Chapter 1

Introduction

This thesis investigates sums such as

$$\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}+\cot^2\frac{4\pi}{7}+\cot^2\frac{5\pi}{7}+\cot^2\frac{6\pi}{7}=10,$$

and

$$\cot\frac{\pi}{7}+\cot\frac{2\pi}{7}-\cot\frac{3\pi}{7}+\cot\frac{4\pi}{7}-\cot\frac{5\pi}{7}-\cot\frac{6\pi}{7}=2\sqrt{7}\ .$$

The first of these sums is an example of the identity

$$\sum_{j=1}^{k-1}\cot^2\frac{\pi j}{k}=\frac{(k-1)(k-2)}{3}, \qquad (1)$$

while the second is an example of the identity

$$\sum_{j=1}^{k-1}\chi(j)\cot\frac{\pi j}{k}=2\sqrt{k}\,h(-k), \qquad (2)$$

where χ denotes a real, nonprincipal, primitive, odd character (mod k) and $h(-k)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-k})$.

Berndt and Yeap [1] investigated classes of trigonometric sums of type (1) and Berndt and Zaharescu [2] investigated classes of trigonometric sums of type (2). In both of these papers the proofs involved contour integration. My first exposure to a proof of (1) was in a course where discrete Fourier Analysis was used, and the question was posed whether it was possible to use Fourier Analysis to prove other trigonometric sums of both types (1) and (2). For many sums of trigonometric functions (and character sums of trigonometric functions) this method was successful.

Chapter 2 presents the main theorems upon which this thesis is centered, the Finite Fourier Series Expansion and Fourier Inversion Theorem and the Convolution Theorem for Finite Fourier Series. Basically what they provide is the method by which one can determine certain trigonometric sums by evaluating the sum of corresponding Fourier coefficients. In the case of many trigonometric functions, this is a relatively simple integer valued function. For example, the identity in (1) above was determined by evaluating $\sum_{n=1}^{k-1}(n/k-1/2)^2$ where n and k are positive integers. In addition to these two main theorems, Chapter 2 includes some useful corollaries as well as determining the Fourier coefficients for all of the basic trigonometric functions.

Applying these theorems, Chapter 3 starts by finding closed forms for sums of the form $\sum_{j=0}^{k-1} f(j)g(j)$, where f and g are elementary trigonometric functions. My starting point was Berndt and Yeap's paper [1]; however, I explored some sums which were not included in their work leading to some interesting results. While investigating sums of the form $\sum_{j=0}^{k-1}\sin^{2a}(2\pi bj/k)$ I discovered that for certain integer values of a, b and k, shifting the function by any real amount, $\sum_{j=0}^{k-1}\sin^{2a}(2\pi bj/k+x)$ does not alter the result. That is $\sum_{j=0}^{k-1}\sin^{2a}(2\pi bj/k+x)$ is equal to a constant for all real numbers x. See

Theorem 3.14. In addition I found an alternative proof of the well known identity

$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$. See Theorem 3.16.

Having successfully used discrete Fourier analysis to determine several trigonometric functions, I turned to Berndt and Zahrescu's paper [2] which looked at character sums of trigonometric functions. These are of the form $\sum_{j=0}^{k-1}\chi(j)f(j)$, where $\chi(j)$ is a real, nonprincipal, primitive, character (mod *k*) and $f(j)$ is a trigonometric function.

Chapter 4 presents some useful background information on characters, which includes necessary definitions and theorems, and then Chapter 5 proceeds to determine closed forms for many types of character sums of trigonometric functions. For many of these sums, the sum of their corresponding Fourier coefficients was considerably more complicated than those in Chapter 3, involving congruencies (mod *k*) and some creative manipulations of binomial coefficients. I was able to successfully prove some of the theorems of Berndt and Zaharescu [2] using discrete Fourier analysis and was even able to make a slight improvement on one in particular. See Theorem 5.13. As is the case in Chapter 3, I also determined identities for sums not considered previously.

Chapter 2

Discrete Fourier Transformations and Convolutions

This thesis involves the use of discrete Fourier analysis in order to prove identities involving the finite sums of trigonometric functions, and finite character sums of trigonometric functions. This chapter presents the two main theorems which are used throughout the rest of the paper; specifically the Finite Fourier series expansion and Fourier Inversion Theorem and the Convolution Theorem, along with some useful corollaries. A discussion of their applications is also provided.

In addition, the Fourier transformations for four of the basic trigonometric functions, sine, cosine, tangent and cotanget, are determined.

Theorem 2.1 (Finite Fourier series expansion and Fourier inversion). *Let f(n) be any periodic function on $\mathbb{Z}$, with period k. Then we have the following finite Fourier series expansion:*

$$f(n) = \sum_{j=0}^{k-1} \hat{f}(j)\omega^{nj} \text{ where the Fourier coefficients are } \hat{f}(j) = \frac{1}{k}\sum_{n=0}^{k-1} f(n)\omega^{-nj},$$

$$\text{with } \omega = e^{2\pi i/k}.$$

Proof: Let $f(n)$ be any periodic function on $\mathbb{Z}$, with period k. Consider the sequence $\{f(n)\}_{n\geq 0}$ and embed this in the generating function:

$$F(x) := \sum_{n\geq 0} f(n)x^n .$$

Due to the periodicity of *f*, this can be rewritten as

$$F(x) = \sum_{m\geq 0} (\sum_{n=0}^{k-1} f(n)x^n)x^{km}$$

$$= \frac{\sum_{n=0}^{k-1} f(n)x^n}{1-x^k}$$

$$= \frac{P(x)}{1-x^k},$$

where the last step simply defines the polynomial $P(x) = \sum_{n=0}^{k-1} f(n)x^n$. Now expanding the rational function $F(x)$ into its partial fraction decomposition over the kth roots of unity gives:

$$F(x) = \sum_{j=0}^{k-1} \frac{\widehat{f}(j)}{1-\omega^j x} .$$

By expanding each of the terms $\frac{1}{1-\omega^j x}$ as a geometric series, and substituting into the sum above,

$$F(x) = \sum_{n\geq 0} f(n)x^n$$

$$= \sum_{j=0}^{k-1} \frac{\widehat{f}(j)}{1-\omega^j x}$$

$$= \sum_{j=0}^{k-1} \widehat{f}(j) \sum_{n\geq 0} \omega^{jn} x^n$$

$$= \sum_{n \geq 0} \left(\sum_{j=0}^{k-1} \widehat{f}(j) \omega^{jn} \right) x^n \ .$$

Comparing the coefficients of any fixed x^n gives the finite Fourier series for $f(n)$

$$f(n) = \sum_{j=0}^{k-1} \widehat{f}(j) \omega^{jn} \ .$$

To find the formula for the Fourier coefficients $\widehat{f}(j)$, we have

$$P(x) = \sum_{m=0}^{k-1} \widehat{f}(m) \frac{1 - x^k}{1 - \omega^m x}$$

$$= \sum_{m=0}^{k-1} \widehat{f}(m) \prod_{1 \leq b \leq k, b \neq m} (1 - \omega^b x) \ ,$$

where we have used the factorization $1 - x^k = \prod_{b=1}^{k} (1 - \omega^b x)$.

By evaluating $P(x)$ at $x = \omega^{-j}$, we get $P(\omega^{-j}) = \sum_{m=0}^{k-1} \hat{f}(m) \prod_{1 \leq b \leq k, b \neq m} (1 - \omega^{b-j})$.

For $m \neq j$, $\prod_{1 \leq b \leq k, b \neq m} (1 - \omega^{b-j}) = (1 - \omega^{j-j}) \prod_{\substack{1 \leq b \leq k, \\ b \neq m, j}} (1 - \omega^{b-j}) = 0$.

For $m = j$, since $\prod_{r=1}^{k-1} (1 - \omega^r x) = \frac{1 - x^k}{1 - x} = \sum_{n=0}^{k-1} x^n$, then

$$\prod_{1 \leq b \leq k, b \neq m} (1 - \omega^{b-j}) = \prod_{r=1}^{k-1} (1 - \omega^r) = \sum_{r=1}^{k-1} 1 = k \ .$$

Hence, all of the terms vanish in the sum for $P(\omega^{-j})$ except for $m = j$, which yields

$$P(\omega^{-j}) = k\,\hat{f}(j).$$

Therefore, since $P(x) = \sum_{n=0}^{k-1} f(n)x^n$, we get

$$\hat{f}(j) = \frac{1}{k}P(\omega^{-j}) = \frac{1}{k}\sum_{n=0}^{k-1} f(n)\omega^{-nj}.$$

The linear transformation $f(n) = \sum_{j=0}^{k-1} \hat{f}(j)\omega^{nj}$ is usually referred to as the discrete Fourier transformation of *f* and $\hat{f}(j) = (1/k)\sum_{n=0}^{k-1} f(n)\omega^{-nj}$ is called the inverse discrete Fourier transformation of *f*. Since an invertible linear transformation is the inverse of its inverse transformation I will use the term Fourier transformation for both throughout the rest of this paper.

Since every trigonometric function is periodic, if we take its fundamental period *p*, and evaluate it at jp/k for $j = 0, 1, \ldots, k-1$, then it is a periodic function on the integers with period *k* and, thus by Theorem 1.1, has a corresponding Fourier transformation. For certain trigonometric functions it is necessary to restrict the values of *k* in order to avoid having terms that are undefined.

Given two periodic functions, $f(j)$ and $g(j)$, whose Fourier transformations are known, the next theorem provides a means by which one can determine the Fourier transformation for their product, $f(j)g(j)$. And more importantly, the corollary which follows provides a method by which explicit evaluations of trigonometric sums can be determined using their Fourier coefficients.

Theorem 2.2 (Convolution Theorem for finite Fourier series).

Let $f(n)=\frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\omega^{nj}$ *and* $g(n)=\frac{1}{k}\sum_{j=0}^{k-1}\hat{g}(j)\omega^{nj}$ *, where* $\omega=e^{2\pi i/k}$*. Then their convolution, defined by*

$$(f*g)(n)=\sum_{m=0}^{k-1}f(n-m)g(m),$$

satisfies

$$\sum_{m=0}^{k-1}f(n-m)g(m)=\frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\hat{g}(j)\omega^{nj}.$$

Proof: Computing the left hand side:

$$\sum_{m=0}^{k-1}f(n-m)g(m)=\sum_{m=0}^{k-1}\left(\frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\omega^{j(n-m)}\right)\left(\frac{1}{k}\sum_{l=0}^{k-1}\hat{g}(l)\omega^{lm}\right)$$

$$=\frac{1}{k^2}\sum_{j=0}^{k-1}\sum_{l=0}^{k-1}\hat{f}(j)\hat{g}(l)\omega^{nj}\left(\sum_{m=0}^{k-1}\omega^{(l-j)m}\right)$$

$$=\frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\hat{g}(j)\omega^{jn}$$

because the sum $\sum_{m=0}^{k-1}\omega^{(l-j)m}$ vanishes, unless $l=j$. In the case that $l=j$, we have $\sum_{m=0}^{k-1}\omega^{(l-j)m}=k$. أ

Theorems 2.1 and 2.2 are special cases of Fourier inversion and convolution from representation theory where the group is $\mathbb{Z}/k\mathbb{Z}$. A representation ρ of a finite group G is a homomorphism from G to the group of invertible *d* x *d* matrices with complex entries, $GL_d(\mathbb{C})$. The dimension of ρ is *d*, denoted by $\dim_\rho$, and $V = \mathbb{C}^d$ is called the representation space of ρ. A subspace $W \subset V$ is said to be *G*-invariant if for all $s \in G, \rho(s)W \subset W$. A representation ρ is irreducible if *V* has no *G*-invariant subspaces other than the trivial subspace $\{0\}$ and *V*, and reducible otherwise. The total number of irreducible representations is equal to the number of conjugacy classes of *G*. Let $\hat{G}$ denote the set of all irreducible representations of *G*.

Given a finite set *G*, if *f* is a complex valued function on *G* and ρ is a representation of *G*, then the Fourier transform of *f* at ρ is defined by

$$\mathcal{F} f(\rho) = \hat{f}(\rho) = \sum_{s \in g} f(s)\rho(s).$$

The Fourier inversion theorem is then

$$f(s) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} \dim_\rho trace\left(\hat{f}(\rho)\rho(s^{-1})\right),$$

and the convolution theorem is

$$\mathcal{F}(f * g)(\rho) = \mathcal{F} f(\rho) \cdot \mathcal{F} g(\rho),$$

where the multiplication on the right is matrix multiplication.

For $G = \mathbb{Z}/k\mathbb{Z}$, all of the irreducible representations are one-dimensional and are defined by $\rho_j = \exp(2\pi ijn/k)$, where $0 \le j < k - 1$ and $n \in \mathbb{Z}/k\mathbb{Z}$ with the corresponding Fourier transform is as given in Theorem 2.1.

***Corollary* 2.3:** By setting $n = 0$ we get: $\sum_{m=0}^{k-1} f(-m)g(m) = \frac{1}{k}\sum_{j=0}^{k-1} \hat{f}(j)\hat{g}(j)$. أ

Thus for two trigonometric functions $\hat{f}(j)$ and $\hat{g}(j)$, with Fourier coefficients $f(m)$ and $g(m)$ respectively, one can find the sum of their product by evaluating the sum of the product of these coefficients.

Since every trigonometric function is either odd or even, the following theorem will prove to be useful.

Theorem 2.4: Let $f(n) = \frac{1}{k}\sum_{j=0}^{k-1} \hat{f}(j)\omega^{nj}$, then $f(n)$ is odd if and only if $\hat{f}(j)$ is odd, and $f(n)$ is even if and only if $\hat{f}(j)$ is even.

Proof: $f(-n) = \frac{1}{k}\sum_{j=0}^{k-1} \hat{f}(j)\omega^{-nj} = \frac{1}{k}\sum_{j=0}^{k-1} \hat{f}(-j)\omega^{nj}$, thus $f(-n) = -f(n)$ if and only if $\hat{f}(-j) = -\hat{f}(j)$ and $f(-n) = f(n)$ if and only if $\hat{f}(-j) = \hat{f}(j)$. أ

By combining Corollary 2.3 and Theorem 2.4 we get the following.

***Corollary* 2.5:** Let $f(n)$ and $g(n)$ be defined as in Theorem 2.2, then

(i) if $\hat{f}(j)$ is odd, $\frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\hat{g}(j) = -\sum_{m=0}^{k-1} f(m)g(m)$ and

(ii) if $\hat{f}(j)$ is even, $\frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\hat{g}(j) = \sum_{m=0}^{k-1} f(m)g(m).$

As a starting point the following Lemmas 2.6 through 2.9 determine the Fourier coefficients for four of the basic trigonometric functions. Throughout the rest of the paper we will define $\omega = e^{2\pi i/k}$. We will also be using Euler's identity, $e^{i\theta} = \cos\theta + i\sin\theta$, to rewrite the trigonometric functions in terms of ω.

Lemma 2.6:

Let $1 \le a < k, a \ne k/2$, *and* $f(n) = \begin{cases} 1/2 & \text{if } k \mid a+n, \\ -1/2 & \text{if } k \mid a-n, \\ 0 & \text{otherwise,} \end{cases}$ *then* $f(n) = \frac{1}{k}\sum_{j=0}^{k-1}(i\sin\frac{2\pi aj}{k})\omega^{nj}$.

Proof: $\frac{1}{k}\sum_{j=0}^{k-1}(i\sin\frac{2\pi aj}{k})\omega^{nj} = \frac{1}{2k}\sum_{j=0}^{k-1}(\omega^{aj} - \omega^{-aj})\omega^{nj} = \frac{1}{2k}\sum_{j=0}^{k-1}(\omega^{(a+n)j} - \omega^{(n-a)j})$.

If $k \mid a+n$ and $k \mid a-n$, then $k \mid 2a$. Since $1 \le a < k$, this implies $k = 2a$, which contradicts the hypothesis. Therefore, k can never divide both $a + n$ and $a - n$. Since $\sum_{j=0}^{k-1}\omega^{mj}$ vanishes unless $k|m$, in which case it is equal to k, then

$$\frac{1}{k}\sum_{j=0}^{k-1}(i\sin\frac{2\pi aj}{k})\omega^{nj} = \begin{cases} 1/2 & \textit{if } k \mid a+n, \\ -1/2 & \textit{if } k \mid a-n, \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 2.7: *Let* $1 \le a < k, a \ne k/2$, *and* $f(n) = \begin{cases} 1/2 & \textit{if } k \mid a+n \textit{ or } k \mid a-n, \\ 0 & \textit{otherwise,} \end{cases}$ *then*

$$f(n) = \frac{1}{k}\sum_{j=0}^{k-1}(\cos\frac{2\pi aj}{k})\omega^{nj}.$$

The proof follows with the same methods as Lemma 2.

Lemma 2.8: *For k odd, let* $f(n) = \left(\left(\frac{n}{k}\right)\right) := \begin{cases} \left\{\frac{n}{k}\right\} - \frac{1}{2} & \textit{if } k \nmid n, \\ 0 & \textit{otherwise,} \end{cases}$

with $\{x\} = x - \lfloor x \rfloor$, *then* $f(n) = \frac{1}{k}\sum_{j=1}^{k-1}(\frac{i}{2}\cot\frac{\pi j}{k})\omega^{nj}$.

Proof: From Theorem 2.1, we know that $f(n)$ has a finite Fourier series

$$\left(\left(\frac{n}{k}\right)\right) = \sum_{j=0}^{k-1}\hat{f}(j)\omega^{nj} \quad \text{where} \quad \hat{f}(j) = \frac{1}{k}\sum_{n=0}^{k-1}\left(\left(\frac{n}{k}\right)\right)\omega^{-nj}.$$

We first compute $\hat{f}(0)=\frac{1}{k}\sum_{n=0}^{k-1}\left(\left(\frac{n}{k}\right)\right)=\frac{1}{k}\sum_{n=1}^{k-1}\left(\frac{n}{k}-\frac{1}{2}\right)=\frac{1}{k}\left(\frac{k-1}{2}-\frac{k-1}{2}\right)=0$.

For $j\neq 0$ we have $\hat{f}(j)=\frac{1}{k}\sum_{n=1}^{k-1}\left(\frac{n}{k}-\frac{1}{2}\right)\omega^{-nj}=\left(\frac{1}{k^2}\sum_{n=1}^{k-1}n\left(\omega^{-j}\right)^n\right)-\frac{1}{2k}\sum_{n=1}^{k-1}\omega^{-nj}.$ (2.8.1)

In order to simplify the first term on the right side of (2.8.1), we start with the identity $\sum_{n=0}^{k-1}x^n=\frac{1-x^k}{1-x}$, and then to both sides of this equality we first differentiate with respect to x and then multiply by x, arriving at $\sum_{n=1}^{k-1}nx^n=\frac{-kx^k}{(1-x)}+\frac{(1-x^k)}{(1-x)^2}$. Since $(\omega^{-j})^k=\omega^{-jk}=1$, then

$$\frac{1}{k^2}\sum_{n=1}^{k-1}n\left(\omega^{-j}\right)^n=\frac{1}{k^2}\cdot\frac{-k}{(1-\frac{1}{\omega^j})}=\frac{1}{k}\cdot\frac{\omega^j}{(1-\omega^j)}. \tag{2.8.2}$$

To simplify the second term on the right side of (2.8.1), note that $\sum_{n=0}^{k-1}\omega^{-nj}=0$, thus

$$-\frac{1}{2k}\sum_{n=1}^{k-1}\omega^{-nj}=-\frac{1}{2k}\left(\sum_{n=0}^{k-1}\omega^{-nj}-1\right)=\frac{1}{2k}. \tag{2.8.3}$$

Therefore, by combining (2.8.1), (2.8.2) and (2.8.3),

$$\hat{f}(j)=\frac{1}{k}\left(\frac{\omega^j}{1-\omega^j}+\frac{1}{2}\right)=\frac{1}{2k}\cdot\frac{1+\omega^j}{1-\omega^j}=\frac{i}{2k}\cot\frac{\pi j}{k}.$$ أ

Lemma 2.9: *For k odd, let* $f(n) = \begin{cases} (-1)^{n \bmod k} & \text{if } k \nmid n, \\ 0 & \text{otherwise}, \end{cases}$ *then*

$$f(n) = \frac{1}{k}\sum_{n=0}^{k-1}(i \tan\frac{\pi j}{k})\omega^{nj}.$$

Proof: From Theorem 2.1, $f(n)$ has a finite Fourier series

$$f(n) = \sum_{j=0}^{k-1}\hat{f}(j)\omega^{nj}, \quad \text{where} \quad \hat{f}(j) = \frac{1}{k}\sum_{n=0}^{k-1} f(n)\omega^{-nj}.$$

Since $f(0) = 0$, $\widehat{f}(j) = \frac{1}{k}\sum_{n=1}^{k-1}(-1)^n\omega^{-nj} = \frac{1}{k}\sum_{n=1}^{k-1}(-\omega^{-j})^n = \frac{1}{k}\left(\frac{1-(-\omega^{-j})^k}{1+\omega^{-j}} - 1\right)$.

And since k is odd, $(-\omega^{-j})^k = -\omega^{-kj} = -1$, which gives

$$\hat{f}(j) = \frac{1}{k}\left(\frac{2}{1+\omega^{-j}} - 1\right) = \frac{1}{k}\left(\frac{2\omega^j}{1+\omega^j} - 1\right) = \frac{1}{k}\cdot\frac{\omega^j - 1}{1+\omega^j}.$$

$$= \frac{i}{k}\tan\frac{\pi j}{k},$$

where the last step comes from the fact that

$$\tan\frac{\pi j}{k} = \frac{1}{i}\cdot\frac{e^{\pi ji/k} - e^{-\pi ji/k}}{e^{\pi ji/k} + e^{-\pi ji/k}} = \frac{1}{i}\cdot\frac{e^{2\pi ji/k} - 1}{e^{2\pi ji/k} + 1} = \frac{1}{i}\cdot\frac{\omega^j - 1}{\omega^j + 1}.$$ أ

Chapter 3

Explicit Evaluations of Finite Trigonometric Sums Using Discrete Fourier Transformations

Using the theorems, corollaries and lemmas from chapter 2, the first part of this chapter finds closed forms for the finite sums of functions which are products of two of the elementary trigonometric functions, sine, cosine, tangent, cotangent, cosecant and secant. Certain sums of trigonometric products, such as $\sum_{j=0}^{k-1}\cos(2\pi aj/k)\cot(\pi j/k)$ are not included since they are trivially equal to 0. The second part of this chapter investigates trigonometric sums of powers higher than two.

As a starting point, I used Berndt and Yeap's paper [5], which used contour integration to find closed forms for certain classes of trigonometric sums. I was able to prove some of their corollaries and I make the appropriate references where applicable. In their paper they make reference to other publications in which these identities were proven. Where applicable I refer only to those corollaries from [5]. For further references please see [5].

In addition to proving identities previously known, I investigated other sums. In the course of my investigations, I found some interesting results as seen in Theorems 3.14 and 3.16.

Lemma 3.1: Let $1 \le a < k$, $a \neq k/2$, then $\sum_{j=0}^{k-1} \sin^2 \frac{2\pi aj}{k} = \frac{k}{2}$.

Proof: Let $f(n) = \begin{cases} 1/2 & \text{if } k \mid a+n, \\ -1/2 & \text{if } k \mid a-n, \\ 0 & \text{otherwise.} \end{cases}$

thus by Lemma 2.6 and Corollary 2.5 (i); and since $f(m)$ vanishes except for $f(a) = -1/2$ and $f(k-a) = 1/2$, then

$$-\frac{1}{k}\sum_{j=0}^{k-1} \sin^2 \frac{2\pi aj}{k} = \frac{1}{k}\sum_{j=0}^{k-1}\left(i\sin\frac{2\pi aj}{k}\right)\left(i\sin\frac{2\pi aj}{k}\right) = -\sum_{m=0}^{k-1}\left(f(m)\right)^2 = -\left(\frac{1}{4}+\frac{1}{4}\right) = -\frac{1}{2}.$$ أ

Using a similar argument we get the next lemma.

Lemma 3.2: Let $1 \le a < k$, $a \neq k/2$, then $\sum_{j=0}^{k-1} \cos^2 \frac{2\pi aj}{k} = \frac{k}{2}$. أ

The next Lemma was proven by Berndt and Yeap [5, p.365] as a corollary to a general theorem.

Theorem 3.3: $$\sum_{j=1}^{k-1} \cot^2 \frac{\pi j}{k} = \frac{(k-2)(k-1)}{3}.$$

Proof: Let $$f(n) = \left(\left(\frac{n}{k}\right)\right) := \begin{cases} \left\{\frac{n}{k}\right\} - \frac{1}{2} & \textit{if } k \nmid n, \\ 0 & \textit{otherwise}, \end{cases}$$

then by Lemma 2.8 and Corollary 2.5 (i),

$$\begin{aligned} -\frac{1}{4k} \sum_{j=1}^{k-1} \cot^2 \frac{\pi j}{k} = \frac{1}{k} \sum_{j=1}^{k-1} \left(\frac{i}{2} \cot \frac{\pi j}{k} \right)^2 &= -\sum_{m=0}^{k-1} (f(m))^2 \\ &= -\sum_{m=1}^{k-1} \left(\frac{m}{k} - \frac{1}{2} \right)^2 \\ &= -\sum_{m=1}^{k-1} \left(\frac{m^2}{k} - \frac{m}{k} + \frac{1}{4} \right) \\ &= -\frac{(k-1)(2k-1)}{6k} - \frac{k-1}{2} + \frac{k-1}{4} \\ &= -\frac{(k-2)(k-1)}{12k}. \end{aligned}$$

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The next Lemma was given by Berndt and Yeap [5, p.368] as a corollary to a general theorem.

Lemma 3.4: *For k odd,* $\displaystyle\sum_{j=0}^{k-1} \tan^2 \frac{\pi j}{k} = k(k-1)$.

Proof: Let $f(n) = \begin{cases} (-1)^{n \bmod k} & \textit{if } k \nmid n, \\ 0 & \textit{otherwise}, \end{cases}$ then by Lemma 2.9 and Corollary 2.5 (i);

$$-\frac{1}{k}\sum_{j=0}^{k-1} \tan^2 \frac{\pi j}{k} = \frac{1}{k}\sum_{j=0}^{k-1}\left(i \tan \frac{\pi j}{k}\right)^2 = -\sum_{m=0}^{k-1} (f(m))^2$$

$$= -\sum_{m=1}^{k-1} (-1)^{2m}$$

$$= -(k-1).$$

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Lemma 3.5: *For k odd,* $\displaystyle\sum_{j=1}^{k-1} \csc^2 \frac{2\pi j}{k} = \frac{k^2-1}{3}$.

Proof: Using Lemmas 3.3 and 3.4 and the identity $\csc 2\theta = \frac{1}{2}(\tan\theta + \cot\theta)$,

$$\sum_{j=0}^{k-1} \csc^2 \frac{2\pi j}{k} = \sum_{j=1}^{k-1}\left(\frac{1}{2}(\cot\frac{\pi j}{k} + \tan\frac{\pi j}{k})\right)^2$$

$$= \frac{1}{4}\sum_{j=1}^{k-1}\cot^2\frac{\pi j}{k} + \frac{1}{4}\sum_{j=1}^{k-1}\tan^2\frac{\pi j}{k} + \frac{1}{2}\sum_{j=1}^{k-1} 1$$

$$= \frac{(k-1)(k-2)}{12} + \frac{k(k-1)}{4} + \frac{k-1}{2}$$

$$= \frac{k^2-1}{3}.$$

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Lemma 3.6: *For k odd,* $\sum_{j=0}^{k-1} \sec^2 \frac{2\pi j}{k} = k^2$.

Proof: $\sum_{j=0}^{k-1} \sec^2 \frac{2\pi j}{k} = \sum_{j=0}^{k-1} (1 + \tan^2 \frac{2\pi j}{k}) = k + \sum_{j=0}^{k-1} \tan^2 \frac{2\pi j}{k} = k + \sum_{j=0}^{k-1} \tan^2 \frac{\pi j}{k}$,

where the last step is due to the fact that k is odd. Thus using Lemma 3.4,

$$\sum_{j=0}^{k-1} \sec^2 \frac{2\pi j}{k} = k + k(k-1) = k^2 .$$

The next Lemma was given by Berndt and Yeap [5, p.375] as a corollary to a general theorem.

Lemma 3.7: *For* $1 \leq a < k$, $\sum_{j=1}^{k-1} \cot \frac{\pi j}{k} \sin \frac{2\pi aj}{k} = k - 2a$.

Proof: If $a = k/2$, then $\sin \frac{2\pi aj}{k} = \sin \pi j = 0$, thus $\sum_{j=0}^{k-1} \cot \frac{\pi j}{k} \sin \frac{2\pi aj}{k} = 0 = k - 2(k/2)$.

If $a \neq k/2$, then let $f(n) = \left(\left(\frac{n}{k}\right)\right)$ and $g(n) = = \begin{cases} 1/2 & \textit{if } k \mid a+n, \\ -1/2 & \textit{if } k \mid a-n, \\ 0 & \textit{otherwise.} \end{cases}$

Thus by Lemma 2.6, Lemma 2.8 and Corollary 2.5 (i),

$$-\frac{1}{2k}\sum_{j=0}^{k-1}\cot\frac{\pi j}{k}\sin\frac{2\pi aj}{k}=\frac{1}{k}\sum_{j=1}^{k-1}(\frac{i}{2}\cot\frac{\pi j}{k})(i\sin\frac{2\pi aj}{k})=-\sum_{m=0}^{k-1}f(m)g(m).$$

Since $g(m)$ vanishes except for $g(a) = -1/2$ and $g(k-a) = 1/2$,

$$-\frac{1}{2k}\sum_{j=1}^{k-1}\cot\frac{\pi j}{k}\sin\frac{2\pi aj}{k}=-\left(\frac{a}{k}-\frac{1}{2}\right)\left(-\frac{1}{2}\right)-\left(\frac{k-a}{k}-\frac{1}{2}\right)\left(\frac{1}{2}\right)=-\frac{k-2a}{2k}.$$ أ

Lemma 3.8: *For k odd, 1≤ a<k,* $\displaystyle\sum_{j=0}^{k-1}\tan\frac{\pi j}{k}\sin\frac{2\pi aj}{k}=(-1)^{a+1}k$.

Proof: Let $f(n)=\begin{cases}(-1)^{n \bmod k} & \text{if } k\nmid n,\\ 0 & \text{otherwise},\end{cases}$ and $g(n)=\begin{cases}1/2 & \text{if } k\mid a+n,\\ -1/2 & \text{if } k\mid a-n,\\ 0 & \text{otherwise},\end{cases}$

then by Lemma 2.6, Lemma 2.9 and Corollary 2.5 (i),

$$-\frac{1}{k}\sum_{j=0}^{k-1}\tan\frac{\pi j}{k}\sin\frac{2\pi aj}{k}=\frac{1}{k}\sum_{j=0}^{k-1}(i\tan\frac{\pi j}{k})(i\sin\frac{2\pi aj}{k})=-\sum_{m=0}^{k-1}f(m)g(m).$$

Since $g(m)$ vanishes except for $g(a) = -1/2$ and $g(k-a) = 1/2$ we get

$$-\frac{1}{k}\sum_{j=0}^{k-1}\tan\frac{\pi j}{k}\sin\frac{2\pi aj}{k}=-\frac{(-1)^{a+1}}{2}-\frac{(-1)^{k-a}}{2}=-(-1)^{a+1},$$

where the last step comes from the fact that since k is odd, $(-1)^{a+1}=(-1)^{k-a}$. أ

Lemma 3.9: *For k odd, 1 ≤ a < k,* $\sum_{j=0}^{k-1} \sin\frac{2\pi aj}{k}\csc\frac{2\pi j}{k} = \begin{cases} k-a & \text{if } a \text{ is odd}, \\ -a & \text{if } a \text{ is even.} \end{cases}$

Proof: Using the identity $\csc 2\theta = (\cot\theta + \tan\theta)/2$ and Lemmas 3.7 and 3.8,

$$\sum_{j=1}^{k-1} \sin\frac{2\pi ak}{k}\csc\frac{\pi j}{k} = \frac{1}{2}\sum_{j=1}^{k-1} \sin\frac{2\pi aj}{k}\left(\cot\frac{\pi j}{k} + \tan\frac{\pi j}{k}\right)$$

$$= \frac{1}{2}\left[\left(k-2a\right) + (-1)^{a+1}k\right]$$

$$= \begin{cases} k-a & \text{if } a \text{ is odd}, \\ -a & \text{if } a \text{ is even.} \end{cases}$$ أ

Lemma 3.10: *For k odd,* $\sum_{j=1}^{k-1} \tan\frac{\pi j}{k}\csc\frac{2\pi j}{k} = \frac{k^2-1}{2}$.

Proof: Using the identity $\csc 2\theta = \frac{1}{2}(\cot\theta + \tan\theta)$ and Lemma 3.4,

$$\sum_{j=1}^{k-1} \tan\frac{\pi j}{k}\csc\frac{2\pi j}{k} = \frac{1}{2}\sum_{j=1}^{k-1} \tan\frac{\pi j}{k}\left(\tan\frac{\pi j}{k} + \cot\frac{\pi j}{k}\right) = \frac{1}{2}\sum_{j=1}^{k-1}(\tan^2\frac{\pi j}{k} + 1)$$

$$= \frac{k(k-1)}{2} + \frac{k-1}{2} = \frac{k^2-1}{2}.$$ أ

Lemma 3.11: *For k odd,* $\sum_{j=1}^{k-1} \cot\frac{\pi j}{k}\csc\frac{2\pi j}{k} = \frac{k^2-1}{6}$.

Proof: Using the identity $\csc 2\theta = \frac{1}{2}(\cot\theta + \tan\theta)$ and Lemma 3.3,

$$\sum_{j=1}^{k-1} \cot\frac{\pi j}{k}\csc\frac{2\pi j}{k} = \frac{1}{2}\sum_{j=1}^{k-1} \cot\frac{\pi j}{k}(\cot\frac{\pi j}{k} + \tan\frac{\pi j}{k}) = \frac{1}{2}\sum_{j=1}^{k-1}(\cot^2\frac{\pi j}{k} + 1)$$

$$= \frac{1}{2}\left(\frac{(k-2)(k-1)}{3} + k - 1\right) = \frac{k^2-1}{6}.$$

This next section determines explicit evaluations for sums involving higher powers of trigonometric functions. For Theorems 3.12 and 3.13, the Fourier Transformation for $\tan^2(\pi j/k)$ and $\cot^2(\pi j/k)$ were found by using Theorem 2.2, and then Corollary 2.5 (ii) was used in order to find the sums for $\tan^4(\pi j/k)$ and $\cot^4(\pi j/k)$.

Theorem 3.12: *For k odd,* $\sum_{j=0}^{k-1} \tan^4\frac{\pi j}{k} = \frac{k(k-1)(k^2+k-3)}{3}$.

Proof: Let $g(n) = \frac{1}{k}\sum_{j=0}^{k-1} \tan^2\frac{\pi j}{k}\omega^{nj}$. From Lemma 2.9 and Theorem 2.2, if we let

$$f(n) = \begin{cases} (-1)^{n \bmod k} & \text{if } k \nmid n, \\ 0 & \text{otherwise.} \end{cases} \quad \text{then} \quad g(n) = \sum_{m=0}^{k-1} f(n-m)f(m), \qquad (3.12.1)$$

Since $f(n)$ has period k, $g(n)$ has period k as well. Thus for simplicity we will assume $0 \leq n < k$.

For $n = 0$, by Lemma 3.4 we have $g(0) = k - 1$. (3.12.2)

For $n \neq 0$, since $f(0) = 0$,

$$g(n) = \sum_{m=1}^{n-1} f(n-m)f(m) + \sum_{m=n+1}^{k-1} f(n-m)f(m). \tag{3.12.3}$$

By definition of f, and since k is odd by hypothesis, $f(n)$ is not necessarily equal to $(-1)^n$ for all $n \not\equiv 0 \bmod k$. However, for $0 < n < k$, we have $f(n) = (-1)^n$. Keeping this in mind we get the following:

Since $0 < m < k$, then $f(m) = (-1)^m$.

For $0 < m < n$, we have $0 < n - m < k$, thus $f(n-m) = (-1)^{n-m}$.

For $n < m < k$, we have $-k < n - m < 0$, or equivalently $0 < m - n < k$. Since tangent is an odd function, f is odd (Theorem 2.4), thus $f(n-m) = -f(m-m) = -(-1)^{m-n}$.

Therefore (3.12.3) becomes,

$$\begin{aligned} g(n) &= \sum_{m=1}^{n-1} (-1)^{n-m}(-1)^m - \sum_{m=n+1}^{k-1} (-1)^{m-n}(-1)^m \\ &= (-1)^n \left(\sum_{m=1}^{n-1} 1 - \sum_{m=n+1}^{k-1} 1 \right) \\ &= (-1)^n (n-1-(k-1-n) \\ &= (-1)^n (2n-k). \end{aligned} \tag{3.12.4}$$

Combining (3.12.2) and (3.12.4),

$$g(n) = \begin{cases} (-1)^{(n+1)\bmod k}(k - 2(n \bmod k) & \textit{if } n \not\equiv 0 \bmod k, \\ k-1 & \textit{otherwise.} \end{cases}$$

Now we are ready to use Corollary 2.5 (i),

$$\frac{1}{k}\sum_{j=0}^{k-1} \tan^4 \frac{\pi j}{k} = \sum_{m=0}^{k-1} (g(m))^2$$

$$= (k-1)^2 + \sum_{m=1}^{k-1} (k-2m)^2$$

$$= \frac{(k-1)(k^2+k-3)}{3}.$$

The next Theorem was given by Berndt and Yeap [5, p. 365].

Theorem 3.13: $$\sum_{j=1}^{k-1} \cot^4 \frac{\pi j}{k} = \frac{(k-1)(k-2)(k^2+3k-13)}{45}.$$

Proof: Let $g(n) = \frac{1}{4k}\sum_{j=1}^{k-1} \cot^2 \frac{\pi j}{k} \omega^{nj}$, then by Lemma 2.8, Theorem 2.1 and since $\left(\left(0\right)\right) = 0$,

$$g(n) = \frac{1}{4k}\sum_{j=1}^{k-1} \cot^2 \frac{\pi j}{k} = -\frac{1}{k}\sum_{j=1}^{k-1}\left(\frac{i}{2}\cot\frac{\pi j}{k}\right)^2 = -\sum_{m=0}^{k-1}\left(\left(\frac{n-m}{k}\right)\right)\left(\left(\frac{m}{k}\right)\right).$$

Since $g(n)$ has period k, will assume for simplicity that $0 \le n < k$.

From Lemma 3.3, $g(0) = \dfrac{(k-1)(k-2)}{12k}$. (3.13.1)

For $n \neq 0$, if $m = n$, $\left(\left(\dfrac{n-m}{k}\right)\right) = 0$, and

if $m < n$, $\left(\left(\dfrac{n-m}{k}\right)\right) = \dfrac{n}{k} - \dfrac{m}{k} - \dfrac{1}{2}$, and

if $m > n$, $\left(\left(\dfrac{n-m}{k}\right)\right) = \left(\left(\dfrac{n-m+k}{k}\right)\right) = \dfrac{n}{k} - \dfrac{m}{k} - \dfrac{1}{2} + 1$.

Thus,

$$g(n) = -\left(\sum_{m=1}^{k-1}\left(\frac{n}{k} - \frac{m}{k} - \frac{1}{2}\right)\left(\frac{m}{k} - \frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{n}{k} - \frac{1}{2}\right) + \sum_{m=n+1}^{k-1}\left(\frac{m}{k} - \frac{1}{2}\right)\right)$$
$$= \frac{6n(n-k) + 2 + k^2}{12k}.$$

(3.13.2)

Combining (3.13.1) and (3.13.2), then

$$g(n) = \begin{cases} \dfrac{(k-1)(k-2)}{12k} & \textit{if } n \equiv 0 \bmod k, \\ \dfrac{6(n \bmod k)((n \bmod k) - k) + 2 + k^2}{12k} & \textit{otherwise.} \end{cases} \quad (3.13.3)$$

Now using Corollary 2.5 (i),

$$\frac{1}{16k}\sum_{j=1}^{k-1}\cot^4\frac{\pi j}{k}=\sum_{m=0}^{k-1}(g(m))^2$$

$$=\left(\frac{(k-1)(k-2)}{12k}\right)^2+\sum_{m=1}^{k-1}\left(\frac{6m(m-k)+2+k^2}{12k}\right)^2$$

$$=\frac{(k-1)(k-2)(k^2+3k-13)}{720k}.$$

Note the increased complexity of the closed form for each of the two above sums as compared to Lemmas 3.4 and 3.5 which involve only powers of two. This leads one to believe that to try to determine a general formula for the sum of the functions $\sum_{j=1}^{k-1}\tan^{2a}(\pi j/k)$ and $\sum_{j=1}^{k-1}\cot^{2a}(\pi j/k)$ for any integer a using the theorems and corollaries from Chapter 2 would be difficult.

Using induction on Theorem 2.2 to find an explicit evaluation for any function $(f(n))^a$, it is easy to show

$$\sum_{j=0}^{k-1}\left(f(j)\right)^a=k\sum_{0\le n_1,n_2,\ldots n_{a-1}\le k-1}\hat{f}(-n_1-n_2-\cdots-n_{a-1})\hat{f}(n_1)\hat{f}(n_2)\cdots\hat{f}(n_{a-1})$$

where the $\hat{f}(j)$ are the Fourier coefficients for f. Thus it is clear that this method would prove to be very complicated, if not impossible, to find closed forms when the functions are tangent and cotangent.

Chu and Marini investigated sums of these types [8] using generating functions and found the formulas

$$\sum_{j=0}^{k-1}\tan^{2p}\frac{\pi j}{k}=k(-1)^p+(-1)^p\sum_{m=0}^{2p}\frac{k}{2^m}\sum_{r=0}^{m}\binom{m}{r}\times(-1)^r\sum_{l=1}^{p}\binom{2kr}{2l-1}\binom{kr+p-l}{p-l}, \qquad [8, p.131]$$

$$\begin{aligned}\sum_{j=1}^{k-1}\cot^{2p}\frac{\pi j}{k}=k(-1)^p-\frac{1}{2}\sum_{m=0}^{2p}\frac{(-1)^p}{(2k)^m}\binom{1+2p}{1+m}\sum_{k=0}^{m+1}\binom{1+m}{k}\\ \times(-1)^n\frac{1+m-2n}{1+m}\sum_{l\geq 0}\binom{kn+l}{l}\binom{2kn}{m+2p-2l},\end{aligned} \qquad [8, p.137]$$

and Berndt and Yeap, using contour integration, found the formula

$$\frac{1}{k}\sum_{j=1}^{k-1}\cot^{2a}\left(\frac{\pi j}{k}\right)=(-1)^a-(-1)^a2^{2a}\sum_{\substack{j_0,j_1,\dots,j_{2a}\geq 0\\ j_0+j_1+\dots+j_{2a}=a}}k^{2j_0-1}\prod_{r=0}^{2r}\frac{B_{2j_r}}{(2j_r)!}, \qquad [5, p.363]$$

where B_j, $j\geq 0$ denotes the j^{th} Bernoulli number.

However, for sums involving higher even positive powers of sine and cosine the Fourier transformations are relatively easy to calculate. For certain positive integers a, b and k and any real number x, I found some interesting results.

Theorem 3.14: *For a, b positive integers and* $1\leq ab<k$, *and for all real numbers x,*

$$\sum_{j=0}^{k-1}\sin^{2a}\left(\frac{b\pi j}{k}+x\right)=\frac{k}{2^{2a}}\binom{2a}{a}.$$

Proof: Let $f_{a,b,x}(n):=\frac{1}{k}\sum_{j=0}^{k-1}\left(i\sin\left(\frac{b\pi j}{k}+x\right)\right)^a\omega^{nj}$, then

$$f_{a,b,x}(n) = \frac{1}{2^a k}\sum_{j=0}^{k-1}(\omega^{bj/2}e^{xi} - \omega^{-bj/2}e^{-xi})^a\,\omega^{nj}$$

$$= \frac{1}{2^a k}\sum_{j=0}^{k-1}\sum_{m=0}^{a}(-1)^m\binom{a}{m}\omega^{(a-m)bj/2}e^{(a-m)xi}\omega^{-mbj/2}e^{-mxi}\omega^{nj}$$

$$= \frac{1}{2^a}\sum_{m=0}^{a}(-1)^m\binom{a}{m}e^{(a-2m)xi}\left(\frac{1}{k}\sum_{j=0}^{k-1}\omega^{(n+(a-2m)b/2)j}\right).$$

Thus, $$f_{2a,b,x}(n) = \frac{1}{2^{2a}}\sum_{m=0}^{2a}(-1)^m\binom{2a}{m}e^{2xi(a-m)}\left(\frac{1}{k}\sum_{j=0}^{k-1}\omega^{j(n+b(a-m))}\right).$$

Since $\sum_{j=0}^{k-1}\omega^{jn} = \begin{cases} k & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n, \end{cases}$ then $f_{2a,b,x}(n) = \frac{1}{2^{2a}}\sum_{\substack{0\le m\le 2a,\\ k\mid n+b(a-m)}}(-1)^m\binom{2a}{m}.$

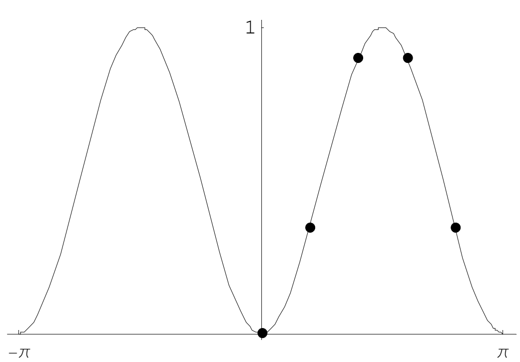
Since $0 \le m \le 2a$, then $-ab \le b(a-m) \le ab$. And since $ab < k$, then $k \mid b(a-m)$ only when $m = a$. Therefore,

$$\frac{(-1)^a}{k}\sum_{j=0}^{k-1}\sin^{2a}\frac{2\pi j}{k} = f_{2a,b,x}(0) = \frac{(-1)^a}{2^{2a}}\binom{2a}{a}.$$ أ

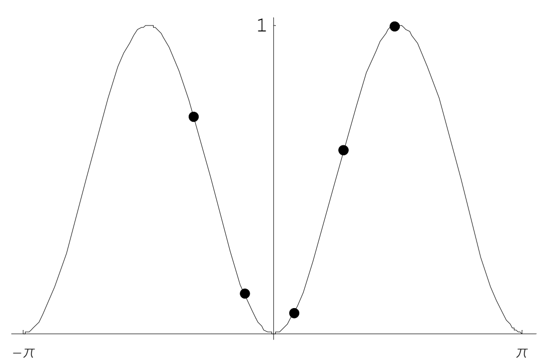
It should be noted that there are sometimes other k's less than ab such that the theorem still holds. For example, if $a = 4$, $b = 2$, and $k = 5$, then $-8 \le 2(a - m) \le 8$, which means $5 \mid 2a - 2m$ only when $m = 4$. But for $a = 4$, $b = 2$, and $k = 6$, then

$6 \mid 2(4 - m)$ when $m = 1, 4$ and 7. There are many other examples in which the Lemma holds (or does not hold), but in order to eliminate all of the many special conditions for a, b and k, I chose to use $1 \le ab < k$, which is the simplest condition that always ensures the resulting conclusion.

According to this theorem, then for any interval of length $b\pi$, if we evaluate the function $f(x) = \sin^{2a}(bx)$ at k equally spaced values over this interval (provided that $1 \le ab < k$), then the sum is a constant. As an illustration, consider the function $f(x) = \sin^2(x)$, evaluated at 5 equally spaced values over an interval of length π. In Graph a we are evaluating $\sin^2(x)$ over the interval $[0, \pi]$ and in graph b we are evaluating $\sin^2(x)$ over the interval $[-1, \pi - 1]$.



Graph a



Graph b

The values for $\sin^2(x)$ are different in the two graphs; however, their sums are equal.

***Corollary* 3.15:** *For a, b positive integers and* $1 \le ab < k$*, and for all real numbers x, then*

$$\sum_{j=0}^{k-1} \cos^{2a}\left(\frac{b\pi j}{k} + x\right) = \frac{k}{2^{2a}}\binom{2a}{a}.$$

Proof: Replace x with $x + \dfrac{\pi}{2}$ in Theorem 3.14. أ

Up to this point Fourier analysis was not required in order to find the closed forms for sums of even powers of sines and cosines. However, by using the Fourier transformation for $\sin^a(2\pi j/k)$ along with Corollary 2.5, this leads to a proof of a well known identity.

Theorem 3.16: $\displaystyle\sum_{m=0}^{a}\binom{a}{m}^2 = \binom{2a}{a}.$

Proof: Given any $a \ge 1$, choose a k such that $2a < k$. Let $f(n) = \dfrac{1}{k}\displaystyle\sum_{j=0}^{k-1}\left(i\sin\frac{2\pi j}{k}\right)^a \omega^{nj}$.

In the proof for Lemma 3.14 it was shown that $f(n) = f_{a,2,0}(n) = \dfrac{1}{2^a}\displaystyle\sum_{\substack{0\le m\le a,\\ k\mid n+a-2m}} (-1)^m\binom{a}{m}$.

Since $0 \le m \le a$ and $1 \le 2a < k$, then $-k/2 < a - 2m < k/2$. Thus given any n, there exists at most one m such that $k \mid n + a - 2m$. Therefore,

$$f(n)=\begin{cases}(-1)^m\binom{a}{m}\Big/2^a & \textit{if } k\mid n+a-2m,\ \textit{for some } 0\le m\le a,\\ 0 & \textit{otherwise.}\end{cases}$$

Since $\sin^a(-\theta)=(-1)^a\sin(\theta)$, then by Theorem 2.4, $f(-n)=(-1)^a f(n)$.

Now since for all m such that $0 \le m < a/2$, when $n = k - (a - 2m)$, then $n + a - 2m = k$, and for all m such that $a/2 \le m \le a$, when $n = -(a - 2m)$, then $n + a - 2m = 0$. This means given any $m = 0, 1, 2, \ldots, a$, there exists an n between 0 and $k - 1$ such that $k \mid n + a - 2m$. Therefore, from Theorem 3.14 and Corollary 2.3,

$$\frac{(-1)^a\binom{2a}{a}}{2^{2a}}=\frac{(-1)^a}{k}\sum_{j=0}^{k-1}\sin^{2a}\frac{2\pi j}{k}=\sum_{n=0}^{k-1}f(-n)f(n)=\frac{(-1)^a}{2^{2a}}\sum_{m=0}^{a}\binom{a}{m}^2.$$

***Corollary* 3.18**: $$\binom{2a}{a}=\sum_{m=0}^{a}(-1)^m 2^{2(a-m)}\binom{2m}{m}\binom{a}{m}.$$

Proof: Given any b, choose a k such that $2a < k$. Then from Lemmas 3.14 and 3.15 we get:

$$\sum_{j=0}^{k-1}\cos^{2a}\frac{2\pi j}{k}=\sum_{j=0}^{k-1}\sin^{2a}\frac{2\pi j}{k}=k\binom{2a}{a}\Big/2^{2a}.$$

Since $\cos^{2a}\frac{2\pi j}{k}=(1-\sin^{2}\frac{2\pi j}{k})^{a}=\sum_{m=0}^{a}(-1)^{m}\binom{a}{m}\sin^{2m}\frac{2\pi j}{k}$, then

$$\frac{k\binom{2a}{a}}{2^{2a}}=\sum_{j=0}^{k-1}\cos^{2a}\frac{2\pi j}{k}=\sum_{m=0}^{a}(-1)^{m}\binom{a}{m}\frac{k\binom{2m}{m}}{2^{2m}}.$$ أ

Chapter 4

Characters

The two trigonometric identities

$$\frac{\sin(2\pi/7)}{\sin^2(3\pi/7)} - \frac{\sin(\pi/7)}{\sin^2(2\pi/7)} + \frac{\sin(3\pi/7)}{\sin^2(\pi/7)} = 2\sqrt{7} \tag{4.1}$$

and

$$\frac{\sin^2(3\pi/7)}{\sin(2\pi/7)} - \frac{\sin^2(2\pi/7)}{\sin(\pi/7)} + \frac{\sin^2(\pi/7)}{\sin(3\pi/7)} = 0 \tag{4.2}$$

were presented as corollaries in Berndt and Zahrescu's [6] evaluation of large classes of trigonometric sums in terms of class numbers of imaginary quadratic fields. These identities involve the sums of products of specific characters and trigonometric functions. As in chapter 3, the question was posed whether the methods of discrete Fourier Analysis could be used in order to prove some of these identities. In some instances I was successful. In addition I discovered identities of my own.

But prior to presenting these theorems it is necessary to provide some background information on characters. Definitions and theorems necessary to the subsequent chapter are provided. This thesis is not meant to be an in depth investigation into characters, thus only the results of some of the theorems are presented, with references for where their proofs can be found, as they are beyond the scope of this paper.

Definition: A *character* χ (mod *k*) is a nonzero map from ¢ / *k*¢ into the complex numbers such that:

$$\chi(ab) = \chi(a)\chi(b) \text{ and } \chi(a) = 0 \text{ when } \gcd(a, k) > 1.$$

By definition χ (mod *k*) is a group homomorphism on $(\mathbb{Z}/k\mathbb{Z})^*$, thus it must map the identity element in $(\mathbb{Z}/k\mathbb{Z})^*$ to the identity element in £ ; that is χ(1) = 1. In addition, since for all *a* in $(\mathbb{Z}/k\mathbb{Z})^*$, $a^{\phi(k)} = 1$, where $\phi(k)$ is the totient function, then χ must be a $\phi(k)^{th}$ root of unity. Thus, if χ is real valued, then the only values that χ can assume are 0, 1, and −1.

Definition: The *principal character* is the function that maps every element relatively prime to *k* to 1. If χ is *nonprincipal* this means there exists an element *a* in $(\mathbb{Z}/k\mathbb{Z})^*$ such that χ(*a*) ≠ 1.

Theorem 4.1: For all χ (mod *k*), $\sum_{j=0}^{k-1} \chi(j) = \begin{cases} \varphi(k) & \textit{if } \chi \textit{ is the principal character,} \\ 0 & \textit{otherwise.} \end{cases}$

Proof: For ease of notation let $G = (¢ / k¢)^*$. Then $\sum_{j=0}^{k-1} \chi(j) = \sum_{r \in G} \chi(r)$, since χ vanishes for all *j* such that gcd(*j*, *k*) > 1. If χ is the principal character then the theorem is obvious. If χ is a nonprincipal character then there exists an $a \in G$ such that $\chi(a) \neq 1$. For any $a \in G$,

$$\chi(a)\sum_{r\in G}\chi(r) = \sum_{r\in G}\chi(a)\chi(r) = \sum_{r\in G}\chi(ar) = \sum_{r\in G}\chi(r),$$

where the last step comes from the fact that for any a in G, $\{r : r \in G\} = \{ar : r \in G\}$. Thus,

$$(1-\chi(a))\sum_{r\in G}\chi(r) = 0.$$

Since $\chi(a) \neq 1$, this implies $\sum_{r\in G}\chi(r) = 0$.

Defintion: A character χ (mod k) is said to be *induced* by a character χ' (mod m') if $m' \mid m$ and $\chi(n) = \chi'(n)$. A character which is not induced by any other character is called *primitive*.

The next theorem establishes for which k there exist real, primitive characters and gives their explicit forms. This theorem refers to the Kronecker symbol, which is an extension of the Legendre symbol, which in turn is an extension of the Jacobi symbol; thus following this theorem are their definitions and some of their properties from which further theorems are derived.

Theorem 4.2: The only real, primitive characters are $\chi(n) = (\frac{d}{n})$, where $(\frac{d}{n})$ is the Kronecker symbol and d is a product of relatively prime factors of the form −4, 8, −8, $(-1)^{(p-1)/2}p$ $(p > 2)$; and the symbol is a real primitive character to the modulus $|d|$.

Proof: See [3, p. 40]

Definition: For odd prime p, the Legendre symbol is defined by

$$\left(\frac{n}{p}\right) = \begin{cases} 0 & \textit{if } p \mid n, \\ 1 & \textit{if } n \textit{ is a quadratic residue of } p, \\ -1 & \textit{if } n \textit{ is a quadratic nonresidue of } p, \end{cases}$$

where n is a quadratic residue if $x \equiv a^2 \bmod p$ is solvable, and a quadratic nonresidue otherwise.

An extension of the Legendre symbol to odd positive integers is the Jacobi Symbol.

Definition: Let $k = \prod p_i^{a_i}$, for p_i's odd primes. The Jacobi symbol $\left(\frac{n}{k}\right)$ is defined by

$$\left(\frac{n}{k}\right) = \prod \left(\frac{n}{p_i}\right)^{a_i}$$, where the factors $\left(\frac{n}{p_i}\right)$ are the Legendre symbols.

In the case where k is prime, the Jacobi symbol reduces to the Legendre symbol.

Definition: The Kronecker symbol is an extension of the Jacobi symbol $\left(\frac{n}{m}\right)$ to all integers *m* with the addition of definitions for $\left(\frac{n}{2}\right)$ and $\left(\frac{n}{-1}\right)$. These added definitions are not necessary to the subsequent theorems involving primitive characters, for in each of them we restrict *k* to be odd, thus $k = |d|$, where $d = \prod (-1)^{(p_i-1)/2} p_i$ for odd primes p_i; in which case the Kronecker symbol $\left(\frac{d}{n}\right)$ reduces to the Legendre symbol $\left(\frac{n}{|d|}\right)$.

As an illustration, if χ denotes a real, nonprimitve character mod *k*, then for $k = 5$, $\chi(1) = \chi(4) = 1$ and $\chi(2) = \chi(3) = -1$, and for $k = 7$, $\chi(1) = \chi(2) = \chi(4) = 1$ and $\chi(3) = \chi(5) = \chi(6) = -1$. Note that for $k = 5$, χ is even and for $k = 7$, χ is odd. It turns out that we can determine whether a character is odd or even by whether $k \equiv 1 \bmod 4$ or $k \equiv 3 \bmod 4$.

The Legendre symbol obeys the identities,

$$\left(\frac{n}{k}\right)\left(\frac{m}{k}\right) = \left(\frac{nm}{k}\right), \quad \text{and} \quad \left(\frac{-1}{k}\right) = (-1)^{(k-1)/2}, \text{ thus}$$

$$\left(\frac{-n}{k}\right) = \left(\frac{-1}{k}\right)\left(\frac{n}{k}\right) = (-1)^{(k-1)/2}\left(\frac{n}{k}\right) = \begin{cases} \left(\frac{n}{k}\right) & \text{if } k \equiv 1 \bmod 4, \\ -\left(\frac{n}{k}\right) & \text{if } k \equiv 3 \bmod 4. \end{cases}$$

This gives us the following theorem.

Theorem 4.3: For *k* odd, if χ denotes a nonprincipal, real, primitive character modulo *k*, then

$$\chi(-n) = \begin{cases} \chi(n) & \text{if } k \equiv 1 \bmod 4, \\ -\chi(n) & \text{if } k \equiv 3 \bmod 4. \end{cases}$$

Another identity of the Legendre Symbol is $\left(\frac{2}{k}\right) = \begin{cases} 1 & \text{for } k \equiv \pm 1 \bmod 8, \\ -1 & \text{for } k \equiv \pm 3 \bmod 8, \end{cases}$

which leads to the following theorem:

Theorem 4.4: For *k* odd, if χ denotes a nonprincipal, real, primitive character modulo *k*, then

$$\chi(2) = \begin{cases} 1 & \text{if } k \equiv 1, 7 \bmod 8, \\ -1 & \text{if } k \equiv 3, 5 \bmod 8. \end{cases}$$

The next two theorems provide identities which will be of use in the subsequent chapter. These theorems involve class numbers of imaginary quadratic fields whose definition is as follows:

Definition: Let *K* be a number field, then each fractional ideal *I* of *K* belongs to an equivalence class [*I*] consisting of all fractional ideals *J* satisfying *I* = *aJ* for some nonzero element *a* of *K*. The number of equivalent classes of fractional ideals of *K* is a finite number, known as the class number of *K*.

Theorem 4.5: Let χ denote a nonprincipal, real, primitive, odd character modulo k, where $k \geq 7$, then

$$h(-k) = -\frac{1}{k}\sum_{j=1}^{k-1} j\chi(j)$$

where $h(-k)$ is the class number of the imaginary quadratic field $Q(\sqrt{-k})$

Proof: See [7, p. 344]

Theorem 4.6: Let χ denote a nonprincipal, real, primitive, odd character modulo k, where $k \geq 7$, then

$$h(-k) = \frac{1}{2-\chi(2)}\sum_{j=1}^{(k-1)/2} \chi(j).$$

Proof: see [7, p. 346]

By its very definition, a character $\chi(n)$ (mod k) is a periodic function on the integers with period k. Thus by Theorem 2.1, there exists a Fourier transformation of $\chi(n)$. By restricting ourselves to characters (mod k) which are real and primitive, their Fourier coefficients have a relatively simple form.

In order to find the Fourier transformation for χ, where χ is a real, primitive character (mod k), we first need the following theorems:

Theorem 4.7: Let χ be a character (mod *k*). If we define the Gauss sum $G(z,\chi)$ for any complex number *z* by

$$G(z,\chi) := \sum_{j=0}^{k-1} \chi(j)\omega^{zj} ,$$

then for each integer *n*, χ is real and primitive if and only if

$$G(n,\chi) = \chi(n)G(1,\chi) =: \chi(n)G(\chi) .$$

Proof: See [4, p.22]

Theorem 4.8: Let χ be a real, primitive character (mod *k*). Then

$$G(\chi) = \begin{cases} \sqrt{k} & \text{if } \chi \text{ is even,} \\ i\sqrt{k} & \text{if } \chi \text{ is odd.} \end{cases}$$

Proof: See [7, p. 349]

Theorem 4.9: Let χ be a real, primitive character (mod *k*), then

$$\frac{1}{k}\sum_{j=0}^{k-1} \chi(j)\omega^{nj} = \begin{cases} \frac{\sqrt{k}}{k}\chi(n) & \text{if } \chi \text{ is even,} \\ i\frac{\sqrt{k}}{k}\chi(n) & \text{if } \chi \text{ is odd.} \end{cases}$$

Proof: Combining Theorems 4.7 and 4.8 yields

$$\frac{1}{k}\sum_{j=0}^{k-1} \chi(n)\omega^{nj} = \frac{1}{k}G(n,\chi) = \frac{1}{k}\chi(n)G(\chi) = \begin{cases} \frac{\sqrt{k}}{k}\chi(n) & \text{if } \chi \text{ is even,} \\ i\frac{\sqrt{k}}{k}\chi(n) & \text{if } \chi \text{ is odd.} \end{cases}$$

Chapter 5

Explicit Evaluations of Character Sums of Trigonometric Functions Using Discrete Fourier Transformations

Using the theorems from chapter 4, we are now able to find character sums of trigonometric functions using Theorems 2.2 and 2.3 and corollary 2.5. Some of these theorems were identities proven by Berndt and Zahrescu [6], while some of them were from my own investigations. Where applicable I make reference to [6]. For further references see their paper.

Theorem 5.1: Let χ denote a nonprincipal, real, primitive, odd character modulo k, a and b are positive integers such that a is odd and $1 \leq ab < k$, and x is any real number.

Then

$$\sum_{j=0}^{k-1} \chi(j) \sin^a\left(\frac{2\pi bj}{k} + x\right) = \frac{\sqrt{k}}{2^{a-1}} \sum_{\substack{n,m\geq 0 \\ n+2mb=ab}} (-1)^{m-(a-1)/2} \binom{a}{m} \cos\left(\left(a-2m\right)x\right)\chi(n).$$

Proof: Let $f(n) = \frac{1}{k}\sum_{j=0}^{k-1}\left(i\sin\left(\frac{2b\pi j}{k} + x\right)\right)^a \omega^{nj}$, then

$$f(n) = \frac{1}{2^a k}\sum_{m=0}^{a}(-1)^m \binom{a}{m} e^{(a-2m)xi} \sum_{j=0}^{k-1} \omega^{(n+(a-2m)b)j}$$

$$= \frac{1}{2^a} \sum_{\substack{0\leq m\leq a, \\ k|n+(a-2m)b}} (-1)^m \binom{a}{m} e^{(a-2m)xi},$$

where the last step comes from the fact that $\sum_{r=0}^{k-1}\omega^{n+r}=k$ if $k|n+r$ and 0 otherwise. If we let $s_m=(a-2m)b$, then $s_{a-m}=(2m-a)b=-s_m$. In addition, $\binom{a}{m}=\binom{a}{a-m}$; and since a is odd, $(-1)^m=-(-1)^{a-m}$. Therefore, we can rewrite the above equation as

$$f(n)=\frac{1}{2^a}\sum_{\substack{0\le m<a/2,\\ k|n+s_m}}(-1)^m\binom{a}{m}e^{(a-2m)xi}-\sum_{\substack{0\le m<a/2,\\ k|n-s_m}}(-1)^m\binom{a}{m}e^{-(a-2m)xi}.$$

Let $g(n)=i\frac{\sqrt{k}}{k}\chi(n)$, then by Corollary 2.5 (i) and Theorem 4.9

$$\frac{i^a}{k}\sum_{j=0}^{k-1}\chi(j)\sin^a\left(\frac{2\pi bj}{k}\right)=-\sum_{n=0}^{k-1}f(n)g(n)$$

$$=-\frac{i\sqrt{k}}{2^a k}\sum_{n=0}^{k-1}\chi(n)\left[\sum_{\substack{0\le m<a/2,\\ k|n+s_m}}(-1)^m\binom{a}{m}e^{(a-2m)xi}-\sum_{\substack{0\le m<a/2,\\ k|n-s_m}}(-1)^m\binom{a}{m}e^{-(a-2m)xi}\right]$$

$$=-\frac{i\sqrt{k}}{2^a k}\left[\sum_{\substack{0\le n<k.\\ 0\le m<a/2,\\ k|n+s_m}}(-1)^m\binom{a}{m}\chi(n)e^{(a-2m)xi}-\sum_{\substack{0\le n<k,\\ 0\le m<a/2,\\ k|n-s_m}}(-1)^m\binom{a}{m}\chi(n)e^{-(a-2m)xi}\right].$$

Since $1\le ab\le k$, and $0\le m<a/2$, then $0<s_m\le ab<k$. And since $0\le n<k$, then for all m, $0<n+s_m<2k$ and $-k<n-s_m<k$. Thus, $k|n+s_m$ only when $n=k-s_m$ and $k|n-$

s_m only when $n = s_m$. Since for every m there exists exactly one n such that $n = s_m$, and $\chi(k - s_m) = -\chi(s_m)$, then

$$\frac{i^a}{k}\sum_{j=0}^{k-1}\chi(j)\sin^a\left(\frac{2\pi bj}{k}\right) = -\frac{i\sqrt{k}}{2^a k}\sum_{0\le m<a/2}(-1)^m\binom{a}{m}[\chi(k-s_m)e^{(a-2m)xi} - \chi(s_m)e^{-(a-2m)xi}]$$

$$= \frac{i\sqrt{k}}{2^a k}\sum_{o\le m<a/2}(-1)^m\binom{a}{m}\chi(s_m)(e^{(a-2m)xi} + e^{-(a-2m)xi})$$

$$= \frac{i\sqrt{k}}{2^{a-1} k}\sum_{\substack{n,m\ge 0,\\ n+2mb=ab}}(-1)^m\binom{a}{m}\cos((a-2m)x)\chi(n),$$

where the last step comes from the fact that $n = s_m = (a - 2m)b$ and for $0 \le m < a/2$,

$0 \le 2mb < ab$. أ

***Corollary* 5.2:** Let χ denote a nonprincipal, real, primitive, odd character modulo k, b is a positive integer such that $1 \le b < k$. Then

$$\sum_{j=0}^{k-1}\chi(j)\sin\frac{2\pi bj}{k} = \sqrt{k}\chi(b).$$

Proof: Let $a = 1$, $x = 0$ in Theorem 5.4, then $n + 2mb = b$ only when $n = b$ and $m = 0$, thus the result follows. أ

Theorem 5.3: Let χ denote a nonprincipal, real, primitive, even character modulo k, where a and b are positive integers such that $1 \le ab < k$ and x is any real number. Then

$$\sum_{j=0}^{k-1} \chi(j) \cos^a\left(\frac{2\pi bj}{k} + x\right) = \frac{\sqrt{k}}{2^{a-1}} \sum_{\substack{n,m\ge 0 \\ n+2mb=ab}} \binom{a}{m} \cos\big((a-2m)x\big)\chi(n).$$

Proof: Let $f(n) = \frac{1}{k}\sum_{j=0}^{k-1} \cos^a\left(\frac{2\pi bj}{k}\right)\omega^{nj}.$

Since $\binom{a}{m} = \binom{a}{a-m}$, $-(a-2m) = (a-2(a-m))$, and $\sum_{n=0}^{k-1}\omega^{n+r} = k$ if $k|n+r$ and 0 otherwise, then

$$f(n) = \frac{1}{2^a k}\sum_{j=0}^{k-1}\sum_{m=0}^{a}\binom{a}{m}\omega^{(n+(a-2m)b)j}e^{(a-2m)xi}$$

$$= \frac{1}{2^a k}\left(\sum_{0\le m<a/2}\binom{a}{m}e^{(a-2m)xi}\sum_{j=0}^{k-1}\omega^{(n+(a-2m)bj} + \sum_{0\le m<a/2}\binom{a}{m}e^{-(a-2m)xi}\sum_{j=0}^{k-1}\omega^{(n-(a-2m)bj}\right)$$

$$= \frac{1}{2^a}\left(\sum_{\substack{0\le m<a/2, \\ k|n+s_m}}\binom{a}{m}e^{(a-2m)xi} + \sum_{\substack{0\le m<a/2, \\ k|n-s_m}}\binom{a}{m}e^{-(a-2m)xi}\right),$$

where $s_m = (a-2m)b$.

Let $g(n) = \frac{\sqrt{k}}{k}\chi(n)$, then by Theorem 4.9 and Corollary 2.5 (ii),

$$\frac{1}{k}\sum_{j=0}^{k-1}\chi(j)\cos^a\left(\frac{2\pi bj}{k}+x\right) = \sum_{n=0}^{k-1} f(n)g(n)$$

$$= \frac{\sqrt{k}}{2^a}\sum\left(\sum_{\substack{0\le m<a/2,\\ k|n+s_m}}\binom{a}{m}\chi(n)e^{(a-2m)xi} + \sum_{\substack{0\le m<a/2,\\ k|n-s_m}}\binom{a}{m}\chi(n)e^{-(a-2m)xi}\right).$$

If $m = a/2$, $s_m = 0$, and since $0 \le n < k$, then $k|n + s_m$ only for $n = 0$. But $\chi(0) = 0$, so this term will vanish in the sum above. Thus we need only consider $0 \le m < a/2$, in which case $0 < s_m < ab < k$. Since $0 \le n < k$, then using the same arguments as in Theorem 5.6, given each m, $k|n + s_m$ only for $n = k - s_m$, and $k|n - s_m$ only for $n = s_m$. Since $\chi(k - s_m) = \chi(s_m)$, then

$$\frac{1}{k}\sum_{j=0}^{k-1}\chi(j)\cos^a\left(\frac{2\pi bj}{k}+x\right) = \frac{\sqrt{k}}{2^a}\sum_{0\le m<a/2}\binom{a}{m}\left(\chi(k-s_m)e^{(a-2m)xi} + \chi(s_m)e^{-(a-2m)xi}\right)$$

$$= \frac{\sqrt{k}}{2^a}\sum_{0\le m<a/2}\binom{a}{m}\chi(s_m)(e^{(a-2m)xi} + e^{-(a-2m)xi})$$

$$= \frac{\sqrt{k}}{2^a}\sum_{\substack{n,m\ge 0,\\ n+2mb=ab}}\binom{a}{m}\chi(n)\cos((a-2m)x),$$

where the last step uses the identity $e^{xi} + e^{-xi} = 2\cos x$ and replaces $0 \le m < a/2$ and

$s_m = (a - 2m)b = n$ with its equivalent $n, m \geq 0$, $n + 2mb = ab$. أ

***Corollary* 5.4:** Let χ denote a nonprincipal, real, primitive, even character modulo k, where b is an integer such that $1 \leq b < k$. Then

$$\sum_{j=0}^{k-1} \chi(j) \cos \frac{2\pi bj}{k} = \sqrt{k}\,\chi(b).$$

Proof: Let $a = 1$, $x = 0$ in Theorem 5.3, then $n + 2mb = b$ only when $n = b$ and $m = 0$, thus the result follows. أ

The next theorem was presented as a corollary to a general theorem by Berndt and Zaharescu [6, p.557].

Theorem 5.5: Let χ denote a nonprincipal, real, primitive, odd character modulo k, where k is odd and $k \geq 7$. Then

$$\sum_{j=1}^{k-1} \chi(j) \cot \frac{\pi j}{k} = 2\sqrt{k}\,h(-k).$$

Proof: Let $f(n) = \left(\left(\frac{n}{k}\right)\right)$ and $g(n) = i\frac{\sqrt{k}}{k}\chi(n)$. Then by Lemma 2.8, Theorem 4.9 and Corollary 2.5 (i), and since $f(0) = 0$,

$$\frac{i}{2k}\sum_{j=1}^{k-1}\chi(j)\cot\frac{\pi j}{k} = -\sum_{m=1}^{k-1} f(m)g(m)$$

$$= \frac{i\sqrt{k}}{k}\sum_{m=1}^{k-1}\left(-\frac{1}{k}\sum_{m=1}^{k-1} m\chi(m) + \frac{1}{2}\sum_{m=1}^{k-1}\chi(m)\right)$$

$$= \frac{i\sqrt{k}}{k} h(-k).$$

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The next corollary was proven by Berndt and Zaharescu [6, p. 571] for the case where $b = 2$. I have extended it to included any positive integer b such that $(b, k) = 1$.

***Corollary* 5.6** Let χ denote a nonprincipal, real, primitive, odd character modulo k, where k is odd and $k \geq 7$, and let b be a positive integer such that $(b, k) = 1$. Then

$$\sum_{j=1}^{k-1}\chi(j)\cot\frac{b\pi j}{k} = \chi(b)\sqrt{k}h(-k).$$

Proof: By Theorem 5.5 and since $(b, k) = 1$, then

$$\sqrt{k}h(-k) = \sum_{j=1}^{k-1}\chi(j)\cot\frac{\pi j}{k} = \sum_{j=1}^{k-1}\chi(bj)\cot\frac{b\pi j}{k} = \chi(b)\sum_{j=1}^{k-1}\chi(j)\cot\frac{b\pi j}{k}.$$

Since $\chi(b) = \pm 1$, multiplying both sides of the above equality by $\chi(b)$ yields the desired result.

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The next theorem was a corollary of a general theorem proven by Berndt and Zaharescu [6, p. 568].

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Theorem 5.7: Let χ denote a nonprincipal, real, primitive, odd character modulo k, where k is odd and $k \geq 7$. Then

$$\sum_{j=1}^{k-1} \chi(j) \tan \frac{\pi j}{k} = \sqrt{k}(2-4\chi(2))h(-k) = \begin{cases} -2\sqrt{k}h(-k) & \text{if } k \equiv 7 \bmod 8, \\ 6\sqrt{k}h(-k) & \text{if } k \equiv 3 \bmod 8. \end{cases}$$

Proof: Let $f(n) = \begin{cases} (-1)^{n \bmod k} & \text{if } k \nmid n, \\ 0 & \text{otherwise}, \end{cases}$ and $g(n) = i\frac{\sqrt{k}}{k}\chi(n)$.

Then by Lemma 2.9, Theorem 4.9 and Corollary 2.5 (i), and since $f(0)=0$,

$$\begin{aligned} \frac{i}{k}\sum_{j=1}^{k-1} \chi(j) \tan \frac{\pi j}{k} &= -\sum_{n=1}^{k-1} f(n)g(n) \\ &= -\frac{i\sqrt{k}}{k}\left(\sum_{m=1}^{k-1} \chi(2m) - \sum_{m=1}^{k-1} \chi(2m-1)\right) \\ &= -\frac{i\sqrt{k}}{k}\left(2\chi(2)\sum_{m=1}^{(k-1)/2} \chi(m)\right) \\ &= -\frac{i\sqrt{k}}{k}\left(2\chi(2)(2-\chi(2))h(-k)\right) \\ &= \frac{i\sqrt{k}}{k}(2-4\chi(2))h(-k) \\ &= \begin{cases} -2\sqrt{k}h(-k) & \text{if } k \equiv 7 \bmod 8, \\ 6\sqrt{k}h(-k) & \text{if } k \equiv 3 \bmod 8, \end{cases} \end{aligned}$$

where we have used Theorems 4.6 and 4.4. أ

Theorem 5.8: Let χ denote a nonprincipal, real, primitive, odd character modulo *k*, where *k* is odd and $k \geq 7$. Then

$$\sum_{j=1}^{k-1} \chi(j) \csc \frac{2\pi j}{k} = 2\sqrt{k}h(-k)(1-\chi(2)) = \begin{cases} 4\sqrt{k}h(-k) & \text{if } k \equiv 3 \bmod 8, \\ 0 & \text{if } k \equiv 7 \bmod 8. \end{cases}$$

Proof: Using the identity $\csc 2\theta = \frac{1}{2}(\tan\theta + \cot\theta)$ and Theorems 5.5 and 5.7 yields

$$\begin{aligned} \sum_{j=1}^{k-1} \chi(j) \csc \frac{2\pi j}{k} &= \frac{1}{2}\left(2\sqrt{k}h(-k) - \sqrt{k}(4\chi(2)-2)h(-k)\right) \\ &= 2\sqrt{k}h(-k)(1-\chi(2)). \end{aligned}$$

Theorem 4.4 provides the final result. أ

Theorem 5.9: Let χ denote a nonprincipal, real, primitive, odd character modulo *k*, where *k* is odd and $k \geq 7$. Then

$$\frac{1}{k}\sum_{j=1}^{k-1} \chi(j) \cot \frac{\pi j}{k} \omega^{nj} = \frac{\sqrt{k}}{k}\left(2h(-k) + \chi(n) - 2\sum_{m=0}^{n} \chi(m)\right).$$

Proof: Let $f(n) = \frac{i}{2k}\sum_{j=1}^{k-1} \cot \frac{\pi j}{k} \omega^{nj}$ and $g(n) = \sum_{j=1}^{k-1} \chi(j)\omega^{nj}$, then by Theorem 2.2, Lemma 2.8, and Theorem 5.5,

$$\frac{i}{2k}\sum_{j=1}^{k-1}\chi(j)\cot\frac{\pi j}{k}\omega^{nj} = \sum_{m=0}^{k-1} f(n-m)g(m) = \frac{i\sqrt{k}}{k}\sum_{m=0}^{k-1}\chi(m)\left(\left(\frac{n-m}{k}\right)\right).$$

For simplicity assume $0 \le n < k$, then since $0 \le m < k$, we get

$$\left(\left(\frac{n-m}{k}\right)\right) = \begin{cases} \frac{n}{k}-\frac{m}{k}-\frac{1}{2} & \text{if } m < n, \\ 0 & \text{if } m = n, \\ \frac{n}{k}-\frac{m}{k}-\frac{1}{2}+1 & \text{if } m > n. \end{cases}$$

Thus, we can rewrite the right hand side of the above equation as

$$= \frac{i\sqrt{k}}{k}\left[\left(\sum_{m=0}^{k-1}\chi(m)\left(\frac{n}{k}-\frac{m}{k}-\frac{1}{2}\right)\right) - \chi(n)\left(\frac{n}{k}-\frac{n}{k}-\frac{1}{2}\right) + \sum_{m=n+1}^{k-1}\chi(m)\right]$$

$$= \frac{i\sqrt{k}}{k}\left(\left(\frac{n}{k}-\frac{1}{2}\right)\sum_{m=0}^{k-1}\chi(m) - \frac{1}{k}\sum_{m=0}^{k-1} m\chi(m) + \frac{1}{2}\chi(n) + \sum_{m=n+1}^{k-1}\chi(m)\right)$$

$$= \frac{i\sqrt{k}}{k}\left(h(-k) + \frac{1}{2}\chi(n) - \sum_{m=0}^{n}\chi(m)\right),$$

where the last step uses Theorems 4.1 and 4.5. أ

For χ, a nonprincipal, real, primitive, odd character modulo k, where k is odd and $k \ge 7$, a an odd positive integer, and b an even positive integer such that $ab - a - 3 < 2k$,

$$\sum_{j=1}^{k-1}\chi(j)\frac{\sin^a(b\pi j/k)}{\sin^{a+1}(\pi j/k)} = 2\sqrt{k}(b^a h(-k) - 2\sum_{\substack{n,m,r\ge 0\\ n+bm+r=(ab-a-1)/2}}(-1)^m\chi(m)\binom{a}{m}\binom{a+r}{r}, \tag{5.1}$$

was proven by Berndt and Zaharescu [6, p.553]. I was not able to prove the general case; however, I was able to prove two of their corollaries, the case for $b = 2$ and a odd [6, p556], and for $a = 1$ and b is an even positive integer [6, p.557].

For the first case, $b = 2$, $\dfrac{\sin^a(2\pi j/k)}{\sin^{a+1}(\pi j/k)} = 2^a \cot\dfrac{\pi j}{k}\cos^{a-1}\dfrac{\pi j}{k}$. The next two theorems determine the Fourier transformations for $\cos^{a-1}(\pi j/k)$ and $\chi(j)\cot(\pi j/k)$, and then the proof for this corollary of the above theorem follows. It should be noted that my final solution was a slight improvement on (5.1).

Theorem 5.10: For positive integers a and k,

$$\frac{1}{k}\sum_{j=0}^{k-1}\cos^{a-1}\frac{\pi j}{k}\omega^{nj} = \frac{1}{2^{a-1}}\left(\sum_{\substack{0<m\le r,\\ k\mid n+m}}\binom{a-1}{r-m} + \sum_{\substack{0\le m\le r,\\ k\mid n-m}}\binom{a-1}{r-m}\right),\text{ where } r = (a-1)/2.$$

Proof:

$$\frac{1}{k}\sum_{j=0}^{k-1}\cos^{a-1}\frac{\pi j}{k}\omega^{nj} = \frac{1}{2^{a-1}k}\sum_{j=0}^{k-1}\left(\omega^{j/2}+\omega^{-j/2}\right)^{a-1}\omega^{nj}$$

$$= \frac{1}{2^{a-1}k}\sum_{j=0}^{k-1}\left(\sum_{m=0}^{a-1}\binom{a-1}{m}\omega^{(a-1-2m)j/2}\right)\omega^{nj}$$

$$= \frac{1}{2^{a-1}k}\sum_{j=0}^{k-1}\left(\sum_{m=0}^{2r}\binom{2r}{m}\omega^{(r-m)j}\right)\omega^{nj},$$

where $r = (a-1)/2$. By using a change of variables twice and noting that $\binom{2r}{r-m} = \binom{2r}{r+m}$ the inner sum above can be rewritten as

$$\sum_{m=0}^{2r}\binom{2r}{m}\omega^{(r-m)j}=\sum_{m=-r}^{r}\binom{2r}{r-m}\omega^{mj}$$

$$=\sum_{m=-r}^{0}\binom{2r}{r+m}\omega^{mj}+\sum_{m=1}^{r}\binom{2r}{r-m}\omega^{mj}$$

$$=\sum_{m=0}^{r}\binom{2r}{r-m}\omega^{-mj}+\sum_{m=1}^{r}\binom{2r}{r-m}\omega^{mj}.$$

Therefore,

$$\frac{1}{k}\sum_{j=0}^{k-1}\cos^{a-1}\frac{\pi j}{k}\omega^{nj}=\frac{1}{2^{a-1}k}\sum_{j=0}^{k-1}\left(\sum_{m=1}^{r}\binom{2r}{r-m}\omega^{mj}+\sum_{m=0}^{r}\binom{2r}{r-m}\omega^{-mj}\right)\omega^{nj}$$

$$=\frac{1}{2^{a-1}}\left(\sum_{m=1}^{r}\binom{2r}{r-m}\frac{1}{k}\sum_{j=0}^{k-1}\omega^{(n+m)j}+\sum_{m=0}^{r}\binom{2r}{r-m}\frac{1}{k}\sum_{j=0}^{k-1}\omega^{(n-m)j}\right)$$

$$=\frac{1}{2^{a-1}}\left(\sum_{\substack{0<m\le r,\\ k|n+m}}\binom{2r}{r-m}+\sum_{\substack{0\le m\le r,\\ k|n-m}}\binom{2r}{r-m}\right),$$

where the last step comes from the fact that $\sum_{j=0}^{k-1}\omega^{nj}$ vanishes except when $k|n$, in which case it equals k. أ

Theorem 5.11: Let χ denote a nonprincipal, real, primitive, odd character modulo *k*, where *k* is odd and $k \geq 7$ and *a* is an odd positive integer such that $a - 3 < 2k$. Then

$$\sum_{j=1}^{k-1} \chi(j) \cot \frac{\pi j}{k} \cos^{a-1} \frac{\pi j}{k} = 2\sqrt{k}\left(h(-k) - 2^{1-a} \sum_{\substack{n,m,s \geq 0, \\ n+2m+2=(a-1)/2}} \chi(n) \binom{a+1}{s} \right).$$

Proof: Let $f(n) = 2h(-k) + \chi(n) - 2\sum_{s=0}^{n} \chi(s)$ and

$$g(n) = \frac{1}{2^{a-1}}\left(\sum_{\substack{0<m\leq r, \\ k|n+r+m}} \binom{a-1}{r-m} + \sum_{\substack{0\leq m\leq r, \\ k|n+r-m}} \binom{a-1}{r-m} \right),$$ where $r = (a-1)/2$,

then by Theorem 5.5, Theorem 5.10 and Corollary 2.5 (ii) we get

$$\begin{aligned}
\frac{i}{k}\sum_{j=1}^{k-1} \chi(j) \cot \frac{\pi j}{k} \cos^{a-1} \frac{\pi j}{k} &= \frac{i\sqrt{k}}{k}\sum_{n=0}^{k-1} f(n) g(n) \\
&= \frac{i\sqrt{k}}{2^{a-1}k}\sum_{n=0}^{k-1}\left(2h(-k) + \chi(n) - 2\sum_{s=0}^{k-1}\chi(s) \right)\left(\sum_{\substack{0<m\leq r, \\ k|n+m}} \binom{a-1}{r-m} + \sum_{\substack{0\leq m\leq r, \\ k|n-m}} \binom{a-1}{r-m} \right) \\
&= \frac{i\sqrt{k}}{2^{a-1}k}\left(\begin{aligned} & 2h(-k)\sum_{n=0}^{k-1}\left(\sum_{\substack{0<m\leq r, \\ k|n+m}} \binom{a-1}{r-m} + \sum_{\substack{0\leq m\leq r, \\ k|n-m}} \binom{a-1}{r-m} \right) \\ & + \sum_{n=0}^{k-1}\chi(n)\left(\sum_{\substack{0<m\leq r, \\ k|n+m}} \binom{a-1}{r-m} + \sum_{\substack{0\leq m\leq r, \\ k|n-m}} \binom{a-1}{r-m} \right) \\ & - 2\sum_{n=0}^{k-1}\sum_{s=0}^{n}\chi(s)\left(\sum_{\substack{0<m\leq r, \\ k|n+m}} \binom{a-1}{r-m} + \sum_{\substack{0\leq m\leq r, \\ k|n-m}} \binom{a-1}{r-m} \right) \end{aligned} \right).
\end{aligned} \tag{5.11.1}$$

In the first of the sums in (5.11.1) above, since for every m between 0 and r, there exists an n such that $k|n+m$ and there exists an n such that $k|n-m$, then every binomial coefficient of $a-1$ will appear in the sum exactly once. Thus

$$2h(-k)\sum_{n=0}^{k-1}\left(\sum_{\substack{0\le m\le r,\\ k|n+m}}\binom{a-1}{r-m}+\sum_{\substack{0\le m\le r,\\ k|n-m}}\binom{a-1}{r-m}\right)=2h(-k)\sum_{m=0}^{a-1}\binom{a-1}{m}=2^a h(-k). \qquad (5.11.2)$$

By hypothesis a is a positive odd integer such that $a-3<2k$. Thus, $0\le(a-1)/2<k+1$.

Since $r=(a-1)/2$, then $0\le r\le k$. If $r=k$, then for $n=0$, $k|n+m$ and $k|n-m$ for

$m=0, k$ and if $r<k$, $k|n+m$ and $k|n-m$ only for $m=0$. Since for $n=0$, $\chi(0)=0$ these terms will vanish in the second sum on the right hand side of (5.11.1), thus we need only consider $0<m<k$. For each of these values of m, $k|n+m$ only for $n=k-m$, and $k|n-m$ only for $n=m$, thus if we let $s=r$ if $r<k$, or $k-1$ if $r=k$, then the second equation in (5.11.1) becomes

$$\sum_{n=0}^{k-1}\chi(n)\left(\sum_{\substack{0\le m\le r,\\ k|n+m}}\binom{a-1}{r-m}+\sum_{\substack{0\le m\le r,\\ k|n-m}}\binom{a-1}{r-m}\right)=\sum_{m=1}^{s}\binom{a-1}{r-m}\big(\chi(m)+\chi(k-m)\big)=0, \qquad (5.11.3)$$

where the last step uses the fact that since χ is odd and has period k, $\chi(k-n)=-\chi(n)$.

To simplify the last of the sums in (5.11.1), first note for $n=0$, $\sum_{s=0}^{0}\chi(s)=\chi(0)=0$, thus for $m = 0$, these terms will vanish. For every $m > 0$, if $r < k$ then, since $k|n + m$ only for $n = k - m$ and $k|n - m$ only for $n = m$, the last sum in (5.11.1) becomes

$$2\sum_{n=0}^{k-1}\sum_{s=0}^{n}\chi(s)\left(\sum_{\substack{0<m\le r,\\ k|n+m}}\binom{a-1}{r-m}+\sum_{\substack{0\le m\le r,\\ k|n-m}}\binom{a-1}{r-m}\right)=2\left(\sum_{m=1}^{r}\binom{a-1}{r-m}\sum_{s=0}^{k-m}\chi(s)+\binom{a-1}{r-m}\sum_{s=0}^{m}\chi(s)\right).$$

(5.11.4)

But since χ is odd, $\sum_{s=0}^{k-1}\chi(s)=0$, and $\chi(k) = 0$, then

$$\sum_{s=0}^{k-m}\chi(s)=-\sum_{s=0}^{k-m}\chi(k-s)=-\sum_{s=m}^{k}\chi(s)=\sum_{s=0}^{m-1}\chi(s),$$

thus (5.11.4) can be rewritten as

$$\begin{aligned}2\sum_{n=0}^{k-1}\sum_{s=0}^{n}\chi(s)\left(\sum_{\substack{0<m\le r,\\ k|n+m}}\binom{a-1}{r-m}+\sum_{\substack{0\le m\le r,\\ k|n-m}}\binom{a-1}{r-m}\right)&=2\left(\sum_{m=1}^{r}\left(\binom{a-1}{r-m}\sum_{s=0}^{m-1}\chi(s)+\binom{a-1}{r-m}\sum_{s=0}^{m}\chi(s)\right)\right)\\
&=2\left(\sum_{m=1}^{r-1}\binom{a-1}{r-m-1}\sum_{s=0}^{m}\chi(s)+\sum_{s=0}^{r}\binom{a-1}{r-m}\sum_{s=0}^{m}\chi(s)\right)\\
&=2\left(\sum_{m=1}^{r-1}\left(\binom{a-1}{r-m-1}+\binom{a-1}{r-m}\right)\sum_{s=0}^{m}\chi(s)+\binom{a-1}{0}\sum_{s=0}^{r}\chi(s)\right)\\
&=2\left(\sum_{m=1}^{r}\binom{a}{r-m}\sum_{s=0}^{m}\chi(s)\right),\end{aligned}$$

(5.11.5)

where the last step uses $\binom{a-1}{m}+\binom{a-1}{m-1}=\binom{a}{m}$ and $\binom{a}{0}=\binom{a-1}{0}$.

If $r = k$, then $\sum_{s=0}^{k-1}\chi(s)=\sum_{s=0}^{k}\chi(s)=0$, thus (5.11.5) holds whether $r = k$ or $r < k$.

Since for any functions *f* and *g*

$$\begin{aligned}\sum_{m=1}^{r} f(m)\sum_{s=1}^{m} g(s) &= f(1)g(1)+f(2)\big(g(1)+g(2)\big)+\cdots+f(r)\big(g(1)+g(2)+\cdots+g(r)\big)\\ &= g(1)\big(f(1)+f(2)+\cdots+f(r)\big)+g(2)\big(f(2)+\cdots+f(r)\big)+\cdots+g(r)f(r)\\ &= \sum_{m=1}^{r} g(n)\sum_{s=m}^{r} f(s),\end{aligned}$$

then

$$\begin{aligned}\sum_{m=1}^{r}\binom{a}{r-m}\sum_{s=1}^{m}\chi(s) &= \sum_{m=1}^{r}\chi(m)\sum_{s=m}^{r}\binom{a}{r-s}\\ &= \sum_{m=1}^{r}\chi(m)\sum_{s=0}^{r-m}\binom{a}{s}.\end{aligned}$$

Since χ(0) = 0 and

$$\begin{aligned}\sum_{s=0}^{r-n}\binom{a}{s} &= \binom{a}{0}+\binom{a}{1}+\binom{a}{2}+\binom{a}{3}+\cdots+\binom{a}{r-n}\\ &= \begin{cases}\binom{a+1}{1}+\binom{a+1}{3}+\cdots+\binom{a+1}{r-n} & \text{if } 2\mid r-n+1,\\ \binom{a+1}{0}+\binom{a+1}{2}+\cdots+\binom{a+1}{r-n} & \text{otherwise}\end{cases}\\ &= \sum_{\substack{s,m\ge 0,\\ s+2m=r-n}}\binom{a+1}{s},\end{aligned}$$

then
$$\sum_{n=1}^{r}\chi(n)\sum_{s=0}^{r-n}\binom{a}{s}=\sum_{n=0}^{r}\chi(n)\sum_{s=0}^{r-n}\binom{a}{s}=\sum_{\substack{n,m,s\ge 0,\\ n+2m+s=(a-1)/2}}\chi(n)\binom{a+1}{s} \qquad (5.11.6)$$

Combining the results of equations (5.11.2), (5.11.3) and (5.11.6), we can simplify (5.11.1) as

$$\frac{1}{k}\sum_{j=0}^{k-1}\chi(j)\cot\frac{\pi j}{k}\cos^{a-1}\frac{\pi j}{k}=\frac{\sqrt{k}}{2^{a-1}k}\left(2^{a}h(-k)-2\sum_{\substack{n,m,s\geq 0\\ n+2m+s=(a-1)/2}}\chi(n)\binom{a+1}{s}\right)$$

$$=\frac{2\sqrt{k}}{k}\left(h(-k)-2^{1-a}\sum_{\substack{n,m,s\geq 0,\\ n+2m+s=(a-1)/2}}\chi(n)\binom{a+1}{s}\right).$$

The next two theorems involve the second special case $a = 1$, b even in (5.1). Note that instead of saying b even, I amended it to $2b$, b a positive integer. The first theorem finds the Fourier transformation for $\cot^2(\pi j/k)$ and the second uses this to prove this case.

Theorem 5.12: Let χ denote a nonprincipal, real, primitive, even character modulo k, where k is odd and $k \geq 7$. Then

$$\sum_{j=1}^{k-1}\chi(j)\cot^{2}\frac{\pi j}{k}\omega^{nj}=\frac{i\sqrt{k}}{k}\left(4nh(-k)-\chi(n)+4\sum_{m=0}^{n}\chi(m)(m-n)\right).$$

Proof: Let $f(n)=\begin{cases}\dfrac{(k-1)(k-2)}{3k} & \text{if } n\equiv 0 \bmod k,\\ \dfrac{2(n \bmod k)(n(\bmod k)-k}{k}+\dfrac{k^{2}+2}{3k} & \text{otherwise},\end{cases}$ and

$$g(n)=\frac{i\sqrt{k}}{k}\chi(n),$$

then by Theorem 3.13, Theorem 4.9 and Theorem 2.2

$$\frac{1}{k}\sum_{j=1}^{k-1}\chi(j)\cot^2\frac{\pi j}{k}\omega^{nj}=\frac{\sqrt{k}}{k}\sum_{m=0}^{k-1}f(n-m)g(m).$$

For simplicity, since $f(n)$ has period k, we will assume $0 \le n < k$. Let $C_k = \frac{2+k^2}{3k}$, then

$$\text{for } m < n,\quad f(n-m)=\frac{2(n-m)^2}{k}-2(n-m)+C_k=\frac{2n^2-4nm+2m^2}{k}-2n+2m+C_k,$$

$$\text{for } m = n,\quad f(n-m)=C_k-1,\ \text{ and}$$

$$\begin{aligned}\text{for } m > n,\quad f(n-m)&=\frac{2(k+n-m)^2}{k}-2(k+n-m)+C_k\\&=\frac{2n^2-4nm+2m^2}{k}+2n-2m+C_k.\end{aligned}$$

Therefore,

$$\frac{1}{k}\sum_{j=1}^{k-1}\chi(j)\cot^2\frac{\pi j}{k}\omega^{nj}=\frac{i\sqrt{k}}{k}\left(\begin{array}{l}\left(\sum_{m=0}^{k-1}\chi(m)\left(\frac{2n^2}{k}-\frac{4nm}{k}+\frac{2m^2}{k}-2n+2m+C_k\right)\right)\\-\chi(n)\left(\frac{2n^2}{k}-\frac{4n^2}{k}+\frac{2n^2}{k}-2n+2n\right)-\chi(n)\\+4\sum_{m=n+1}^{k-1}\chi(m)(n-m)\end{array}\right).\quad(5.12.1)$$

Since n, k, and C_k are independent of m and $\sum_{m=0}^{k-1}\chi(m)=0$, the right hand side of (5.12.1) can be simplified to

$$=\frac{i\sqrt{k}}{k}\left(-\frac{4n}{k}\sum_{m=0}^{k-1}m\chi(m)+\frac{2}{k}\sum_{m=0}^{k-1}m^2\chi(m)+2\sum_{m=0}^{k-1}m\chi(m)-\chi(n)+4n\sum_{m=n}^{k-1}\chi(m)-\sum_{m=n+1}^{k-1}m\chi(m)\right).\quad(5.12.2)$$

Since χ is odd, with period *k*, then

$$\begin{aligned}\sum_{m=0}^{k-1} m^2\chi(m) &= \sum_{m=0}^{k-1}(k-m)^2\chi(k-m)\\ &= -\sum_{m=0}^{k-1}(k^2-2mk+m^2)\chi(m)\\ &= -k^2\sum_{m=0}^{k-1}\chi(m)+2k\sum_{m=0}^{k-1}m\chi(m)-\sum_{m=0}^{k-1}m^2\chi(m)\\ &= 2k\sum_{m=0}^{k-1}m\chi(m)-\sum_{m=0}^{k-1}m^2\chi(m).\end{aligned} \tag{5.12.3}$$

Thus $\sum_{m=0}^{k-1} m^2\chi(m) = k\sum_{m=0}^{k-1} m\chi(m)$. This means we can further simplify (5.12.2) as

$$\begin{aligned}&= \frac{i\sqrt{k}}{k}\left(-\frac{4n}{k}\sum_{m=0}^{k-1}m\chi(m)+4\left(\sum_{m=0}^{k-1}m\chi(m)-\sum_{m=n+1}^{k-1}m\chi(m)\right)+4n\sum_{m=n+1}^{k-1}\chi(m)-\chi(n)\right)\\ &= \frac{i\sqrt{k}}{k}\left(4nh(-k)+4\sum_{m=0}^{n}m\chi(m)-4n\sum_{m=0}^{n}\chi(m)-\chi(n)\right)\\ &= \frac{i\sqrt{k}}{k}\left(4nh(-k)+4\sum_{m=0}^{n}\chi(m)(m-n)-\chi(n)\right).\end{aligned}$$

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The next theorem was proven by Berndt and Zaharescu [5, p.558].

Theorem 5.13: Let χ denote a nonprincipal, real, primitive, even character modulo *k*, where *k* is odd and *k* ≥ 7. Then

$$\sum_{j=1}^{k-1}\chi(j)\cot^2\frac{\pi j}{k}\sin\frac{2\pi aj}{k} = \frac{i\sqrt{k}}{k}\left(4ah(-k)-\chi(a)-2\sum_{m=0}^{a-1}\chi(m)(a-m)\right).$$

Proof: Let $f(n)=\begin{cases}1/2 & \textit{if } k\mid a+n,\\ -1/2 & \textit{if } k\mid a-n,\\ 0 & \textit{otherwise.}\end{cases}$ and

$$g(n)=\frac{i\sqrt{k}}{k}\left(4nh(-k)-\chi(n)+4\sum_{m=0}^{n}\chi(m)(m-n)\right),$$

then by Lemma 2.6, Theorem 4.9 and Corollary 2.5 (i)

$$\frac{i}{k}\sum_{j=1}^{k-1}\chi(j)\cot^2\frac{\pi j}{k}\sin\frac{2\pi aj}{k}=\sum_{m=0}^{k-1}f(-m)g(m)=-\sum_{m=0}^{k-1}f(m)g(m)\,.$$

Since $f(n)$ vanishes except for $f(a) = -1/2$ and $f(k - a) = 1/2$, the right hand side becomes

$$=-\frac{i\sqrt{k}}{k}\begin{pmatrix}-2ah(-k)+\dfrac{\chi(a)}{2}-2\displaystyle\sum_{m=0}^{a}\chi(m)(m-a)\\ +2(k-a)h(-k)-\dfrac{\chi(k-a)}{2}+2\displaystyle\sum_{m=0}^{k-a}\chi(m)(m-(k-a))\end{pmatrix}$$

$$=\frac{i\sqrt{k}}{k}\left(4ah(-k)-\chi(a)-2kh(-k)+2\sum_{m=0}^{a}\chi(m)(m-a)-2\sum_{m=0}^{k-a}\chi(m)(m-(k-a))\right).$$

Since χ is odd, $\chi(m) = -\chi(k - m)$, thus the last term in the sum above can be rewritten as:

$$\sum_{m=0}^{k-a}\chi(m)(m-k+a)=-\sum_{m=0}^{k-a}\chi(k-m)(m-k+a)=\sum_{m=0}^{k-a}\chi(k-m)(k-m-a)=\sum_{m=a}^{k-1}\chi(m)(m-a)\,,$$

where the last step is just a change in variables and noting $\chi(k) = 0$. Remembering $-kh(-k) = \sum_{m=0}^{k-1} m\chi(m)$, we can make these two substitutions and rearrange terms to get

$$= \frac{i\sqrt{k}}{k}\left(4ah(-k) - \chi(a) + 2\left(\sum_{m=0}^{k-1} m\chi(m) + \sum_{m=0}^{a} m\chi(m) - \sum_{m=a}^{k-1} m\chi(m)\right) - 2a\left(\sum_{m=0}^{a}\chi(m) - \sum_{m=0}^{k-a}\chi(m)\right)\right)$$
$$= \frac{i\sqrt{k}}{k}\left(4ah(-k) - \chi(a) + 2\sum_{m=0}^{a-1} m\chi(m) + 2a\chi(a) - 2a\left(\sum_{m=0}^{a}\chi(m) - \sum_{m=0}^{k-a}\chi(m)\right)\right).$$

Note that $\sum_{m=0}^{k-a}\chi(m) = -\sum_{m=0}^{k-a}\chi(k-m) = -\sum_{m=a}^{k-1}\chi(m) = \sum_{m=0}^{a-1}\chi(m)$, thus the last term above yields

$$\sum_{m=0}^{a}\chi(m) - \sum_{m=0}^{k-a}\chi(m) = \sum_{m=0}^{a}\chi(m) - \sum_{m=0}^{a-1}\chi(m) = \chi(a).$$

Therefore, we can further simplify to get

$$\frac{i}{k}\sum_{j=1}^{k-1}\chi(j)\cot^2\frac{\pi j}{k}\sin\frac{2\pi aj}{k} = \frac{i\sqrt{k}}{k}\left(4ah(-k) - \chi(a) - 2\sum_{m=0}^{a-1}\chi(m)(a-m)\right).$$

The next theorem was presented as a general theorem by Berndt and Zaharescu [2, p.570].

Theorem 5.17: Let χ denote a nonprincipal, real, primitive, even character modulo *k*, where *k* is odd and $k \geq 7$, and *b* is a positive integer such that $b \leq k$, then

$$\sum_{j=1}^{k-1} \chi(j) \cos\frac{2b\pi j}{k} \cot\frac{\pi j}{k} = \sqrt{k}\,(2h(-k) - \chi(b) - 2\sum_{n=1}^{b-1} \chi(n)) .$$

Proof: Let $f(n) = \frac{\sqrt{k}}{k}\left(2h(-k) + \chi(n) - 2\sum_{m=0}^{n-1} \chi(m) \right)$ and let

$$g(n) = \begin{cases} 1/2 & \textit{if } k \mid n-b \textit{ or } k \mid n+b, \\ 0 & \textit{otherwise.} \end{cases}$$

Thus by Lemma 2.7, Theorem 5.12, and Corollary 2.5 (ii) and since $\chi(b) = -\chi(k-b)$, and $g(n)$ vanishes except for $g(b) = g(k-b) = ½$, then

$$\begin{aligned} \frac{1}{k}\sum_{j=1}^{k-1} \chi(j) \cot\frac{\pi j}{k} \cos\frac{2b\pi j}{k} &= \sum_{n=0}^{k-1} f(n) g(-n) \\ &= \sum_{n=0}^{k-1} f(n) g(n) \\ &= \frac{\sqrt{k}}{2k}\left(4h(-k) + \chi(b) + \chi(k-b) - 2(\sum_{m=0}^{b} \chi(m) + \sum_{m=0}^{k-b} \chi(m)) \right) \\ &= \frac{\sqrt{k}}{k}\left(2h(-k) - \sum_{m=0}^{b} \chi(m) - \sum_{m=0}^{k-b} \chi(m) \right). \end{aligned}$$

Note $-\sum_{m=0}^{k-b}\chi(m)=\sum_{m=0}^{k-b}\chi(k-m)=\sum_{m=b}^{k}\chi(m)=-\sum_{m=1}^{b-1}\chi(m)$, therefore,

$$=\frac{\sqrt{k}}{k}\left(2h(-k)-2\sum_{m=0}^{b-1}\chi(m)-\chi(b)\right).$$ ĭ

Another general theorem of Berndt and Zaharescu [2, p.558] is for χ as defined in the preceding theorem and *a* and *b* positive integers, *a* even,

$$\sum_{j=1}^{k-1}\chi(j)\frac{\sin^a(b\pi j/k)}{\sin(4\pi j/k)}=\frac{3\sqrt{k}}{2}\left(\chi(2)-1\right)h(-k).$$

I was able to prove this same theorem, but it is a corollary to three other theorems which I present below. As done previously, some of these theorems find the Fourier transformations for different functions prior to finding the closed forms for their sums.

Theorem 5.18: Let *k* be a positive integer, and

$$f(n)=\begin{cases}1/2 & \text{if } k\mid n,\\ -1/4 & \text{if } k\mid n+1 \text{ or } k\mid n-1,\\ 0 & \text{otherwise},\end{cases} \qquad \text{then} \qquad f(n)=\frac{1}{k}\sum_{j=0}^{k-1}\sin^2\frac{\pi j}{k}\,\omega^{nj}.$$

Proof:

$$\frac{1}{k}\sum_{j=0}^{k-1}\sin^2\frac{\pi j}{k}\,\omega^{nj}=\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\omega^{j/2}-\omega^{-j/2}}{2i}\right)^2\omega^{nj}$$
$$=\frac{1}{k}\sum_{j=0}^{k-1}\left(-\frac{1}{4}\omega^{(n+1)j}+\frac{1}{2}\omega^{nj}-\frac{1}{4}\omega^{(n-1)j}\right).$$

Since $\sum_{j=0}^{k-1}\omega^{rj}$ vanishes unless $k|r$, in which case it is equal to *k*, the result follows. ĭ

Theorem 5.19: Let χ denote a nonprincipal, real, primitive, odd character modulo *k*, where *k* is an odd integer and $k \geq 7$. Then

$$\sum_{j=1}^{k-1} \chi(j)\sin^2\frac{\pi j}{k}\cot\frac{\pi j}{k} = \frac{\sqrt{k}}{2}.$$

Proof: Let $f(n) = \frac{\sqrt{k}}{k}\left(2h(-k) + \chi(n) - 2\sum_{m=0}^{n}\chi(m)\right)$ and

$$g(n) = \begin{cases} 1/2 & \text{if } k \mid n, \\ -1/4 & \text{if } k \mid n+1 \text{ or } k \mid n-1, \\ 0 & \text{otherwise.} \end{cases}$$

then by Corollary 2.5 (ii), Theorem 5.12, and Theorem 4.9

$$\begin{aligned} \frac{1}{k}\sum_{j=1}^{k-1}\chi(j)\sin^2\frac{\pi j}{k}\cot\frac{\pi j}{k} &= \sum_{m=0}^{k-1} f(n)g(-m) \\ &= \frac{1}{2}f(0) - \frac{1}{4}\big(f(1) + f(k-1)\big) \\ &= \frac{1}{2}\big(f(0) - f(1)\big), \end{aligned}$$

where the last step uses Theorem 2.4, since $\sin^2\theta$ is an even function, $f(1) = f(k-1)$.

$$f(0) = \frac{2h(-k)\sqrt{k}}{k} \quad \text{and} \quad f(1) = \frac{\sqrt{k}}{k}\big(2h(-k) - \chi(1)\big) = \frac{\sqrt{k}}{k}\big(2h(-k) - 1\big),$$

therefore, $\frac{1}{2}(f(0) - f(1)) = \frac{\sqrt{k}}{2k}$. أ

Theorem 5.20: Let χ denote a nonprincipal, real, primitive, odd character modulo *k*, where *k* is an odd integer and $k \geq 7$. Then

$$\sum_{j=0}^{k-1} \chi(j)\sin^2\frac{\pi j}{k}\tan\frac{\pi j}{k} = \sqrt{k}\left(-\frac{1}{2}+(2-4\chi(2))h(-k)\right)$$

$$= \begin{cases} \sqrt{k}\left(-\frac{1}{2}+6h(-k)\right) & \text{if } k \equiv 7 \bmod 8, \\ \sqrt{k}\left(-\frac{1}{2}-2h(-k)\right) & \text{if } k \equiv 3 \bmod 8. \end{cases}$$

Proof: Let $g(n) = \begin{cases} 0 & \text{if } k \mid n, \\ (-1)^{n \bmod k} & \text{otherwise,} \end{cases}$ $\quad h(n) = \dfrac{i\sqrt{k}}{k}\chi(n)$, and

$t(n) = \dfrac{i}{k}\sum_{j=0}^{k-1}\chi(j)\tan\dfrac{\pi j}{k}\omega^{nj}$, then by Theorem 2.2, Theorem 4.9, and Theorem 5.18

$$t(n) = \sum_{m=0}^{k-1} g(n-m)h(m) = \frac{i\sqrt{k}}{k}\sum_{m=0}^{k-1} g(n-m)\chi(m).$$

Let $\quad f(n) = \begin{cases} 1/2 & \text{if } k \mid n, \\ -1/4 & \text{if } k \mid n+1 \text{ or } k \mid n-1, \\ 0 & \text{otherwise.} \end{cases}$

Then using Lemma 2.6 and Corollary 2.5(ii), and since *f* is an even function and *f* vanishes except for $n = 0$, 1 and $k - 1$,

$$\frac{i}{k}\sum_{j=1}^{k-1}\chi(j)\sin^2\frac{\pi j}{k}\tan\frac{\pi j}{k}=\sum_{n=0}^{k-1}f(-n)t(n)$$
$$=\frac{1}{2}t(0)-\frac{1}{4}\big(t(1)+t(k-1)\big)$$
$$=\frac{1}{2}\big(t(0)-t(1)\big),$$

where the last step uses Theorem 2.4, since $\chi(j)\tan(\pi j/k)$ is an even function, $t(1) = t(k-1)$.

From Theorem 5.10, $t(0)=-\frac{i\sqrt{k}}{k}\big(4\chi(2)-2\big)h(-k),$ (5.20.1)

Since g is an odd function, $g(1-m) = -g(m-1)$, and since $\chi(0) = g(0) = 0$, we get

$$t(1)=\frac{i\sqrt{k}}{k}\sum_{m=0}^{k-1}g(1-m)\chi(m)$$
$$=-\frac{i\sqrt{k}}{k}\sum_{m=2}^{k-1}g(m-1)\chi(m)$$
$$=-\frac{i\sqrt{k}}{k}\sum_{m=2}^{k-1}(-1)^{m-1}\chi(m)$$
$$=\frac{i\sqrt{k}}{k}\left(\chi(1)+\sum_{m=1}^{k-1}(-1)^m\chi(m)\right).$$

Since for each odd m, $k-m$ is even, and $-\chi(m) = \chi(k-m)$, we can rewrite the right hand sum above as twice the sum of all the even terms. Thus

$$
\begin{aligned}
t(1) &= \frac{i\sqrt{k}}{k}\left(\chi(1)+2\sum_{m=1}^{(k-1)/2}\chi(2m)\right)\\
&= \frac{i\sqrt{k}}{k}\left(\chi(1)+2\chi(2)\sum_{m=1}^{(k-1)/2}\chi(m)\right)\\
&= \frac{i\sqrt{k}}{k}\big(1+2\chi(2)(2-\chi(2))h(-k)\big)\\
&= \frac{i\sqrt{k}}{k}\big(1+(4\chi(2)-2)h(-k)\big),
\end{aligned}
$$

where the last two steps use $\chi(2) \neq 0$ (since k is odd) and $\sum_{m=1}^{(k-1)/2}\chi(m)=(2-\chi(2))h(-k)$.

Thus

$$
\begin{aligned}
\frac{i}{k}\sum_{j=1}^{k-1}\chi(j)\sin^2\frac{\pi j}{k}\tan\frac{\pi j}{k} &= \frac{1}{2}\big(t(0)-t(1)\big)\\
&= \frac{i\sqrt{k}}{k}\left(-\frac{1}{2}+(2-4\chi(2))h(-k)\right)\\
&= \begin{cases} \dfrac{i\sqrt{k}}{k}\left(-\dfrac{1}{2}+6h(-k)\right) & \textit{if } k\equiv 7 \bmod 8,\\[2ex] \dfrac{i\sqrt{k}}{k}\left(-\dfrac{1}{2}-2h(-k)\right) & \textit{if } k\equiv 3 \bmod 8,\end{cases}
\end{aligned}
$$

where the last step uses Theorem 4.4. أ

Theorem 5.21: Let *k* be a positive odd integer such that $k \equiv 3 \bmod 4$, and let

$$f(n) = \begin{cases} 0 & \textit{if } n = 0, \\ 1 & \textit{if } n \equiv 0,1 \bmod 4 \quad \textit{and} \quad n \not\equiv 0 \bmod k, \\ -1 & \textit{if } n \equiv 2,3 \bmod 4, \quad \textit{and} \quad n \not\equiv 0 \bmod k, \end{cases} \qquad \text{then}$$

$$f(n) = \frac{i}{k}\sum_{j=0}^{k-1} \tan\frac{2\pi j}{k}\,\omega^{nj} \; .$$

Proof: Since *f* has period *k*, then by Theorem 2.1 its Fourier transformation satisfies

$$\hat{f}(j) = \frac{1}{k}\sum_{n=0}^{k-1} f(n)\omega^{-nj} = \sum_{n-1}^{k-1} f(n)\omega^{-nj}.$$

Since *k* is odd, $(-\omega^{-2j})^k = -1$; thus,

$$\begin{aligned} \frac{1}{k}\sum_{n=1}^{k-1}(-\omega^{-2j})^n &= \frac{1}{k}\left(\frac{1-(-\omega^{-2j})^k}{1-(-\omega^{-2j})} - 1\right) \\ &= \frac{1}{k}\left(\frac{2\omega^{2j}}{1+\omega^{2j}} - 1\right) \\ &= \frac{1}{k}\cdot\frac{\omega^{2j}-1}{\omega^{2j}+1} \\ &= \frac{i}{k}\tan\frac{2\pi j}{k}. \end{aligned}$$

Therefore, we need only show $\sum_{n=1}^{k-1}(-\omega^{-2j})^n = \sum_{n=1}^{k-1} f(n)\omega^{-nj}$.

Since $k \equiv 3 \bmod 4$, then $(k-1)/2$ is odd and $(k+1)/2$ is even; thus,

$$\sum_{n=1}^{k-1}(-\omega^{-2j})^n = \sum_{n=1}^{(k-1)/2}(-\omega^{-2j})^n + \sum_{n=(k+1)/2}^{k-1}(-\omega^{-2j})^n$$
$$= (-\omega^{-2j} + \omega^{-4j} - \cdots - \omega^{-(k-1)}) + (\omega^{-j} - \omega^{-3j} + \cdots + \omega^{-(k-2)j}) .$$

Thus for $m \equiv 0,1 \bmod 4$ the coefficient of ω^{-mj} is 1 and for $m \equiv 2,3 \bmod 4$ the coefficient of ω^{-mj} is -1. □

Theorem 5.22: Let χ denote a nonprincipal, real, primitive, odd character modulo k, where k is an odd integer and $k \geq 7$. Then

$$\sum_{j=1}^{k-1}\chi(j)\tan\frac{2\pi j}{k} = \sqrt{k}(2\chi(2)-4)h(-k) = \begin{cases} -2\sqrt{k}h(-k) & \text{if } k \equiv 7 \bmod 8, \\ -6\sqrt{k}h(-k) & \text{if } k \equiv 3 \bmod 8. \end{cases}$$

Proof: Since k is an odd integer and χ is odd, then $k \equiv 3 \bmod 4$. Thus if we let

$$f(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n \equiv 0,1 \bmod 4 \text{ and } n \not\equiv 0 \bmod k, \\ -1 & \text{if } n \equiv 2,3 \bmod 4 \text{ and } n \not\equiv 0 \bmod k, \end{cases} \quad \text{and}$$

$$g(n) = \frac{i\sqrt{k}}{k}\chi(n),$$

then by Theorem 5.18, Theorem 5.21, and Corollary 2.5(i),

$$\begin{aligned}\frac{i}{k}\sum_{j=0}^{k-1}\chi(j)\tan\frac{2\pi j}{k} &= -\sum_{n=0}^{k-1} f(n)g(n) \\ &= -\frac{i\sqrt{k}}{k}\sum_{n=1}^{k-1}(-1)^n\chi(2n) \\ &= -\frac{i\sqrt{k}}{k}\chi(2)\sum_{n=1}^{k-1}(-1)^n\chi(n),\end{aligned}$$

where the second step is using a similar argument as in Theorem 5.21 which showed

$\sum_{n=1}^{k-1} f(n)\omega^{-nj} = \sum_{n=1}^{k-1}(-\omega^{-2j})^n$ due to $k \equiv 3 \bmod 4$.

Since for each odd n, $k - n$ is even and $\chi(n) = -\chi(k - n)$ we can rewrite the sum on the right above as twice the sum of the even terms, thus

$$\begin{aligned}\frac{i}{k}\sum_{j=1}^{k-1}\chi(j)\tan\frac{2\pi j}{k} &= -\frac{i\sqrt{k}}{k}2\chi(2)\sum_{n=1}^{(k-1)/2}\chi(2n) \\ &= -\frac{i\sqrt{k}}{k}2(\chi(2))^2\sum_{n=1}^{(k-1)/2}\chi(n) \\ &= -\frac{i\sqrt{k}}{k}2(2-\chi(2))h(-k) \\ &= \frac{i\sqrt{k}}{k}(2\chi(2)-4)h(-k),\end{aligned}$$

where we have used the facts that since k is odd, then $\chi(2) \neq 0$ and Theorem 2.7,

$\sum_{n=1}^{(k-1)/2}\chi(n) = (2-\chi(2))h(-k)$. أ

Theorem 5.23: Let χ denote a nonprincipal, real, primitive, odd character modulo *k*, where *k* is an odd integer and *k* ≥ 7. Then

$$\sum_{j=1}^{k-1} \chi(j)\sin^2\frac{\pi j}{k}\tan\frac{2\pi j}{k} = \sqrt{k}\left(\frac{1}{2}+(\chi(2)-2)h(-k)\right)$$

$$= \begin{cases} \sqrt{k}\left(\dfrac{1}{2}-h(-k)\right) & \text{if } k\equiv 7 \bmod 8, \\ \sqrt{k}\left(\dfrac{1}{2}-3h(-k)\right) & \text{if } k\equiv 3 \bmod 8. \end{cases}$$

Proof: Let $f(n)=\begin{cases} 1/2 & \text{if } k\mid n, \\ -1/4 & \text{if } k\mid n+1 \text{ or } k\mid n-1, \\ 0 & \text{otherwise,} \end{cases}$ and $g(n)=\sum_{j=1}^{k-1}\chi(j)\tan\frac{2\pi j}{k}\omega^{nj}$.

Then by Theorem 5.18, Theorem 5.21 and Corollary 2.5 (ii)

$$\frac{i}{k}\sum_{j=1}^{k-1}\chi(j)\sin^2\frac{\pi j}{k}\tan\frac{2\pi j}{k} = \sum_{m=0}^{k-1} f(-n)g(m)$$

$$= \sum_{m=0}^{k-1} f(n)g(m)$$

$$= \frac{g(0)}{2}-\frac{g(1)+g(k-1)}{4}.$$

We have from Theorem 5.22 that $g(0)=\dfrac{i\sqrt{k}}{k}2h(-k)(\chi(2)+2).$ (5.23.1)

From Corollary 2.3 and Theorem 5.18,

$$g(n) = \frac{i\sqrt{k}}{k}\sum_{m=0}^{k-1} h(n-m)\chi(n), \text{ where } h(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n \equiv 0,1 \bmod 4 \text{ and } n \not\equiv 0 \bmod k, \\ -1 & \text{if } n \equiv 2,3 \bmod 4 \text{ and } n \not\equiv 0 \bmod k, \end{cases}$$

thus since $h(0) = 0$ and $\chi(0) = 0$, and since tangent is odd, $h(n)$ is odd, then

$$\begin{aligned} g(1) + g(k-1) &= \frac{i\sqrt{k}}{k}\left(\sum_{m=0}^{k-1}\chi(m)h(1-m) + \sum_{m=0}^{k-1}\chi(m)h(k-1-m)\right) \\ &= -\frac{i\sqrt{k}}{k}\left(\sum_{m=2}^{k-1}\chi(m)h(m-1) + \sum_{m=1}^{k-2}\chi(m)h(m+1)\right) \\ &= -\frac{i\sqrt{k}}{k}\left(\chi(k-1) - \chi(1) + \sum_{m=2}^{k-2}\chi(m)\big(h(m+1) + h(m-1)\big)\right) \\ &= \frac{i2\sqrt{k}}{k}, \end{aligned}$$

where the last step uses $-\chi(1) = \chi(k-1) = -1$ and $h(m + 1) + h(m - 1) = 0$ for

$2 \leq m \leq k - 2$.

Therefore,

$$\begin{aligned} \frac{i}{k}\sum_{j=1}^{k-1}\chi(j)\sin^2\frac{\pi j}{k}\tan\frac{2\pi j}{k} &= \frac{g(0)}{2} - \frac{g(1)+g(k-1)}{4} \\ &= \frac{i\sqrt{k}}{k}\left(\frac{(2\chi(2)-4)h(-k)}{2} - \frac{2}{4}\right) \\ &= \frac{i\sqrt{k}}{k}\left(-\frac{1}{2} + (\chi(2)-2)h(-k))\right). \end{aligned}$$

أ

Corollary 5.24: Let χ be the same as above, then

$$\sum_{j=1}^{k-1}\chi(j)\frac{\sin^2(\pi j/k)}{\sin(4\pi j/k)}=\frac{3\sqrt{k}}{2}(\chi(2)-1)h(-k)=\begin{cases}\frac{3\sqrt{k}}{2}h(-k) & \textit{if } k\equiv 7 \bmod 8,\\ 0 & \textit{if } k\equiv 3 \bmod 8.\end{cases}$$

Proof: Since $\frac{1}{\sin(2\theta)}=\frac{1}{2}(\cot\theta+\tan\theta)$, and $\cot(2\theta)=\frac{1}{2}(\cot\theta-\tan\theta)$, then

$$\begin{aligned}\sum_{j=1}^{k-1}\chi(j)\frac{\sin^2(\pi j/k)}{\sin(4\pi j/k)}&=\sum_{j=1}^{k-1}\sin^2\frac{\pi j}{k}\left(\frac{1}{4}\chi(j)\cot\frac{\pi j}{k}-\frac{1}{4}\chi(j)\tan\frac{\pi j}{k}+\frac{1}{2}\chi(j)\tan\frac{2\pi j}{k}\right)\\ &=\frac{1}{4}\sum_{j=1}^{k-1}\chi(j)\sin^2\frac{\pi j}{k}\cot\frac{\pi j}{k}-\frac{1}{4}\sum_{j=1}^{k-1}\chi(j)\sin^2\frac{\pi j}{k}\tan\frac{\pi j}{k}\\ &\quad+\frac{1}{2}\sum_{j=1}^{k-1}\chi(j)\sin^2\frac{\pi j}{k}\tan\frac{2\pi j}{k}.\end{aligned}$$

Using Theorems 5.19, 5.20 and 5.23 we get

$$\begin{aligned}\sum_{j=1}^{k-1}\chi(j)\frac{\sin^2(\pi j/k)}{\sin(4\pi j/k)}&=\sqrt{k}\left(\frac{1}{4}\cdot\frac{1}{2}-\frac{1}{4}\left(-\frac{1}{2}+(2-4\chi(2))h(-k)\right)+\frac{1}{2}\left(-\frac{1}{2}+(\chi(2)-2)h(-k)\right)\right)\\ &=\frac{3\sqrt{k}(1-\chi(2))h(-k)}{2}.\end{aligned}$$

From Theorem 4.4, this further simplifies to,

$$\sum_{j=1}^{k-1}\chi(j)\frac{\sin^2(\pi j/k)}{\sin(4\pi j/k)}=\begin{cases}\frac{3\sqrt{k}}{2}h(-k) & \textit{if } k\equiv 7 \bmod 8,\\ 0 & \textit{if } k\equiv 3 \bmod 8.\end{cases}$$

Using these methods it is possible to find similar proofs for other general theorems discovered by Berndt and Zaharescu [2] as long as the function in the summation does not involve trigonometric functions with denominators higher than power 2.

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