MAGIC COUNTING WITH INSIDE-OUT POLYTOPES

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Contents

Chapter 1

Introduction

If someone is to ask how many 3×3 magic squares are there, one may say the answer is eight, and they are

8	1	6
3	5	7
4	9	2

6	1	8
7	5	3
2	9	4

8	3	4
1	5	9
6	7	2

4	3	8
9	5	1
2	7	6

6	7	2
1	5	9
8	3	4

2	7	6
9	5	1
4	3	8

4	9	2
3	5	7
8	1	6

2	9	4
7	5	3
6	1	8

However, this answer made a few assumptions. First, we assumed squares that are rotationally or reflectively images of each other are counted as different magic squares. Also, we assumed that we are restricted to entries $1, 2, \ldots, 9$. What if we are to use any integer from 0 to some integer t? What if we can use any positive entries, except the magic sum is not 15 but some other number? We can also open up the possibility of repeated entries.

Let a generic magic square be labeled in the style of a matrix. In this paper, we will consider the eight magic squares in the example as distinct even though they are all the same up to rotation and reflection. On the issue of repeated entries, we will specify which one of three conditions — **strong** ($a_{ij} \neq a_{i'j'}$ if $(i, j) \neq (i', j')$), **Latin** ($a_{ij} \neq a_{ij'}$ if $(i, j) \neq (i, j')$ and $a_{ij} \neq a_{i'j}$ if $(i, j) \neq (i', j)$), or **weak** (no restrictions on repeated entries) — we will be using in each chapter. As for the range of entries, only nonnegative entries are allowed, and we will provide two different counts: **cubical count** ($a_{ij} < t$ for all i and j) and **affine count** (magic sum equals s) [7]. For example,

9	2	7
4	6	8
5	10	3

is counted when we do the cubical count with $t \geq 11$, or when we do the affine count with $s = 18$.

An $n \times n$ **pandiagonal magic square** is defined as a square array where the **row sums** ($\sum_{j=1}^{n} a_{kj}$ for $k \in \{1, 2, \ldots, n\}$), the **column sums** ($\sum_{i=1}^{n} a_{ik}$ for $k \in \{1, 2, \ldots, n\}$), and the **(wrap-around) diagonal sums** ($\sum_{i+j \equiv k \pmod{n}} a_{ij}$ for $k \in \{1, 2, \ldots, n\}$ and $\sum_{i-j \equiv k \pmod{n}} a_{ij}$ for $k \in \{1, 2, \ldots, n\}$) are equal. A **magilatin rectangle** has equal row sums and column sums while using the Latin condition.

Through the use of inside-out polytopes (details in Chapter 3) and other techniques, this paper will focus on the cubical count and affine count of strong 4×4 pandiagonal magic squares (Theorems 4.8 and 4.9 below), the structure of strong 5×5 pandiagonal magic squares (Theorem 5.3), the affine count of weak $2 \times n$ magic rectangles (Theorems 6.1, Corollary 6.7, and Thorem 6.9), and the affine count of $2 \times n$ magilatin rectangles (Theorems 7.1, 7.2, 7.3, and 7.4).

Chapter 2

Background

When most people think of magic squares, they likely think of recreational mathematics. However, some people also use magic squares in art. In 2014, Macau issued nine stamps of values one to nine pataca, printed on which are famous magic squares and word squares [8]. For instance, on the four pataca stamp is the magic square on German Renaissance artist Dürer Albrecht's engraving *Melencolia I* [1].

While fun and art are good reasons to study magic squares, they have practical applications as well. In cryptography, magic squares can be used to encrypt images [16, 18]. Statisticians use stochastic matrices (matrices of nonnegative entries with row sum, column sum, or both row and column sums equal to 1), a specialized form of magic squares [10].

Throughout the centuries, many studies have been done on the topic of magic squares. In the early days of magic square investigation, people mostly focused on the construction of various magic squares. For example, De la Loubére introduced the Siamese method for generating $(2n + 1) \times (2n + 1)$ magic squares [15]. Benjamin Franklin came up with an 8×8 magic square and a 16×16 magic square with interesting properties [4]. Édouard Lucas, the mathematician famous for the Lucas sequence, devised a generalization of all 3×3 magic squares. Some other construction-related magic square problems include finding magic squares with prime number entries [2] and finding magic squares with perfect square entries [13].

Though much research focuses on construction, mathematicians also enumerated magic squares. If we use entries 1–16 (or 1–25 in the case of 5×5) and not count magic squares that are the same up to reflection or rotation, there are 880 magic squares of size 4×4 and $275,305,224$ magic squares of size 5×5 [11]. The count of bigger magic squares is not known. As for cubical count and affine count, Matthias Beck and Thomas Zaslavsky enumerated the number of magic, semimagic (diagonal sums irrelevant), and magilatin 3×3 squares [7].

Chapter 3

Methodology

We will turn the problem of counting magic squares into a problem of combinatorial geometry. As such, we need to first define some geometry-related terms. A **hyperplane** $H_{\vec{a},b}$ is defined as

$$H_{\vec{a},b} := \{\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \vec{a} \cdot \vec{x} = b\},$$

for some $\vec{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d \setminus (0, 0, \ldots, 0)$ and $b \in \mathbb{R}$. A hyperplane splits $\mathbb{R}^d$ into two **open halfspaces**

$$H^{>}_{\vec{a},b} := \{\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \vec{a} \cdot \vec{x} > b\} \text{ and}$$
$$H^{<}_{\vec{a},b} := \{\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \vec{a} \cdot \vec{x} < b\}.$$

$H_{\vec{a},b}$ is the boundary of the open halfspaces $H^{>}_{\vec{a},b}$ and $H^{<}_{\vec{a},b}$. Taking the union of open halfspaces and their boundaries, we get the **closed halfspaces**

$$H^{\geq}_{\vec{a},b} := \{\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \vec{a} \cdot \vec{x} \geq b\} \text{ and}$$
$$H^{\leq}_{\vec{a},b} := \{\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \vec{a} \cdot \vec{x} \leq b\}.$$

A **convex*** **polytope** P is the bounded intersection of finitely many closed halfspaces. In other words,

$$P = \bigcap_{i=1}^{n_1} H_{\vec{a_i},b_i} \cap \bigcap_{i=1}^{n_2} H^{\geq}_{\vec{c_i},d_i} \cap \bigcap_{i=1}^{n_3} H^{\leq}_{\vec{e_i},f_i}.$$

The **affine hull** $\mathrm{aff}(P)$ of P is $\bigcap\{H_{\vec{a},b} : P \subseteq H_{\vec{a},b}\}$. The **dimension** of a polytope is the dimension of its affine hull. The **relative interior of a polytope** $\mathrm{int}(P)$ is the interior of P relative to $\mathrm{aff}(P)$. The set $\mathrm{int}(P)$ has the representation

$$\mathrm{int}(P) = \bigcap_{i=1}^{n_1} H_{\vec{a_i},b_i} \cap \bigcap_{i=1}^{n_2} H^{>}_{\vec{c_i},d_i} \cap \bigcap_{i=1}^{n_3} H^{<}_{\vec{e_i},f_i}.$$

*All polytopes in this paper will be convex, so we will just call them polytopes

The dimension of the relative interior of a polytope is the same as the dimension of the polytope. The **relative boundary of a polytope** is

$$\partial P = P \setminus \text{int}(P).$$

We can also represent a polytope as the convex hull of a finite set of points [5], where the **convex hull** of $\overrightarrow{p_1}, \overrightarrow{p_2}, \ldots, \overrightarrow{p_n} \in \mathbb{R}^d$ is

$$\left\{k_1\overrightarrow{p_1} + k_2\overrightarrow{p_2} + \cdots + k_n\overrightarrow{p_n} : k_j \geq 0 \text{ and } k_1 + k_2 + \cdots + k_n = 1\right\}.$$

The set of **vertices** of a polytope is the smallest set of points whose convex hull is said polytope. The vertices of the relative interior of a polytope are the vertices of the polytope. For more information on polytopes, [12] is a good resource.

With these definitions in mind, we return to magic squares. 3×3 magic squares correspond to points $(a_{11}, a_{12}, \ldots, a_{33})$ in $\mathbb{R}^9$. Note that the entries of a magic square are all nonnegative integers, so the coordinates $a_{11}, a_{12}, \ldots, a_{33} \in \mathbb{Z}_{\geq 0}$. Points with integral coordinates are called **lattice points**. All we need to do now is to count the lattice points $(a_{11}, a_{12}, \ldots, a_{33})$ that satisfy the conditions of the magic square structure.

Under the cubical count, each entry is nonnegative and less than t, so one set of conditions that we have are the inequalities $0 \leq a_{ij} < t$ for all i and j. Of course, just bounding the entries does not make the square magic, so we need to equate the sums of each **line** (entries that we sum such as ones in a row). In the case of a 3×3 magic square, the equations are $a_{11} + a_{12} + a_{13} = a_{21} + a_{22} + a_{23} = \cdots = a_{13} + a_{22} + a_{31}$. Finally, we need to forbid duplicate entries. We can do so by setting nonequalities pairwise between coordinates (like $a_{11} \neq a_{12}$).

For the affine count, one set of equations is to set each line sum to s, i.e., $a_{11} + a_{12} + a_{13} = s$, $a_{21} + a_{22} + a_{23} = s$, etc. We also need the set of inequalities $a_{ij} \geq 0$ for all i and j. Lastly, just as the cubical count, we need nonequalities to take care of duplicate entries.

With the strong condition, the magic square can only have at most one 0. If we designate an $a_{i^\star j^\star}$ to be 0, then the inequalities $a_{ij} \geq 0$ become strict inequalities $a_{ij} > 0$ for all $(i, j) \neq (i^\star, j^\star)$. We will discuss the logistics of designating a 0 later in the paper.

In general, not just for 3×3 magic squares, without applying the nonequalities for duplicate entries, the equations and strict inequalities will result in the relative interior of a rational polytope (vertices have rational coordinates). Let the polytope be P_t^c for the cubical count and P_s^a for the affine count. Equations such as $a_{11} = a_{12}$ are hyperplanes. Let $\mathcal{H}$ be the set of hyperplanes that accounts for the duplicate entries. Forcing $a_{11} \neq a_{12}$ is the same as removing a hyperplane from the relative interior of a rational polytope. Therefore, we need to find $|(\text{int}(P_t^c) \setminus \bigcup \mathcal{H}) \cap \mathbb{Z}^d|$ and $|(\text{int}(P_s^a) \setminus \bigcup \mathcal{H}) \cap \mathbb{Z}^d|$. The pairs $(P_t^c, \mathcal{H})$ and $(P_s^a, \mathcal{H})$ are **inside-out polytopes** [5].

As we change the value of the maximum entry limit t, the set $\text{int}(P_t^c) \setminus \bigcup \mathcal{H}$ changes. The equations for the line sums and the noninequalities for duplicate entries do not change, but the inequalities $0 \leq a_{ij} < t$ do. All of these restrictions dilate about the origin with t. In other words,

$$\operatorname{int}(P^c_t) \setminus \bigcup \mathcal{H} = t\left(\operatorname{int}(P^c_1) \setminus \bigcup \mathcal{H}\right) = \left\{t\vec{x} \in \mathbb{R} : \vec{x} \in \operatorname{int}(P^c_1) \setminus \bigcup \mathcal{H}\right\}.$$

This is similarly true when we increase the magic sum s as we do the affine count. For simplicity, we will forgo the subscript 1, i.e., $P^c = P^c_1$ and $P^a = P^a_1$.

The notation for the number of lattice points in a set S dilated by n is $L(S, n) = |nS \cap \mathbb{Z}^d|$. The associated generating function is the **Ehrhart series**

$$\operatorname{Ehr}_S(x) = 1 + \sum_{n \geq 1} L(S, n)x^n.$$

For $\operatorname{int}(S)$, the interior of a polytope, the Ehrhart series is

$$\operatorname{Ehr}_{\operatorname{int}(S)}(x) = \sum_{n \geq 1} L(\operatorname{int}(S), n)x^n.$$

In general, the **generation function** of a sequence a_n is $\sum_{n \geq 0} a_n x^n$.

Proven by Eugéne Ehrhart (the namesake of the Ehrhart series), the number of lattice points in a dilated rational polytope nP follow a quasipolynomial

$$L(P, n) = c_{\dim P}(n)n^{\dim P} + \cdots + c_1(n)n + c_0(n),$$

where $c_0, c_1, \ldots, c_{\dim P}$ are periodic functions, the period of each c_j divides the least common denominator of the coordinates of all vertices, and $c_{\dim P}(n)$ is nonzero for some n [9]. As a quasipolynomial with n as a variable, $L(P, n)$ can be evaluated at $n < 0$. The function $L(\operatorname{int}(P), n)$ is also a quasipolynomial, and Ian G. Macdonald showed that [17]

$$L(P, -n) = (-1)^{\dim P} L(\operatorname{int}(P), n).$$

As for their Ehrhart series,

$$\operatorname{Ehr}_{\operatorname{int}(P)}(x) = (-1)^{1+\dim P}\operatorname{Ehr}_P\left(\frac{1}{x}\right). \tag{3.1}$$

Unfortunately, $\operatorname{int}(nP) \setminus \bigcup \mathcal{H}$ is not a dilated polytope. However,

$$\begin{aligned}
&\operatorname{int}(nP) \setminus \bigcup \mathcal{H} \\
&= n\left(\bigcap_{i=1}^{n_1} H_{\vec{a_i}, b_i} \cap \bigcap_{i=1}^{n_2} H^{\geq}_{\vec{c_i}, d_i} \cap \bigcap_{i=1}^{n_3} H^{\leq}_{\vec{e_i}, f_i}\right) \setminus \left(\bigcup_{i=1}^{n_4} H_{\vec{g_i}, h_i}\right) \\
&= n\left(\bigcap_{i=1}^{n_1} H_{\vec{a_i}, b_i} \cap \bigcap_{i=1}^{n_2} H^{\geq}_{\vec{c_i}, d_i} \cap \bigcap_{i=1}^{n_3} H^{\leq}_{\vec{e_i}, f_i} \cap \left(\bigcup_{i=1}^{n_4} H^{>}_{\vec{g_i}, h_i} \cup \bigcup_{i=1}^{n_4} H^{<}_{\vec{g_i}, h_i}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= n\left(\left(\bigcap_{i=1}^{n_1} H_{\vec{a_i},b_i} \cap \bigcap_{i=1}^{n_2} H^{>}_{\vec{c_i},d_i} \cap \bigcap_{i=1}^{n_3} H^{<}_{\vec{e_i},f_i} \cap \left(H^{>}_{\vec{g_1},h_1} \cup H^{>}_{\vec{g_2},h_2} \cup \cdots \cup H^{>}_{\overrightarrow{g_{n_4}},h_{n_4}}\right)\right)\right) \\
&\quad \cup n\left(\left(\bigcap_{i=1}^{n_1} H_{\vec{a_i},b_i} \cap \bigcap_{i=1}^{n_2} H^{>}_{\vec{c_i},d_i} \cap \bigcap_{i=1}^{n_3} H^{<}_{\vec{e_i},f_i} \cap \left(H^{>}_{\vec{g_1},h_1} \cup H^{>}_{\vec{g_2},h_2} \cup \cdots \cup H^{<}_{\overrightarrow{g_{n_4}},h_{n_4}}\right)\right)\right) \\
&\quad \vdots \\
&\quad \cup n\left(\left(\bigcap_{i=1}^{n_1} H_{\vec{a_i},b_i} \cap \bigcap_{i=1}^{n_2} H^{>}_{\vec{c_i},d_i} \cap \bigcap_{i=1}^{n_3} H^{<}_{\vec{e_i},f_i} \cap \left(H^{<}_{\vec{g_1},h_1} \cup H^{<}_{\vec{g_2},h_2} \cup \cdots \cup H^{<}_{\overrightarrow{g_{n_4}},h_{n_4}}\right)\right)\right)
\end{aligned}
$$

is the union of finitely many disjoint relative interiors of dilated rational polytopes. Let us call each of these disjoint relative interiors of rational polytopes a **region** (if it is nonempty). The closure of each region is some polytope R. The observation above gives us

$$
L\left(\operatorname{int}(P) \setminus \bigcup \mathcal{H}, n\right) = \sum L(\operatorname{int}(R), n),
$$

summing over all regions. As for its Ehrhart series,

$$
\begin{aligned}
\operatorname{Ehr}_{\operatorname{int}(P)\setminus\bigcup\mathcal{H}}(x) &:= \sum_{n\geq 1} L\left(\operatorname{int}(P) \setminus \bigcup \mathcal{H}, n\right) x^n \\
&= \sum_{n\geq 1}\sum_{R} L(\operatorname{int}(R), n)x^n \\
&= \sum_{R}\sum_{n\geq 1} L(\operatorname{int}(R), n)x^n \\
&= \sum_{R} \operatorname{Ehr}_{\operatorname{int}(R)}(x).
\end{aligned}
$$

Counting the lattice points in each region is challenging; the dimension of each region is $\dim(P)$. If we apply (3.1) to each region $\operatorname{int}(R)$ and plug in $\frac{1}{x}$, we obtain

$$
\sum_{R} \operatorname{Ehr}_{\operatorname{int}(R)}\left(\frac{1}{x}\right) = (-1)^{1+\dim P} \sum_{R} \operatorname{Ehr}_{R}(x).
$$

Note that all regions have the same dimension as P, so factoring out $(-1)^{1+\dim P}$ is justified. $\sum \operatorname{Ehr}_R(x)$ counts the lattice points in each region along with their boundaries. Since the regions have overlapping boundaries, some lattice points are counted more than once. For example, lattice points on a hyperplane $H_{\vec{a},b}$ in $\mathcal{H}$ in $\operatorname{int}(P)$ are counted at least twice: once for boundaries of regions in the halfspace $H^{>}_{\vec{a},b}$ and once for boundaries of regions in the halfspace $H^{<}_{\vec{a},b}$. Lattice points on the intersection of two hyperplanes in the relative interior of the polytope are counted four times (the lattice points are on the boundaries of regions in the $>$ or $<$ side of the two hyperplanes). Taking all the extra counts in consideration, we have

$$
\sum_{R} \operatorname{Ehr}_{R}(x) = \sum_{S\subseteq[n]} \lambda(S)\operatorname{Ehr}_{\bigcap_{i\in S} H_i \cap P}(x)
$$

where $\mathcal{H} = \{H_i : 1 \leq i \leq n\}$, $[n] = \{x \in \mathbb{Z} : 1 \leq x \leq n\}$, and $\lambda(S)$ is a coefficient to account for the number of times an intersection $\bigcap_{i \in S} H_i \cap P$ should be counted. All but one term of the sum count the lattice points of a polytope of dimension less than $\dim(P)$.

By the inclusion-exclusion principle, Matthias Beck and Thomas Zaslavsky demonstrated that [6]

$$\sum_{R} \mathrm{Ehr}_{\mathrm{int}(R)}(x) = \sum_{u \in \mathcal{L}} \mu(u) \mathrm{Ehr}_u(x).$$

To unpack this equation, let us start with the partially ordered set $\mathcal{L}$. As sets, $\mathcal{L} = \{\bigcap_{i \in S \subseteq [n]} H_i \cap \mathrm{int}(P)\}^{\dagger}$. The order $\prec$ is defined by reverse inclusion, i.e., for any $A, B \in \mathcal{L}$, $A \prec B$ if and only if $B \subset A$.

The function $\mu : \mathcal{L} \to \mathbb{Z}$ is the **Möbius function**

$$\mu(A) := \begin{cases} 1 & \text{if } \mathrm{int}(P) = A, \\ -\sum_{B \prec A} \mu(B) & \text{if } \mathrm{int}(P) \prec A. \end{cases}$$

Applying 3.1 to each $\mathrm{Ehr}_u(x)$ and Putting everything together, we get

$$-\mathrm{Ehr}_{\mathrm{int}(P) \setminus \bigcup \mathcal{H}} \left(\frac{1}{x} \right) = \sum_{u \in \mathcal{L}} |\mu(u)| \mathrm{Ehr}_{\bar{u}}(x),$$

where $\bar{u}$ is the topological closure of u.

†$\mathrm{int}(P) \in \mathcal{L}$ because $\mathrm{int}(P) = \bigcap_{i \in \emptyset} H_i \cap \mathrm{int}(P)$.

Chapter 4

Strong 4×4 Pandiagonal Magic Squares

4.1 Structure

If we are to directly apply the counting method outlined in Chapter 2 to compute the number of 4×4 magic squares, the most obvious difficulty stems from the number of entries that 4×4 magic squares have; we would need a 16-dimensional polytope! This is a rather daunting task. Instead, we will make use of the structure and symmetries of 4×4 magic squares to make the calculation much simpler.

Lemma 4.1. *In a 4×4 pandiagonal magic square, the* ***diagonal skip-over sums*** *satisfy $a_{11} + a_{33} = a_{13} + a_{31}$, $a_{12} + a_{34} = a_{14} + a_{32}$, $a_{21} + a_{43} = a_{23} + a_{41}$, and $a_{22} + a_{44} = a_{24} + a_{42}$.*

Proof. In a 4×4 pandiagonal magic square, $a_{11} + a_{22} + a_{33} + a_{44} = a_{13} + a_{22} + a_{31} + a_{44}$. Subtracting $a_{22} + a_{44}$ and we get $a_{11} + a_{33} = a_{13} + a_{31}$. The other equations can be proved the same way. □

Lemma 4.2. *In a 4×4 pandiagonal magic square with magic sum s, all diagonal skip-over sums equal $\frac{s}{2}$.*

Proof. Let $a_{11} + a_{33} = a_{13} + a_{31} = a$ and $a_{12} + a_{34} = a_{14} + a_{32} = b$ (valid by Lemma 4.1). Consequently,

$$\begin{aligned} 2s &= a_{11} + a_{33} + a_{13} + a_{31} + a_{12} + a_{34} + a_{14} + a_{32} \\ &= 2a + 2b, \end{aligned}$$

so $s = a + b$. Similarly, if $a_{21} + a_{43} = a_{23} + a_{41} = c$, then $s = a + c$. Therefore, $b = c$. Next, consider the diagonal on entries $a_{14}, a_{23}, a_{32}, a_{41}$.

$$\begin{aligned} s &= a_{14} + a_{23} + a_{32} + a_{41} \\ &= b + c \\ &= 2b. \end{aligned}$$

Thus, all diagonal skip-over sums are $\frac{s}{2}$. □

Lemma 4.3. *In a 4×4 pandiagonal magic square with magic sum s, all 2×2 arrays of adjacent entries also have sum s.*

Proof. Due to symmetry, it is enough to show that the center entries sum to s. The sum of the middle two columns and the middle two rows is $4s$. Taking away the wrap-around diagonals on the entries $a_{13}, a_{24}, a_{31}, a_{42}$ and $a_{12}, a_{21}, a_{34}, a_{43}$, we get twice the sum of the center entries equaling $2s$. □

Lemma 4.4. *In a 4×4 pandiagonal magic square with magic sum s, if $a_{11} = 0$, then $a_{23} + a_{32} + a_{34} + a_{43} = \frac{s}{2}$.*

Proof. Consider the sum $a_{22} + a_{23} + a_{32} + 2a_{33} + a_{34} + a_{43} + a_{44}$. On one hand, it equals $2s$ since it is comprised of two 2×2 squares. On the other hand, $a_{22} + a_{44} = a_{11} + a_{33} = a_{33} = \frac{s}{2}$. Therefore, $a_{23} + a_{32} + a_{34} + a_{43} = \frac{s}{2}$. □

Theorem 4.5. *Any 4×4 pandiagonal magic square with $a_{11} = 0$ is of the form*

0	$\alpha + \beta + \delta$	$\beta + \gamma$	$\alpha + \gamma + \delta$
$\alpha + \beta + \gamma$	$\gamma + \delta$	α	$\beta + \delta$
$\alpha + \delta$	β	$\alpha + \beta + \gamma + \delta$	γ
$\beta + \gamma + \delta$	$\alpha + \gamma$	δ	$\alpha + \beta$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_{\geq 0}$. Consequently, a_{33} is the maximum entry of the square.

Proof. If we let $a_{23} = \alpha$, $a_{32} = \beta$, $a_{34} = \gamma$, $a_{43} = \delta$, and the magic sum be s, then, by Lemmas 4.2 and 4.4,

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= \frac{s}{2} \\ &= a_{11} + a_{33} \\ &= a_{33}. \end{aligned}$$

We can then use Lemma 3 and row sums to fill out the rest of the square. □

With Theorem 4.5, we reduced the number of variables to four.

As a side note, we can use this structure to construct interesting pandiagonal magic squares.

For example, here is one for any computer scientist or robot reading this paper:

0	1101	1010	111
1011	110	1	1100
101	1000	1111	10
1110	11	100	1001

4.2 Cubical Count

Given any strong 4×4 pandiagonal magic square, we can subtract the smallest entry of the square from every entry to get a strong 4×4 pandiagonal magic square with 0 as an entry, so let us first compute the number of strong 4×4 pandiagonal magic squares with $a_{11} = 0$.

Thanks to Theorem 4.5, we know the structure of a magic square with $a_{11} = 0$. Even though α, β, γ, and δ can be in any order, due to symmetry (all entries that use α has a counterpart that use β, etc.), we can freely permute α, β, γ, and δ (and sums that involve them). Let $r_c(t)$ be the number of strong 4×4 pandiagonal magic squares with $a_{11} = 0$, $\alpha < \beta < \gamma < \delta$, and $\alpha+\beta+\gamma+\delta < t$. Let $\mathbf{r}_c(x) = \sum_{t\geq 0} r_c(t)x^t$ be its generating function. The accompanied inside-out polytope $(P, \mathcal{H})$ consists of

$$P = \{(\alpha,\beta,\gamma,\delta) : 0 \leq \alpha \leq \beta \leq \gamma \leq \delta, \alpha+\beta+\gamma+\delta \leq 1\} \text{ and}$$
$$\mathcal{H} = \left\{ \begin{array}{l} \pi_1 = \{(\alpha,\beta,\gamma,\delta) : \gamma = \alpha+\beta\}, \text{ (from } a_{34} \neq a_{44}) \\ \pi_2 = \{(\alpha,\beta,\gamma,\delta) : \delta = \alpha+\beta\}, \text{ (from } a_{43} \neq a_{44}) \\ \pi_3 = \{(\alpha,\beta,\gamma,\delta) : \delta = \alpha+\gamma\}, \text{ (from } a_{43} \neq a_{42}) \\ \pi_4 = \{(\alpha,\beta,\gamma,\delta) : \delta = \beta+\gamma\}, \text{ (from } a_{43} \neq a_{13}) \\ \pi_5 = \{(\alpha,\beta,\gamma,\delta) : \delta = \alpha+\beta+\gamma\}, \text{ (from } a_{43} \neq a_{21}) \\ \pi_6 = \{(\alpha,\beta,\gamma,\delta) : \alpha+\delta = \beta+\gamma\}, \text{ (from } a_{31} \neq a_{13}) \end{array} \right\}.$$

The vertices of P are

$$O = (0,0,0,0),\ A = (0,0,0,1),\ B = \left(0,0,\tfrac{1}{2},\tfrac{1}{2}\right),\ C = \left(0,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right),\ D = \left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}\right).$$

Now we need to find the intersections of elements of $\mathcal{H}$ in P and their Möbius function values. If we label the following points

$$\begin{array}{lll} E = (0,\frac{1}{4},\frac{1}{4},\frac{1}{2}) \in AC, & F = (\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{2}{5}) \in AD, & F' = (\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2}) \in AD, \\ G = (\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3}) \in BD & H = (\frac{1}{7},\frac{1}{7},\frac{2}{7},\frac{3}{7}) \in ABD, & H' = (\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{2}) \in ABD, \\ I = (\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{3}{8}) \in ACD, & J = (\frac{1}{10},\frac{1}{5},\frac{3}{10},\frac{2}{5}), & \end{array}$$

then the intersections are:

Zero hyperplanes:
$\varnothing$: $OABCD$

One hyperplane:
π_1: $OACG$
π_2: $OCFG$
π_3: $OBCF$
π_4: $OBEF$
π_5: $OBEF'$
π_6: $OBDE$

Two hyperplanes:
$\pi_1 \cap \pi_3$: OCH
$\pi_1 \cap \pi_4$: OEH
$\pi_1 \cap \pi_5$: OEH'
$\pi_1 \cap \pi_6$: OEG
$\pi_2 \cap \pi_6$: OGI
$\pi_3 \cap \pi_6$: OBI

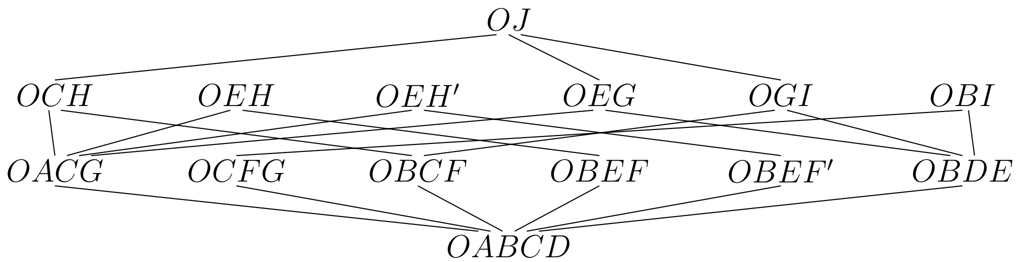
(Unlisted intersections such as $\pi_1 \cap \pi_2$: $OCG \in OBCD$ are not in $\mathrm{int}(OABCD)$, so their Möbius function value is 0.)

Three hyperplanes:
$\pi_1 \cap \pi_3 \cap \pi_6$: OJ

For four or more hyperplanes, OE is the intersection of π_1, π_4, π_5, and π_6. However,

$$\begin{aligned}
\mu(OE) &= -\mu(\pi_1 \cap \pi_4) - \mu(\pi_1 \cap \pi_5) - \mu(\pi_1 \cap \pi_6) - \mu(\pi_1) \\
&\quad - \mu(\pi_4) - \mu(\pi_5) - \mu(\pi_6) - \mu(OABCD) \\
&= \mu(\pi_1) + \mu(\pi_4) + \mu(OABCD) + \mu(\pi_1) + \mu(\pi_5) + \mu(OABCD) + \mu(\pi_1) \\
&\quad + \mu(\pi_6) + \mu(OABCD) - \mu(\pi_1) - \mu(\pi_4) - \mu(\pi_5) - \mu(\pi_6) - \mu(OABCD) \\
&= 2\mu(\pi_1) + 2\mu(OABCD) \\
&= -2\mu(OABCD) + 2\mu(OABCD) \\
&= 0.
\end{aligned}$$

For visualization and computational ease of μ, we arranage the intersections in a lattice:



Each element of the lattice has Möbius function value -1 or 1.

The last main piece of the puzzle is computing the generating functions of the quasipolynomials for the number of lattice points for each intersection. We will start with $OABCD$. To count the lattice points in $t \cdot OABCD$, considering $t \cdot ABCD$ first would make it simpler. All lattice points in $t \cdot ABCD$ can be represented as

$$(n_1, n_2, n_3, n_4) = at(0,0,0,1) + +bt\left(0,0,\tfrac{1}{2},\tfrac{1}{2}\right) + ct\left(0,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right) + dt\left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}\right)$$

where $n_1, n_2, n_3, n_4 \in \mathbb{Z}$, $a, b, c, d \geq 0$, and $a + b + c + d = 1$. From that, we can extract the equations

$$dt = 4n_1,\ ct = 3(n_2 - n_1),\ bt = 2(n_3 - n_2), \text{ and } at = n_4 - n_3. \tag{4.1}$$

Since a, b, c, d, and t are nonnegative, so are n_1, $n_2 - n_1$, $n_3 - n_2$, and $n_4 - n_3$. Adding the equations in (4.1), we have

$$t = (a + b + c + d)t = 4n_1 + 3(n_2 - n_1) + 2(n_3 - n_2) + (n_4 - n_3).$$

This suggests that all lattice points (n_1, n_2, n_3, n_4) uniquely corresponds to a representation of t as $4e + 3f + 2g + h$ for some $e, f, g, h \in \mathbb{Z}_{\geq 0}$, and vice versa. These are exactly the partitions of t using 1, 2, 3, and 4. Therefore, the number of lattice points in $t \cdot ABCD$ are the coefficients of x^t in the generating function

$$(1 + x + \cdots)(1 + x^2 + \cdots)(1 + x^3 + \cdots)(1 + x^4 + \cdots) = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

More information about generating functions of partitions can be found in [3]. The Ehrhart series of each intersection with vertex O removed is

$$\frac{1}{\prod_{\vec{v}}(1 - x^{d_{\vec{v}}})}$$

where $d_{\vec{v}}$ is the denominator of the first nonzero coordinate of each vertex $\vec{v}$. The Ehrhart series for most of the intersections can be proved the same way as $\mathrm{Ehr}_{ABCD}(x)$, since the nonorigin vertices of all intersections have unit fractions as their first nonzero coordinate (for $ABCD$, A leads with a 1, B leads with a $\frac{1}{2}$, etc.), all coordinates of individual vertices have the same denominator, and most intersections have vertices with their first nonzero coordinate at different coordinates (for $ABCD$, D's first nonzero coordinate is the first coordinate of D, C's first nonzero coordinate is the second coordinate of C, etc.).

The only two that do not fit the second criterion are CFG and GI. However, their Ehrhart series are still of the same form. All we need to do is to slightly tweak the first step. For CFG, let $((n_1, n_2, n_3, n_4)) \in t(CFG)$. We can write the equation

$$(n_1, n_2, n_3, n_4) = at\left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + bt\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}\right) + ct\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}\right)$$

for some $a, b, c \geq 0$ such that $a + b + c = 1$. This gives us $at = 3(n_2 - n_1)$, $bt = 5(n_4 - n_3)$, and $ct = 6(n_3 - n_2)$. With those three equations, we can apply the same procedure as the one for $ABCD$. This is similarly true for GI.

To go from Ehrhart series such as $\mathrm{Ehr}_{ABCD}(x)$ to $\mathrm{Ehr}_{OABCD}(x)$, we will use the following claims:

Lemma 4.6. *Suppose the rational hyperplane $H_{\vec{a},b} \in \mathbb{R}^d$ satisfies the following conditions:*

- *$b \neq 0$,*
- *$H_{\vec{a},b}$ contains a lattice point,*
- *the distance between $H_{\vec{a},b}$ and O is less than or equal to the distance between $H_{\vec{a},b}$ and any lattice point not on $H_{\vec{a},b}$.*[*]

Then $\mathbb{Z}^d \subseteq \bigcup_{k \in \mathbb{Z}} kH_{\vec{a},b}$.

Proof. Let $H_{\vec{a},b}$ be a hyperplane that satisfies the conditions in the lemma. Let $\vec{p}$ be a lattice point in $H_{\vec{a},b}$. The distance between H and $\vec{p}$ is 0, so $\vec{a} \cdot \vec{p} = b$. Let lattice point $\vec{q} \notin H_{\vec{a},b}$. The distance between $H_{\vec{a},b}$ and $\vec{q}$ is $\frac{|\vec{a} \cdot \vec{q} - b|}{||\vec{a}||} = \frac{k_1 b}{||\vec{a}||}$ for some $k_1 \in \mathbb{R}$. $\vec{q} - n\vec{p}$ is also a lattice point for any $n \in \mathbb{Z}$, and the distance between H and $\vec{q} - n\vec{p}$ is $\frac{(k_1 - n)b}{||\vec{a}||}$. Since $\frac{|b|}{||\vec{a}||}$, the distance between H and O, is less than or equal to $\frac{(k_1 - n)b}{||\vec{a}||}$ for all $n \in \mathbb{Z}$, k_1 must be an integer. This tells us $\vec{a} \cdot \vec{q} = kb$ for some $k \in \mathbb{Z}$. By definition, $\vec{q} \in kH_{\vec{a},b}$. ∎

Theorem 4.7. *For any $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in H_{\vec{a},b}$, where $H_{\vec{a},b}$ satisfies the conditions of Lemma 4.6,*

$$L(O\vec{v_1}\vec{v_2}\ldots\vec{v_n}, t) = \sum_{k=0}^{t} L(\vec{v_1}\vec{v_2}\ldots\vec{v_n}, k).$$

Proof. Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in H_{\vec{a},b}$, where $H_{\vec{a},b}$ satisfies the conditions of Lemma 4.6. Fix $t \in \mathbb{Z}_{>0}$. Any lattice point $\vec{p} \in t(O\vec{v_1}\vec{v_2}\ldots\vec{v_n})$ can be represented as $t(k_0\vec{0} + k_1\vec{v_1} + k_2\vec{v_2} + \cdots + k_n\vec{v_n})$, where $k_j \geq 0$ and $k_0 + k_1 + \cdots + k_n = 1$. This implies, if we let $s = \sum_{i=1}^{n} k_i$,

$$\vec{p} = st\left(\frac{k_1}{s}\vec{v_1} + \frac{k_2}{s}\vec{v_2} + \cdots + \frac{k_n}{s}\vec{v_n}\right) \in st(\vec{v_1}\vec{v_2}\ldots\vec{v_n}).$$

[*]The distance between the hyperplane $H_{\vec{a},b}$ and the point $\vec{p}$ is $\frac{|\vec{a} \cdot \vec{p} - b|}{||\vec{a}||}$.

From $0 \leq s \leq 1$, we can conclude $0 \leq st \leq t$. Moreover, Lemma 4.6 shows us that $st \in \mathbb{Z}$. Ergo, all lattice points in $L(O\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}, t)$ are accounted for by $\sum_{k=0}^{t} L(\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}, k)$. Furthermore, $n_1 H_{\vec{a},b}$ and $n_2 H_{\vec{a},b}$ are disjoint for $n_1 \neq n_2$, so none of $L(\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}, k)$ count the same lattice point for distinct k. □

The hyperplane $H_{(1,1,1,1),1}$ contains $(1,0,0,0)$. For any lattice point $\vec{p}$, the distance between $H_{(1,1,1,1),1}$ and $\vec{p}$ is $\frac{k}{2}$ for some $k \in \mathbb{Z}_{>0}$. This distance is greater than or equal to $\frac{1}{2}$, the distance between $H_{(1,1,\ldots,1),1}$ and O. Since all intersections, discounting the vertex O, have vertices in $H_{(1,1,1,1),1}$, we can apply Theorem 4.7 to get the Ehrhart series

$$\begin{aligned}
\mathrm{Ehr}_{O\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}}(x) &= 1 + \sum_{t\geq 1} L(O\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}, t)x^t \\
&= 1 + \sum_{t\geq 1}\sum_{k=0}^{t} L(\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}, k)x^t \\
&= \sum_{i\geq 0} x^i + L(\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}, 1)x\sum_{i\geq 0} x^i + \cdots \\
&= \sum_{i\geq 0} x^i \left(1 + \sum_{t\geq 1} L(\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}, t)x^t\right) \\
&= \frac{1}{1-x}\mathrm{Ehr}_{\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}}(x) \\
&= \frac{1}{(1-x)\prod_{\vec{v}}(1-x^{d_{\vec{v}}})}
\end{aligned}$$

for each intersection where $d_{\vec{v}}$ is the denominator of the first nonzero coordinate of each non-origin vertex $\vec{v}$.[†]. We now have the ingredients for $\mathbf{r}_c(x)$.

$$\begin{aligned}
-\mathbf{r}_c\left(\frac{1}{x}\right) = {} & \mathrm{Ehr}_{OABCD}(x) + \mathrm{Ehr}_{OACG}(x) + \mathrm{Ehr}_{OCFG}(x) + \mathrm{Ehr}_{OBCF}(x) + \mathrm{Ehr}_{OBEF}(x) \\
& + \mathrm{Ehr}_{OBEF'}(x) + \mathrm{Ehr}_{OBDE}(x) + \mathrm{Ehr}_{OCH}(x) + \mathrm{Ehr}_{OEH}(x) + \mathrm{Ehr}_{OEH'}(x) \\
& + \mathrm{Ehr}_{OEG}(x) + \mathrm{Ehr}_{OGI}(x) + \mathrm{Ehr}_{OBI}(x) + \mathrm{Ehr}_{OJ}(x)
\end{aligned}$$

[†]From the first line to the second line, $1 = L(\overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}, 0)$ since O is the one and only point in $0 \cdot \overrightarrow{v_1}\overrightarrow{v_2}\ldots\overrightarrow{v_n}$.

$$
\begin{aligned}
&= \frac{1}{(1-x)^2(1-x^2)(1-x^3)(1-x^4)} + \frac{1}{(1-x)^2(1-x^3)(1-x^6)} \\
&\quad + \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^6)} + \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^5)} \\
&\quad + \frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)} + \frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^6)} \\
&\quad + \frac{1}{(1-x)(1-x^2)(1-x^4)^2} + \frac{1}{(1-x)(1-x^3)(1-x^7)} \\
&\quad + \frac{1}{(1-x)(1-x^4)(1-x^7)} + \frac{1}{(1-x)(1-x^4)(1-x^8)} \\
&\quad + \frac{1}{(1-x)(1-x^4)(1-x^6)} + \frac{1}{(1-x)(1-x^6)(1-x^8)} \\
&\quad + \frac{1}{(1-x)(1-x^2)(1-x^8)} + \frac{1}{(1-x)(1-x^{10})},
\end{aligned}
$$

so

$$
\begin{aligned}
\mathbf{r}_c(x) &= \frac{\begin{matrix} x^{16}+x^{18}-2x^{19}+x^{20}-3x^{21}+x^{22}-3x^{23} \\ -x^{24}+2x^{25}-3x^{26}+4x^{27}+5x^{28}+5x^{29}-3x^{30} \\ +2x^{31}-7x^{32}+7x^{33}+3x^{34}-5x^{35}-9x^{36}+2x^{37} \\ -9x^{38}+9x^{40}-13x^{41}+11x^{42}+x^{43}+3x^{44}+8x^{45} \\ +11x^{46}-11x^{47}-2x^{48}-8x^{49}-12x^{50}+14x^{51} \end{matrix}}{(1-x)^2(1-x^2)(1-x^3)(1-x^4)^2(1-x^5)(1-x^6)(1-x^7)(1-x^8)(1-x^{10})} \\
&= \frac{x^{16}(1-x^2)\begin{pmatrix} 1+2x+6x^2+8x^3+17x^4+20x^5+36x^6 \\ +38x^7+58x^8+57x^9+76x^{10}+68x^{11}+84x^{12}+70x^{13} \\ +81x^{14}+57x^{15}+59x^{16}+34x^{17}+38x^{18}+16x^{19}+14x^{20} \end{pmatrix}}{(1-x^3)(1-x^4)(1-x^6)(1-x^7)(1-x^8)(1-x^{10})}.
\end{aligned}
$$

Recall that $\mathbf{r}_c(x)$ enumerates strong 4×4 pandiagonal magic squares with $\alpha<\beta<\gamma<\delta$, $a_{11}=0$, and $a_{33}<t$. For a strong 4×4 pandiagonal magic square with nonnegative entries, there are three things to tweak:

(1): the smallest entry does not have to be 0,
(2): the smallest entry could have been any one of the sixteen a_{ij}, and
(3): α, β, γ, and δ could be ordered in one of 24 permutations.

Fixes (2) and (3) are easy to make; we only need to multiply the count by 384. Let us reason through fix (1). To make the smallest entry nonzero, we can add one to every a_{ij}. We could also have added two to every a_{ij}. Any integer added to all a_{ij} would result in a new magic square. Adding one pushes the maximum entry by one, so we multiply the generating function by x; Adding two pushes the maximum entry by two, so we multiply the generating function by x^2; so on and so forth. This means we need to multiply $\mathbf{r}_c(x)$ by $1+x+x^2+\cdots=\frac{1}{1-x}$. This gives us the final count.

Theorem 4.8. *The generating function for the cubical count of strong* 4×4 *pandiagonal magic squares is*

$$\frac{384x^{16}(1+x)\begin{pmatrix} 1+2x+6x^2+8x^3+17x^4+20x^5+36x^6 \\ +38x^7+58x^8+57x^9+76x^{10}+68x^{11}+84x^{12}+70x^{13} \\ +81x^{14}+57x^{15}+59x^{16}+34x^{17}+38x^{18}+16x^{19}+14x^{20} \end{pmatrix}}{(1-x^3)(1-x^4)(1-x^6)(1-x^7)(1-x^8)(1-x^{10})}$$

Using the generating function gives us the numbers of strong 4×4 pandiagonal magic squares with nonnegative entries with strict upper bound t, as shown in Table 4.1.

t	16	17	18	19	20	21	22	23	24	25
$\frac{\text{magic count}}{384}$	1	3	8	15	29	48	80	121	182	260

Table 4.1: Number of strong 4×4 pandiagonal magic squares with strict upper bound t.

4.3 Affine Count

We will again make use of Theorem 4.5 and first consider only strong 4×4 pandiagonal magic squares with $a_{11} = 0$. Let $r_a(s)$ be the number of strong 4×4 pandiagonal magic squares with, $\alpha < \beta < \gamma < \delta$, $a_{11} = 0$, and magic sum s. Let $\mathbf{r}_a(x)$ be its generating function. According to Theorem 4.5, strong 4×4 pandiagonal magic squares with $a_{11} = 0$ and magic sum s have maximum entry $a_{33} = \frac{s}{2}$. Since there is a link between the maximum entry and the magic sum in the case that one of the entries is 0, we can piggyback on the previous section.

Note that the t of $r_c(t)$ from the previous section was not the maximum entry, but rather a strict upper bound for every entry. Fortunately, making that adjustment is easy; $r_c(t+1) - r_c(t)$ gives the number of strong 4×4 pandiagonal magic squares with $\alpha < \beta < \gamma < \delta$, $a_{11} = 0$, and maximum entry t; hence its generating function is

$$\sum_{t\geq 0}(r_c(t+1) - r_c(t))x^t = \sum_{t\geq 0} r_c(t+1)x^t - \sum_{t\geq 0} r_c(t)x^t = \frac{1-x}{x}\mathbf{r}_c(x).$$

The magic sum is $s = 2a_{33} = 2t$, so

$$\begin{aligned} \mathbf{r}_a(x) &= \frac{1-x^2}{x^2}\mathbf{r}_c(x^2) \\ &= \frac{x^{30}(1-x^2)(1-x^4)\begin{pmatrix} 1+2x^2+6x^4+8x^6+17x^8+20x^{10} \\ +36x^{12}+38x^{14}+58x^{16}+57x^{18}+76x^{20} \\ +68x^{22}+84x^{24}+70x^{26}+81x^{28}+57x^{30} \\ +59x^{32}+34x^{34}+38x^{36}+16x^{38}+14x^{40} \end{pmatrix}}{(1-x^6)(1-x^8)(1-x^{12})(1-x^{14})(1-x^{16})(1-x^{20})}. \end{aligned}$$

To turn this into the generating function for the number of strong 4×4 pandiagonal magic squares with nonnegative entries and magic sum s, we can apply the three fixes in the cubical count section. The only difference is that adding one to all entry results in adding four to the magic sum, so the multiplier is $1 + x^4 + x^8 + \cdots = \frac{1}{1-x^4}$ instead of $\frac{1}{1-x}$. Thus, the magic count is

Theorem 4.9. *The generating function for the affine count of strong* 4×4 *pandiagonal magic squares is*

$$\frac{384x^{30}(1-x^2)\begin{pmatrix} 1+2x^2+6x^4+8x^6+17x^8+20x^{10}+36x^{12} \\ +38x^{14}+58x^{16}+57x^{18}+76x^{20}+68x^{22}+84x^{24}+70x^{26} \\ +81x^{28}+57x^{30}+59x^{32}+34x^{34}+38x^{36}+16x^{38}+14x^{40} \end{pmatrix}}{(1-x^6)(1-x^8)(1-x^{12})(1-x^{14})(1-x^{16})(1-x^{20})}.$$

Table 4.2 shows the number of strong 4×4 pandiagonal magic squares with nonnegative entries and magic sum s.

s	30	32	34	36	38	40	42	44	46	48
$\frac{\text{magic count}}{384}$	1	1	4	3	11	8	24	17	44	34

Table 4.2: Number of strong 4×4 pandiagonal magic squares with magic sum s.

Chapter 5

Strong 5×5 Pandiagonal Magic Squares

5.1 Structure

Like the 4×4 pandiagonal magic squares, 5×5 pandiagonal magic squares have a nice structure.

Lemma 5.1. *In a 5×5 pandiagonal magic square with magic sum s, "+" patterns have sum s. In other words, for any a_{ij}, $a_{i-1,j} + a_{i,j-1} + a_{ij} + a_{i,j+1} + a_{i+1,j} = s$.*

Proof. Due to symmetry, it is enough to show $a_{23} + a_{32} + a_{33} + a_{34} + a_{43} = s$.

$$\begin{aligned} 10s =& (a_{11} + a_{25} + a_{34} + a_{43} + a_{52}) + (a_{15} + a_{21} + a_{32} + a_{43} + a_{54}) \\ &+ (a_{12} + a_{23} + a_{34} + a_{45} + a_{51}) + (a_{14} + a_{23} + a_{32} + a_{41} + a_{55}) \\ &+ (a_{21} + a_{22} + a_{23} + a_{24} + a_{25}) + 3(a_{31} + a_{32} + a_{33} + a_{34} + a_{35}) \\ &+ (a_{41} + a_{42} + a_{43} + a_{44} + a_{45}) + (a_{12} + a_{22} + a_{32} + a_{42} + a_{52}) \\ &+ 3(a_{13} + a_{23} + a_{33} + a_{43} + a_{53}) + (a_{14} + a_{24} + a_{34} + a_{44} + a_{54}) \\ &- (a_{13} + a_{24} + a_{35} + a_{41} + a_{52}) - (a_{13} + a_{22} + a_{31} + a_{45} + a_{54}) \\ &- (a_{12} + a_{21} + a_{35} + a_{42} + a_{53}) - (a_{14} + a_{25} + a_{31} + a_{42} + a_{53}). \end{aligned}$$

The entire magic square consists of five rows, so the sum of all entries is $5s$. Subtracting all entries and dividing both sides by 5, we get $s = a_{23} + a_{32} + a_{33} + a_{34} + a_{43}$. □

Lemma 5.2. *In a 5×5 pandiagonal magic square with magic sum s, if $a_{11} = 0$, then $a_{23} + a_{24} + a_{32} + a_{35} + a_{42} + a_{45} + a_{53} + a_{54} = s$.*

Proof. The sum of all entries is $5s$. Subtract the first row, the first column, the diagonal $a_{11} + a_{22} + a_{33} + a_{44} + a_{55}$, and the diagonal $a_{11} + a_{25} + a_{34} + a_{43} + a_{52}$, we get $s = a_{23} + a_{24} + a_{32} + a_{35} + a_{42} + a_{45} + a_{53} + a_{54}$. □

Theorem 5.3. *Any 5×5 pandiagonal magic square with $a_{11} = 0$ is of the form*

0	$\alpha_3+\beta_1$	$\alpha_4+\beta_4$	$\alpha_2+\beta_2$	$\alpha_1+\beta_3$
$\alpha_4+\beta_2$	$\alpha_2+\beta_3$	α_1	β_1	$\alpha_3+\beta_4$
$\alpha_1+\beta_1$	β_4	$\alpha_3+\beta_2$	$\alpha_4+\beta_3$	α_2
$\alpha_3+\beta_3$	α_4	$\alpha_2+\beta_1$	$\alpha_1+\beta_4$	β_2
$\alpha_2+\beta_4$	$\alpha_1+\beta_2$	β_3	α_3	$\alpha_4+\beta_1$

for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z}_{>0}$.

Proof. Let $a_{23} = \alpha_1$, $a_{24} = \beta_1$, $a_{35} = \alpha_2$, $a_{45} = \beta_2$, $a_{54} = \alpha_3$, $a_{53} = \beta_3$, $a_{42} = \alpha_4$, and $a_{32} = \beta_4$. The row sum $a_{11}+a_{12}+a_{13}+a_{14}+a_{15}$ equals the "+" pattern sum $a_{54}+a_{13}+a_{14}+a_{15}+a_{24}$, which implies $a_{12} = \alpha_3 + \beta_1$. Equating row/column sums with "+" pattern sums as such gives us the entries of the first row and the first column in terms of α_j's and β_j's. Lemma 5.2 tells us magic sum $s = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_3 + \beta_4$, so the rest of the entries can be filled out using the "+" patterns. □

5.2 Outline for the Cubical and Affine Count

As with strong 4×4 pandiagonal magic squares, we first consider strong 5×5 pandiagonal magic squares with $a_{11} = 0$.

Theorem 5.3 tells us that strong 5×5 pandiagonal magic squares with $a_{11} = 0$ have entries of the form 0, α_j, β_j, or $\alpha_j + \beta_k$. Notice that the square stays pandiagonally magic after interchanging α_1, α_2, α_3 and α_4 for all the entries each α_j appears. The same is true for interchanging β_1, β_2, β_3, and β_4. Because of this symmetry, we will impose the order $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and $\beta_1 < \beta_2 < \beta_3 < \beta_4$. We can also swap all α and β to maintain values, so we will suppose $\alpha_4 < \beta_4$. This supposition also lets us assert that $\alpha_4 + \beta_4$ is the maximum entry.

Let $r_c(t)$ be the number of strong 5×5 pandiagonal magic squares with $a_{11} = 0$, $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \beta_4$, $\beta_1 < \beta_2 < \beta_3 < \beta_4$, and $\alpha_4 + \beta_4 < t^*$. Let $\mathbf{r}_c(x) = \sum_{t \geq 0} r_c(t) x^t$ be its generating function. The accompanied inside-out polytope $(P, \mathcal{H})$ consists of

*For the affine count, the inequality $\alpha_4+\beta_4 < t$ would be replaced with $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\beta_1+\beta_2+\beta_3+\beta_4 = s$.

$$P = \left\{ (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) : \begin{array}{c} 0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \beta_4, \\ 0 \leq \beta_1 \leq \beta_2 \leq \beta_3 \leq \beta_4, \alpha_4 + \beta_4 \leq 1 \end{array} \right\} \text{ and}$$

$$\mathcal{H} = \left\{ \begin{array}{l} \pi_1 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) : \alpha_1 = \beta_1\}, \text{ (from } a_{23} \neq a_{24}) \\ \vdots \\ \pi_{12} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) : \alpha_4 = \beta_3\}, (from a_{42} \neq a_{53}) \\ \pi_{13} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) : \alpha_2 = \alpha_1 + \beta_1\}, (from a_{35} \neq a_{31}) \\ \vdots \\ \pi_{30} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) : \alpha_4 = \alpha_3 + \beta_3\}, (from a_{42} \neq a_{41}) \\ \pi_{31} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) : \beta_2 = \alpha_1 + \beta_1\}, (from a_{45} \neq a_{31}) \\ \vdots \\ \pi_{54} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) : \beta_4 = \alpha_4 + \beta_3\}, (from a_{42} \neq a_{34}) \\ \pi_{55} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) : \alpha_1 + \beta_2 = \alpha_2 + \beta_1\}, (from a_{52} \neq a_{43}) \\ \vdots \\ \pi_{90} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) : \alpha_3 + \beta_4 = \alpha_4 + \beta_3\}, (from a_{25} \neq a_{34}) \end{array} \right\}.$$

The trouble here is with the number of hyperplanes. While 90 rational expressions are manageable, we have to also find the intersections of the hyperplanes and their generating functions, as well as computing their Möbius function values. As such, we only outlined a method for calculating $r_c(t)$.

Chapter 6

Weak $2 \times n$ Magic Rectangles

Shifting our attention from pandiagonal magic squares to weak $2 \times n$ magic rectangles, we need to address a few matters. First, row sums cannot equal to column sums for $n \neq 2$, since if they are both s, then the sum of all entries would be both $2s$ and ns. Therefore, instead of having a unified magic sum, a rectangular array qualifies as magic if all row sums are equal and all column sums are equal. Second, we will focus on the affine count, which will be based on the column sum.

6.1 Weak $2 \times 2n$ Magic Rectangles

Given a column sum s, the first row of a $2 \times 2n$ magic rectangle has sum ns and uniquely determines the entries of the second row. Since each a_{ij} is nonnegative and must sum with the entry above or below, each a_{ij} is bounded by s. As such, we need to count the number of ways for the sum $a_{11} + \cdots + a_{1,2n}$ to equal ns where a_{ij} are nonnegative integers less than or equal to s. For simplicity of notation, we will relabel a_{1j} as α_j.

Consider the series $\sum_{\alpha_j \geq 0}^{s} x^{\alpha_1 + \cdots + \alpha_{2n}}$. The coefficient of x^{ns} is the number of ways the sum $\sum_{j=1}^{2n} \alpha_j$ is ns. Let $[x^k]f(x)$ denote the coefficient of x^k of the series $f(x)$. This definition allows us to phrase the following theorem.

Theorem 6.1. *For a given column sum s, there are*

$$[x^{ns}] \left(\frac{1 - x^{s+1}}{1 - x} \right)^{2n}$$

weak $2 \times 2n$ magic rectangles with $a_{ij} \geq 0$.

Proof. Let $m_{2n}(s)$ be the number of weak $2 \times 2n$ magic rectangles with $a_{ij} \geq 0$ with column sum s.

$$\begin{aligned} m_{2n}(s) &= [x^{ns}] \sum_{a_{1j} \geq 0}^{s} x^{\alpha_1 + \cdots + \alpha_{2n}} \\ &= [x^{ns}] \sum_{\alpha_1 \geq 0}^{s} x^{\alpha_1} \cdots \sum_{\alpha_{2n} \geq 0}^{s} x^{\alpha_{2n}} \\ &= [x^{ns}] \left(\frac{1 - x^{s+1}}{1 - x} \right)^{2n} \end{aligned} \qquad \square$$

Using the expression with $n = 1$ gives us Table 6.1.

s	0	1	2	3	4	5	6	7	8	9
magic count	1	2	3	4	5	6	7	8	9	10

Table 6.1: Number of weak 2×2 magic rectangles with column sum s.

This straightforward pattern of $m_2(s) = s + 1$ is not surprising because for any a_{11} between 0 and s inclusive, there is one unique weak 2×2 magic rectangle (square) with column sum s. Table 6.2 and 6.3 show the magic count of weak 2×4 and 2×6 magic rectangles respectively.

s	0	1	2	3	4	5	6	7	8	9
magic count	1	6	19	44	85	146	231	344	489	670

Table 6.2: Number of weak 2×4 magic rectangles with column sum s.

s	0	1	2	3	4	5	6	7	8	9
magic count	1	20	141	580	1751	4332	9331	18152	32661	55252

Table 6.3: Number of weak 2×6 magic rectangles with column sum s.

Looking at the enumeration through a polytopal lens, the conditions $0 \leq a_{1j} \leq 1$ and $\sum_{j=1}^{2n} \alpha_j = n$ make the polytope associated with $m_{2n}(s)$. Let P be this polytope. A polytope of the form

$$\{(x_1, \ldots, x_d) : 0 \leq x_j \leq 1 \text{ and } x_1 + \cdots + x_d = k\}$$

where k is a positive integer is called a (d, k)-**hypersimplex** [5]. With this definition, P is a $(2n, n)$-hypersimplex.

Recall from Chapter 3 that the Ehrhart quasipolynomial has degree equal to the dimension of P, which is $2n - 1$ due to the sum equation $\sum_{j=1}^{2n} \alpha_j = n$.

The period of $m_{2n}(s)$ divides the least common denominator of the coordinates of all vertices of P, so we need to know what the vertices of P are.

Theorem 6.2. *Let $P \in \mathbb{R}^{2n}$ be the polytope defined by $0 \leq \alpha_j \leq 1$ and $\sum_{j=1}^{2n} \alpha_j = n$, and let V be the set of all $2n$-dimensional points where n of the coordinates are ones and the other n of the coordinates are zeroes. Then P is the convex hull of the points in V.*

This is a known theorem, but we will provide an alternative proof because we can apply the same technique in the $2n+1$ case. Also, we will use Lemma 6.6 in Chapter 7.

Before we begin the proof, we need the four following lemmas.

Lemma 6.3. *If $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$ have equal coordinates except for the jth and kth coordinate, and $a_j + a_k = b_j + b_k$, then the convex hull of $\vec{a}$ and $\vec{b}$ are points whose non$j$th and non$k$th coordinates match with $\vec{a}$, the jth coordinate is between a_j and b_j, the kth coordinate is between a_k and b_k, and the sum of its jth and kth coordinate is $a_j + a_k$.*

Proof. For any linear combination $k_1\vec{a} + k_2\vec{b}$ where $k_1 + k_2 = 1$, if $a_i = b_i$, then $k_1 a_i + k_2 b_i = k_1 a_i + k_2 a_i = a_i$.

Suppose $a_j > b_j$. Using again $k_1 + k_2 = 1$, we get

$$b_j = k_1 b_j + k_2 b_j \leq k_1 a_j + k_2 b_j \leq k_1 a_j + k_2 a_j = a_j.$$

If $a_j + a_k = b_j + b_k$, then $a_k < b_k$ and $b_k \leq k_1 a_k + k_2 b_k \leq a_k$. The sum of the two coordinates of the linear combination is

$$\begin{aligned} k_1 a_j + k_2 b_j + k_1 a_k + k_2 b_k = k_1 &= k_1(a_j + a_k) + k_2(b_j + b_k) \\ &= k_1(a_j + a_k) + k_2(a_j + a_k) \\ &= (k_1 + k_2)(a_j + a_k) \\ &= a_j + a_k. \end{aligned}$$

□

As an example, $\left(0, \frac{2}{5}, 1, 1, 0, \frac{3}{5}\right)$ is in the convex hull of $(0, 1, 1, 1, 0, 0)$ and $(0, 0, 1, 1, 0, 1)$.

Lemma 6.4. *If a point in the convex hull of the points in V (as defined in Theorem 6.2), then the point is still in the convex hull after its coordinates are permuted. In other words, for any permutation s of $\{1, \ldots, 2n\}$, if $(a_1, \ldots, a_{2n})$ is in the convex hull of the points in V, then so is $(a_{s(1)}, \ldots, a_{s(2n)})$.*

Proof. Let s be a permutation of $\{1, \ldots, 2n\}$. Since V is the set of all $2n$-dimensional points where n of the coordinates are ones and the other n of the coordinates are zeroes, for any $\vec{v} = (v_1, \ldots, v_{2n})$ in V, the point $\vec{v}_s := (v_{s(1)}, \ldots, v_{s(2n)})$ is also in V. Therefore, if $(a_1, \ldots, a_{2n}) = \sum_{\vec{v} \in V} k_{\vec{v}} \vec{v}$ is in the convex hull, then so is $(a_{s(1)}, \ldots, a_{s(2n)}) = \sum_{\vec{v} \in V} k_{\vec{v}} \vec{v}_s$. □

As an example, if $\left(0, \frac{2}{5}, 1, 1, 0, \frac{3}{5}\right)$ is in the convex hull of the points in V, then so is $\left(0, \frac{3}{5}, 1, \frac{2}{5}, 1, 0\right)$.

Lemma 6.5. *All points in the convex hull of the points in V have coordinate sum n.*

Proof. Let $\vec{v}_j = (v_{j,1}, \ldots, v_{j,2n})$ for $j = 1, \ldots, \binom{2n}{n}$ be the points in V. All points in the convex hull can be represented as

$$\sum_{j=1}^{\binom{2n}{n}} k_{v_j} \vec{v}_j.$$

The coordinate sum is, therefore,

$$\sum_{i=1}^{2n} \sum_{j=1}^{\binom{2n}{n}} k_{v_j} v_{j,i} = \sum_{j=1}^{\binom{2n}{n}} k_{v_j} \sum_{i=1}^{2n} v_{j,i} = \sum_{j=1}^{\binom{2n}{n}} k_{v_j} n = n. \qquad \square$$

Lemma 6.6. *If $\vec{d}$ is in the convex hull of $\vec{a}$ and $\vec{b}$, and $\vec{e}$ is in the convex hull of $\vec{c}$ and $\vec{d}$, then $\vec{e}$ is in the convex hull of $\vec{a}$, $\vec{b}$, and $\vec{c}$.*

Proof. By the definition of convex hull,

$$\vec{d} = k_1\vec{a} + k_2\vec{b} \qquad \text{and} \qquad \vec{e} = k_3\vec{c} + k_4\vec{d}$$

for some k_1, k_2, k_3, k_4 such that $0 \leq k_j \leq 1$ and $k_1 + k_2 = k_3 + k_4 = 1$. Therefore, $\vec{e} = k_1k_4\vec{a} + k_2k_4\vec{b} + k_3\vec{c}$. The coefficients have sum 1 and are bounded by 0 and 1. $\square$

With these four lemmas, we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. Lemma 6.5 ensures that the convex hull of the points in V is a subset of P. To prove the theorem, we need to show that P is a subset of the convex hull of the points in V.

With Lemma 6.3 and 6.4, we have another way to think about the convex hull of the points in V. Imagine a row of $2n$ switches representing the coordinates of a point, up being 1 and down being 0 (see Figure 6.1).

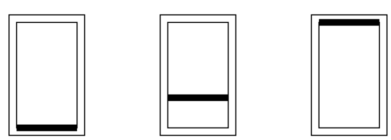


Figure 6.1: The first setting represents 0, the second represents $\frac{2}{7}$, and the last represents 1.

The points in V have settings where n of the switches are up and n of the switches are down. Lemma 6.3 and 6.4 allow us to select any two switches and move the higher one down and the lower one up by the same amount without passing each other. The resulting point is still in the convex hull. To justify this, suppose in a point, coordinate $\alpha_j < \alpha_k$. By Lemma 6.4, if we swap α_j and α_k, the new point is still in the convex hull. We can then apply Lemma 6.3. As it is, we can only perform this move once, but Lemma 6.6 allows us to perform this move repeatedly.

With this move, we can show that any point $(\alpha_1, \ldots, \alpha_{2n})$ in P is in the convex hull of the points in V. First, we identify the order of a_{1j} and set the n higher coordinates to up and the n lower coordinates to down. This represents a point in V, which is in the convex hull. Without loss of generality, suppose $\alpha_1 \leq \cdots \leq \alpha_{2n}$. This implies we start with the setting in Figure 6.2.

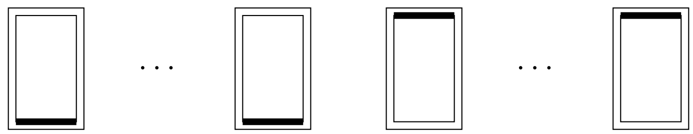


Figure 6.2: Starting position.

From this setting, we move the first and last switch until at least one of them hits α_1 or α_{2n}. When one of them hits α_1 or α_{2n}, we pair the next switch with the one that did not hit their target. For instance, if the $2n$th switch hits α_{2n} first, we will move the first and $(2n-1)$st switch next. When both of them hit their target values, we move on to the next pair. This process takes at most $2n-1$ moves to terminate. Terminating means either no switches are undervalued, no switches are overvalued, or both. By Lemma 6.5, the switches cannot be purely overvalued or purely undervalued. Thus, $(\alpha_1, \ldots, \alpha_{2n})$ is attained and is in the convex hull. □

Corollary 6.7. *The number of weak $2 \times 2n$ magic rectangles evaluates to a polynomial in s.*

The information about the degree and period allows us to find the polynomials $m_{2n}(s)$ for any given n using values we computed from Theorem 6.1. The polynomials for weak 2×4 and 2×6 magic rectangles are

$$m_4(s) = \frac{2}{3}s^3 + 2s^2 + \frac{7}{3}s + 1 \quad \text{and}$$
$$m_6(s) = \frac{11}{20}s^5 + \frac{11}{4}s^4 + \frac{24}{4}s^3 + \frac{25}{4}s^2 + \frac{37}{10}s + 1.$$

6.2 Weak $2 \times (2n+1)$ Magic Rectangles

In a weak $2 \times (2n+1)$ magic rectangle with column sum s and row sum r, the sum of all entries in the whole rectangle is $(2n+1)s = 2r$. Therefore, the column sum s must be even. We can derive the expression for the magic count the same way as the previous section.

Theorem 6.8. *For a given column sum s, there are*

$$\left[x^{(2n+1)\frac{s}{2}}\right] \left(\frac{1 - x^{s+1}}{1 - x}\right)^{2n+1}.$$

weak $2 \times (2n+1)$ magic rectangles with $a_{ij} \geq 0$.

This allows us to calculate the number of weak $2 \times (2n+1)$ magic rectangles. Table 6.4, 6.5, and 6.6 show the magic count for weak 2×3, 2×5, and 2×7 magic rectangle respectively.

s	0	2	4	6	8	10	12	14	16	18
magic count	1	7	19	37	61	91	127	169	217	271

Table 6.4: Number of weak 2×3 magic rectangles with even column sum s.

s	0	2	4	6	8	10	12	14	16	18
magic count	1	51	381	1451	3951	8801	17151	30381	50101	78151

Table 6.5: Number of weak 2×5 magic rectangles with even column sum s.

s	0	2	4	6	8	10	12	14
magic count	1	393	8135	60691	273127	908755	2473325	5832765

Table 6.6: Number of weak 2×7 magic rectangles with even column sum s.

The polytope associated with the counting quasipolynomial is defined by $0 \leq \alpha_j \leq 1$ and $\sum_{j=1}^{2n+1} \alpha_j = \frac{2n+1}{2}$.

Theorem 6.9. *The number of weak* $2 \times (2n+1)$ *magic rectangles evaluates to a quasipolynomial (in* s*) of period* 2.

Proof. With a similar proof as the previous section, we can show that the polytope has $(2n+1)$-dimensional vertices where n coordinates are ones, n coordinates are zeroes, and one coordinate is $\frac{1}{2}$. The main difference between this polytope and the $(2n, n)$-hypersimplex is that $\frac{2n+1}{2}$ is not an integer. This is why, unlike the $(2n, n)$-hypersimplex, each vertex of this polytope has a single coordinate being $\frac{1}{2}$. All the lemmas leading up to the proof of Theorem 6.2 still apply to this set of points we claim to be the vertices of this polytope.

The least common multiple of the denominator of the vertices is 2, so the period of the quasipolynomial divides 2. The quasipolynomial evaluated at any odd integer s is 0 because the column sum must be even. Thus, the period of the quasipolynomial is exactly 2. ☐

Just as in the last section, the equation $\sum_{j=1}^{2n+1} \alpha_j = \frac{2n+1}{2}$ cuts down the dimension by one, and the degree of the quasipolynomial is $2n$.

Examples for the quasipolynomials are

$$m_3(s) = \begin{cases} \frac{3}{4}s^2 + \frac{3}{2}s + 1 & \text{if } s \equiv 0 \mod 2, \\ 0 & \text{if } s \equiv 1 \mod 2; \end{cases}$$

$$m_5(s) = \begin{cases} \frac{115}{192}s^4 + \frac{115}{48}s^3 + \frac{185}{48}s^2 + \frac{35}{12}s + 1 & \text{if } s \equiv 0 \mod 2, \\ 0 & \text{if } s \equiv 1 \mod 2; \text{ and} \end{cases}$$

$$m_7(s) = \begin{cases} \frac{5887}{11520}s^6 + \frac{5887}{1920}s^5 + \frac{2275}{288}s^4 + \frac{357}{32}s^3 + \frac{6643}{720}s^2 + \frac{259}{60}s + 1 & \text{if } s \equiv 0 \mod 2, \\ 0 & \text{if } s \equiv 1 \mod 2. \end{cases}$$

Chapter 7

$2 \times n$ Magilatin Rectangles

The conditions for weak $2 \times n$ magic rectangles as outlined in Chapter 6 still apply to $2 \times n$ magilatin rectangles, except the magilatin conditions also require entries in each row/column not to repeat. Due to symmetry, we can rearrange the columns of a magilatin rectangle such that $\alpha_1 < \cdots < \alpha_n$ and preserve magilatinness. Because of that, we will impose the inequalities $\alpha_1 < \cdots < \alpha_n$ and multiply the count by $n!$ at the end. Let magilatin rectangles that satisfy the inequalities be called **progressive**.

We will first consider $2 \times 2n$ progressive magilatin rectangles. The progressive condition solves the issue of repeating row entries. However, if the column sum s is even, then we have to be careful not to count rectangles with the entry $\frac{s}{2}$ in order to avoid duplicate column entries. We will set this caveat aside for a moment.

Let $p(M, N, n)$ denote the number of partitions of n into N parts using the elements of $\{0, \ldots, M\}$ for each part. In other words, $p(M, N, n)$ counts the number of $(m_1, \ldots, m_N)$ such that $\sum_{j=1}^{n} m_j = n$ and $0 \le m_1 \le \cdots \le m_N \le M$. The generating function for $p(M, N, n)$ is the **Gaußian binomial coefficient** [3]

$$\binom{M+N}{M}_x = \frac{\prod_{j=1}^{M+N}(1-x^j)}{\prod_{j=1}^{M}(1-x^j)\prod_{j=1}^{N}(1-x^j)}.$$

We need, however, that $0 \le m_1 < \cdots < m_N \le M$. We can form a bijection between partitions into possibly repeating parts and partitions into unique parts by adding 1 to m_2, adding 2 to m_3, etc. Under this conversion scheme, the maximum part m_N increased by $N-1$ and the sum is increased by $\frac{N(N-1)}{2}$. Let $p^\star(M, N, n)$ denote the number of partitions of n into N parts using unique elements of $\{0, \ldots, M\}$ for each part. We have

$$\begin{aligned}\sum_{n \ge 0} p^\star(M, N, n) x^n &= x^{\frac{N(N-1)}{2}} \sum_{n \ge 0} p(M-N+1, N, n) x^n \\ &= \frac{x^{\frac{N(N-1)}{2}} \prod_{j=1}^{M+1}(1-x^j)}{\prod_{j=1}^{M-N+1}(1-x^j)\prod_{j=1}^{N}(1-x^j)}.\end{aligned}$$

Theorem 7.1. *The number of $2 \times 2n$ magilatin rectangle with the odd column sum s (and row sum ns) is*

$$(2n)![x^{ns}]\sum_{k\geq 0} p^\star(s,2n,k)x^k = (2n)![x^{ns}]\frac{x^{\frac{2n(2n-1)}{2}}\prod_{j=1}^{s+1}(1-x^j)}{\prod_{j=1}^{s-2n+1}(1-x^j)\prod_{j=1}^{2n}(1-x^j)}.$$

This gives us the number of 2×2, 2×4, and 2×6 magilatin rectangles in Tables 7.1, 7.2, and 7.3.

s	1	3	5	7	9	11	13	15	17	19
$\frac{\text{magic count}}{2}$	1	2	3	4	5	6	7	8	9	10

Table 7.1: Number of 2×2 magilatin rectangles with odd column sum s.

s	3	5	7	9	11	13	15	17	19	21
$\frac{\text{magic count}}{24}$	1	3	8	18	33	55	86	126	177	241

Table 7.2: Number of 2×4 magilatin rectangles with odd column sum s.

s	5	7	9	11	13	15	17	19	21	23
$\frac{\text{magic count}}{720}$	1	4	18	58	151	338	676	1242	2137	3486

Table 7.3: Number of 2×6 magilatin rectangles with odd column sum s.

When the column sum s is even, we need to be mindful about duplicate entries in a column. Say one of the columns is comprised of two $\frac{s}{2}$. Removing the column result in a $2 \times (2n-1)$ progressive magilatin rectangle with column sum s and row sum $ns - \frac{1}{2}s$. Subtracting the number of such rectangles before multiplying $(2n)!$ fixes our count, so let us quickly venture to $2 \times (2n+1)$ progressive magilatin rectangles.

As shown in Section 6.2, the column sum of a $2 \times (2n+1)$ magilatin rectangle must be even. As with $2 \times 2n$ magilatin rectangles, we take

$$[x^{(2n+1)\frac{s}{2}}]\sum_{k\geq 0} p^\star(s,2n+1,k)x^k = [x^{(2n+1)\frac{s}{2}}]\frac{x^{\frac{2n(2n+1)}{2}}\prod_{j=1}^{s+1}(1-x^j)}{\prod_{j=1}^{s-2n}(1-x^j)\prod_{j=1}^{2n+1}(1-x^j)},$$

subtract the number of $2 \times 2n$ progressive magilatin rectangles with column sum s and row sum $(2n+1)\frac{s}{2} - \frac{s}{2} = ns$, and then multiply $(2n+1)!$.

Theorem 7.2. *Let $m_n(s)$ be the number of $2 \times n$ magilatin rectangles with column sum s. For any even s,*

$$\frac{m_n(s)}{n!} = [x^{\frac{ns}{2}}]\frac{x^{\frac{n(n-1)}{2}}\prod_{j=1}^{s+1}(1-x^j)}{\prod_{j=1}^{s-n+1}(1-x^j)\prod_{j=1}^{n}(1-x^j)} - \frac{m_{n-1}(s)}{(n-1)!}$$

Using the recursion, we get Table 7.4, 7.5, 7.6, and 7.7.

s	2	4	6	8	10	12	14	16	18	20
$\frac{\text{magic count}}{2}$	1	2	3	4	5	6	7	8	9	10

Table 7.4: Number of 2×2 magilatin rectangles with even column sum s.

s	6	8	10	12	14	16	18	20	22	24
$\frac{\text{magic count}}{6}$	2	4	8	12	18	24	32	40	50	60

Table 7.5: Number of 2×3 magilatin rectangles with even column sum s.

s	4	6	8	10	12	14	16	18	20	22
$\frac{\text{magic count}}{24}$	1	3	8	16	31	51	80	118	167	227

Table 7.6: Number of 2×4 magilatin rectangles with even column sum s.

s	8	10	12	14	16	18	20	22	24
$\frac{\text{magic count}}{120}$	4	16	42	90	172	296	482	740	1092

Table 7.7: Number of 2×5 magilatin rectangles with even column sum s.

These sequences show up in A188122 of the On-Line Encyclopedia of Integer Sequences [14]. The table A188122 shows the number of set of n nonzero integers from the interval $[-n-k+2, n+k-2]$ with sum zero. If we add $n+k-2$ to every integer in the set, we get a set of n integers from the interval $[0, 2(n+k-2)]$ sans $n+k-2$ that add to $n(n+k-2)$. Letting $2(n+k-2)$ equal column sum s gives us exactly the what we want; the number of sets of n integers from $[0, s]$ without using $\frac{s}{2}$ and with the sum of $\frac{ns}{2}$.

Finally, let us look at the degrees and periods of the quasipolynomials. Without the nonequalities, the magilatin conditions $0 < \alpha_1 < \cdots < \alpha_n < 1$ and $\sum_{j=1}^{n} \alpha_j = \frac{n}{2}$ result in the relative interior of a polytope. Let the closure be the polytope P. The nonequalities are the forbidden hyperplanes $\alpha_j = \frac{1}{2}$. What we know about the degrees and periods about polytopes still hold true for polytopes with hyperplanes removed, but we have to also consider the vertices of the intersections of the polytope and the hyperplanes [6].

As in the last chapter, the equation $\sum_{j=1}^{n} \alpha_j = \frac{n}{2}$ causes the dimension, and thus the degree of $m_n(s)$, to be $n-1$. To find the period, let us start by examining the vertices of P.

Theorem 7.3. *Let $P \in \mathbb{R}^n$ be the polytope defined by $0 \le \alpha_1 \le \cdots \le \alpha_n \le 1$ and $\sum_{j=1}^{n} \alpha_j = \frac{n}{2}$. The vertices are of the form*

$$\left(0, \ldots, 0, \tfrac{n-2n_2}{2(n-n_1-n_2)}, \ldots, \tfrac{n-2n_2}{2(n-n_1-n_2)}, 1, \ldots, 1\right),$$

where $0 \le n_1, n_2 \le \frac{n}{2}$, the first n_1 coordinates are 0, and the last n_2 coordinates are 1.

Proof. We will prove this by induction using base cases $n = 1$ and $n = 2$.

For $n = 1$, the vertex we claim in the theorem and the only point in P are both $\left(\frac{1}{2}\right)$.

For $n = 2$, we claim that the vertices of P are $(0, 1)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$. Let (α_1, α_2) be in P. All points in P satisfies $\alpha_1 + \alpha_2 = 1$, so

$$(\alpha_1, \alpha_2) = (1 - 2\alpha_1)(0, 1) + 2\alpha_1 \left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

Since $\alpha_1 \le \alpha_2$ and $\alpha_1 + \alpha_2 = 1$, the coordinate α_1 must be at most $\frac{1}{2}$. This means (α_1, α_2) is in the convex hull of $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$. Also, if $k_1, k_2 \ge 0$ and $k_1 + k_2 = 1$, then $k_1(0, 1) + k_2(\frac{1}{2}, \frac{1}{2})$ satisfies both conditions of P. Therefore, the vertices of P are $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$.

For the induction step, suppose the polytope $P \in \mathbb{R}^{n-2}$ defined by $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{n-2} \le 1$ and $\sum_{j=1}^{n-2} \alpha_j = \frac{n-2}{2}$ has vertices of the form

$$\left(0, \ldots, 0, \tfrac{n-2-2n_2}{2(n-2-n_1-n_2)}, \ldots, \tfrac{n-2-2n_2}{2(n-2-n_1-n_2)}, 1, \ldots, 1\right),$$

where $0 \le n_1, n_2 \le \frac{n-2}{2}$, the first n_1 coordinates are 0, and the last n_2 coordinates are 1. Let V be this set of points. We need to show that the polytope $P' \in \mathbb{R}^n$ defined by $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$ and $\sum_{j=1}^{n} \alpha_j = \frac{n}{2}$ has vertices of the form

$$\left(0, \ldots, 0, \tfrac{n-2n_2}{2(n-n_1-n_2)}, \ldots, \tfrac{n-2n_2}{2(n-n_1-n_2)}, 1, \ldots, 1\right),$$

where $0 \le n_1, n_2 \le \frac{n}{2}$, the first n_1 coordinates are 0, and the last n_2 coordinates are 1. Let V' be this set of points.

Let $(\alpha_1, \ldots, \alpha_n)$ be in P'. We need to first determine if α_1 or $1 - \alpha_n$ is smaller. If α_1 is smaller, we use the point $(1 - \alpha_n, \ldots, 1 - \alpha_1)$ instead, since $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$ if and only if $0 \le 1 - \alpha_n \le 1 - \alpha_{n-1} \le \cdots \le 1 - \alpha_1 \le 1$; and $\sum_{j=1}^{n} \alpha_j = \frac{n}{2}$ if and only if $\sum_{j=1}^{n} (1 - \alpha_{n+1-j}) = \frac{n}{2}$. Therefore, we can assume $1 - \alpha_n \le \alpha_1$ without loss of generality.

We will work backward to prove that $(\alpha_1, \ldots, \alpha_n)$ is in the convex hull of the points we claimed to be the vertices. First, we will find a point $\vec{p}$ and coefficients a, b such that

- $a\vec{p} + b\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) = (\alpha_1, \ldots, \alpha_n)$,
- the last coordinate of $\vec{p}$ is 1, and
- $a + b = 1$.

From the last coordinate, we get the equation $a + \frac{1}{2}b = \alpha_n$. Solving it along with $a + b = 1$ gives us $a = 2\alpha_n - 1$ and $b = 2 - 2\alpha_n$. Using a and b, we have $\vec{p} = \left(\frac{\alpha_1+\alpha_n-1}{a}, \dots, \frac{\alpha_{n-1}+\alpha_n-1}{a}, 1\right)$. Note that since $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$ and $\sum_{j=1}^n \alpha_j = \frac{n}{2}$, the value of α_n is bounded by $\frac{1}{2}$ and 1, and a and b are bounded by 0 and 1. This means $(\alpha_1, \dots, \alpha_n)$ is in the convex hull of $\vec{p}$ and $\left(\frac{1}{2}, \dots, \frac{1}{2}\right)$. Furthermore, $0 \le \frac{\alpha_1+\alpha_n-1}{a} \le \cdots \le \frac{\alpha_{n-1}+\alpha_n-1}{a} \le 1$ and

$$\begin{aligned}\sum_{j=1}^{n} \frac{\alpha_j + \alpha_n - 1}{a} &= \frac{\frac{n}{2} + n\alpha_n - n}{a} \\ &= n\frac{2\alpha_n - 1}{2a} \\ &= \frac{n}{2}.\end{aligned}$$

Now that we pulled the last coordinate to 1, we will find a point $\vec{q} = (q_1, \dots, q_n)$ and coefficients c, d such that

- $c\vec{q} + d\left(\frac{n-2}{2n-2}, \dots, \frac{n-2}{2n-2}, 1\right) = \vec{p} = \left(\frac{\alpha_1+\alpha_n-1}{a}, \dots, \frac{\alpha_{n-1}+\alpha_n-1}{a}, 1\right)$,
- $q_1 = 0$, and
- $c + d = 1$.

With $n_1 = 0$ and $n_2 = 1$, the point $\left(\frac{n-2}{2n-2}, \dots, \frac{n-2}{2n-2}, 1\right)$ is in V'. Using the first coordinate, we get $d = \frac{(\alpha_1+\alpha_n-1)(2n-2)}{a(n-2)}$. With the assumption earlier that $1 - \alpha_n \le \alpha_1$, the coefficient d must be nonnegative. Also, if $\frac{\alpha_1+\alpha_n-1}{a} > \frac{n-2}{2n-2}$, then

$$\begin{aligned}\frac{n}{2} &= \sum_{j=1}^{n} \frac{\alpha_j + \alpha_n - 1}{a} \\ &= 1 + \sum_{j=1}^{n-1} \frac{\alpha_j + \alpha_n - 1}{a} \\ &> 1 + \sum_{j=1}^{n-1} \frac{n-2}{2n-2} \\ &= 1 + (n-1)\frac{n-2}{2n-2} \\ &= \frac{n}{2},\end{aligned}$$

which is a contradiction. Therefore, $0 \le d \le 1$, and so is c due to $c + d = 1$, implying that $\left(\frac{a_1+a_n-1}{a}, \dots, \frac{a_{n-1}+a_n-1}{a}, 1\right)$ is in the convex hull of $\vec{q}$ and $\left(\frac{n-2}{2n-2}, \dots, \frac{n-2}{2n-2}, 1\right)$. Since $\left(\frac{1}{2}, \dots, \frac{1}{2}\right), \left(\frac{n-2}{2n-2}, \dots, \frac{n-2}{2n-2}, 1\right) \in V'$, if we can prove that $\vec{q}$ is in the convex hull of the points in V', then $(\alpha_1, \dots, \alpha_n)$ is in the convex hull of points in V by Lemma 6.6.

For $1 \leq j \leq n-1$, the coordinate $q_j = \frac{1}{c}\left(\frac{\alpha_j+\alpha_n-1}{a} - \frac{\alpha_1+\alpha_n-1}{a}\right) = \frac{\alpha_j-\alpha_1}{ac}$, which means $0 = q_1 \leq q_2 \leq q_2 \leq \cdots \leq q_{n-1} \leq q_n = 1$, and the coordinate sum of $\vec{q}$ ignoring q_1 and q_n is

$$\begin{aligned}
\sum_{j=2}^{n-1} q_j &= \sum_{j=1}^{n-1} \frac{\alpha_j - \alpha_1}{ac} \\
&= \frac{\frac{n}{2} - \alpha_n - (n-1)\alpha_1}{a(1-d)} \\
&= \frac{\frac{n}{2} - \alpha_n - (n-1)\alpha_1}{2\alpha_n - 1 - \frac{(\alpha_1+\alpha_n-1)(2n-2)}{n-2}} \\
&= \frac{(n-2)(n-2\alpha_n-2(n-1)\alpha_1)}{2[(n-2)(2\alpha_n-1)-(\alpha_1+\alpha_n-1)(2n-2)]} \\
&= \frac{(n-2)(n-2\alpha_n-2(n-1)\alpha_1)}{2(n-2\alpha_n-2(n-1)\alpha_1)} \\
&= \frac{n-2}{2}.
\end{aligned}$$

By the inductive hypothesis, $(q_2, \ldots, q_{n-1})$ is in the convex hull of points in V, i.e.,

$$(q_2, \ldots, q_{n-1}) = \sum_{\vec{v} \in V} k_{\vec{v}} \vec{v}$$

for some $k_{\vec{v}} \geq 0$ where $\sum_{\vec{v} \in V} k_{\vec{v}} = 1$. Let $\uparrow\colon \mathbb{R}^n \to \mathbb{R}^{n+2}$ be the function such that

$$\uparrow((x_1, \ldots, x_n)) = (0, x_1, \ldots, x_n, 1).$$

If $v \in V$, then $\uparrow(\vec{v}) \in V'$. Therefore,

$$\vec{q} = \sum_{\vec{v} \in V} k_{\vec{v}} \uparrow(\vec{v})$$

is in the convex hull of the points in V'. Thus, $(\alpha_1, \ldots, \alpha_n)$ is in the convex hull of the points in V', finishing the inductive step. □

The hyperplanes we need to remove are of the form

$$H_j = \left\{(\alpha_1, \ldots, \alpha_n) : \alpha_j = \tfrac{1}{2}\right\}.$$

for $j = 1, \ldots, n$. No point inside $\mathrm{int}(P)$ has $\frac{1}{2}$ for two coordinates due to strict inequalities between coordinates, so the hyperplanes do not intersect each other inside $\mathrm{int}(P)$. Let us investigate the vertices of $H_j \cap P$.

Theorem 7.4. *Let $Q_j \in \mathbb{R}^n$ be the polytope defined by $0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq 1$, $\sum_{j=1}^{n} \alpha_j = \frac{n}{2}$, and $\alpha_j = \frac{1}{2}$. The vertices are of the form*

$$\left(0, \ldots, 0, \frac{n-n_2-2n_3}{2(n-n_1-n_2-n_3)}, \ldots, \frac{n-n_2-2n_3}{2(n-n_1-n_2-n_3)}, \frac{1}{2}, \ldots, \frac{1}{2}, 1, \ldots, 1\right) \quad \textit{or}$$
$$\left(0, \ldots, 0, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{n-n_2-2n_3}{2(n-n_1-n_2-n_3)}, \ldots, \frac{n-n_2-2n_3}{2(n-n_1-n_2-n_3)}, 1, \ldots, 1\right),$$

where $0 \le n_1, n_3 \le \frac{n}{2}$*;* $0 \le n_2 \le n$*; the first* n_1 *coordinates are* 0*;* n_2 *consecutive coordinates, including* α_j*, are* $\frac{1}{2}$*; and the last* n_3 *coordinates are* 1*.*

Proof. The proof for Theorem 7.4 is essentially the same as the one for Theorem 7.3.

The base case of $n = 1$ is exactly the same as before. For $n = 2$, if we fix a coordinate to be $\frac{1}{2}$, since the coordinate sum is 1, the other coordinate is also $\frac{1}{2}$. Therefore, the vertex we claim in the theorem and the only point in Q are both $(\frac{1}{2}, \frac{1}{2})$.

For the induction step, $(\frac{1}{2}, \dots, \frac{1}{2})$ is still a vertex we claim, so the first part of the induction step for pulling the last coordinate to 1 is still the same. For the second part of the induction step, instead of using $(\frac{n-2}{2n-2}, \dots, \frac{n-2}{2n-2}, 1)$, we use the point $(\frac{j-2}{j-1}, \dots, \frac{j-2}{j-1}, \frac{1}{2}, \dots, \frac{1}{2}, 1)$, where all coordinates before the jth one are $\frac{j-2}{j-1}$. □

At least one coordinate is $\frac{1}{2}$, so $n_2 \ge 1$. The vertex with $n_1 = n_3 = 0$ and $n_2 = 1$ is just $(\frac{1}{2}, \dots, \frac{1}{2})$. Therefore, the denominators of the vertices are even integers from 2 to $2(n-2)$ if unsimplified. When a coordinate simplifies, by adding or subtracting one from n_1 and doing the opposite to n_2, we can get another point where the denominator of $\frac{n-n_2-2n_3}{2(n-n_1-n_2-n_3)}$ is the same and the numerator is off by 1. Any factor that was simplified before does not simplify.

Corollary 7.5. *The number of* $2 \times n$ *magilatin rectangles evaluates to a quasipolynomial (in* s*) with period that divides* $2\,lcm(2, \dots, n-1)$*.*

The information about the degree and period lets us construct the quasipolynomials

$$m_2(s) = \begin{cases} s & \text{if } s \equiv 0 \mod 2, \\ s+1 & \text{if } s \equiv 1 \mod 2; \end{cases}$$

$$m_3(s) = \begin{cases} \frac{3}{4}s^2 - 3s & \text{if } s \equiv 0 \mod 4, \\ \frac{3}{4}s^2 - 3s + 3 & \text{if } s \equiv 2 \mod 4, \\ 0 & \text{if } s \equiv 1 \mod 2; \text{ and} \end{cases}$$

$$m_4(s) = \begin{cases} \frac{2}{3}s^3 - 4s^2 + 14s & \text{if } s \equiv 0 \mod 12, \\ \frac{2}{3}s^3 - s^2 + 2s - \frac{5}{3} & \text{if } s \equiv 1 \mod 12, \\ \frac{2}{3}s^3 - 4s^2 + 14s - \frac{78}{3} & \text{if } s \equiv 2 \mod 12, \\ \frac{2}{3}s^3 - s^2 + 2s + 9 & \text{if } s \equiv 3 \mod 12, \\ \frac{2}{3}s^3 - 4s^2 + 14s - \frac{32}{3} & \text{if } s \equiv 4 \mod 12, \\ \frac{2}{3}s^3 - s^2 + 2s + \frac{11}{3} & \text{if } s \equiv 5 \mod 12, \\ \frac{2}{3}s^3 - 4s^2 + 14s - 12 & \text{if } s \equiv 6 \mod 12, \\ \frac{2}{3}s^3 - s^2 + 2s + \frac{5}{3} & \text{if } s \equiv 7 \mod 12, \\ \frac{2}{3}s^3 - 4s^2 + 14s - \frac{16}{3} & \text{if } s \equiv 8 \mod 12, \\ \frac{2}{3}s^3 - s^2 + 2s + 9 & \text{if } s \equiv 9 \mod 12, \\ \frac{2}{3}s^3 - 4s^2 + 14s - \frac{68}{3} & \text{if } s \equiv 10 \mod 12, \\ \frac{2}{3}s^3 - s^2 + 2s + \frac{11}{3} & \text{if } s \equiv 11 \mod 12. \end{cases}$$

Chapter 8

Cliffhanger (a.k.a. Future Researches)

- In Chapter 5, we did not compute the Ehrhart series for strong 5×5 pandiagonal magic squares. If the vertices of the intersections are as nice as the ones for the 4×4 square, then we can write a program for the generating function.
- Strong 6×6 may have a nice structure.
- We conjecture that for any integers m and n, the affine count of weak $m \times mn$ magic rectangles follows a polynomial in s, the magic sum, just as the affine count of weak $2 \times 2n$ magic rectangles.

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