GENERATING FUNCTIONS FOR k-REPRESENTABLE INTEGERS WITH TWO PARAMETERS

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CERTIFICATION OF APPROVAL

I certify that I have read *GENERATING FUNCTIONS FOR k-REPRESENTABLE INTEGERS WITH TWO PARAMETERS* by Leonardo Bardomero and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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Given relatively prime integers $a_1, a_2, \ldots, a_d$, what is the largest positive integer g_0 that *cannot* be expressed as an integral linear combination of the given integers? This query is called the *Frobenius problem*, and the integer g_0 is called the Frobenius number. For $d = 2$ it is known that $g_0 = a_1a_2 - a_1 - a_2$, but the problem remains widely open for $d \geq 3$.

We explore *k-representable* integers with two parameters: For $k \geq 1$ and relatively prime integers a_1 and a_2, we say that another positive integer n is representable if it can be expressed as an integral linear combination of a_1 and a_2; and k-representable if it can be expressed in *exactly* k ways using a_1 and a_2. We use generating functions to provide explicit formulas which count the number of integers with exactly k representations, and the number of integers with *at most* k representations; as well as explicit formulas which sum all integers with exactly k representations, and those integers with at most k representations.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

Given positive relatively prime integers $a_1, a_2, \ldots, a_d$, we say that $n \in \mathbb{Z}_{\geq 0}$ is *representable* if

$$n = m_1 a_1 + \cdots + m_d a_d,$$

for some $m_1, m_2, \cdots m_d \in \mathbb{Z}_{\geq 0}$. The number of all partitions of n using exclusively elements of $A = \{a_1, a_2, \ldots, a_d\}$ as parts is given by

$$p_A(n) := \#\{(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : \ m_j \geq 0, \ n = m_1 a_1 + \cdots + m_d a_d\}.$$

1.1 k-representable Integers

Definition 1.1. Let $k > 0$. We say that n is *k-representable* if $p_A(n) = k$; that is, n can be expressed in exactly k ways using integers of the set A.

Definition 1.2. We define $g_k = g_k(a_1, a_2, \cdots, a_d)$ to be the largest k-representable integer, and $g_k^* = g_k^*(a_1, a_2, \cdots, a_d)$ to be the largest integer with *at most* k representations.We also call the integer $g_k = g_k(a_1, a_2, \cdots, a_d)$ the *k-Frobenius number* [3].

The *linear Diophantine problem of Frobenius* consists of finding the largest integer that is not representable. That integer is called the *Frobenius number*, and it is denoted by $g = g(a_1, a_2, \ldots, a_d)$. Note that $g = g_0 = g_0^*$.

1.1.1 Generating Functions

Let $(a_k)_{k=0}^{\infty}$ be a sequence of numbers, and suppose we are interested in determining whether there exist identities involving a_k's, the k^{th} terms of this infinite sequence. Generating functions allow us to transform this problem pertaining a sequence into a problem about functions. This is certainly great news, for there are available to us a variety of mathematical machineries for manipulating functions. To determine whether there is a formula for a_k as a function of k, we let the terms of the infinite sequence $(a_k)_{k=0}^{\infty}$ be the coefficients of a formal power series, and then we apply the corresponding machinery, which is contingent upon the sequence in question. Thus, a generating function for the infinite sequence $(a_k)_{k=0}^{\infty}$ is the power series [5]

$$F(z) = \sum_{k \geq 0} a_k z^k.$$

The infinite sequence of our interest has as elements representable integers with parameters a and b. We collect them in a set, and we denote it by

$$R := \{am + bn : m, n \in \mathbb{Z}_{\geq 0}\}.$$

Theorem 1.1. *The generating function associated with R is precisely*

$$\sum_{k \in R} z^k = \frac{1 - z^{ab}}{(1 - z^a)(1 - z^b)}. \tag{1.1}$$

Embedded in this rational generating function resides everything we need to know about the Frobenius problem of a and b. This result appears as Theorem 1 in [16], but it is reported in [9] that it was also known by Sylvester. Nonetheless, for the sake of completeness we'll provide a proof of this identity using the technique described in [[5], Chapter 1] (see page 6). Furthermore, this generating function allows us to prove, by means of L'Hôpital's rule, *Sylvester's theorem* (see page 7).

Theorem 1.2 (Sylvester's theorem). *Let a and b be relatively prime integers. Exactly half of the integers between 1 and $(a-1)(b-1)$ are representable by a and b.*

Our main object of study is a subset of R, namely $T_k := \{r \in R : p_{\{a,b\}}(r) > k\}$, and we will derive a closed formula for the generating function

$$\sum_{r \in T_k} x^r$$

(see Theorem 4.1). This is one of our original results.

The generating function of the set of all k-representable integers gives us a polynomial (see Corollary 4.1) whose degree is precisely the largest k-representable integer (see Corollary 4.2). Within the constraint $k \geq 2$, we compute the number of k-representable integers (see proof of Theorem 2.2). We then give a formula for computing the sum of all k-representable integers. This is another of our original results (see Theorem 4.2).

Subsequently, we consider those integers with at most k representations,

$$g_k^* = g_k^*(a_1, a_2, a_3, \ldots, a_d).$$

Lastly, we derive the sum of all integers with at most k representations. (See Theorem 5.1 and Corollary 5.1).

Chapter 2

History and Background

The Diophantine problem of Frobenius is perhaps one of the most well-known problems in combinatorial number theory, and its study dates back to the 1800s. The following Theorem have been known since 1844 (see [15]).

Theorem 2.1. *Let a and b be relatively prime integers. Then the Frobenius number of a and b is exactly $g(a, b) = ab - a - b$.*

Brown and Shiue [7] derived a formula which gives the sum of all non-representable integers. We'll later rederive the same result by means of generating functions (see Theorem 3.3).

As is often the case with problems in number theory, the Frobenius problem is very appealing due to the fact that it can be easily described. For instance, in terms of coins of denominations $a_1, a_2, \ldots, a_k$, the Frobenius number is the largest amount of money that cannot be obtained using these coins. For more information about the Frobenius problem, see [14].

The Frobenius problem has been of interest to many mathematicians, and recent work on this respect includes the k-Frobenius number. Beck and Robins [4] showed that, given $k \geq 2$,

$$g_k = (k+1)a_1a_2 - a_1 - a_2, \tag{2.1}$$

which is a generalization of Theorem 2.1.

Furthermore, we know that $g_k = g_k^* = (k+1)a_1a_2 - a_1 - a_2$, and thus there is not any difference between the $g_k -$ numbers and the $g_k^* -$ numbers for two parameters (see Corollary 5.1).

Theorem 2.2. *For $k \geq 2$, there are exactly ab k-representable integers with relatively prime parameters a and b. That is,*

$$\sum_{r \in R_k} 1 = ab.$$

Given that Exercises 1.23 and 1.25 in [5] contain information about k-representable integers with two parameters, and since our objective is to explore such integers, we make them into Lemmas.

Lemma 2.3. *For relatively prime positive integers a and b,*

$$p_{\{a,b\}}(ab+n) = p_{\{a,b\}}(n) + 1.$$

Lemma 2.4. *Let $d = 2$. Then the following statements are all true.*

1. *For every $k \in \mathbb{Z}_{\geq 0}$, there is an N such that all integers larger than N have at least k representations (and hence $g_k(a, b)$ is well defined).*

2. *Given $k \geq 2$, the smallest k-representable integer is $ab(k-1)$.*

3. *The smallest interval containing all uniquely representable integers is*

$$[\min(a, b), g_1(a, b)].$$

4. *Given $k \geq 2$, the smallest interval containing all k-representable integers is*

$$[g_{k-2}(a, b) + a + b, g_k(a, b)].$$

5. *There are exactly $ab - 1$ integers that are uniquely representable.*

2.1 Barlow-Popoviciu Formula

The following formula is an indispensable tool in our disquisition about k-representable integers. This formula was first presented by Peter Barlow [1], and over a century later, it was rediscovered by Tiberiu Popoviciu [12] in the form in which we state it here.

Theorem 2.5. (Barlow-Popoviciu formula). *If a and b are relatively prime, then*

$$p_{\{a,b\}}(n) = \frac{n}{ab} - \left\{\frac{b^{-1}n}{a}\right\} - \left\{\frac{a^{-1}n}{b}\right\} + 1,$$

where $b^{-1}b \equiv 1 \mod a$ *and* $a^{-1}a \equiv 1 \mod b$.

Here the two sets of braces denote the **fractional-part function.** So $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the **greatest integer function**, and it denotes the greatest integer less than or equal to x. This directly implies that $\{-x\} = 1 - \{x\}$ for $x \notin \mathbb{Z}$ [5].

Chapter 3

Known Results

The following Lemmas are a direct consequence of the Barlow-Popoviciu formula. For a proof of Lemma 3.1, see [5].

Lemma 3.1. *The representable integers between* 1 *and* $ab-1$ *are uniquely representable.*

Lemma 3.2. *If* a *and* b *are relatively prime positive integers exceeding* 1 *and* $n \in [1, ab-1]$ *is not a multiple of* a *or* b*, then*

$$p_{\{a,b\}}(ab-n) + p_{\{a,b\}}(n) = 1.$$

That is, if $n \in [1, ab-1]$ *is not a multiple of* a *or* b*, exactly one of the two numbers* n *and* $ab-n$ *is uniquely representable in terms of* a *and* b*, and the other is not representable.*

3.1 Generating Function Associated with the Set of All Representable Integers

Our later proofs rely substantially on Theorem 1.1. Thus, for the sake of self-containment, we offer a proof.

Proof of Theorem 1.1. Our goal consists on forcing the right-hand side of (1) to look exactly as its left-hand side. Upon examining the right-hand side, we notice that

$$\begin{aligned}
\frac{1-z^{ab}}{(1-z^a)(1-z^b)} &= \frac{1}{(1-z^a)(1-z^b)} - \frac{z^{ab}}{(1-z^a)(1-z^b)} \\
&= \left(\frac{1}{1-z^a}\right)\left(\frac{1}{1-z^b}\right) - z^{ab}\left(\frac{1}{1-z^a}\right)\left(\frac{1}{1-z^b}\right) \\
&= \sum_{k\geq 0}\sum_{l\geq 0} z^{ak}z^{bl} - \sum_{k\geq 0}\sum_{l\geq 0} z^{ak}z^{bl}z^{ab} \\
&= \sum_{n\geq 0} p_{\{a,b\}}(n)z^n - \sum_{n\geq 0} p_{\{a,b\}}(n)z^{n+ab} \\
&= \sum_{n\geq 0} p_{\{a,b\}}(n)z^n - \sum_{n\geq ab} p_{\{a,b\}}(n-ab)z^n \\
&= \sum_{0\leq n\leq ab-1} p_{\{a,b\}}(n)z^n + \left(\sum_{n\geq ab} p_{\{a,b\}}(n)z^n - \sum_{n\geq ab} p_{\{a,b\}}(n-ab)z^n\right) \\
&= \sum_{0\leq n\leq ab-1} p_{\{a,b\}}(n)z^n + \sum_{n\geq ab}\left(p_{\{a,b\}}(n) - p_{\{a,b\}}(n-ab)\right) z^n. \quad (3.1)
\end{aligned}$$

By Lemma 3.1, it follows that $p_{\{a,b\}}(n) - p_{\{a,b\}}(n-ab) = 1$. Thus,

$$\sum_{n \geq ab} \left(p_{\{a,b\}}(n) - p_{\{a,b\}}(n-ab)\right) z^n = \sum_{n \geq ab} z^n.$$

By Lemma 3.2, if $n \in [1, ab-1]$ is a representable integer, then it is uniquely representable. That is,

$$p_{\{a,b\}} = \begin{cases} 1 & \text{if} \quad n \in [1, ab-1] \text{ is representable,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, (3.1) equals

$$\sum_{n \in R} z^n.$$

This concludes our proof. □

Proof of Theorem 1.2. We have

$$\sum_{s \in \mathbb{Z}_{\geq 0}} z^s = \sum_{s \in R} z^s + \sum_{s \in \mathbb{Z}_{\geq 0} \setminus R} z^s, \tag{3.2}$$

and by Theorem 1.1 it follows that

$$\begin{aligned} h(z) := \sum_{s \in \mathbb{Z}_{\geq 0} \setminus R} z^s &= \frac{1}{1-z} - \frac{1 - z^{a_1 a_2}}{(1-z^{a_1})(1-z^{a_2})} \\ &= \frac{(1-z^{a_1})(1-z^{a_2}) - (1-z)(1-z^{a_1 a_2})}{(1-z)(1-z^{a_1})(1-z^{a_2})}. \end{aligned} \tag{3.3}$$

Now, $h(1) = \sum_{s \in \mathbb{Z}_{\geq 0} \setminus R} 1$ counts the number of non-representable integers in terms of $\{a_1, a_2\}$. We apply L'Hôpital's rule thrice (three being the degree of the singularity of $h(z)$ at $z = 1$):

$$\begin{aligned}
h(1) = \sum_{s \in \mathbb{Z}_{\geq 0} \setminus R} 1 = \lim_{z \to 1} h(z) &= \lim_{z \to 1} \frac{(1 - z^{a_1})(1 - z^{a_2}) - (1 - z)(1 - z^{a_1 a_2})}{(1 - z)(1 - z^{a_1})(1 - z^{a_2})} \\
&= \frac{3a_1^2 a_2 - 3a_1 a_2 + 3a_1 a_2^2 - 3a_1 a_2 - 3a_1^2 a_2^2 + 3a_1 a_2}{-6a_1 a_2} \\
&= \frac{-3a_1^2 a_2^2 + 3a_1^2 a_2^2 + 3a_1 a_2^2 - 3a_1 a_2}{-6a_1 a_2} \\
&= \frac{a_1^2 a_2^2 - a_1^2 a_2^2 - a_1 a_2^2 + a_1 a_2}{2a_1 a_2} \\
&= \frac{1}{2}(a_1 a_2 - a_1 - a_2 + 1) \\
&= \frac{1}{2}(a_1 - 1)(a_2 - 1),
\end{aligned}$$

as desired. □

3.2 Non-representable Integers

Theorem 3.3. *The sum of all non-representable integers in terms of relatively prime integers a_1 and a_2, is given by*

$$\sum_{s \in \mathbb{Z}_{\geq 0} \setminus R} s = \frac{1}{12}(a_1 - 1)(a_2 - 1)(2a_1 a_2 - a_1 - a_2 - 1). \tag{3.4}$$

Proof. If we differentiate (6) once, we obtain

$$h'(z) = \sum_{s\in\mathbb{Z}_{\geq 0}\setminus R} sz^{s-1}.$$

Thus,

$$h'(1) = \sum_{s\in\mathbb{Z}_{\geq 0}\setminus R} s$$

gives us the sum of all non-representable integers in terms of $\{a_1, a_2\}$.

We use *Wolfram Mathematica* to apply L'Hôpital's rule six times, to obtain

$$\sum_{s\in\mathbb{Z}_{\geq 0}\setminus R} s = \lim_{z\to 1} h^{'}(z) = \frac{1}{12}(a_1-1)(a_2-1)(2a_1a_2-a_1-a_2-1).$$

This concludes our proof. □

Example 3.1. Set $S(a_1, a_2) = \sum_{s\in\mathbb{Z}_{\geq 0}\setminus R} s$. Then for $a_1 = 4$ and $a_2 = 11$ we have

$$\begin{aligned} S(4,11) &= \frac{1}{12}(4-1)(11-1)[2(4)(11)-4-11-1] \\ &= \frac{1}{12}(30)(72) \\ &= 180. \end{aligned}$$

The number of non-representable integers in terms of $\{4, 11\}$ is $\frac{1}{2}(4-1)(11-1) = 15$, and their sum is 180.

Proof of Theorem 2.1. Since $\mathbb{Z}_{\geq 0} \setminus R < \infty$, it follows that

$$h(z) = \sum_{s \in \mathbb{Z}_{\geq 0} \setminus R} z^s = \frac{(1 - z^{a_1})(1 - z^{a_2}) - (1 - z)(1 - z^{a_1 a_2})}{(1 - z)(1 - z^{a_1})(1 - z^{a_2})}$$

is a polynomial whose degree is precisely the Frobenius number. So $\deg(h(z)) = a_1 a_2 + 1 - (1 + a_1 + a_2) = a_1 a_2 - a_1 - a_2$. Hence, $g(a_1, a_2) = \max(\mathbb{Z}_{\geq 0} \setminus R) = a_1 a_2 - a_1 - a_2$, as desired. □

Chapter 4

Generating Functions for k-representable Integers with Parameters a and b

Theorem 4.1. *For relatively prime positive integers a and b, let*

$$T_k := \{r \in R : p_{\{a,b\}}(r) > k\},$$

the set of integers whose number of representations is bigger than k, where $k \geq 0$. Then the generating function associated with T_k is

$$\sum_{r \in T_k} x^r = \frac{x^{abk}(1 - x^{ab})}{(1 - x^a)(1 - x^b)}. \tag{4.1}$$

Proof. We use mathematical induction.

Base case: Let $k = 0$. Then $T_0 = R$, and the result follows from Theorem 1.1.

Induction Step: By Lemma 2.3, $r \in T_j$ if and only if $r + ab \in T_{j+1}$, and so

$$\begin{aligned}
\sum_{r \in T_{j+1}} x^r = \sum_{r \in T_j} x^{r+ab} &= \sum_{r \in T_j} x^r \cdot x^{ab} \\
&= \frac{x^{ab} \cdot x^{abj}(1 - x^{ab})}{(1 - x^a)(1 - x^b)} \quad \text{(by induction hypothesis)} \\
&= \frac{x^{ab(j+1)}(1 - x^{ab})}{(1 - x^a)(1 - x^b)}.
\end{aligned}$$

Thus, (7) holds true for $k = j + 1$, and the proof of the induction hypothesis is complete. Therefore, by the principle of mathematical induction, (7) holds true for all $k \in \mathbb{Z}_{\geq 0}$. □

Corollary 4.2. *Let $R_k := \{r \in R : p_{\{a,b\}}(r) = k\}$, the set of all integers with exactly k representations in terms of relatively prime integers a and b, where $k \in \mathbb{Z}_{>0}$. Then the generating function associated with R_k is*

$$\sum_{r \in R_k} x^r = \frac{x^{ab(k+1)} - 2x^{abk} + x^{ab(k-1)}}{(1 - x^a)(1 - x^b)}. \tag{4.2}$$

Proof. We consider the sets $T_{k-1} = \{r \in R : p_{\{a,b\}}(r) > k - 1\}$ and $T_k = \{r \in R : p_{\{a,b\}}(r) > k\}$. Then $T_{k-1} \setminus T_k = R_k$, and so

$$\sum_{r \in R_k} x^r = \sum_{r \in T_{k-1}} x^r - \sum_{r \in T_k} x^r. \tag{4.3}$$

Therefore, it follows by Theorem 4.1 that

$$\begin{aligned}
\sum_{r\in R_k} x^r &= \sum_{r\in T_{k-1}} x^r - \sum_{r\in T_k} x^r \\
&= \frac{x^{ab(k-1)}(1-x^{ab})}{(1-x^a)(1-x^b)} - \frac{x^{ab(k)}(1-x^{ab})}{(1-x^a)(1-x^b)} \\
&= \frac{x^{ab(k-1)}(1-x^{ab})^2}{(1-x^a)(1-x^b)} \\
&= \frac{x^{ab(k+1)} - 2x^{abk} + x^{ab(k-1)}}{(1-x^a)(1-x^b)}.
\end{aligned}$$

This concludes our proof. □

Remark. We notice that

$$\begin{aligned}
\sum_{r\in R_k} x^r &= \frac{x^{ab(k-1)}(1-x^{ab})^2}{(1-x^a)(1-x^b)} \\
&= x^{ab(k-1)} \times \frac{1-x^{ab}}{1-x^a} \times \frac{1-x^{ab}}{1-x^b} \\
&= x^{ab(k-1)} \times \frac{(1-x^a)\left(1+x^a+x^{2a}+\cdots+x^{a(b-1)}\right)}{1-x^a} \\
&\times \frac{(1-x^b)\left(1+x^b+x^{2b}+\cdots+x^{b(a-1)}\right)}{1-x^b} \\
&= x^{ab(k-1)}\left(1+x^a+x^{2a}+\cdots+x^{a(b-1)}\right)\left(1+x^b+x^{2b}+\cdots+x^{b(a-1)}\right).
\end{aligned}$$

Let $a = 3$ and $b = 5$. Then $R_k = \{r \in p_{\{3,5\}} = k \geq 1\}$, and so

$$\begin{aligned}\sum_{r \in R_k} x^r &= x^{15(k-1)}\left(1 + x^3 + x^6 + x^9 + x^{12}\right)\left(1 + x^5 + x^{10}\right)\\ &= x^{15(k-1)}\left(1 + x^3 + x^5 + x^6 + x^8 + x^9 + x^{10} + x^{11}\right)\\ &+ x^{15(k-1)}\left(x^{12} + x^{13} + x^{14} + x^{16} + x^{17} + x^{19} + x^{22}\right).\end{aligned}$$

This suggests that

$$R_k = \left\{\begin{array}{ccccc} 15(k\text{-}1), & 15(k\text{-}1)+3, & 15(k\text{-}1)+5, & 15(k\text{-}1)+6, & 15(k\text{-}1)+8, \\ 15(k\text{-}1)+9, & 15(k\text{-}1)+10, & 15(k\text{-}1)+11, & 15(k\text{-}1)+12, & 15(k\text{-}1)+13, \\ 15(k\text{-}1)+14, & 15(k\text{-}1)+16, & 15(k\text{-}1)+17, & 15(k\text{-}1)+19, & 15(k\text{-}1)+22 \end{array}\right\}.$$

Below is a short list of integers with exactly $k \geq 1$ representations

It is worth noting that the set R_k is finite, so

$$f(x) = \sum_{r \in R_k} x^r \tag{4.4}$$

is a polynomial whose total degree is precisely (3). This gives rise to the following Corollary.

Corollary 4.3. $g_k(a, b) = ab(k + 1) - a - b.$

R_k	Integers with Exactly $k \geq 1$ Representations									
R_1	0	3	5	6	8–14	16	17	19	22	
R_2	15	18	20	21	23	24–29	31	32	34	37
R_3	30	33	35	36	38–44	46	47	49	52	
R_4	45	48	50	51	53–59	61	62	64	67	
R_5	60	63	65	66	68–74	76	77	79	82	
R_6	75	78	80	81	83	84–89	91	92	94	97
R_7	90	93	95	96	98–104	106	107	109	112	
R_8	105	108	110	111	113–119	121	122	124	127	

Proof of Theorem 2.2. We have $f(x) = \sum_{r \in R_k} x^r$, and so $f(1) = \sum_{r \in R_k} 1$ counts the number of k-representable integers in terms of relatively prime parameters a and b. We consider

$$\lim_{z \to 1} f(x) = \lim_{z \to 1} \frac{x^{ab(k+1)} - 2x^{abk} + x^{ab(k-1)}}{(1 - x^a)(1 - x^b)}.$$

Upon applying L'Hôpital twice, we obtain

$$\sum_{r\in R_k} 1 = \lim_{z\to\ 1} \frac{x^{ab(k+1)} - 2x^{abk} + x^{ab(k-1)}}{(1-x^a)(1-x^b)}$$
$$= \frac{2ab^2 + 2abk - 2abk^2 - 2abk + 2ab^2k^2}{2ab}$$
$$= -\frac{(abk-1)abk}{ab} - k + ab(k^2+1)$$
$$= -abk^2 + k - k + abk^2 + ab$$
$$= ab,$$

as desired □

4.1 Sum of all k-representable integers, with relatively prime parameters a and b

Theorem 4.4. *Let $k \geq 1$. The sum of all k-representable integers, with relatively prime parameters a and b, is precisely*

$$\sum_{r\in R_k} r = \frac{1}{2}ab(2abk - a - b).$$

Proof. If we differentiate (10) once, we obtain $f'(x) = \sum_{r\in R_k} rx^{r-1}$. Thus, $f(1) = \sum_{r\in R_k} r$ gives the sum of all the integers with exactly k representations. So we have

$$f'(x) = \sum_{r \in R_k} r x^{r-1} = \frac{(1-x^{ab})\left(ab(k-1)x^{ab(k-1)} - abkx^{abk-1}\right)}{(1-x^a)(1-x^b)}$$
$$+ \frac{bx^{b-1}(1-x^{ab})\left(x^{ab(k-1)} - x^{abk}\right)}{(1-x^a)(1-x^b)^2} - \frac{abx^{ab-1}\left(x^{ab(k-1)} - x^{abk}\right)}{(1-x^a)(1-x^b)}$$
$$+ \frac{ax^{a-1}(1-x^{ab})\left(x^{ab(k-1)} - x^{abk}\right)}{(1-x^a)^2(1-x^b)}.$$

We use *Wolfram Mathematica* to obtain

$$\sum_{r \in R_k} r = \lim_{x \to 1} f'(x) = \frac{1}{2}ab(2abk - a - b),$$

as desired. □

Example 4.1. If $\{a, b\} = \{4, 11\}$, then

$$\sum_{r \in R_k} r = 1936k - 330.$$

Chapter 5

Variations of k-representable Integers with Two parameters

Theorem 5.1. *The generating function associated with all integers with at most k representations in terms of relative prime parameters a and b is*

$$\sum_{r\in\mathbb{Z}_{\geq 0}\setminus T_k} x^r = \frac{(1-x^a)(1-x^b) - x^{abk}(1-x)(1-x^{ab})}{(1-x)(1-x^a)(1-x^b)}. \tag{5.1}$$

Proof. We have

$$\begin{aligned}
\sum_{r\in\mathbb{Z}_{\geq 0}\setminus T_k} x^r &= \sum_{r\in\mathbb{Z}_{\geq 0}} x^r - \sum_{r\in T_k} x^r \\
&= \frac{1}{1-x} - \frac{x^{abk}(1-x^{ab})}{(1-x^a)(1-x^b)} \\
&= \frac{(1-x^a)(1-x^b) - x^{abk}(1-x)(1-x^{ab})}{(1-x)(1-x^a)(1-x^b)},
\end{aligned}$$

as desired. □

The set $\mathbb{Z}_{\geq 0} \setminus T_k$ is finite, for it only contains integers whose representations in terms of a or b do not exceed k. Hence, the total degree of (11) gives rise to the following Corollary.

Corollary 5.2. $g_k^*(a,b) = (k+1)ab - a - b = g_k(a,b)$.

Theorem 5.3. *There are precisely*

$$\frac{1}{2}\left[(a-1)(b-1) + 2abk\right]$$

integers with at most k representations, with relatively prime parameters a and b.

Proof. Let

$$\omega(x) := \sum_{r \in \mathbb{Z}_{\geq 0} \setminus T_k} x^r. \tag{5.2}$$

Then $\omega(1) = \sum_{r \in \mathbb{Z}_{\geq 0} \setminus T_k} 1$ counts the number of representable integers with at most

k representations. Thus, we have

$$\sum_{r\in\mathbb{Z}_{\geq 0}\setminus T_k} 1 = \lim_{x\to 1}\omega(x) = \lim_{x\to 1}\frac{(1-x^a)(1-x^b)-x^{abk}(1-x)(1-x^{ab})}{(1-x)(1-x^a)(1-x^b)}$$
$$= \frac{ab^2+ab(-1+2abk)-ab(-2+a+b)}{2ab}$$
$$= -\frac{1}{2}+\frac{1}{2}ab+abk-\frac{1}{2}a-\frac{1}{2}b+1$$
$$= \frac{1}{2}\left(ab-a-b+1+2abk\right)$$
$$= \frac{1}{2}\left[(a-1)(b-1)+2abk\right],$$

as desired. □

Remark. Theorem 5.2 is a consequence of Sylvester's theorem and Lemma 2.3; and for $k=0$, Sylvester's theorem is a corollary of Theorem 5.2.

5.1 Sum of All Integers with at Most k Representations

Theorem 5.4. *Let $k \geq 0$. The sum of all integers with at most k representations, with relatively prime parameters a and b, is given by*

$$\sum_{r\in\mathbb{Z}_{\geq 0}\setminus T_k} r = \frac{1}{12}(-1+b^2-3ab(-1+b+2bk)+a^2(1-3b(1+2k)+b^2(2+6k+6k^2))).$$

Proof. Upon differentiating (12) once, we obtain $\omega^{'}(x) = \sum_{r\in\mathbb{Z}_{\geq 0}\setminus T_k} rx^{r-1}$, and so

$$\begin{aligned}
\sum_{r\in\mathbb{Z}_{\geq 0}\setminus T_k} r &= \lim_{x\to 1}\omega'(x) \\
&= -\frac{1}{12}+\frac{a^2}{12}+\frac{ab}{4}-\frac{a^2b}{4}+\frac{b^2}{12}-\frac{ab^2}{4}+\frac{a^2b^2}{6}-\frac{1}{2}a^2bk-\frac{1}{2}ab^2k \\
&+\frac{1}{2}a^2b^2k+\frac{1}{2}a^2b^2k^2 \\
&= \frac{1}{12}(-1+b^2-3ab(-1+b+2bk)+a^2(1-3b(1+2k))) \\
&+\frac{1}{12}(b^2(2+6k+6k^2)) \\
&= \frac{1}{12}(a^2(1-3b(1+2k))+b^2(2+6k+6k^2+1)-3ab(2bk+b-1)-1),
\end{aligned}$$

as desired. □

Example 5.1. If $\{a, b\} = \{4, 11\}$, we have

$$\sum_{r\in\mathbb{Z}_{\geq 0}\setminus T_k} r = 180 + 638k + 968k^2,$$

where $T_k = \{r \in R : p_{\{4,11\}}(r) > k\}$.

Remark. If $k = 0$, then $T_0 = \{4m + 11n : m, n \in \mathbb{Z}_{\geq 0}\}$, and so

$$\sum_{r\in\mathbb{Z}_{\geq 0}\setminus T_0} r = S(4, 11) = 180,$$

which is precisely the sum of all non-representable integers in terms of $\{4, 11\}$, already computed in Example 3.1. That is, Theorem 3.3 is a corollary of Theorem 5.3 when $k = 0$, and so

$$\begin{aligned}\sum_{r\in\mathbb{Z}_{\geq 0}\setminus T_0} r &= \frac{1}{12}(-1+b^2-3ab(-1+b+2b(0))+a^2(1-3b(1+2(0))) \\ &+ \frac{1}{12}(b^2(2+6(0)+6(0)^2)) \\ &= \frac{1}{12}(-1-3a(-1+b)b+b^2+a^2(1-3b+2b^2)) \\ &= \frac{1}{12}(a-1)(b-1)(2ab-a-b-1),\end{aligned}$$

as expected.

Appendix A: The Frobenius Problem with Three or More Parameters

The Frobenius problem for $d \geq 3$ is significantly harder. Although serious attempts have been made to its inquiry, the Frobenius problem for $d \geq 3$ still remains unresolved. For readers interested in the Frobenius problem, as well as related open problems, see [14], which surveys the references to near every article concerning the aforementioned problem.

Let $R := \{a_1m_1+a_2m_2+\cdots+a_dm_d : m_1, m_2, \ldots, m_d \in \mathbb{Z}_{\geq 0}\}$, where $a_1, a_2, \ldots, a_d$ are relatively prime positive integers. Then the generating function associated with R is precisely (see [5], Exercise 1.36)

$$j(x) := \sum_{r \in R} x^r = \frac{\kappa(x)}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_d})}$$

for some polynomial κ. Hence, the Frobenius number of $\{a_1, a_2, \ldots, a_d\}$ narrows down to finding the total degree of the rational function

$$\frac{1-(1-x)j(x)}{1-x},$$

which is possible once the polynomial κ has been determined. Marcel Morales [10, 11] and Graham Denham [8] descovered that for $d = 3$, the polynomial κ has either four or six terms, of which they provide explicit formulas. On the other hand, Henrik Bresinsky [6] proved that for $d \geq 4$, there is no absolute bound on the number of terms of the polynomial κ, and Alexander Barvinok and Kevin Woods [2] proved that if we fix d, then the rational generating function $j(x)$ can be written as a “short” sum of rational functions. Lastly, Jorge L. Ramírez Alfonsín [13] proved that trying to compute the Frobenius number efficiently is impossible if d is left as a variable.

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