ASPHERIC ORIENTATIONS OF SIMPLICIAL COMPLEXES

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Logan Godkin

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CERTIFICATION OF APPROVAL

I certify that I have read *ASPHERIC ORIENTATIONS OF SIMPLICIAL COMPLEXES* by Logan Godkin and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

Matthias Beck
Professor of Mathematics

Felix Breuer
Professor of Mathematics

Jeremy Martin
Associate Professor of Mathematics
University of Kansas

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Logan Godkin
San Francisco State University
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We generalize the notions of colorings, flows, and tensions of graphs to simplicial complexes, and then to cell complexes, by viewing a graph as a one dimensional simplicial complex. We look at both integral and modular colorings, flows, and tensions of cell complexes. The functions that count colorings, flows, and tensions for graphs are known to be polynomials. We show that these counting functions for cell complexes are in general quasipolnomials and for the modular counting functions we give sufficent conditions for these functions to be polynomials. Furthermore, we show that these modular counting functions are evaluations of the Tutte polynomial. We show that for certain cell complexes we can generalize the deletion-contraction operation for graphs and use it to compute these modular counting functions. We generalize the reciprocity results for these integral counting fucntions of graphs to cell complexes via inside-out polytopes and Ehrhart–Macdonald reciprocity.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Chapter 1

Introduction

1.1 Motivation

In this section we give some motivation for how we define colorings, tensions, and flows for cell complexes. These definitions are based on the boundary matrix of a cell complex and since we wish to extend ideas from graph theory to cell complexes we start by constructing the boundary matrix of a graph.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Let $e = vu$ be an edge of G, where v and u are the vertices incident to e. Then we can give the edge e an orientation by writing its vertices as an ordered pair $e = (v, u)$. We say that e is directed from v to u, and we say that v is the tail and u is the head. Giving an orientation to each edge of G we obtain an oriented graph G'. We construct the boundary matrix B of G', a $|V| \times |E|$-matrix whose rows are indexed by the vertices of G' and whose columns are indexed by the edges of G', by setting the ve-entry equal to -1 if v is the tail of e; 1 if v is the head of e; 0 otherwise (see [4] for more).

Now let G be an oriented graph. A path of G is a sequence of vertices $\{v_1, \ldots, v_n\}$ such that v_1v_2, v_2v_3, $\ldots$, $v_{n-1}v_n$ are edges of the graph and the v_i are all distinct. A cycle C of a graph is a closed path, that is, if the sequence $\{v_1, v_2, \ldots, v_n\}$ represents a cycle of G, then the cycle has edges v_1v_2, v_2v_3, $\ldots$, $v_{n-1}v_n$, v_nv_1. If G is an oriented graph, then an oriented cycle C^* of G is a sequence of vertices $\{v_1, v_2, \ldots, v_n\}$ such that (v_1, v_2), (v_2, v_3), $\ldots$, (v_{n-1}, v_n), (v_n, v_1) are oriented edges of G. It can be shown that any cycle C of G can be represented as an element s in $\{-1, 0, 1\}^E$ such that $Bs = 0$ and, vice versa, any element s in $\{-1, 0, 1\}^E$ such that $Bs = 0$ represents a cycle C of G. Furthermore, any oriented cycle C^* of G can be represented as an element s^* in $\{0, 1\}^E$ such that $Bs^* = 0$ and, vice versa, any element s^* in $\{0, 1\}^E$ such that $Bs^* = 0$ represents an oriented cycle C^* of G. It can be shown that the set of cycle vectors of a graph form a basis for the null space (over $\mathbb{R}$) of B (see [4] and [5, Chapter 14.2] for more).

An A-coloring of G is a labeling of the vertices of G with elements of A, where A is a commutative ring with unity. A proper A-coloring of a graph is a labeling of the vertices such that there do not exist adjacent vertices labeled by the same element of A. We can think of an A-coloring as an element c in A^V. Thus c is a proper A-coloring if cB is nowhere-zero.

An A-tension of G is a labeling of the edges of G with elements of A such that each oriented sum of edge labelings on each cycle of G is zero. We say that an A-tension is nowhere-zero if none of the edge labels are zero. We can represent an A-tension as an element ψ in A^E such that $\psi \cdot s = 0$ for every cycle s of G, and ψ is nowhere-zero if none of the entries of ψ are zero.

An A-flow of G is a labeling of the edges of G with elements of A such that the oriented sum of edge labelings at each vertex is equal to zero, that is, the sum of the edge labelings whose edges are directed to the vertex v equals the edge labelings whose edges are directed away from v. We say that an A-flow is nowhere-zero if none of the edge labels are zero. We can represent an A-flow as an element ϕ in A^E such that $B\phi = 0$, and ϕ is nowhere-zero if none of the entries of ϕ are zero.

An oriented graph G is said to be acyclic if G does not contain any oriented cycles or equivalently if the null space of B (over $\mathbb{R}$) does not contain any nonzero vector in $\{0,1\}^E$. We say that G is totally cyclic if every edge of G belongs to some oriented cycle or equivalently if for every edge e there is a vector in $\{0,1\}^E$ in the null space of B whose e-entry is nonzero.

1.2 Simplicial Complexes

In this section we will construct the bounday matrix of a simplicial complex. We will use the boundary matrix of a simplicial complex to define colorings, tensions, and flows of cell complexes as we did for graphs in the previous section.

A **simplicial complex** X is a collection of finite nonempty sets (the **faces** of X)

such that if Y is an element of X, then so is every nonempty subset of Y. An element of cardinality $d+1$ is called a d-**dimensional simplex** or more simply d-**simplex**. We say X is d-**dimensional** if the maximum cardinality of any element of X is $d+1$. For any d-dimensional simplicial complex X, let F denote the set of d-dimensional simplices called **facets**, let R denote the set of $(d-1)$-dimensional simplices called **ridges**, and let V denote the 0-dimensional simplices called **vertices** (see [6] for more on simplicial complexes).

The **boundary matrix**, denoted $[\partial_d]$, of a d-dimensional simplical complex X is an $|R| \times |F|$-matrix where the rows are indexed by the ridges and the columns are indexed by the facets of X. The entries of $[\partial_d]$ are obtained from the following setup.

1. Fix a total ordering of the vertices of X. Given that $|V| = k$, we fix a total ordering via a bijection $V \to [k]$, where $[k] := \{1, 2, \ldots, k\}$.

2. Orient each face of X by writing its vertices in an increasing order, that is, a d-face D with vertex set $\{v_0, v_1, \ldots, v_d\}$ will produce a **chain** $[D] = [v_0, v_1, \ldots, v_d]$. The **chain group**, denoted $C_d(X)$, is the free $\mathbb{Z}$-module of $\mathbb{Z}$-linear combinations of d-dimensional simplices, represented by their chains.

3. Define the d-th boundary map (see, for instance, [6]) $\partial_d : C_d(X) \to C_{d-1}(X)$ by
$$\partial_d([v_0, v_1, \ldots, v_d]) = \sum_{j=0}^{d} (-1)^j [v_0, ..., v_{j-1}, v_{j+1}, ..., v_d].$$

If $f = [v_0, v_1, \ldots, v_d]$ is a facet of X and $r = [v_0, ..., v_{j-1}, v_{j+1}, ..., v_d]$ is a ridge of f, then the rf-entry in $[\partial_d]$ is $(-1)^j$ and otherwise 0.

Remark. From the construction of the boundary matrix of a simplicial complex we get that the nonzero entries in each column must alternate in sign. Thus, since there are two possibilities for how the nonzero entires alternate in sign, we see that each facet of a simplicial complex can have two orientations. Hence a reorientation of a facet can be given by multiplying the appropriate column of the boundary matrix by -1.

Let X be a simplicial complex with boundary matrix $[\partial]$. We define a **boundary matrix orientation** of X to be an element o of $\{-1, 1\}^F$. We **reorient** X by scalar multiplying every column f_i of the boundary matrix $[\partial]$ of X by the corresponding entry o_i in the orientation vector o. Note that the initial boundary matrix has the orientation o in $\{1\}^F$.

We call the vectors belonging to the null space (over $\mathbb{R}$) of $[\partial_d]$ the **cycles** of X and from now on we will refer to this null space as the **cycle space**. We define a **sphere** (**simple cycle**) of X to be an element s of $\mathbb{Z}^F$ belonging to the cycle space of $[\partial_d]$ and an **oriented sphere** (**oriented cycle**) of X to be an element s_* of $\mathbb{N}^F$ belonging to the cycle space of $[\partial_d]$, where $\mathbb{N}$ is the set of nonnegative integers. A simplicial complex X is **aspheric** (**acyclic**) if there does not exist a nonzero element s_* of $\mathbb{N}^F$ in the cycle space of $[\partial_d]$ and we say X is **totally spheric** (**totally cyclic**) if for every facet f_i of X we have an element s_* of $\mathbb{N}^F$ with a nonzero ith-entry in the cycle space of $[\partial_d]$.

An A-**coloring** of X is an element c of A^R, where A is a commutative ring with unity, and we say the coloring is **proper** if $c \cdot [\partial_d]$ is nowhere-zero in A. A k-**coloring** (**integral coloring**) of X is a coloring that is an element of $[-k, k]^R$, where $A = \mathbb{Z}$ and $[-k, k] := \{-k, \ldots, -1, 0, 1, \ldots, k\}$. A $\mathbb{Z}_k$-coloring (**modular coloring**) of X is a coloring, where $A = \mathbb{Z}_k$.

Note. As far as we know, the concept of proper colorings has not been studied for simplicial complexes.

An element ϕ of A^F is an A-**flow** if $[\partial_d] \cdot \phi = 0$ in A. Thus, the null space (over A) of $[\partial_d]$ contains all of the flow vectors of a simplicial complex. A k-**flow** of X is a nonzero element ϕ of $[-k, k]^F$ that is a flow in $\mathbb{Z}$, and a $\mathbb{Z}_k$-**flow** of X is a nonzero element ϕ of $\mathbb{Z}_k^F$ that is a flow in $\mathbb{Z}_k$. We define the **flow space** of X to be the null space of $[\partial_d]$ over A.

An A-**tension** is an element ψ of A^F such that $\psi \cdot q = 0$ in A for every vector q in the cycle space of $[\partial_d]$. Thus every A-linear combination of row vectors from $[\partial_d]$ is an A-tension. A k-**tension** of X is a nonzero element ψ of $[-k, k]^F$ such that $\psi \cdot q = 0$ in $\mathbb{Z}$ and a $\mathbb{Z}_k$-**tension** of X is a nonzero element ψ of $\mathbb{Z}^F$ such that $\psi \cdot q = 0$ in $\mathbb{Z}_k$, for every vector q in the cycle space of $[\partial_d]$.

We define the **order** of a simplicial complex X to be the number of ridges in X, and the **degree** of a ridge is the number of facets that have the ridge as a boundary.

Proposition 1.1. *In a simplicial complex X, the sum of degrees of the ridges is* *equal to $(d+1)$ times the number of facets, where d is the dimension of X.*

Proof. Let
$$S = \sum_{r \in R} \deg(r)$$
and let the dimension of X be d. Note that we count each facet exactly $d+1$ times. Thus, $S = (d+1)|F|$. $\square$

Corollary 1.2. *Let X be a d-dimensional simplicial complex. If d is odd, then the number of ridges with odd degree is even, and if d is even, then the number of ridges with odd degree is odd if $|F|$ is odd and even if $|F|$ is even.*

1.3 Homology

We begin with an introduction to homology (taken from [6]), and then proceed to give a construction of cell complexes and their corresponding boundary matrices.

We start with a sequence of homomorphism of abelian groups

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n and $\partial_0 = 0$. We refer to such a sequence as a **chain complex**. It follows that the image of ∂_{n+1}, denoted $\operatorname{Im} \partial$, is a subset of the kernel of ∂_n, denoted $\ker \partial_n$. Then the **n-th homology group** of the chain complex is the quotient group $H_n = \ker \partial_n / \operatorname{Im} \partial_{n+1}$.

When X is a simplicial complex the C_i are groups whose elements are $\mathbb{Z}$-linear combinations of i-dimensional faces (as described in Section 1.2).

We define the **reduced homology group** $\tilde{H}_n(X)$ to be the homology groups of the augmented chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\varepsilon\left(\sum_i n_i \sigma_i\right) = \sum_i n_i$, σ_i is a chain in C_0, and n_i is an integer.

Given a space X and a subspace $Y \subset X$, let $C_n(X,Y)$ be the quotient group $C_n(X)/C_n(Y)$. Since the boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(Y)$ to $C_{n-1}(Y)$ we have the induced quotient boundary map $\partial : C_n(X,Y) \to C_{n-1}(X,Y)$. Then we have a chain complex

$$\cdots \to C_{n+1}(X,Y) \xrightarrow{\partial_{n+1}} C_n(X,Y) \to \cdots \to C_1(X,Y) \xrightarrow{\partial_1} C_0(X,Y) \xrightarrow{\partial_0} 0.$$

The **relative homology group** $H_n(X,Y)$ is $\ker \partial_n / \operatorname{Im} \partial_{n+1}$.

An n-**cell** e^n is an open n-dimensional ball (up to homeomorphism). We construct a **cell complex** X in the following way using the construction from [6]:

1. Start with a finite set of 0-cells denoted X^0.

2. Inductively, form the n-**skeleton** X^n from X^{n-1} by attaching n-cells e_i^n via maps $\alpha_i : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the

disjoint union $X^{n-1} \coprod_i D_i^n$ of X^{n-1} with a collection of n-disks D_i^n under the identifications $x \equiv \alpha_i(x)$ for $x \in \partial D_i^n$. As a set, $X^n = X^{n-1} \coprod_i e_i^n$.

3. We stop at some finite n and set $X = X^n$.

We say a cell complex X is n-**dimensional** if $X = X^n$. Given an n-dimensional cell complex we will refer to the n-cells as **facets** and the $(n-1)$-cells as **ridges**.

Given a map $f : S^n \to S^n$, the induced map $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ is a homomorphism of the form $f_*(x) = \alpha x$, where α is some integer that depends only on f [6]. This integer α is called the **degree** of f [6].

The **cellular boundary formula** is given [6] by $d_n(e_f^n) = \sum_r d_{rf} e_r^{n-1}$ where d_{rf} is the degree of the map

$$S_f^{n-1} = \partial D_f \to X^{n-1} \to \frac{X^{n-1}}{X^{n-1} - e_r^{n-1}} = S_r^{n-1},$$

that is, the composition of the attaching maps of e_f^n with the quotient map collapse $X^{n-1} \setminus e_r^{n-1}$ to a point [6].

We form the n-**th boundary matrix** $[d_n]$ of a cell complex X by indexing the rows by the ridges and the columns by the facets of X. We set the rf-entry in $[d_n]$ to d_{rf}, where d_{rf} is given by the cellular boundary formula.

Remark. A boundary matrix of a cell complex is an integer matrix since the d_{rf} entry in $[d_n]$ is always an integer and any integer matrix is the boundary matrix of some cell complex since we can construct a cell complex where the e_n cell f is wrapped around the boundary of the e_{n-1} cell r any integer number of times.

Since our definitions in section 1.2 only depend on the boundary matrix of our simplicial complex we will extend those definitions by replacing the "boundary matrix of a simplicial complex" with that of the "boundary matrix of a cell complex".

Proposition 1.3. *Let X be a d-dimensional cell complex. We have a proper A-coloring c if and only if we have a nowhere-zero A-tension ψ, where ψ is an A-linear combination of row vectors of $[\partial_d]$.*

Proof. Assume c is a proper A-coloring. Then $c[\partial_d] = \psi$ is nowhere-zero and since ψ is an A-linear combination of row vectors of $[\partial_d]$ we have that $\psi \cdot q = 0$, where q is any cycle. Assume ψ is an A-tension that is an A-linear combination of row vectors of $[\partial_d]$, that is,

$$\sum_{r \in R} c_r b_r = \psi,$$

where b_r is a row of $[\partial_d]$ and $c_r \in A$ for all $r \in R$. Thus $c = (c_r)_{r \in R}$ is a proper coloring of X since ψ is nowhere-zero. $\square$

Chapter 2

Deletion-Contraction and Inclusion-Exclusion

We define deletion and contraction of a facet of a cell complex in terms of operations on the boundary matrix and note that the contraction operation is not always possible.

We define the **deletion of a facet** f of a cell complex X to be the cell complex X without the facet f. In terms of the boundary matrix $[\partial_n]$ of $X^n = X$, the deletion of the facet f corresponds to the removal of the column in $[\partial_n]$ corresponding to the facet f. We denote the cell complex obtained from the deletion of f by $X \setminus f$.

We define the **contraction of a facet** f of a cell complex X to be the following **pivot operation** on the boundary matrix of X, where the pivot operation of a matrix B selects a nonzero entry b_{rf}, a **pivot**, in B and performs the following steps:

1. For each row $i \neq r$ of B, replace row i by row $i - \left(\frac{b_{if}}{b_{rf}}\right)$row r,

2. Multiply row r by $\frac{1}{b_{rf}}$.

Remark. The contraction of a facet f of a cell complex is, in general, not unique since we may any ridge r of f.

Remark. It is possible that the contraction of a facet f of a cell complex does not result in an integer matrix. For our purposes we restrict our choices of pivot elements to ± 1 entries in our boundary matrix.

A matrix is **totally unimodular** if and only if the determinant of each square sub-matrix is -1, 0, or 1. We have the following theorems from [3]:

Let X be a compact Hausdorff space. A **curved triangle** in X is a subspace A of X and a homeomorphism $h : T \to A$, where T is a closed triangular region in the plane. If e is an edge of T, then $h(e)$ is said to be an edge of A; if v is a vertex of T, then $h(v)$ is said to be a vertex of A. A **triangulation** of X is a collection of curved triangles $A_1, \ldots, A_n$ in X whose union is X such that for $i \neq j$, the intersection $A_i \cap A_j$ is either empty, or a vertex of both A_i and A_j, or an edge of both. Furthermore, if $h_i : T_i \to A_i$ is the homeomorphism associated with A_i, we require that when $A_i \cap A_j$ is an edge e of both, then the map $h_j^{-1} h_i$ defines a linear homeomorphism of the edge $h_i^{-1}(e)$ of T_i with the edge $h_j^{-1}(e)$ of T_j [7].

An n**-dimensional manifold** is a Hausdorff space M such that each point has an open neighborhood homeomorphic to $\mathbb{R}^n$. A **local orientation** of M at a point

x is a choice of generator μ_x of the infinite cyclic group $H_n(M, M - \{x\})$. An **orientation** of an n-dimensional manifold M is a function $x \mapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M, M - \{x\})$, satisfying the condition that each $x \in M$ has a neighborhood $\mathbb{R}^n \subset M$ containing an open ball B of finite radius about x such that all the local orientations μ_y at points $y \in B$ are the images of one generator μ_B of $H_n(M, M - \{x\}) \sim H_n(\mathbb{R}^n, \mathbb{R}^n - B)$ under the natural maps $H_n(M, M - B) \to H_n(M, M - \{y\})$. If an orientation exists for M, then M is called **orientable** [6].

Theorem 2.1. *[3, Theorem 4.1] For a finite simplicial complex X triangulating a $(d+1)$-dimensional compact orientable manifold, the boundary matrix of X is always totally unimodular.*

A **pure simplicial complex** of dimension d is a simplicial complex formed from a collection of d-simplices and their proper faces and a **pure subcomplex** is a subcomplex that is a pure simplicial complex.

For any finite simplicial complex X we have from the fundamental theorem of finitely generated abelian groups $H_i(X) = Z \oplus T$, where $Z = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ and $T = \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_m}$ with $k_j > 1$ and k_j dividing k_{j+1}. The subgroup T of $H_i(X)$ is called the **torsion** of $H_i(X)$. When $T = 0$ then $H_i(X)$ is said to be **torsion-free**.

Theorem 2.2. *[3, Theorem 5.2] $[\partial_{d+1}]$ is totally unimodular if and only if $H_d(L, L_0)$ is torsion-free, for all pure subcomplexes L_0, L of X of dimension d and $d+1$ respectively, where $L_0 \subset L$.*

Theorem 2.3. *[3, Theorem 5.7] Let X be a finite simplicial complex embedded in $\mathbb{R}^{d+1}$. Then, $H_d(L, L_0)$ is torsion-free for all pure subcomplexes L_0 and L of dimension d and $d+1$ respectively, such that $L_0 \subset L$.*

Remark. Applying a pivot operation to a totally unimodular matrix results in a totally unimodular matrix (see, for instance [8]). Thus the three above theorems give conditions for a simplicial complex to be contracted to a single column and hence allows us to easly compute $\chi_X^*(k)$ for a simplicial complex X.

We define a **totally unimodular complex** to be any cell complex whose boundary matrix is totally unimodular. We denote the contraction operation on a cell complex X by X/rf, where f is the facet and r is the ridge selected in the operation.

2.1 The Modular Coloring Function

We show that the modular coloring function is a polynomial for any totally unimodular complex, for certain cell complexes, and in general we have a quasipolynomial.

Recall that a $\mathbb{Z}_k$-coloring of a cell complex X is an element c of $\mathbb{Z}_k^R$ and is said to be proper if $c \cdot [\partial]$ is nowhere-zero in $\mathbb{Z}_k$, where $[\partial]$ is the boundary matrix of X. We define

$$\chi_X^*(k) := \text{number of proper } \mathbb{Z}_k\text{-colorings of } X.$$

Lemma 2.4. *Let X be a cell complex with $|F| = 0$. Then $\chi_X^*(k) = k^{|R|}$.*

Proof. If X has no facets, then there are no restrictions on the colors of the ridges of X and so each ridge can be colored with k colors. Thus, $\chi^*_X(k) = k^{|R|}$. □

Theorem 2.5. *Let X be a totally unimodular complex. Then we have*

$$\chi^*_X(k) = \chi^*_{X\backslash f}(k) - \chi^*_{X/rf}(k),$$

where f is any facet of X and r is any ridge of f.

Proof. Let X be a totally unimodular complex with boundary matrix $[\partial]$ and without loss of generality let $b_{rf} = 1$. Let $c = (c_i)_{i\in R}$ be a proper coloring of $X\backslash f$, and let $\psi = c[\partial]$. Then all entries of ψ, except possibly ψ_f, are zero.

If indeed we have $\psi_f = 0$, then $c_r + \sum_{i\in R-\{r\}} c_i b_{if} = 0$ so $-c_r = \sum_{i\in R-\{r\}} c_i b_{if}$. Let g be a facet of X different from f. Then we have two cases when the b_{rg} entry is nonzero; either $\sum_{i\in R-\{r\}} c_i b_{ig} + c_r \neq 0$ or $\sum_{i\in R-\{r\}} c_i b_{ig} - c_r \neq 0$. Performing the pivot operation we get for each case

$$\sum_{i\in R-\{r\}} c_i b_{ig} - \sum_{i\in R-\{r\}} c_i b_{if} = \sum_{i\in R-\{r\}} c_i b_{ig} + c_r \neq 0$$

respectively

$$\sum_{i\in R-\{r\}} c_i b_{ig} + \sum_{i\in R-\{r\}} c_i b_{if} = \sum_{i\in R-\{r\}} c_i b_{ig} - c_r \neq 0.$$

Thus we see that $c = (c_i)_{i\in R-\{r\}}$ is a proper coloring of X/rf.

Assume we have a proper coloring $c = (c_i)_{i\in R-\{r\}}$ of X/rf. Working in reverse we see that there is a unique coloring c_r, namely $-\sum_{i\in R-\{r\}} c_i b_{if}$, that makes $c = (c_i)_{i\in R}$ a proper coloring of X except for $c[\partial] = \psi$ we have $\psi_f = 0$.

Thus we see that $\chi^*_{X\backslash f}(k)$ counts the number of $\mathbb{Z}_k$-colorings $c[\partial] = \psi$ such that ψ_f can be anything and $\psi_g \neq 0$ where $g \neq f \in F$, and $\chi^*_{X/rf}(k)$ counts the number of $\mathbb{Z}_k$-colorings $c[\partial] = \psi$ such that $\psi_f = 0$ and $\psi_g \neq 0$ where $g \neq f \in F$. Hence $\chi^*_{X\backslash f}(k) - \chi^*_{X/rf}(k)$ counts the number of proper $\mathbb{Z}_k$-colorings of X. □

Remark. The proof above illustrates why the contraction operation does not work for k-colorings, that is, if we start with a proper k-coloring c of X/rf we cannot always expect to find a number c_r in $[-k, k]$ such that $c_r = \sum_{i\in R-\{r\}} c_i b_{if}$.

Remark. The deletion-contraction operation on cell complexes specializes to the deletion-contraction operation on graphs when our cell complex is a graph.

Corollary 2.6. *Let X be a cell complex whose boundary matrix $[\partial]$ has at least one entry $b_{rf} = \pm 1$. Then we have*

$$\chi^*_X(k) = \chi^*_{X\backslash f}(k) - \chi^*_{X/rf}(k).$$

Proof. The proof is similar to the proof of Theorem 2.5 except for one slight adjustment.

Assume $c = (c_i)_{i\in R}$ is a proper coloring of $X\backslash f$, and let $\psi = c[\partial]$. Then all entries of ψ, except possibly ψ_f, are zero. Then $c_r + \sum_{i\in R-\{r\}} c_i b_{if} = 0$ so $-c_r = \sum_{i\in R-\{r\}} c_i b_{if}$. Then when the b_{rg} entry is nonzero for $g \neq f$ we have $\sum_{i\in R-\{r\}} c_i b_{ig}+$

$ac_r \neq 0$, where $a = b_{rg} \neq 0$. Performing the pivot operation we get

$$\sum_{i \in R-\{r\}} c_i b_{ig} - a \sum_{i \in R-\{r\}} c_i b_{if} = \sum_{i \in R-\{r\}} c_i b_{ig} + ac_r \neq 0.$$

Thus we see that $c = (c_i)_{i \in R-\{r\}}$ is a proper coloring of X/rf.

This completes the adjustment. □

Proposition 2.7. *Let X be a cell complex whose boundary matrix $[\partial]$ has at least one zero column. Then $\chi^*_X(k) = 0$.*

Proof. Let X be a cell complex whose boundary matrix $[\partial]$ has a zero column. Then there does not exists a coloring $c \in \mathbb{Z}_K^R$ such that $c[\partial]$ is nowhere-zero. □

Corollary 2.8. *Let X be a totally unimodular complex. Then $\chi^*_X(k)$ is a polynomial.*

Proof. Let X be a totally unimodular complex. We induct on the number of facets of X.

Base Case: Assume X has no facets. Then $\chi^*_X(k) = k^{|R|}$.

Inductive Step: Assume the claim is true for any totally unimodular complex with $|F| < m$. Suppose X is a totally unimodular complex with $|F| = m$. Then we have that $\chi^*_X(k)$ is a polynomial since

$$\chi^*_X(k) = \chi^*_{X \setminus f}(k) - \chi^*_{X/rf}(k)$$

and both $\chi^*_{X \setminus f}(k)$ and $\chi^*_{X/rf}(k)$ are polynomials. □

Corollary 2.9. *Let X be a totally unimodular complex whose boundary matrix does not contain a zero column. Then the following are true:*

1. *the degree of $\chi_X^*(k)$ is the number of ridges of X,*

2. *the leading coefficient of $\chi_X^*(k)$ is 1, and*

3. *the coefficients of $\chi_X^*(k)$ alternate in sign.*

Proof. We induct on the number of facets of X.
Base Case : Assume X has no facets. Then $\chi_X^*(k) = k^{|R|}$, thus claims 1 through 3 are true.
Inductive Step: Assume claims 1 through 3 are true for totally unimodular complexes with less than m facets. Then we have

$$\chi_{X\setminus f}^*(k) = k^{|R|} - a_{|R|-1}k^{|R|-1} + a_{|R|-2}k^{|R|-2} - \cdots - a_1 + a_0$$

and

$$\chi_{X/rf}^*(k) = k^{|R|-1} - b_{|R|-2}k^{|R|-2} + \cdots + b_1 - b_0,$$

where $a_i, b_i \in \mathbb{Z}$ for all i. Thus

$$\begin{aligned}\chi^*_X(k) &= \chi^*_{X\setminus f}(k) - \chi^*_{X/rf}(k) \\ &= k^{|R|} - a_{|R|-1}k^{|R|-1} + a_{|R|-2}k^{|R|-2} - \cdots - a_1 + a_0 \\ &- (k^{|R|-1} - b_{|R|-2}k^{|R|-2} + \cdots + b_1 - b_0) \\ &= k^{|R|} - (a_{|R|-1} + b_{|R|-2})k^{|R|-1} + \cdots + (a_0 + b_0).\end{aligned}$$

Thus claims 1 through 3 are true. $\square$

2.1.1 Deletion-Contraction Examples

Example 2.1. The constant term of $\chi^*_X(k)$ is not always zero. Consider the cell complex X whose boundary matrix is

$$[\partial]_X = \begin{array}{c c} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{array}; \quad [\partial]_{X\setminus f_2} = \begin{array}{c c} & f_1 \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}, \quad [\partial]_{X/r_2f_2} = \begin{array}{c c} & f_1 \\ r_1 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

$$\implies \chi^*_X(k) = \chi^*_{X\setminus f_2}(k) - \chi^*(k)_{X/r_2f_2} = k(k-1) - (k-1) = k^2 - 2k + 1.$$

Example 2.2. The absolute value of the coefficient of the second leading term is not always equal to the number of facets of the cell complex X. Consider

$$[\partial]_X = \begin{array}{c c} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}; \quad [\partial]_{X\setminus f_2} = \begin{array}{c c} & f_1 \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}, \quad [\partial]_{X/r_2 f_2} = \begin{array}{c c} & f_1 \\ r_1 & \begin{pmatrix} 0 \end{pmatrix} \end{array}$$

$$\implies \chi^*_X(k) = \chi^*_{X\setminus f_2}(k) - \chi^*(k)_{X/r_2 f_2} = k(k-1) - 0 = k^2 - k.$$

Remark. Examples 2.1 and 2.2 demonstrate that not all of the nice properties of the chromatic polynomial of a graph are shared with the modular coloring function of a cell complex.

Example 2.3. Consider the Möbius strip M as the cell complex in Figure 2.1.

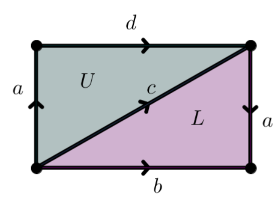

Figure 2.1: The Möbius strip M with facets L and U both oriented clockwise.

$$[\partial]_M = \begin{array}{c c} & \begin{array}{cc} L & U \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \end{array}; \quad [\partial]_{M\setminus L} = \begin{array}{c c} & U \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \end{array}, \quad [\partial]_{M/bL} = \begin{array}{c c} & U \\ \begin{array}{c} a \\ c \\ d \end{array} & \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{array}$$

$$\implies \chi^*_M(k) = \chi^*_{M\setminus L}(k) - \chi^*_{M/bL}(k) = k^3(k-1) - k^2(k-1) = k^4 - 2k^3 + k^2.$$

Example 2.4. Consider $\mathbb{R}P^2$ as the cell complex in Figure 2.2.

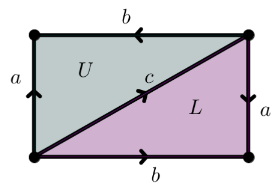

Figure 2.2: $\mathbb{R}P^2$ with facets L oriented clockwise and U oriented counter-clockwise.

$$[\partial]_{\mathbb{R}P^2} = \begin{array}{c c} & \begin{array}{cc} L & U \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}; \quad [\partial]_{\mathbb{R}P^2\setminus L} = \begin{array}{c c} & U \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{array}, \quad [\partial]_{\mathbb{R}P^2/bL} = \begin{array}{c c} & U \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} \end{array}$$

$$\implies \chi^*_{\mathbb{R}P^2}(k) = \chi^*_{\mathbb{R}P^2\setminus L}(k) - \chi^*_{\mathbb{R}P^2/bL}(k) = k^2(k-1) - kp(k), \text{ where}$$

$$p(k) = \begin{cases} k-1 & \text{if } k \text{ is odd;} \\ k-2 & \text{if } k \text{ is even.} \end{cases}$$

Thus,

$$\chi^*_{\mathbb{R}P^2}(k) = \begin{cases} k^3 - 2k^2 + k & \text{if } k \text{ is odd;} \\ k^3 - 2k^2 + 2k & \text{if } k \text{ is even.} \end{cases}$$

Example 2.5. For the complete graph K_3 we have the boundary matrix,

$$[\partial]_{K_3} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \end{array} \begin{array}{c} \begin{array}{ccc} [12] & [13] & [23] \end{array} \\ \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \end{array};$$

$$[\partial]_{K_3\setminus[12]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \end{array} \begin{array}{c} \begin{array}{cc} [13] & [23] \end{array} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} \end{array}, \quad [\partial]_{K_3/[1][12]} = \begin{array}{c} \\ [2] \\ [3] \end{array} \begin{array}{c} \begin{array}{cc} [13] & [23] \end{array} \\ \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \end{array}$$

$$\implies \chi^*_{K_3}(k) = \chi^*_{K_3\setminus[12]}(k) - \chi^*_{K_3/[1][12]}(k) = k(k-1)^2 - k(k-1) = k^3 - 3k^2 + 2k.$$

Given an $n \times m$-matrix M with rank s over any principal ideal domain, there exist invertible matrices U (an $n \times n$-matrix) and W (an $m \times m$-matrix) such that $UMW = S$, where S is the **Smith normal form** of M, that is, a diagonal matrix of the form

$$\begin{pmatrix} a_1 & 0 & 0 & & \cdots & & 0 \\ 0 & a_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & a_s & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{pmatrix}$$

where the diagonal elements, called **invariant factors**, satisfy the condition that a_i divides a_{i+1} for $1 \leq i \leq s-1$. The invariant factors are unique up to multiplication by a unit, and if M is a square matrix then $|\det(M)| = |\det(S)|$. It is also known that the product of the invariant factors is equal to the greatest common divisor of all $q \times q$ minors of M, where $q \leq s$ (see, e.g., [9]).

Lemma 2.10. $|\ker M| = |\ker S|$, *where M is any integer matrix and S is the Smith normal form of M.*

Proof. Let M be an $n \times m$ integer matrix. Then we have $UMW = S$, where U is an invertible $n \times n$ integer matrix, W is an invertible $m \times m$ integer matrix, and S is the Smith normal form of M. Then

$$Sx = 0 \iff UMWx = 0 \iff MWx = 0$$

and since W is an invertible linear map we have a bijection between the kernel of

M and the kernel of S. $\square$

A **row-induced** (**column-induced**) submatrix Z of M is a matrix formed from a selection of rows (columns) of M.

Let M be an integer matrix. We define $|M|$ be the number of columns of the matrix M and we denote $Z \subseteq M$ to mean that Z is the matrix formed from a selection of columns of M, that is, Z is a column-induced submatrix of M. We denote the transpose of M by M^T.

Theorem 2.11. *Let X be a cell complex with boundary matrix $[\partial]$. If for every column-induced submatrix Z of $[\partial]$ we have that*

$$\gcd(q \times q \textit{ minors of } Z^T) = 1,$$

where $q = \operatorname{rank}(Z)$, then the following are true:

1. *$\chi_X^*(k)$ is a polynomial whose degree is equal to the number of ridges of X,*
2. *the leading coefficient of $\chi_X^*(k)$ is equal to 1.*

Proof. Let X be a cell complex whose boundary matrix is an $n \times m$ integer matrix

$[\partial]$. Using the principle of inclusion-exclusion we have

$$\begin{aligned}
\chi_X^*(k) &= \sum_{Z\subseteq[\partial]} (-1)^{|Z|} \#\{c \in \mathbb{Z}_k^R : c[\partial] = \psi \ \& \ \psi_f = 0, \ \forall f \in F_Z\} \\
&= \sum_{Z\subseteq[\partial]} (-1)^{|Z|} \#\{c \in \mathbb{Z}_k^R : cZ = 0\} \\
&= \sum_{Z\subseteq[\partial]} (-1)^{|Z|} \#\{c \in \mathbb{Z}_k^R : Z^T c^T = 0\} \\
&= \sum_{Z\subseteq[\partial]} (-1)^{|Z|} \left|\ker Z^T\right| = \sum_{Z\subseteq[\partial]} (-1)^{|Z|} \left|\ker S_{Z^T}\right|,
\end{aligned}$$

where $F_Z \subseteq F$ are the column labels of Z, and S_{Z^T} is the Smith normal form of Z^T. Note that by assumption the invariant factors of S_{Z^T} are all 1 for every Z. Thus we have

$$|\ker S_{Z^T}| = k^{\left|[\partial]^T\right| - \operatorname{rank} Z},$$

and so $\chi_X^*(k)$ is a sum of monomial terms. □

We will call a cell complex that satisfies the assumptions of Theorem 2.11 an **IECP-cell complex**, where IECP stands for "inclusion-exclusion coloring polynomial".

Corollary 2.12. *Let X be an IECP-cell complex with an $n \times m$ boundary matrix $[\partial]$. Then*

$$\chi_X^*(k) = \sum_{Z\subseteq[\partial]} (-1)^{|Z|} k^{\left|[\partial]^T\right| - \operatorname{rank} Z}.$$

A **matroid** M is an ordered pair (E, I) consisting of a finite set E and a collection I of subsets of E satisfying the following three conditions [8]:

1. $\emptyset \in I$.

2. If $J \in I$ and $J' \subseteq J$, then $J' \in I$.

3. If J_1 and J_2 are in I and $|J_1| < |J_2|$, then there is an element $e \in J_2 - J_1$ such that $J_1 \cup \{e\} \in I$.

The **rank** of a matroid M, denoted $\operatorname{rank} M$, is the cardinality of the largest subset of E that belongs to I.

Given a finite set of vectors we can form a matroid by letting E be the set of vectors and let I be the set of all linearly independent subsets of E over some field, say $\mathbb{R}$. Thus we can view any matrix as a matriod by either taking the set of column vectors as our set E or the set of row vectors as our set E.

Given a matroid M the **rank generating polynomial** [11] $S(M; x, y)$ of M is defined by

$$S(M; x, y) = \sum_{Z \subseteq E} x^{\operatorname{rank} E - \operatorname{rank} Z} y^{|Z| - \operatorname{rank} Z}.$$

For our purposes we consider only the case where M is an integer matrix; thus $\operatorname{rank} E$ is simply the rank of the matrix M and Z is column-induced submatrix of M.

The **Tutte polynomial** $T(M;x,y)$ of a matroid M is defined to be

$$T(M;x,y) = S(M;x-1,y-1).$$

The **cardinality-corank polynomial** $S_{KC}(M;x,y)$ of a matroid M is defined to be

$$S_{KC}(M;x,y) = \sum_{Z \subseteq E} x^{|Z|} y^{\operatorname{rank} E - \operatorname{rank} Z}$$

and we have

$$S_{KC}(M;x,y) = x^{\operatorname{rank} M} T\left(M; \frac{x+y}{x}, x+1\right).$$

Corollary 2.13. *Let X be an IECP-cell complex with $n \times m$ boundary matrix $[\partial]$. Then*

$$\chi_X^*(k) = (-1)^{\operatorname{rank}[\partial]} k^{n-\operatorname{rank}[\partial]} T(X; 1-k, 0).$$

Proof. Let X be an IECP-cell complex with $n \times m$ boundary matrix $[\partial]$. Then

$$\chi_X^*(k) = \sum_{Z \subseteq [\partial]} (-1)^{|Z|} k^{\left|[\partial]^T\right| - \operatorname{rank} Z} = \sum_{Z \subseteq [\partial]} (-1)^{|Z|} k^{n-\operatorname{rank} Z}$$

and the cardinality-corank polynomial of the matroid X is

$$S_{KC}(X;x,y) = \sum_{Z \subseteq [\partial]} x^{|Z|} y^{\operatorname{rank}[\partial] - \operatorname{rank} Z}.$$

Setting $x = -1$ and $y = k$ we have

$$\chi^*_X(k) = k^{n-\text{rank}[\partial]} S_{KC}(X; -1, k) = k^{n-\text{rank}[\partial]} \sum_{Z \subseteq [\partial]} (-1)^{|Z|} k^{\text{rank}[\partial] - \text{rank}\, Z}.$$

Thus

$$\chi^*_X(k) = (-1)^{\text{rank}[\partial]} k^{n-\text{rank}[\partial]} T(X; 1-k, 0) = \sum_{Z \subseteq [\partial]} (-1)^{|Z|} k^{n-\text{rank}\, Z}. \qquad \square$$

In general the modular coloring function of a cell complex is a **quasipolynomial**, where a quasipolynomial is a function

$$q(k) = \sum_{i=0}^{n} \alpha_i k^i$$

with coefficients α_i that are periodic functions of k; so $q(k)$ is a polynomial on each residue class modulo some integer, called a **period** (see, for instance, Example 2.4).

Lemma 2.14. *Let M be any $n \times m$ integer matrix. Then*

$$|\ker M| = k^{m-\text{rank}\, M} \prod_{i=1}^{\text{rank}\, M} \gcd(k, a_i),$$

where $\ker M$ is over $\mathbb{Z}_k$ and $a_1, \ldots, a_{\text{rank}\, M}$ are the invariant factors of the Smith normal form of M.

Proof. Let M be any $n \times m$ integer matrix and let S be the Smith normal form of

M. So
$$|\ker M| = |\ker S| = k^{m-\operatorname{rank} M} \prod_{i=1}^{\operatorname{rank} M} \gcd(k, a_i),$$
where the last equality comes from the fact that for $ax = 0$ there are $\gcd(k, a)$ solution when solving for x in $\mathbb{Z}_k$, and the $k^{m-\operatorname{rank} M}$ factor comes from the fact that there are $m - \operatorname{rank} M$ zero columns in S. $\square$

Corollary 2.15. *Let X be a cell complex with $n \times m$ boundary matrix $[\partial]$. Then the modular coloring function is a quasipolynomial and*
$$\chi_X^*(k) = \sum_{Z \subseteq [\partial]} (-1)^{|Z|} k^{n-\operatorname{rank} Z} \prod_{i=1}^{\operatorname{rank} M} \gcd(k, a_i),$$
where the a_i are invariant factors of the Smith normal form of Z.

Proof. Let X be a cell complex with $n \times m$-boundary matrix $[\partial]$. Repeating the proof of Theorem 2.11 we see that at each step we have a sum of $|\ker Z|$. It follows from Lemma 2.14 that the modular coloring function is a sum of quasipolynomials and thus a quasipolynomial itself. $\square$

2.1.2 Inclusion-Exclusion Examples

Example 2.6. Let X be a cell complex with a boundary matrix $[\partial] = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$.

Then

$$\begin{aligned}
\chi_X^*(k) &= k^2 - \left|\ker\begin{pmatrix} 1 & 2 \end{pmatrix}\right| - \left|\ker\begin{pmatrix} 3 & 5 \end{pmatrix}\right| + \left|\ker\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\right| \\
&= k^2 - \left|\ker\begin{pmatrix} 1 & 0 \end{pmatrix}\right| - \left|\ker\begin{pmatrix} 1 & 0 \end{pmatrix}\right| + \left|\ker\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| \\
&= k^2 - k - k + 1.
\end{aligned}$$

Thus $\chi_X^*(k) = k^2 - 2k + 1$.

Example 2.7. Let X be a cell complex with a boundary matrix $[\partial] = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

Then

$$\begin{aligned}
\chi_X^*(k) &= k^2 - \left|\ker\begin{pmatrix} 1 & 2 \end{pmatrix}\right| - \left|\ker\begin{pmatrix} 3 & 4 \end{pmatrix}\right| + \left|\ker\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right| \\
&= k^2 - \left|\ker\begin{pmatrix} 1 & 0 \end{pmatrix}\right| - \left|\ker\begin{pmatrix} 1 & 0 \end{pmatrix}\right| + \left|\ker\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\right| \\
&= k^2 - k - k + \gcd(2, k).
\end{aligned}$$

Thus $\chi_X^*(k) = k^2 - 2k + \gcd(2,k)$.

Example 2.8. Let X be a cell complex with a boundary matrix $[\partial] = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$.

Then

$$\begin{aligned}
\chi_X^*(k) &= k^2 - \left|\ker \begin{pmatrix} 1 & 2 \end{pmatrix}\right| - \left|\ker \begin{pmatrix} 3 & 6 \end{pmatrix}\right| + \left|\ker \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\right| \\
&= k^2 - \left|\ker \begin{pmatrix} 1 & 0 \end{pmatrix}\right| - \left|\ker \begin{pmatrix} 3 & 0 \end{pmatrix}\right| + \left|\ker \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right| \\
&= k^2 - k - k\gcd(3,k) + k.
\end{aligned}$$

Thus $\chi_X^*(k) = k^2 - k\gcd(3,k)$.

2.2 The Modular Tension Function

Define $\tau_X^R(k) :=$ the number of nowhere-zero $\mathbb{Z}_k$-tensions that come from a $\mathbb{Z}_k$-linear combination of rows of the boundary matrix of a cell complex X (the R in $\tau_X^R(k)$ is to differentiate this function from $\tau_X^*(k)$ which counts all $\mathbb{Z}_k$-tensions not just the ones that come from linear combinations for row vectors).

We will refer to the set of vectors formed from $\mathbb{Z}_k$-linear combination of rows of the boundary matrix $[\partial]$ of a cell complex X as the $\mathbb{Z}_k$-**modular row space** of $[\partial]$.

Lemma 2.16. *Let X be an IECP-cell complex with $n \times m$ boundary matrix $[\partial]$*

such that all the 2×2 minors of $[\partial]$ are equal to ± 1. Then there exists a linearly independent set of row vectors of $[\partial]$ that spans that $\mathbb{Z}_k$-modular row space of $[\partial]$.

Proof. Fix a positive integer $k > 1$. The invariant factors of the Smith normal form of $[\partial]$ are all equal to 1. Then the number of invariant factors is equal to the number of linearly independent row vectors over $\mathbb{R}$. Suppose there are $t = \mathrm{rank}[\partial]$ such row vectors. Then there exists t such row vectors that span the $\mathbb{Z}_k$-modular row space. We want to show that this set of t row vectors are linearly independent over $\mathbb{Z}_k$. Let $[\partial]'$ be the $\mathrm{rank}[\partial] \times m$ matrix whose rows are our t row vectors of $[\partial]$. Consider $[x_{1j}, \ldots, x_{tj}] \in \mathbb{Z}_k^R$ as an arbitrary column of $[\partial]'$ and a $\mathbb{Z}_k$-linear combination

$$c_1 x_{1j} + \cdots + c_n x_{nj} \equiv 0 \mod k \iff c_1 x_{1j} + \cdots + c_t x_{tj} = kb$$

for some $b \in \mathbb{Z}$ and $c \in \mathbb{Z}_K^t$ such that $c \neq 0$. Thus each $c_i x_{ij}$ must be divisible by k and so all x_{ij}'s are divisble by some factor of k. Now let $[x_{1l}, \ldots, x_{nl}] \in \mathbb{Z}_k^R$ be a dfferent column of $[\partial]'$. Then by our hypothesis if $x_{ij} \neq 0, \pm 1$, then $x_{il} \neq, 0$ and $\gcd(x_{ij}, x_{il}) = \pm 1$, and if $x_{ij} = 0$, then $x_{il} = \pm 1$. Thus

$$c_1 x_{1l} + \cdots + c_t x_{tl} \neq kb'$$

for some $b' \in \mathbb{Z}$. Hence there exist t row vectors that are linearly independent and span the $\mathbb{Z}_k$-modular row space. $\square$

Lemma 2.17. *Let X be a cell complex whose $n \times n$-boundary matrix $[\partial]$ is invertible* *over $\mathbb{Z}_k$ for all k. Then the number of $\mathbb{Z}_k$-colorings is equal to the number of $\mathbb{Z}_k$-tensions. Furthermore, the number of proper colorings is equal to the number of nowhere-zero tensions, that is, $\chi^*_X(k) = \tau^R_X(k) = (k-1)^n$.*

Proof. We know from the formula $[\partial]^T c^T = \psi$ and Proposition 1.3 that for every (proper) coloring we get a (nowhere-zero) tension and vice-versa. Thus since $[\partial]$ is invertible over $\mathbb{Z}_k$ we have that $[\partial]$ is a bijection. Since $[\partial]$ is invertible over $\mathbb{Z}_k$ for all k we must have that $\det([\partial]) = \pm 1$. Hence all the invariant factors of $[\partial]$ must be ± 1. Thus $\chi^*_X(k) = \tau^R_X(k) = (k-1)^n$. □

Lemma 2.18. *Let X be an IECP-cell complex with an $n \times m$ boundary matrix $[\partial]$ such that all the 2×2 minors of $[\partial]$ are equal to ± 1. Then the number of row vectors that spans the modular row space of $[\partial]$ does not depend on k.*

Proof. Let X be an IECP-cell complex with an $n \times m$ boundary matrix $[\partial]$. Let S be the Smith normal form of $[\partial]$. Then the invariant factors are all equal to 1. Thus the number of row vectors that spans the modular row space does not depend on k. □

Theorem 2.19. *Let X be an IECP-cell complex with an $n \times m$-boundary matrix $[\partial]$ such that all the 2×2 minors of $[\partial]$ are equal to ± 1. Then*

$$\tau^R_X(k) = \chi^*_Y(k),$$

where Y is a cell complex whose boundary matrix is formed by any set of linearly *independent row vectors of* $[\partial]$ *that spans the modular row space of* $[\partial]$*.*

Proof. Let X be a cell complex with an $n \times m$-boundary matrix $[\partial]$ and let Y be a cell complex whose boundary matrix $[\partial]'$ is formed from a set of linearly independent row vectors of $[\partial]$ that spans the modular row space of $[\partial]$. Note that the number of linearly independent row vectors does not change with k.

Then $[\partial]'$ is a $\text{rank}[\partial] \times m$ integer matrix, where $\text{rank}[\partial] \leq m$. Thus $[\partial]'$ gives a unique tension for each coloring and so the number of proper colorings equals the number of nowhere-zero tensions, that is,

$$\chi_Y^*(k) = \tau_Y^R(k).$$

Then since the row vectors of $[\partial]'$ span the row space of $[\partial]$, every tension of Y is a tension of X and vice versa. $\square$

Corollary 2.20. *Let* X *be an IECP-cell complex with an* $n \times m$*-boundary matrix* $[\partial]$ *such that all the* 2×2 *minors of* $[\partial]$ *are equal to* ± 1*. Then*

$$\tau_X^R(k) = k^{\text{rank}[\partial]-n} \chi_X^*(k).$$

Proof. Let X be a cell complex with an $n \times m$-boundary matrix $[\partial]$. Then from Theorem 2.18 we know that

$$\tau_X^R(k) = \chi_Y^*(k),$$

where Y is a cell complex whose boundary matrix is formed from any set of linearly independent row vectors of $[\partial]$ that spans the modular row space of $[\partial]$. From the proof of Theorem 2.11 we have

$$\chi_Y^*(k) = \sum_{Z\subseteq[\partial]'} (-1)^{|Z|} k^{|[\partial]'^T| - \operatorname{rank} Z} = \sum_{Z\subseteq[\partial]'} (-1)^{|Z|} k^{\operatorname{rank}[\partial] - \operatorname{rank} Z}$$

and

$$\chi_X^*(k) = \sum_{Z\subseteq[\partial]} (-1)^{|Z|} k^{n - \operatorname{rank} Z}$$

Thus we see that

$$\chi_Y^*(k) = k^{\operatorname{rank}[\partial]-n} \chi_X^*(k).$$ □

Corollary 2.21. *Let X be a cell complex with $n \times m$ boundary matrix $[\partial]$. Then*

$$\tau_X^R(p) = k^{\operatorname{rank}[\partial]-n} \chi_X^*(p),$$

where p is any prime number.

Proof. Let p be a prime number. Then we can find rank$[\partial]$ number of row vectors of $[\partial]$ that are linearly independent and span the $\mathbb{Z}_p$-modular row space since $\mathbb{Z}_p$ is a field. □

Remark. We cannot in general find a linearly independent set of row vectors of $[\partial]$ that spans the $\mathbb{Z}_k$-modular row space and in some instances we cannot even find a linearly independent row vector for a given k.

2.2.1 A Modular Tension Example

Example 2.9. Let X be a cell complex with boundary matrix $[\partial] = \begin{array}{c c} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \end{array}$.

Then since the Smith normal form of $[\partial]$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we know that r_1 and r_2 are linearly independent over $\mathbb{Z}_k$ for all $k > 1$. Thus $\chi^*_X(k) = k^2 - 2k + 1 = \tau^R_X(k)$.

2.3 The Modular Flow Function

Define $\varphi^*_X(k) :=$ the number of nowhere-zero $\mathbb{Z}_k$-flows on a cell complex X.

Theorem 2.22. *Let X be a cell complex with an $n \times m$ boundary matrix $[\partial]$ such that at least one entry b_{rf} of $[\partial]$ is equal to ± 1. Then*

$$\varphi^*_X(k) = \varphi^*_{X/rf}(k) - \varphi^*_{X \setminus f}(k).$$

Proof. Let X be a cell complex with an $n \times m$ boundary matrix $[\partial]$. Without loss of generality, assume that $b_{rf} = 1$.

Suppose $\phi = (\phi_i)_{i \in F - \{f\}}$ is a nowhere-zero flow of $X \setminus f$. Then $\sum_{i \in F - \{f\}} \phi_i b_{ri} = 0$ for all $r \in R$. Thus when the column f is added back to X we have that

$\sum_{i \in F-\{f\}} \phi_i b_{ri} + b_{rf}\phi_f = 0$ for all $r \in R$. Since $b_{rf} = 1$ we have that $\phi_f = 0$. Conversely, starting with a nowhere-zero flow $\phi = (\phi_i)_{i\in F}$, except for $\phi_f = 0$, of X we see that $\phi = (\phi_i)_{i\in F-\{f\}}$ is a nowhere-zero flow of $X \setminus f$.

Suppose $\phi = (\phi_i)_{i\in F}$ is a nowhere-zero flow of X. Then since pivoting is equivalent to elementary row operations (which preserve the kernel of $[\partial]$), $\phi = (\phi_i)_{i\in F-\{f\}}$ is a nowhere-zero flow of X/rf. However, if $\phi_f = 0$, then $\phi = (\phi_i)_{i\in F-\{f\}}$ would still be a nowhere-zero flow of X/rf.

Then $\varphi^*_{X/rf}(k)$ counts the number of nowhere-zero $\mathbb{Z}_k$-flows $\phi = (\phi_i)_{i\in F}$ of X, except ϕ_f can be any element of $\mathbb{Z}_k$, and $\varphi^*_{X\setminus f}(k)$ counts the number of nowhere-zero $\mathbb{Z}_k$-flows, except $\phi_f = 0$, of X. Thus $\varphi^*_{X/rf}(k) - \varphi^*_{X\setminus f}(k)$ counts the number of nowhere-zero $\mathbb{Z}_k$-flow of X. $\square$

Corollary 2.23. *Let X be a totally unimodular complex. Then $\varphi^*_X(k)$ is a polynomial.*

Proof. Let X be a totally unimodular complex. We induct on the number of facets of X.

Base Case: X has no facets. Then $\varphi^*_X(k) = 0$.

Inductive Step: Assume that $\varphi^*_X(k)$ is a polynomial when X has less than m facets. Now suppose X has m facets. Then by deletion-contraction we have that $\varphi^*_X(k)$ for X is a polynomial. $\square$

Theorem 2.24. *Let X be a cell complex with an $n \times m$ boundary matrix $[\partial]$. If*

$$\gcd(q \times q \text{ minors of } Z) = 1,$$

where Z is a column-induced submatrix of $[\partial]$ and $q \leq \operatorname{rank} Z$, then $\varphi_X^*(k)$ is a polynomial.

Proof. Let X be a cell complex with an $n \times m$ boundary matrix $[\partial]$. We use the principle of inclusion-exclusion. Thus we have

$$\varphi_X^*(k) = \sum_{Z \subseteq [\partial]} (-1)^{m-|Z|} \#\{\phi \in \mathbb{Z}_k^F : [\partial]\phi = 0 \ \&\ \phi_f = 0 \text{ for all } f \in F - F_Z\},$$

where F_Z is the set of facets of Z. Then we have

$$\varphi_X^*(k) = \sum_{Z \subseteq [\partial]} (-1)^{m-|Z|} \#\{\phi' \in \mathbb{Z}_k^{F_Z} : Z\phi' = 0\},$$

since we will show that there exists a bijection between the two sets above. Let $\pi : \phi \to \phi'$ be defined by deleting the entires $\phi_f = 0$ from ϕ. Let $\pi^{-1} : \phi' \to \phi$ be defined by adding the entry ϕ'_f to ϕ' and setting $\phi'_f = 0$ if ϕ_f is an entry of ϕ but not an entry of ϕ', otherwise $\phi'_f = \phi_f$. Then we see

$$\pi(\pi^{-1}(\phi')) = \phi' \quad \text{and} \quad \pi^{-1}(\pi(\phi)) = \phi.$$

Thus

$$\#\{\phi' \in \mathbb{Z}_k^F : Z\phi' = 0\} = |\ker Z| = |\ker S_Z| = k^{|Z|-\operatorname{rank} Z}.$$

Hence $\varphi^*(k)$ is the sum of monomial terms and therefore a polynomial. □

We will call any cell complex that satisfies the assumptions of the Theorem 2.24 an **IEFP-cell complex**, where IEFP stands for "inclusion-exclusion flow polynomial".

Corollary 2.25. *Let X be an IEFP-cell complex with an $n \times m$ boundary matrix $[\partial]$. Then*

$$\varphi_X^*(k) = \sum_{Z \subseteq [\partial]} (-1)^{m-|Z|} k^{|Z|-\operatorname{rank} Z}.$$

Proof. Let X be an IEFP-cell complex with an $n \times m$ boundary matrix $[\partial]$. Let Z be a matrix made from a selection of columns of $[\partial]$. Then we see from the proof of the Theorem 2.24 that

$$\begin{aligned}\varphi_X^*(k) &= \sum_{Z \subseteq [\partial]} (-1)^{m-|Z|} \#\{\phi' \in \mathbb{Z}_k^{F_Z} : Z\phi' = 0\} = \sum_{Z \subseteq [\partial]} (-1)^{m-|Z|} |\ker Z| \\ &= \sum_{Z \subseteq [\partial]} (-1)^{m-|Z|} |\ker S_Z| = \sum_{Z \subseteq [\partial]} (-1)^{m-|Z|} k^{|Z|-\operatorname{rank} Z},\end{aligned}$$

where S_Z is the Smith normal form of Z. □

Corollary 2.26. *Let X be an IEFP-cell complex with an $n \times m$ boundary matrix*

$[\partial]$. *Then*

$$\varphi_X^*(k) = (-1)^{\varepsilon} S(X; -1, -k) = (-1)^{\varepsilon}\, T(X; 0, 1-k),$$

where $\operatorname{rank}[\partial] + \varepsilon = |[\partial]| = m$.

Proof. Let X be an IEFP-cell complex with an $n \times m$ boundary matrix $[\partial]$. Let M be a matroid. Then we have

$$S(M; x, y) = \sum_{Z \subseteq E} x^{\operatorname{rank} E - \operatorname{rank} Z} y^{|Z| - \operatorname{rank} Z}.$$

Then

$$S(M; x, x^{-1}y) = \sum_{Z \subseteq E} x^{\operatorname{rank} E - \operatorname{rank} Z} x^{-|Z| + \operatorname{rank} Z} y^{|Z| - \operatorname{rank} Z} = \sum_{Z \subseteq E} x^{\operatorname{rank} E - |Z|} y^{|Z| - \operatorname{rank} Z}.$$

Hence

$$\varphi_X^*(k) = \sum_{Z \subseteq [\partial]} (-1)^{|[\partial]| - |Z|} k^{|Z| - \operatorname{rank} Z} = (-1)^{\varepsilon} S\left(X; -1, -k\right). \qquad \square$$

Corollary 2.27. *Let X be a cell complex with $n \times m$ boundary matrix $[\partial]$. Then the modular flow function of X is a quasipolynomial and*

$$\varphi_X^*(k) = \sum_{Z \subseteq [\partial]} (-1)^{m - |Z|} k^{|Z| - \operatorname{rank} Z} \prod_{i=1}^{\operatorname{rank} Z} \gcd(a_i, k),$$

where the a_i are the invariant factors of the Smith normal form of Z.

Proof. Let X be a cell complex with $n \times m$ boundary matrix $[\partial]$. We start at an arbitrary step of the inclusion-exclusion used in the proof of Theorem 2.11. Let Z be a matrix formed from a selection of columns of $[\partial]$.

$$\#\{\phi' \in \mathbb{Z}_k^{F_Z} : Z\phi' = 0\} = |\ker Z| = |\ker S_Z| = k^{|Z|-\operatorname{rank} Z} \prod_{i=1}^{\operatorname{rank} Z} \gcd(a_i, k),$$

where S_Z is the Smith normal form of Z and the a_i are the invariant factors of S_Z. $\square$

Chapter 3

Integral Counting Functions and their Reciprocity

3.1 k-Colorings and Aspheric Orientations

In this section we generalize Stanley's theorem, which states that if G is a graph, then $|\chi_G(-1)|$ is equal to the number of acyclic orientations of G [10]. We will show that the above statement is true for k-coloring function of any cell complex, that is, if X is a cell complex, then $|\chi_X(-1)|$ is equal to the number of aspheric orientations of X.

Recall a coloring of a cell complex X is an element c of A^R, where A is a commutative ring with unity and a proper coloring is a coloring c such that $c \cdot [\partial]$ is nowhere-zero in A. In other words, a coloring c is proper if given any facet f, we have

$$\sum_{r \in R} c_r b_{rf} \neq 0,$$

where b_{rf} is an entry in $[\partial]$ and c_r is the color assigned to the ridge r.

Next we will consider c as a point in real affine space $\mathbb{R}^R$; c is proper if it does not lie on any of the hyperplanes

$$h_f := \sum_{r \in R} c_r b_{rf} = 0, \text{ for all } f \in F.$$

Now define the hyperplane arrangement of X to be $H = \{h_f : f \in F \subseteq X\}$. Then counting the number of proper k-colorings of X is the same as counting the lattice points in

$$k[-1,1]^R - \bigcup H,$$

a k-dilate of an **inside-out polytope** [2] that we will denote by C.

We call a polytope **rational** if its vertices lie in $k^{-1}\mathbb{Z}^n$ for some positive integer k. We define

$$E_Q(k) := \#(\mathbb{Z}^n \cap kQ),$$

where Q is a **closed, n-dimensional, rational convex polytope**. Recall that a quasipolynomial is a function

$$q(k) = \sum_{i=0}^{n} \alpha_i k^i$$

with coefficients α_i that are periodic functions of k; so $q(k)$ is a polynomial on each residue class modulo some integer, called a period. We know from Ehrhart theory (see [1] for more) that E_Q is a quasipolynomial whose degree is n, whose period divides the **denominator** of Q (the smallest k such that $k^{-1}\mathbb{Z}^n$ contains every vertex of Q), whose leading coefficient equals the volume of Q, and whose constant term $E_Q(0) = 1$.

Define

$$\chi_X(2k+1) := \text{number of proper } k\text{-colorings of } X.$$

H divides $\mathbb{R}^R$ up into **regions**. An open region is a connected component of $k[-1,1]^R - \bigcup H$ and a closed region the topological closure of an open region. We have the **closed Ehrhart quasipolynomial**

$$E_C(k) := \sum_{i=1}^{m} E_{R_i}(k),$$

where each R_i is a different closed region of H, and we have the **open Ehrhart quasipolynomial**

$$E_C^\circ(k) := \sum_{i=1}^{m} E_{R_i^\circ}(k).$$

From Ehrhart–Macdonald reciprocity [1] we have that

$$(-1)^{|R|} E_C(-k) = E_C^\circ(k).$$

Lemma 3.1. *Let X be a cell complex. The closed and open Ehrhart quasipolyno-*

mials of C satisfy

$$(-1)^{|R|}E_C(-k) = E_C^\circ(k) = \chi_X(2k-1).$$

Proof. The points that are counted by $E_C^\circ(k)$ are the lattice points of $(-k,k)^R$ that do not lie on any forbidden hyperplane, which is exactly the number of proper $(k-1)$-colorings. $\square$

Given an initial orientation o of X and a k-coloring c we get a reorientation z of X by letting

$$z_f = 1 \quad \text{whenever} \quad \sum_{r\in R} c_r b_{rf} \geq 0 \tag{3.1}$$

and

$$z_f = -1 \quad \text{whenever} \quad \sum_{r\in R} c_r b_{rf} \leq 0 \tag{3.2}$$

for all facets $f \in F$. Note that there is a choice for the entry z_f of z when

$$\sum_{r\in R} c_r b_{rf} = 0.$$

We say that c and z are **compatible** if the conditions 3.1 and 3.2 are satisfied.

Lemma 3.2. *Let c be a proper coloring. Let ψ be the tension associated with c as in Proposition 1.3. Then ψ gives a reorientation z, and z is aspheric.*

Proof. Let X be a cell complex with boundary matrix $[\partial]_o$. It's clear that any proper coloring c gives a nowhere-zero tension ψ with respect to $[\partial]_o$, namely, $c \cdot [\partial]_o = \psi$. We then get a new orientation vector z from the conditions 3.1 and 3.2 since

$$\sum_{r \in R} c_r b_{rf} = \psi_f.$$

Let $s \in \mathbb{N}^F$ be any nonzero vector such that $[\partial] \cdot s \neq 0$ iff $\psi \cdot s \neq 0$. Then we want to show that $[\partial] \cdot \mathrm{reo}_z(s) \neq 0$, where $\mathrm{reo}_z\,(s)$ is the component-wise multiplication of z and s. Thus we show $\psi \cdot \mathrm{reo}_z(s) \neq 0$. So we have

$$\psi \cdot \mathrm{reo}_z(s) = \sum_{f \in F} \psi_f \, \mathrm{reo}_z(s_f) > 0$$

since there must exist at least one pair $(\psi_f, s_f) \neq 0$ and we know whenever $\psi_f > 0$ we have $\mathrm{reo}_z(s_f) > 0$ and whenever $\psi_f < 0$ we have $\mathrm{reo}_z(s_f) < 0$. $\square$

Lemma 3.3. *Let X be a cell complex. Every aspheric reorientation z of X has a compatible k-coloring.*

Proof. Assume the aspheric reorientation z of X does not have a compatible reorientation of X. Consider $c[\partial] = \psi$, where c is a k-coloring and let $s \in \mathbb{N}^F$.

First suppose $z_f = \pm 1$ for all $f \in F$. Then

$$\psi \cdot s = \sum_{f \in F} \psi_f s_f$$

and

$$\psi \cdot \mathrm{reo}_z(s) = \sum_{f \in F} \psi_f z_f s_f = \pm \sum_{f \in F} \psi_f s_f.$$

Thus $\psi \cdot s = 0$ iff $\psi \cdot \mathrm{reo}_z(s) = 0$.

Now suppose there exists $z_h = 1$ for some $h \in F$ and $z_g = -1$ for some $g \in F$. Since there does not exist a compatible k-coloring we must have some, without loss of generality assume, $\phi_f > 0$ and $z_f = -1$. Consider s as a point in $\mathbb{R}^F$. Then we can find a $s \in \mathbb{N}^F$ such that

$$\psi \cdot \mathrm{reo}_z(s) = \sum_{i \in F} \psi_i z_i s_i = 0$$

since we can pick s_f such that $\psi_f z_f s_f < 0$. Then since $\psi \cdot s = 0$ and $\psi \cdot \mathrm{reo}_z(s) = 0$ are not the same hyperplane we can then choose a $s \in \mathbb{N}^F$ such that $[\partial]s \neq 0$ but $[\partial]\,\mathrm{reo}_z(s) = 0$.

Thus z cannot be an aspheric orientation of X unless it has at least one compatible k-coloring. $\square$

Theorem 3.4. *The number of pairs (c, z) consisting of an aspheric orientation of a cell complex X and a compatible k-coloring equals $(-1)^{|R|}\chi_X(-(2k+1))$. In particular, $(-1)^{|R|}\chi_X(-1)$ equals the number of aspheric orientations of X.*

Proof. We know that

$$E_C(k) = E_C^{\circ}(-k) = (-1)^{|R|}\chi_X(-(2k+1))$$

and we know $E_C(k)$ counts pairs (c, R), where c is a k-coloring and R is a closed region of H such that $c \in R$. Each open region R° corresponds to some set of proper k-colorings of X and thus to some strict inequality conditions associated to the hyperplanes of H. Hence all of the colorings belonging to a specific open region give nowhere-zero tensions that each give the same orientation vector z in $\{-1, 1\}^F$. We identify each pair (c, R) with the pair (c, z) where z is the orientation associated with R°. Thus, c is compatible with z if and only if $c \in R$. We know from [2] that $E_C(0) = (-1)^{|R|}\chi_X(-1)$ counts the number of regions of H and since each region corresponds to a different aspheric orientation of X we have that $(-1)^{|R|}\chi_X(-1)$ equals the number of aspheric orientations of X. $\square$

3.1.1 A k-coloring Example

Example 3.1. Let X be a cell complex with boundary matrix $[\partial] = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$. Note the X has no cycles. Recall that a k-flow of X is a element $\phi \in [-k, k]^F$ such that $[\partial]\phi = 0$. We compute

$$\begin{aligned}
\chi_X(2k+1) &= (2k+1)^3 - \#_{k\text{-flows}}\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} - \#_{k\text{-flows}}\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} + \#_{k\text{-flows}}\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&= (2k+1)^3 - (2k+1)^2 - (2k+1)p(k) + (2k+1),
\end{aligned}$$

where

$$p(k) = \begin{cases} k & \text{if } k \text{ is odd,} \\ k+1 & \text{if } k \text{ is even.} \end{cases}$$

Thus

$$\chi_X(k) = \begin{cases} 8k^3 + 6k^2 + 3k + 1 & \text{if } k \text{ is odd,} \\ 8k^3 + 6k^2 + k & \text{if } k \text{ is even.} \end{cases}$$

Hence $|\chi_X(-1)| = 4 = |\chi_X^*(-1)|$, where $\chi_X^*(k) = k^3 - 2k^2 + k\gcd(2,k)$. Since X has no cycles, all of the orientations of X are acyclic.

3.2 k-Tensions and Aspheric Orientations

Define $\tau_X(2k+1)$ to be the number of nowhere-zero k-tensions of a cell complex X. Recall that X has a **k-tension** if there exists an element ψ in $[-k,k]^F$ such that $\psi \cdot q = 0$ for any vector q in the cycle space of $[\partial]_o$.

Let X be a cell complex, $\psi \in \mathbb{R}^F$, and $(a_1, \ldots, a_m)$ be a basis for the cycle space of $[\partial]_o$. Then ψ is a tension if

$$\psi \cdot a_i = 0, \text{ for } 1 \leq i \leq m.$$

We form hyperplanes by considering ψ as a point in $\mathbb{R}^F$. Define

$$\gamma_i := \psi \cdot a_i = 0, \text{ for } 1 \leq i \leq m.$$

Define

$$\Gamma := \bigcap_{i=1}^{n} \gamma_i$$

and

$$Y := \{\text{all coordinate hyperplanes in } \mathbb{R}^F\}.$$

Then counting the number of nowhere-zero k-tensions is the same as counting the number of lattice points in

$$k[-1,1]^F \cap \Gamma - Y,$$

a k-dilate of an inside-out polytope we denote by T. Then, as before, Y splits $k[-1,1]^F \cap \Gamma$ into regions. We have the closed Ehrhart quasipolynomial

$$E_T(k) = \sum_{i=1}^{m} E_{R_i}(k)$$

and the open Ehrhart quasipolynomial

$$E_T^\circ(k) = \sum_{i=1}^{m} E_{R_i^\circ}(k),$$

where the R_i are the regions of $k[-1,1]^F \cap \Gamma - Y$. Once again we have

$$(-1)^{|F|} E_T(-k) = E_T^\circ(k)$$

by Ehrhart–Macdonald reciprocity.

Lemma 3.5. *Let X be a cell complex. The closed and open Ehrhart quasipolynomials of T satisfy*

$$(-1)^{|F|}E_T(-k) = E_T^{\circ}(k) = \tau_X(2k-1).$$

Proof. We have

$$(-1)^{|F|}E_T(-k) = E_T^{\circ}(k)$$

from Ehrhart–Macdonald reciprocity. We have

$$E_T^{\circ}(k) = \tau_X(2k-1)$$

since $E_T^{\circ}(k)$ counts only the points off the forbidden hyperplanes. Thus we are counting all nowhere-zero $(k-1)$-tensions of X. $\square$

Let X be a cell complex with some initial orientation, $\psi \in \mathbb{Z}^F$ be a tension and let $z \in \{-1,1\}^F$. Then ψ gives us the orientation z by the following rule:

$$z_f = 1 \text{ whenever } \psi_f \geq 0$$

and

$$z_f = -1 \text{ whenever } \psi_f \leq 0,$$

where z_f and ψ_f are the entries corresponding to facet f. Then we say a tension ψ and an orientation z are **compatible** if the above conditions hold.

Theorem 3.6. *The number of pairs (ψ, z) consisting of an aspheric orientation z of a cell complex X and a compatible k-tension ψ equals $(-1)^{|F|}\tau(-(2k+1))$. In particular, $(-1)^{|F|}\tau(-1)$ equals the number of aspheric orientations of X.*

Proof. We know that

$$E_T(k) = E_T^\circ(-k) = (-1)^{|F|}\tau(-(2k+1))$$

and we know $E_T(k)$ counts pairs (ψ, R), where ψ is a k-tension and R is a closed region of T such that $\psi \in R$. Each open region R° corresponds to some set of nowhere-zero k-tensions of X and thus to some strict inequality conditions on our hyperplanes which puts each region into a single orthant of $\mathbb{R}^F$. Hence all of the tensions belonging to a specific open region are nowhere-zero tensions that each give the same orientation vector $z \in \{-1, 1\}^F$. We identify each pair (ψ, R) with the pair (ψ, z) where z is the orientation associated with R°. Thus, ψ is compatible with z if and only if $\psi \in R$. We know that $E_T(0) = (-1)^{|F|}\tau_X(-1)$ counts the number of regions of T and since each region corresponds to a different aspheric orientation of X we have that $(-1)^{|F|}\tau_X(-1)$ equals the number of aspheric orientations of X. $\square$

3.3 k-Flows and Totally Spheric Orientations

Recall that a vector $\phi \in [-k, k]^F$ is a k-flow if $[\partial]_o \cdot \phi = 0$, where $[\partial]_o$ is the boundary matrix of a cell complex X with some orientation o. Define $\varphi_X(2k+1)$ to be the number of nowhere-zero k-flows of X. We define the hyperplane

$$\omega_r := \sum_{f \in F} x_f b_{rf} = 0,$$

where b_{rf} is the rf-entry in the boundary matrix $[\partial]_o$ and $x \in \mathbb{R}^F$. We define

$$\Omega := \bigcap_{r \in R} \omega_r.$$

Then counting the number of nowhere-zero k-flows is the same as counting the number of lattice points in

$$k[-1,1]^F \cap \Omega - Y,$$

a k-dilate of an inside-out polytope denoted W, where Y is the set of all coordinate hyperplanes. Then, as before, Y splits $[-k,k]^F \cap \Omega$ into regions. We have the closed Ehrhart quasipolynomial

$$E_W(k) = \sum_{i=1}^{m} E_{R_i}(k)$$

and the open Ehrhart quasipolynomial

$$E_W^{\circ}(k) = \sum_{i=1}^{m} E_{R_i^{\circ}}(k),$$

where the R_i are the regions of W. Once again we have

$$(-1)^{|F|}E_W(-k) = E_W^\circ(k)$$

by Ehrhart–Macdonald reciprocity.

Lemma 3.7. *Let X be a cell complex. The closed and open Ehrhart quasipolynomials of W satisfy*

$$(-1)^{|F|}E_W(-k) = E_W^\circ(k) = \varphi_X(2k-1).$$

Proof. We have

$$(-1)^{|F|}E_W(-k) = E_W^\circ(k)$$

from Ehrhart–Macdonald reciprocity. We have

$$E_W^\circ(k) = \varphi_X(2k-1)$$

since $E^\circ W(k)$ counts all the integer points in $(-k,k)^F$ that do not lie on the coordinate axis. Thus we are counting all nowhere-zero $(k-1)$-flows of X. $\square$

We say that a k-flow ϕ and a totally spheric orientation z of a cell complex X are **compatible** if $\phi \geq 0$ when X is given the orientation z.

Theorem 3.8. *The number of pairs (ϕ, z) consisting of a totally spheric orientation*

z of a cell complex X and a compatible k-flow ϕ equals $(-1)^{|F|}\varphi_X(-(2k+1))$. In particular, $(-1)^{|F|}\varphi_X(-1)$ equals the number of totally spheric orientations of X.

Proof. We know that

$$E_W(k) = E_W^{\circ}(-k) = (-1)^{|F|}\varphi_X(-(2k+1))$$

and we know $E_W(k)$ counts pairs (ϕ, R), where ϕ is a k-flow and R is a closed region of W such that $\phi \in R$. Each open region R° corresponds to some set of nowhere-zero k-flows of X and thus to some strict inequality conditions on the hyperplanes of W which puts each region into a single orthant of $\mathbb{R}^F$. Then there exists some orientation $z \in \{-1, 1\}^F$ such that $\phi \geq 0$ for all $\phi \in R^{\circ}$. Each open region then has a distinct orientation vector. Thus we identify the pair (ϕ, R) with (ϕ, z), where z is the orientation associated with R°. Thus, ϕ is compatible with z if and only if $\phi \in R$. We know that $E_W(0) = (-1)^{|F|}\varphi_X(-1)$ counts the number of regions of W and since each region corresponds to a different totally spheric orientation of X we have that $(-1)^{|F|}\varphi(-1)$ equals the number of totally spheric orientations of X. $\square$

Bibliography

[1] Matthias Beck and Sinai Robins, *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Undergraduate Texts in Mathematics, Springer, New York, 2007.

[2] Matthias Beck and Thomas Zaslavsky, *Inside-out polytopes*, Adv. Math. **205** (2006), no. 1, 134–162.

[3] Tamal K. Dey, Anil N. Hirani, and Bala Krishnamoorthy, *Optimal homologous cycles, total unimodularity, and linear programming*, SIAM J. Comput. **40** (2011), no. 4, 1026–1044.

[4] Reinhard Diestel, *Graph theory*, fourth ed., Graduate Texts in Mathematics, vol. 173, Springer, Heidelberg, 2010.

[5] Chris Godsil and Gordon Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001.

[6] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[7] James R. Munkres, *Topology: a first course*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[8] James Oxley, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011.

[9] Henry J. Stephen Smith, *On systems of linear indeterminate equations and congruences*, Philosophical Transactions of the Royal Society of London **151** (1861), pp. 293–326 (English).

[10] Richard P. Stanley, *Acyclic orientations of graphs*, Discrete Math. **5** (1973), 171–178.

[11] Neil White (ed.), *Matroid applications*, Encyclopedia of Mathematics and its Applications, vol. 40, Cambridge University Press, Cambridge, 1992.