)} \textit{ if } i \in Des(\tau^{-1} \circ \pi) \end{array} \right\}.$$

Assuming $\Pi = [c]$ is naturally labeled, the identity permutation, $\tau(i) = i$ for all $i \in [c]$ is in $\mathrm{JH}(\Pi)$. Its corresponding order cone contains the point $\phi(i) = i$ for all $i \in [c]$ which is a convenient point that is generic relative to K_π for all $\pi \in \mathfrak{S}_c$. With this choice for ϕ, the above set simplifies to

$$\mathbb{H}_\phi K_\pi = \left\{ \mathbf{x} \in \mathbb{R}^c \;\middle|\; \begin{array}{l} 0 \le x_{\pi(1)} \le x_{\pi(2)} \le \cdots \le x_{\pi(c)} \\ x_{\pi(i)} < x_{\pi(i+1)} \text{ if } i \in \mathrm{Des}(\pi) \end{array} \right\}.$$

Combining Corollary 2.1, Corollary 2.2, and Lemma 2.3 gives these culminating results in chapter 6 of [BS18].

Theorem 2.2. *Let* $\Pi = [c]$ *be a naturally labeled poset. Then*

$$K_\Pi = \biguplus_{\tau \in JH(\Pi)} \mathbb{H}_\phi K_\tau = \biguplus_{\tau \in JH(\Pi)} \left\{ \boldsymbol{x} \in \mathbb{R}^c \;\middle|\; \begin{array}{l} 0 \leq x_{\tau(1)} \leq x_{\tau(2)} \leq \cdots \leq x_{\tau(c)} \\ \\ x_{\tau(i)} < x_{\tau(i+1)} \text{ if } i \in Des(\tau) \end{array} \right\}.$$

In Figure 2.4, we have that 2 is a descent of τ_2 since $\tau_2(2) \geq \tau_2(3)$. As a result $\mathbb{H}_\phi K_{\tau_2}$ has the facet corresponding to $x_{\tau_2(2)} = x_{\tau_2(3)}$ removed, so $\mathbb{H}_\phi K_{\tau_2}$ is the set of points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_1 \leq x_3 < x_2 \leq x_4$.

2.5 Bringing it All Together

We now have a means of dissecting K_Π into disjoint unimodular cones. Let $\pi \in \mathfrak{S}_c$ and $u_j^\pi = (x_1, \ldots, x_c)$ be the vector such that $x_{\pi(j+1)} = \cdots = x_{\pi(c)} = 1$ and 0 everywhere else. Since every cone in the dissection is unimodular, Lemma 2.3 says each $\mathbb{H}_\phi K_\tau$ is of the form

$$\mathbb{H}_\phi K_\tau = \sum_{j \in \mathrm{Des}(\tau)} \mathbb{R}_{>0} u_j^\tau + \sum_{j \in \{\, 0,\ldots,c-1 \,\} \setminus \mathrm{Des}(\tau)} \mathbb{R}_{\geq 0} u_j^\tau.$$

Applying the integer-point transform and Lemma 2.2,

$$\sigma_{\mathbb{H}_\phi K_\tau}(\mathbf{z}) = \frac{\prod_{j \in \mathrm{Des}(\tau)} \mathbf{z}^{u_j^\tau}}{\prod_{j=0}^{c-1} (1 - \mathbf{z}^{u_j^\tau})}.$$

Now, we are ready for the main result of this entire background section.

Corollary 2.3. *Let $\Pi = [c]$ be a naturally labeled poset. Then,*

$$\sigma_{K_\Pi}(\boldsymbol{z}) = \sum_{\tau \in JH(\Pi)} \frac{\prod_{j \in Des(\tau)} \boldsymbol{z}^{u_j^\tau}}{\prod_{j=0}^{c-1}(1 - \boldsymbol{z}^{u_j^\tau})}.$$

Looking at the poset in Figure 2.4 and using Corollary 2.3, the integer point transform specialized to $z_1 = z_2 = z_3 = z_4 = q$ of its corresponding order cone is given by

$$\frac{1 + q^2}{(1 - q^4)(1 - q^3)(1 - q^2)(1 - q)}.$$

Chapter 3

Main Results

For this section, let $\Pi = [c]$ be a naturally labeled d-fold partition diamond poset. We first seek to understand Π in a way that benefits computing the integer-point transform of its corresponding order cone.

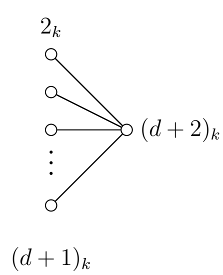

Figure 3.1: The poset $\mathcal{D}_k$.

Let $i_k := i + (k-1)(d+1)$ and consider the poset

$$\mathcal{D}_k := \{\, 2_k, \ldots, (d+2)_k \,\}$$

pictured in Figure 3.1 where $(d+2)_k$ is the maximum element comparable to every element in $\mathcal{D}_k$, and every other element is a minimum element and only comparable to $(d+2)_k$, for $1 \leq k \leq M$. Therefore, $2_k, \ldots, (d+1)_k$ form an anti-chain in $\mathcal{D}_k$.

Definition 3.1. *Let P and Q be disjoint posets. We define the **linear sum** of P and Q to be the partially ordered set*

$$P \oplus Q := (P \uplus Q, \leq_\oplus)$$

where $x \leq_\oplus y$ if and only if

$$\begin{aligned} & x, y \in P \quad \text{and} \quad x \leq_P y, \\ \text{or} \quad & x, y \in Q \quad \text{and} \quad x \leq_Q y, \\ \text{or} \quad & x \in P \quad \text{and} \quad y \in Q. \end{aligned}$$

Notice that the linear sum is associative for pair-wise disjoint posets, so we can iteratively sum numerous posets. As a result,

$$\{ 1 \} \oplus \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_M = \Pi.$$

Definition 3.2. *Let $Q_0, Q_1, \ldots, Q_M$ be naturally labelled posets on $[q_i]$ and let $s_j = q_0 + q_1 + \cdots + q_{j-1}$ with $s_0 = 0$. In addition, let*

$$Q'_j = \{ s_j + 1, s_j + 2, \ldots, s_j + q_j \},$$

with order induced by Q_j, for $0 \leq j \leq M$. Given $\tau_i \in JH(Q_i)$, define the function, $\tau : [q_0 + q_1 + \cdots + q_M] \to [q_0 + q_1 + \cdots + q_M]$ via

$$\tau(i) = \begin{cases} \tau_0(i) & \textit{if } i \in Q_0, \\ \tau_1(i - s_1) + s_3 & \textit{if } i \in Q_1, \\ \tau_2(i - s_2) + s_2 & \textit{if } i \in Q'_2, \\ \tau_3(i - s_3) + s_3 & \textit{if } i \in Q'_3, \\ \vdots & \vdots \\ \tau_M(i - s_M) + s_M & \textit{if } i \in Q'_M, \end{cases}$$

so that $\tau \in JH\left(\bigoplus_{j=0}^{M} Q'_j\right)$.

There is one nuance that we should be aware of when thinking about the Jordan-Hölder set of $\mathcal{D}_i$. In this context, we are considering a natural labeling of $\mathcal{D}_i$ as this is a labeling that can be understood by the Jordan-Hölder set and allows us to use the language of permutations; whereas $\mathcal{D}_i$ is defined in a way that allows us to use the power of the linear sum. In addition, for our purposes we will let $\{ 1 \} = \mathcal{D}_0$. Therefore, $\mathrm{JH}(\mathcal{D}_0)$ contains just the identity permutation in $\mathfrak{S}_1$, $\tau_0(1) = 1$. We now apply Definition 3.2 to the elements of $\mathrm{JH}(\Pi)$. Notice that for $\tau \in \mathrm{JH}(\Pi)$ in the form of the above definition,

$$\mathrm{Des}(\tau) = \biguplus_{i=1}^{M} \{ \, j + s_i \mid j \in \mathrm{Des}(\tau_i) \, \}$$

where $\tau_i \in \mathrm{JH}(\mathcal{D}_i)$. As a result, the numerator in Corollary 2.3 can in this case be rewritten as

$$\prod_{j\in \mathrm{Des}(\tau)} \mathbf{z}^{u_j^\tau} = \prod_{k=1}^{M} \prod_{i\in \mathrm{Des}(\tau_k)} \mathbf{z}^{u_j^\tau} \tag{3.1}$$

where $\mathbf{z}^{u_j^\tau} = z_{\tau_k(i+1)+s_k} \cdots z_{\tau_k(d)+s_k} z_{\tau_k(d+1)+s_k} z_{\tau_{k+1}(1)+s_{k+1}} \cdots z_{\tau_M(d+1)+s_M}$ and $i = j - s_k$.

We now seek to specialize Corollary 2.3 for K_Π. Here we give a two variable specialization where

$$z_i = \begin{cases} a & \text{if } i \not\equiv 1 \bmod (d+1), \\ b & \text{if } i \equiv 1 \bmod (d+1). \end{cases} \tag{3.2}$$

In essence, we assign the variable b to the connecting nodes of Π and the variable a to all elements in the anti-chains of Π. With this specialization, and our definition for u_j^τ, we care less about where a given descent is and more about how many elements come after the descent in the linear order given by the corresponding permutation, so we count those elements with the variables a and b. Given some $\tau \in \mathrm{JH}(\Pi)$ and an index j where $(j+1) \in \mathcal{D}_k$,

$$\mathbf{z}^{u_j^\tau} = z_{\tau_k(j+1-s_k)+s_k} \cdots z_{\tau_k(d)+s_k} z_{\tau_k(d+1)+s_k} z_{\tau_{k+1}(1)+s_{k+1}} \cdots z_{\tau_M(d+1)+s_M}$$

where $\tau_i \in \mathrm{JH}(\mathcal{D}_i)$. In addition, $s_i = 1+(i-1)(d+1)$ for $1 \le i \le M$ and $s_0 = 0$. Notice that since τ and all the τ_i's that constitute it must adhere to the order of Π, $\tau_i(d+1) = d+1$ and $\tau_i(l) \in [d]$ for $l \in [d]$. This implies $\tau_i(d+1)+s_i \equiv 1 \bmod (d+1)$ and $\tau_i(l)+s_i \not\equiv 1 \bmod (d+1)$ for $l \in [d]$. As a result, we set

$$z_{\tau_k(d+1)+s_k} = z_{\tau_{k+1}(d+1)+s_{k+1}} = \cdots = z_{\tau_M(d+1)+s_M} = b.$$

This contributes a factor of b^{1+M-k} in $\mathbf{z}^{u_j^\tau}$. Now, we set each other z_i to be a; we will count how many a's contribute. Let $i = j - s_k$. There are $d - i$ terms before $z_{\tau_k}(d+1) + s_k$; referring to the part of the anti-chain in $\mathcal{D}_k$ after index j. This contributes a factor of a^{d-i}. To conclude, we look at all the z's corresponding to the full anti-chains in $\mathcal{D}_{k+1}, \mathcal{D}_{k+2}, \ldots, \mathcal{D}_M$. This is a total of $M - k$ anti-chains, each contributing d elements. As a result, we contribute an additional factor of $a^{(M-k)d}$ to $\mathbf{z}^{u_j^\tau}$. Putting it together, the numerator in Corollary 2.3 becomes

$$\prod_{j \in \mathrm{Des}(\tau)} \mathbf{z}^{u_j^\tau} = \prod_{k=1}^{M} \prod_{i \in \mathrm{Des}(\tau_k)} \mathbf{z}^{u_j^\tau} = \prod_{k=1}^{M} \prod_{i \in \mathrm{Des}(\tau_k)} a^{d-i} a^{(M-k)d} b^{1+M-k}. \tag{3.3}$$

Looking at the denominator of Corollary 2.3, we get a similar form as the numerator. However, the index corresponding to $j = 0$ gives an extra factor. Consider

$$\mathbf{z}^{u_0^\tau} = z_{\tau_0(1)} z_{\tau_1(1)+s_1} \cdots z_{\tau_1(d+1)+s_1} \cdots z_{\tau_M(d+1)+s_M}.$$

Notice that in this case, we have Md anti-chain elements and $M + 1$ connecting nodes, so the specialization of (3.2) gives

$$\mathbf{z}^{u_0^\tau} = a^{Md} b^{M+1}.$$

As a result, looking at each diamond individually, the denominator of Corollary 2.3 is given by

$$\prod_{j=0}^{c-1} \left(1 - \mathbf{z}^{u_j^\tau}\right) = \left(1 - \mathbf{z}^{u_0^\tau}\right) \prod_{k=1}^{M} \prod_{j=0}^{d} \left(1 - \mathbf{z}^{u_{j+s_k}^\tau}\right) = \left(1 - a^{Md} b^{M+1}\right) \prod_{k=1}^{M} \prod_{j=0}^{d} \left(1 - a^{d-j} a^{(M-k)d} b^{1+M-k}\right).$$

Notice that this denominator is the same regardless of τ, so we can move the summation in Corollary 2.3 to the numerator. Putting everything together from Corollary 2.3 and the specialization of (3.2) gives

$$\sigma_{K_\Pi}(\mathbf{z}) = \frac{\sum_{\tau \in \mathrm{JH}(\Pi)} \prod_{k=1}^{M} \prod_{j \in \mathrm{Des}(\tau_k)} a^{d-j} a^{(M-k)d} b^{1+M-k}}{(1 - a^{Md} b^{M+1}) \prod_{k=1}^{M} \prod_{j=0}^{d} (1 - a^{d-j} a^{(M-k)d} b^{1+M-k})}. \tag{3.4}$$

In summary, we have that the a^{d-j} term comes from $z_{\tau_k(j+1-s_k)+s_k} \ldots z_{\tau_k(d)+s_k}$ referring to the part of the anti-chain in $\mathcal{D}_k$ after the index j. The b^{1+M-k} term comes from the $1+M-k$ connecting nodes above the corresponding anti-chain in Π. Finally the $a^{(M-k)d}$ term comes from the $(M-k)d$ elements in the remaining anti-chains. One thing we made notice of earlier that is helpful to remark here is that each of the τ_i's that make up some $\tau \in \mathrm{JH}(\Pi)$ fix $d+1$. In this, $\tau_i \in \mathrm{JH}(\mathcal{D}_i)$ is completely determined by some permutation in $\mathfrak{S}_d$. As a result $\mathbf{z}^{u_j^\tau}$ in (3.1) can be written as

$$\mathbf{z}^{u_j^\tau} = z_{\pi(j+1-s_k)+s_k} \cdots z_{\pi(d)+s_k} z_{(d+1)+s_k} z_{1+s_{k+1}} \cdots z_{(d+1)+s_M}$$

for some $\pi \in \mathfrak{S}_d$. As a result we can get a factored form of (3.4) summing over all possible permutations in $\mathfrak{S}_d$,

$$\sigma_{K_\Pi}(\mathbf{z}) = \frac{\prod_{k=1}^{M} \sum_{\tau \in \mathfrak{S}_d} \prod_{j \in \mathrm{Des}(\tau)} a^{d-j} a^{(M-k)d} b^{1+M-k}}{(1 - a^{Md} b^{M+1}) \prod_{k=1}^{M} \prod_{j=0}^{d} (1 - a^{d-j} a^{(M-k)d} b^{1+M-k})}.$$

One valuable observation to make is that

$$\{\, \mathrm{Des}(\tau) \,|\, \tau \in \mathfrak{S}_d \,\} = \{\, \mathrm{Asc}(\tau) \,|\, \tau \in \mathfrak{S}_d \,\}.$$

As a result,

$$\begin{aligned}\sigma_{K_\Pi}(\mathbf{z}) &= \frac{\prod_{k=1}^{M}\sum_{\tau\in\mathfrak{S}_d}\prod_{j\in\mathrm{Asc}(\tau)} a^{d-j}a^{(M-k)d}b^{1+M-k}}{(1-a^{Md}b^{M+1})\prod_{k=1}^{M}\prod_{j=0}^{d}(1-a^{d-j}a^{(M-k)d}b^{1+M-k})}\\ &= \frac{\prod_{k=1}^{M}\sum_{\tau\in\mathfrak{S}_d}\prod_{d-j\in\mathrm{Asc}(\tau)} a^{j}a^{(M-k)d}b^{1+M-k}}{(1-a^{Md}b^{M+1})\prod_{k=1}^{M}\prod_{j=0}^{d}(1-a^{d-j}a^{(M-k)d}b^{1+M-k})}.\end{aligned}$$

To simplify further, we define a permutation in $\mathfrak{S}_d$ such that

$$\tau^{\perp}(j) := \tau(d+1-j).$$

With this definition, $d-j \in \mathrm{Asc}(\tau)$ if and only if $j \in \mathrm{Des}(\tau^{\perp})$, but since $\{\, \tau^{\perp} \mid \tau \in \mathfrak{S}_d \,\} = \mathfrak{S}_d$,

$$\sigma_{K_\Pi}(\mathbf{z}) = \frac{\prod_{k=1}^{M}\sum_{\tau\in\mathfrak{S}_d}\prod_{j\in\mathrm{Des}(\tau)} a^{j}a^{(M-k)d}b^{1+M-k}}{(1-a^{Md}b^{M+1})\prod_{k=1}^{M}\prod_{j=0}^{d}(1-a^{d-j}a^{(M-k)d}b^{1+M-k})}. \tag{3.5}$$

To simplify even further, we set $n = M-k$ so that

$$\begin{aligned}\sigma_{K_\Pi}(\mathbf{z}) &= \frac{\prod_{n=0}^{M-1}\sum_{\tau\in\mathfrak{S}_d}\prod_{j\in\mathrm{Des}(\tau)} a^{j}a^{nd}b^{1+n}}{(1-a^{Md}b^{M+1})\prod_{n=0}^{M-1}\prod_{j=0}^{d}(1-a^{d-j}a^{nd}b^{1+n})}\\ &= \frac{\prod_{n=1}^{M}\sum_{\tau\in\mathfrak{S}_d}\prod_{j\in\mathrm{Des}(\tau)} a^{j}a^{(n-1)d}b^{n}}{(1-a^{Md}b^{M+1})\prod_{n=1}^{M}\prod_{j=0}^{d}(1-a^{d-j}a^{(n-1)d}b^{n})}.\end{aligned}$$

The desire to simplify even further is motivation enough for the following definitions. These are thanks to Euler and MacMahon; the same MacMahon responsible for the Ω-operator used in [Doc+24].

Definition 3.3. *The **major index** of $\tau \in \mathfrak{S}_d$ is defined as*

$$maj(\tau) := \sum_{j\in Des(\tau)} j.$$

Definition 3.4. *The **Euler-Mahonian polynomial** is defined as*

$$E_d(x,y) := \sum_{\tau \in \mathfrak{S}_d} x^{des(\tau)} y^{maj(\tau)}.$$

*The **Eulerian polynomial** is defined as*

$$A_d(x) := \sum_{\tau \in \mathfrak{S}_d} x^{des(\tau)}.$$

Notice that $E_d(x,1) = A_d(x)$. Given these definitions, we can write

$$\begin{aligned}\sigma_{K_\Pi}(\mathbf{z}) &= \frac{\prod_{n=1}^{M} \sum_{\tau \in \mathfrak{S}_d} a^{\mathrm{maj}(\tau)} \left(a^{(n-1)d} b^n\right)^{\mathrm{des}(\tau)}}{(1 - a^{Md} b^{M+1}) \prod_{n=1}^{M} \prod_{j=0}^{d} (1 - a^{d-j} a^{(n-1)d} b^n)} \\ &= \frac{\prod_{n=1}^{M} E_d(a^{(n-1)d} b^n, a)}{(1 - a^{Md} b^{M+1}) \prod_{n=1}^{M} \prod_{j=0}^{d} (1 - a^{d-j} a^{(n-1)d} b^n)}.\end{aligned}$$

This gives the desired identity for the integer-point transform of K_Π as stated in Theorem 1.1. Theorem 1.1 provides us with the identities for the generating functions for the variants of d-fold partition diamonds that we discuss in the introduction section.

Proof of Corollary 1.1. We have that

$$\sum_{n=0}^{\infty} r_d(n) q^n = \lim_{M \to \infty} \sum q^{a_1 + a_2 + \cdots + a_c} = \lim_{M \to \infty} \sigma_{K_\Pi}(q, q, \ldots, q).$$

Let $a = b = q$ in Theorem 1.1, which gives

$$\sigma_{K_\Pi}(q, q, \ldots, q) = \frac{\prod_{n=1}^{M} E_d(q^{1+(n-1)(d+1)}, q)}{(1-q)(1-q^2)\cdots(1-q^c)}.$$

Taking $\lim_{M \to \infty} \sigma_{K_\Pi}(q, q, \ldots, q)$ gives,

$$\prod_{n=1}^{\infty} \frac{E_d(q^{1+(n-1)(d+1)}, q)}{1 - q^n}.$$

□

Proof of Corollary 1.2. The parts of the partitions are given by the connecting nodes, therefore

$$\sum_{n=0}^{\infty} s_d(n)q^n = \lim_{M\to\infty} \sigma_{K_\Pi}(q,1,\ldots,1,q,1,\ldots,q)$$

where all the q's correspond to the connecting nodes and the 1's correspond to the anti-chains. Let $a = 1$ and $b = q$ in Theorem 1.1, which gives

$$\sigma_{K_\Pi}(q,1,\ldots,1,q,1,\ldots,q) = \frac{\prod_{n=1}^{M} E_d(q^n,1)}{(1-q^{M+1})\prod_{n=1}^{M}(1-q^n)^{d+1}} = \frac{\prod_{n=1}^{M} A_d(q^n)}{(1-q^{M+1})\prod_{n=1}^{M}(1-q^n)^{d+1}}.$$

Taking $\lim_{M\to\infty} \sigma_{K_\Pi}(q,1,\ldots,1,q,1,\ldots,q)$ gives

$$\prod_{n=1}^{\infty} \frac{A_d(q^n)}{(1-q^n)^{d+1}}.$$

□

We compute some examples that show up in [Doc+24]. First notice that

$$E_2(q^{1+3(n-1)},q) = \sum_{\tau\in\mathfrak{S}_2} q^{\mathrm{maj}(\tau)+\mathrm{des}(\tau)(1+3(n-1))} = 1 + q^{1+1+3(n-1)} = 1 + q^{3n-1},$$

$$\begin{aligned} E_3(q^{1+4(n-1)},q) &= 1 + 2q^{1+1+4(n-1)} + 2q^{2+1+4(n-1)} + q^{1+2+2(1+4(n-1))} \\ &= 1 + 2q^{4n-2} + 2q^{4n-1} + q^{8n-3} \\ &= 1 + 2q^{4n-2}(1+q) + q^{8n-3}. \end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} r_2(n)q^n = \prod \frac{1+q^{3n-1}}{1-q^n},$$
$$\sum_{n=0}^{\infty} r_3(n)q^n = \prod \frac{1+2q^{4n-2}(1+q)+q^{8n-3}}{1-q^n}.$$

In addition,

$$A_2(q^n) = 1 + q^n,$$

$$A_3(q^n) = 1 + 4q^n + q^{2n},$$

since $\mathfrak{S}_3$ has 1 permutation with no descent, 4 permutations with 1 descent, and 1 permutation with 2 descents. As result,

$$\sum_{n=0}^{\infty} s_2(n)q^n = \prod_{n=0}^{\infty} \frac{1+q^n}{(1-q^n)^3},$$
$$\sum_{n=0}^{\infty} s_3(n)q^n = \prod_{n=0}^{\infty} \frac{1+4q^n+q^{2n}}{(1-q^n)^4}.$$

We next think about the situation where each diamond has a different number of folds given by Definition 1.5. In this situation, the anti-chains of the i^{th} diamond of the corresponding poset has d_i elements. Let Ξ be a naturally labeled multifold partition diamond poset of length M corresponding to the sequence $\{d_i\}_{i=1}^{M}$. Let $\omega_k = \sum_{i=k+1}^{M} d_i$. The z_i corresponding to the connecting nodes would be such that $i = 1 + k + \sum_{j=1}^{k} d_j$ for some k. We set those z_i's to b and the rest to a. Modeled after (3.5) and this specialization, the corresponding integer point transform is

$$\begin{aligned}\sigma_{K_\Xi}(\mathbf{z}) &= \frac{\prod_{k=1}^{M} \sum_{\tau\in\mathfrak{S}_{d_k}} \prod_{j\in\mathrm{Des}(\tau)} a^j a^{\omega_k} b^{1+M-k}}{(1-a^{\omega_0}b^{M+1}) \prod_{k=1}^{M} \prod_{j=0}^{d_k} (1-a^{d_k-j}a^{\omega_k}b^{1+M-k})} \\ &= \frac{\prod_{k=1}^{M} E_{d_k}(a^{\omega_k}b^{1+M-k}, a)}{(1-a^{\omega_0}b^{M+1}) \prod_{k=1}^{M} \prod_{j=0}^{d_k} (1-a^{d_k-j}a^{\omega_k}b^{1+M-k})}.\end{aligned}$$

which is as desired in Theorem 1.2.

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