

Generalized Dedekind–Bernoulli Sums

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CERTIFICATION OF APPROVAL

I certify that I have read *Generalized Dedekind–Bernoulli Sums* by Anastasia Chavez and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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The finite arithmetic sum called a Dedekind sum appears in many areas of mathematics, such as topology, geometric combinatorics, algorithmic complexity, and algebraic geometry. Dedekind sums exhibit many beautiful properties, the most famous being Dedekind’s reciprocity law. Since Dedekind, Apostol, Rademacher and others have generalized Dedekind sums to involve Bernoulli polynomials. In 1995, Hall, Wilson and Zagier introduced the 3-variable Dedekind-like sum and proved a reciprocity relation. In this paper we introduce an n -variable generalization of Hall, Wilson and Zagier’s sum called a multivariable Dedekind–Bernoulli sum and prove a reciprocity law that generalizes the generic case of Hall, Wilson and Zagier’s reciprocity theorem. Our proof uses a novel, combinatorial approach that simplifies the proof of Hall, Wilson and Zagier’s reciprocity theorem and aids in proving the general 4-variable extension of Hall, Wilson and Zagier’s reciprocity theorem.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee

Date

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TABLE OF CONTENTS

1	Introduction	1
2	More History, Definitions and Notations	4
2.1	Bernoulli Polynomials and More	4
2.2	Dedekind and Bernoulli meet	9
3	The Multivariate Dedekind–Bernoulli sum	12
4	A New Proof of Hall–Wilson–Zagier’s Reciprocity Theorem	42
5	The 4-variable reciprocity theorem	53
	Bibliography	70

Chapter 1

Introduction

In the 1880's, Richard Dedekind developed the finite sum that today is called the *Dedekind sum* [12].

Definition 1.1 (Classical Dedekind Sum). For any positive integers a and b , the *classical Dedekind sum* $s(a, b)$ is defined to be

$$s(a, b) = \sum_{h \bmod b} \bar{B}_1\left(\frac{h}{b}\right) \bar{B}_1\left(\frac{ha}{b}\right), \quad (1.1)$$

where

$$\bar{B}_1(x) = \begin{cases} \{x\} - 1/2 & (x \notin \mathbb{Z}), \\ 0 & (x \in \mathbb{Z}), \end{cases} \quad (1.2)$$

and we call $x - [x] = \{x\}$ the *fractional part* of x . The sum $\sum_{h \bmod b}$ is interpreted as the periodic sum over all values of h modulo b where $h \in \mathbb{Z}$.

Dedekind arrived at the Dedekind sums while studying the η -*function*. Although Dedekind sums are studied in number theory, they have appeared in many areas of mathematics such as analytical number theory [6], combinatorial geometry [4], topology [9], and algorithmic complexity [8]. Since Dedekind sums have no closed form, calculating their sum can become very time intensive. Thankfully, Dedekind discovered an invaluable tool that simplifies Dedekind sum calculations immensely, called *reciprocity*. Although Dedekind proved the following reciprocity relation by a transcendental method, Rademacher recognized its elementary nature and developed multiple proofs of Dedekind's reciprocity theorem [12] addressing its arithmetic significance.

Theorem 1.1 (Dedekind Reciprocity [13]). *For any positive integers a and b with $(a, b) = 1$,*

$$s(a, b) + s(b, a) = \frac{1}{12} \left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab} \right) - \frac{1}{4}. \quad (1.3)$$

Since Dedekind, many mathematicians such as Rademacher [11], Carlitz [5], and Apostol [1] have introduced Dedekind-like sums that generalize $\bar{B}_1(u)$ to periodized *Bernoulli polynomials*, $\bar{B}_k(u)$ (see Definition 2.4). In suit with Dedekind, reciprocity relations have been proven for the respective Dedekind-like sums.

In 1995, Hall, Wilson and Zagier [7] introduced a generalization of Dedekind sums (see Definition 2.4) that involve three variables and introduce shifts. They proved a reciprocity theorem that is a wonderful example of how these relations express the symmetry inherent in Dedekind-like sums and characterize their potential to be evaluated so swiftly.

Most recent, Bayad and Raouj [3] have introduced a generalization of Hall, Wilson and Zagier's sum and proved a reciprocity theorem that shows the non-generic case of Hall, Wilson and Zagier's reciprocity theorem is a special case.

In this paper we introduce a different generalization of Hall, Wilson and Zagier's sum called the *multivariate Dedekind–Bernoulli sum* (see Definition 3.1) and prove our main result, a reciprocity theorem (see Theorem 3.1). Along with our main result, we show that the generic case of Hall, Wilson and Zagier's reciprocity theorem follows from the proof of our reciprocity theorem and that one can say more about the 4-variable version of Hall, Wilson and Zagier's sum than is covered by the reciprocity theorem given by Bayad and Raouj, as well as our own.

Chapter 2

More History, Definitions and Notations

We now begin working towards rediscovering reciprocity theorems for Dedekind-like sums.

2.1 Bernoulli Polynomials and More

Bernoulli polynomials and numbers are a pervasive bunch, which show up in many areas of mathematics such as number theory and probability [2].

Definition 2.1 (Bernoulli functions). For $u \in \mathbb{R}$, the Bernoulli function $B_k(u)$ is defined through the generating function

$$\frac{ze^{uz}}{e^z - 1} = \sum_{k \geq 0} \frac{B_k(u)}{k!} z^k. \quad (2.1)$$

Definition 2.2 (Bernoulli numbers). The Bernoulli numbers are $B_k := B_k(0)$ and have the generating function

$$\frac{z}{e^z - 1} = \sum_{k \geq 0} \frac{B_k}{k!} z^k. \quad (2.2)$$

It has been proven that Bernoulli functions are in fact polynomials [2, Theorem 12.12].

Theorem 2.1 (k -th Bernoulli polynomial). *The functions $B_k(u)$ are polynomials in u given by*

$$B_k(u) = \sum_{n=0}^k \binom{k}{n} B_n u^{k-n}. \quad (2.3)$$

An important property of k -th Bernoulli polynomials is the Fourier expansion formula [2, Theorem 12.19]

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi i u}}{k^m} = -\frac{(2\pi i)^m}{m!} \bar{B}_m(u). \quad (2.4)$$

In the non-absolutely convergent case $m = 1$, the sum is to be interpreted as a Cauchy principal value.

Property 2.4 defines the unique periodic function $\bar{B}_m(u)$ with period 1 that coincides with $B_m(u)$ on $[0, 1)$, except we set $\bar{B}_1(u) = 0$ for $u \in \mathbb{Z}$ [7].

Lemma 2.2 (Raabe's formula [7]). For $a \in \mathbb{N}, x \in \mathbb{R}$,

$$\sum_{f \bmod a} \bar{B}_m \left(x + \frac{f}{a} \right) = a^{1-m} \bar{B}_m(ax).$$

Proof. Begin by embedding the right-hand side into a generating function:

$$\begin{aligned} \sum_{m=0}^{\infty} a^{1-m} \bar{B}_m(ax) \frac{Y^{m-1}}{m!} &= \sum_{m \geq 0} \bar{B}_m(ax) \frac{(Y/a)^{m-1}}{m!} \\ &= \frac{e^{ax \cdot \frac{Y}{a}}}{e^{\frac{Y}{a}} - 1} \\ &= \frac{e^{xY}}{e^{\frac{Y}{a}} - 1}. \end{aligned}$$

Now, embed the left-hand side into a generating function:

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{f \bmod a} \bar{B}_m \left(x + \frac{f}{a} \right) \frac{Y^{m-1}}{m!} &= \sum_{f=0}^{a-1} \frac{e^{(x+\frac{f}{a})Y}}{e^Y - 1} \\ &= \frac{e^{xY}}{e^Y - 1} + \frac{e^{(x+\frac{1}{a})Y}}{e^Y - 1} + \frac{e^{(x+\frac{2}{a})Y}}{e^Y - 1} + \cdots + \frac{e^{(x+\frac{a-1}{a})Y}}{e^Y - 1} \\ &= \frac{e^{xY} (1 + e^{\frac{1}{a}Y} + e^{\frac{2}{a}Y} + \cdots + e^{\frac{a-1}{a}Y})}{e^Y - 1} \end{aligned}$$

(applying the geometric series identity $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$ where $r = e^{\frac{Y}{a}}$)

$$\begin{aligned} &= \frac{e^{xY}}{e^Y - 1} \cdot \frac{1 - e^{\frac{Y}{a} \cdot a}}{1 - e^{\frac{Y}{a}}} \\ &= \frac{-e^{xY}}{1 - e^{\frac{Y}{a}}} \\ &= \frac{e^{xY}}{e^{\frac{Y}{a}} - 1}. \end{aligned}$$

We have deduced the generating functions of the right and left hand sides of Raabe's formula are equal, so the coefficients of the generating functions are equal.

That is,

$$\sum_{f \bmod a} \bar{B}_m\left(x + \frac{f}{a}\right) = a^{1-m} \bar{B}_m(ax).$$

□

Definition 2.3. Let

$$\beta(\alpha, Y) = \sum_{m=0}^{\infty} \frac{\bar{B}_m(\alpha)}{m!} Y^{m-1}, \quad (2.5)$$

where $\alpha \in \mathbb{R}$.

Lemma 2.3.

$$\beta(\alpha, Y) = \begin{cases} \frac{1}{2} \cdot \frac{e^Y + 1}{e^Y - 1} & \text{for } \alpha \in \mathbb{Z}, \\ \frac{e^{\{\alpha\}Y}}{e^Y - 1} & \text{for } \alpha \notin \mathbb{Z}. \end{cases} \quad (2.6)$$

Proof. Recall that $\bar{B}_m(\alpha)$ coincides with $B_m(\alpha)$ on the interval $[0, 1)$ and is periodic with period 1. Thus, the cases we are interested in are when α is not in \mathbb{Z} and $\alpha = 0$. We will first evaluate $\beta(\alpha, Y)$ for $\alpha \notin \mathbb{Z}$.

$$\begin{aligned} \beta(\alpha, Y) &= \sum_{m=0}^{\infty} \frac{\bar{B}_m(\alpha)}{m!} Y^{m-1} \\ &= \sum_{m=0}^{\infty} \frac{B_m(\{\alpha\})}{m!} Y^{m-1} \\ &= \frac{e^{\{\alpha\}Y}}{e^Y - 1}. \end{aligned}$$

Now, we look at the case $\alpha = 0$. We first write out the first few terms of $\beta(\alpha, Y)$ evaluated at 0, then manipulate the sum to have period 1. We use the well known Bernoulli polynomial identities $B_0(\alpha) = 1$ and $B_m(\alpha) = 0$ for odd $m \geq 3$, as well as

the first few Bernoulli numbers [2].

$$\begin{aligned}
B_0(0)Y^{-1} + B_1(0) + B_2(0)\frac{Y}{2!} + B_3(0)\frac{Y^2}{3!} + B_4(0)\frac{Y^3}{4!} + \dots &= \frac{1}{e^Y - 1} \\
1 - \frac{1}{2} + \frac{1}{6} + 0 - \frac{1}{30} + 0 + \dots &= \frac{1}{e^Y - 1} \\
1 + \frac{1}{6} + 0 - \frac{1}{30} + 0 + \dots &= \frac{1}{e^Y - 1} + \frac{1}{2} \\
&= \frac{2 + e^Y - 1}{2(e^Y - 1)} \\
&= \frac{1}{2} \cdot \frac{e^Y + 1}{(e^Y - 1)}.
\end{aligned}$$

Recall that $\bar{B}_1(\alpha) = 0$ for all $\alpha \in \mathbb{Z}$, and so we have the following identity:

$$\bar{B}_0(0)Y^{-1} + \bar{B}_1(0) + \bar{B}_2(0)\frac{Y}{2!} + \bar{B}_3(0)\frac{Y^2}{3!} + \bar{B}_4(0)\frac{Y^3}{4!} + \dots = \frac{1}{2} \cdot \frac{e^Y + 1}{(e^Y - 1)}.$$

When α is an integer $\{\alpha\} = 0$, thus the above identity holds for $\alpha \in \mathbb{Z}$. \square

2.2 Dedekind and Bernoulli meet

Dedekind-like sums involving higher-order Bernoulli polynomials have been defined by Apostol [1], Carlitz [5], Mikolás [10] and Hall, Wilson and Zagier [7]. The latter introduced the *generalized Dedekind–Rademacher sum*, the function we wish to generalize to n variables.

Definition 2.4. The *generalized Dedekind–Rademacher sum* is

$$S_{m,n} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} = \sum_{h \bmod c} \bar{B}_m \left(a \frac{h+z}{c} - x \right) \bar{B}_n \left(b \frac{h+z}{c} - y \right), \quad (2.7)$$

where $a, b, c \in \mathbb{Z}_{>0}$ and $x, y, z \in \mathbb{R}$.

The generalized Dedekind–Rademacher sum satisfies certain reciprocity relations that mix various pairs of indices (m, n) and so are most conveniently stated in terms of the generating function

$$\Omega \begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} = \sum_{m,n \geq 0} \frac{1}{m!n!} S_{m,n} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} (X/a)^{m-1} (Y/b)^{n-1}, \quad (2.8)$$

where $Z = -X - Y$.

Hall, Wilson and Zagier proved the following reciprocity theorem for the generalized Dedekind–Rademacher sum.

Theorem 2.4. [7] *Let $a, b, c \in \mathbb{Z}_{>0}$ and pairwise relatively prime, $x, y, z \in \mathbb{R}$, and*

X, Y, Z three variables such that $X + Y + Z = 0$. Then

$$\begin{aligned} \Omega \begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} + \Omega \begin{pmatrix} b & c & a \\ y & z & x \\ Y & Z & X \end{pmatrix} + \Omega \begin{pmatrix} c & a & b \\ z & x & y \\ Z & X & Y \end{pmatrix} \\ = \begin{cases} -1/4 & \text{if } (x, y, z) \in (a, b, c)\mathbb{R} + \mathbb{Z}^3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.9)$$

To clarify, we call the result

$$\Omega \begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} + \Omega \begin{pmatrix} b & c & a \\ y & z & x \\ Y & Z & X \end{pmatrix} + \Omega \begin{pmatrix} c & a & b \\ z & x & y \\ Z & X & Y \end{pmatrix} = 0$$

the generic case of the above reciprocity theorem and this occurs most of the time. The one exception is when (x, y, z) are chosen in such a way that the involved Bernoulli periodic polynomials are evaluated at integer values. In the following chapter we introduce our main theorem which generalizes Hall, Wilson and Zagier's reciprocity theorem using a simple combinatorial approach and will make clear how the exceptional integer values come to light.

Chapter 3

The Multivariate Dedekind–Bernoulli sum

We now introduce our main object of study, the *multivariate Dedekind–Bernoulli sum*.

Definition 3.1. For a fixed integer $n \geq 2$, we consider positive integers (p_1, p_2, \dots, p_n) and (a_1, a_2, \dots, a_n) , and real numbers (x_1, x_2, \dots, x_n) . For $1 \leq k \leq n$, set

$$A_k = (a_1, a_2, \dots, \hat{a}_k, \dots, a_n), \quad X_k = (x_1, x_2, \dots, \hat{x}_k, \dots, x_n)$$

$$\text{and } P_k = (p_1, p_2, \dots, \hat{p}_k, \dots, p_n),$$

where \hat{a}_k means we omit the entry a_k . Then the multivariate Dedekind–Bernoulli

sum is

$$S_{P_k} \begin{pmatrix} A_k & a_k \\ X_k & x_k \end{pmatrix} = \sum_{h \bmod a_k} \prod_{\substack{i=1 \\ i \neq k}}^n \bar{B}_{p_i} \left(a_i \frac{h + x_k}{a_k} - x_i \right). \quad (3.1)$$

We see that if $(a_1, a_2, a_3) = (a, b, c)$, $(x_1, x_2, x_3) = (x, y, z)$, and $P_3 = (m, n)$ then $A_3 = (a, b)$, $X_3 = (x, y)$ and thus we recover Hall, Wilson and Zagier's generalized Dedekind–Rademacher sum,

$$S_{m,n} \begin{pmatrix} A_3 & c \\ X_3 & z \end{pmatrix} = \sum_{h \bmod c} \bar{B}_m \left(a \frac{h+z}{c} - x \right) \bar{B}_n \left(b \frac{h+z}{c} - y \right).$$

Moreover, by letting $(x_1, x_2, \dots, x_n) = \vec{0}$ and $P_k = (p_1, p_2, \dots, \hat{p}_k, \dots, p_n)$, we recover another generalization of Hall, Wilson and Zagier's generalized Dedekind–Rademacher sum introduced by Bayad and Raouj in 2009 [3]. Bayad and Raouj's *multiple Dedekind–Rademacher sum* is thus defined

$$S_{P_k} \begin{pmatrix} A_k & a_k \\ P_k & p_k \end{pmatrix} = \sum_{h \bmod a_k} \prod_{\substack{i=1 \\ i \neq k}}^n \bar{B}_{p_i} \left(\frac{a_i h}{a_k} \right). \quad (3.2)$$

Similar to the generalized Dedekind–Rademacher sum, a multivariate Dedekind–Bernoulli sum reciprocity relation mixes various $(n-1)$ -tuples of indices and is most conveniently stated in terms of generating functions. For nonzero variables

(y_1, y_2, \dots, y_n) where $y_n = -y_1 - y_2 - \dots - y_{n-1}$, let $Y_k = (y_1, y_2, \dots, \hat{y}_k, \dots, y_n)$.

Then

$$\begin{aligned} & \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} \\ &= \sum_{p_1} \dots \widehat{\sum_{p_k}} \dots \sum_{p_n} \frac{1}{p_1! p_2! \dots p_{k-1}! p_{k+1}! \dots p_n!} S_{P_k} \begin{pmatrix} A_k & a_k \\ X_k & x_k \end{pmatrix} \prod_{\substack{i=1 \\ i \neq k}}^n \left(\frac{y_i}{a_i} \right)^{p_i-1}. \end{aligned}$$

The series of summands $\sum_{p_1} \dots \widehat{\sum_{p_k}} \dots \sum_{p_n}$ is understood as summations over all non-negative integers $p_1, \dots, \hat{p}_k, \dots, p_n$.

Our main result is the following reciprocity law involving multivariate Dedekind–Bernoulli sums.

Theorem 3.1. *Let $(x_1, x_2, \dots, x_n) \in \mathbb{R}$ and $(p_1, p_2, \dots, p_n), (a_1, a_2, \dots, a_n) \in \mathbb{Z}_{>0}$ where $(a_u, a_v) = 1$ for all $1 \leq u < v \leq n$. For $1 \leq k \leq n$, let*

$$A_k = (a_1, a_2, \dots, \hat{a}_k, \dots, a_n), \quad X_k = (x_1, x_2, \dots, \hat{x}_k, \dots, x_n)$$

$$\text{and } P_k = (p_1, p_2, \dots, \hat{p}_k, \dots, p_n).$$

For nonzero variables y_1, y_2, \dots, y_n , let $Y_k = (y_1, y_2, \dots, \hat{y}_k, \dots, y_n)$ such that $y_1 +$

$y_2 + \cdots + y_n = 0$, then

$$\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} = 0$$

if $\frac{x_u - h_u}{a_u} - \frac{x_v - h_v}{a_v} \notin \mathbb{Z}$ whenever $1 \leq u < v \leq n$ and $h_u, h_v \in \mathbb{Z}$.

We now embark on a journey of transforming $\Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ into a simplified and accessible form that will make proving Theorem 3.1 as easy as a matching game. Before we continue, we first introduce two useful lemmas for the fractional-part function.

Lemma 3.2. Given $a, b, c \in \mathbb{R}$,

$$\{a - b\} - \{a - c\} > 0 \Rightarrow \{a - b\} - \{a - c\} = \{c - b\}.$$

Proof. Our goal is to show

$$\{a - b\} - \{a - c\} - \{c - b\} = 0.$$

By definition $\{x\} = x - *$, where $*$ is an integer and $x \in \mathbb{R}$. Thus, the above

sum can be rewritten as follows,

$$\begin{aligned}
 & \{a - b\} - \{a - c\} - \{c - b\} \\
 &= a - b - *_1 - (a - c - *_2) - (c - b - *_3) \\
 &= - *_1 + *_2 + *_3 \\
 &= *,
 \end{aligned}$$

where all $*$'s are integers.

Given our assumption and the fact that $0 \leq \{x\} < 1$, it follows

$$\begin{aligned}
 & 0 \leq \{a - b\} - \{a - c\} < 1 \\
 \Rightarrow & -1 < \{a - b\} - \{a - c\} - \{c - b\} < 1 \\
 \Rightarrow & -1 < * < 1 \\
 \Rightarrow & * = 0 .
 \end{aligned}$$

□

Lemma 3.3. Given $a, b, c \in \mathbb{R}$,

$$\{a - b\} - \{a - c\} < 0 \Rightarrow \{a - b\} - \{a - c\} = -\{b - c\}.$$

Proof. Our goal is to show

$$\{a - b\} - \{a - c\} + \{b - c\} = 0.$$

By definition $\{x\} = x - *$, where $*$ is an integer and $x \in \mathbb{R}$. Thus, the above sum can be rewritten as follows,

$$\begin{aligned} \{a - b\} - \{a - c\} + \{b - c\} &= a - b - *_1 - (a - c - *_2) + b - c - *_3 \\ &= - *_1 + *_2 - *_3 \\ &= *, \end{aligned}$$

where all $*$'s are integers.

Given our assumption, $0 \leq \{x\} < 1$, and $0 \leq \{b - c\} < 1$, it follows

$$\begin{aligned} -1 &< \{a - b\} - \{a - c\} < 0 \\ \Rightarrow -1 &< \{a - b\} - \{a - c\} + \{b - c\} < 1 \\ \Rightarrow -1 &< * < 1 \\ \Rightarrow * &= 0 . \end{aligned}$$

□

We will now begin manipulating $\Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ using the following identity for multivariate Dedekind–Bernoulli sums.

$$S_{P_k} \begin{pmatrix} A_k & a_k \\ X_k & x_k \end{pmatrix} \prod_{\substack{i=1 \\ i \neq k}}^n a_i^{1-m_i} = \sum_{h \bmod a_k} \prod_{\substack{i=1 \\ i \neq k}}^n \bar{B}_{p_i} \left(a_i \frac{h + x_k}{a_k} - x_i \right) a_i^{1-m_i}$$

(applying 2.2, Raabe’s formula)

$$= \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n \bar{B}_{p_i} \left(\frac{x_k + h_k}{a_k} - \frac{x_i + h_i}{a_i} \right),$$

where $\sum_H = \sum_{h_1 \bmod a_1} \sum_{h_2 \bmod a_2} \cdots \sum_{h_n \bmod a_n}$ includes the original summand h

we now call h_k . Let $r_i = \frac{x_i + h_i}{a_i}$, then

$$\begin{aligned}
& \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} \\
&= \sum_{p_1} \cdots \widehat{\sum_{p_k}} \cdots \sum_{p_n} \frac{1}{p_1! p_2! \cdots p_{k-1}! p_{k+1}! \cdots p_n!} S_{P_k} \begin{pmatrix} A_k & a_k \\ X_k & x_k \end{pmatrix} \prod_{\substack{i=1 \\ i \neq k}}^n \left(\frac{y_i}{a_i} \right)^{p_i-1} \\
&= \sum_H \sum_{p_1} \cdots \widehat{\sum_{p_k}} \cdots \sum_{p_n} \frac{1}{p_1! p_2! \cdots p_{k-1}! p_{k+1}! \cdots p_n!} \prod_{\substack{i=1 \\ i \neq k}}^n \bar{B}_{p_i}(r_k - r_i) \cdot y_i^{p_i-1} \\
&= \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n \beta(r_k - r_i, y_i), \tag{3.3}
\end{aligned}$$

where Definition 2.3 is applied in the final equality. It is now clear that (3.3) depends on the differences $r_k - r_i$ for $1 \leq k < i \leq n$ and all $\beta(r_k - r_i, y_i)$ depend on whether or not these differences are integers (see 2.3). From now on we assume the differences are not integers, which is analogous to the generic case of Theorem 2.4.

Consider the overall sum of $\Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ as k ranges from 1 to n .

$$\begin{aligned} \sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} &= \sum_{k=1}^n \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n \beta(r_k - r_i, y_i) \\ &= \sum_{k=1}^{n-1} \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n \beta(r_k - r_i, y_i) + \sum_H \prod_{j=1}^{n-1} \beta(r_n - r_j, y_j) \end{aligned}$$

(since $\{r_k - r_i\} \neq 0$ for each $1 \leq k < i \leq n$, we apply Lemma 2.3)

$$\begin{aligned} &= \sum_{k=1}^{n-1} \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n \frac{e^{\{r_k - r_i\}y_i}}{e^{y_i} - 1} + \sum_H \prod_{j=1}^{n-1} \frac{e^{\{r_n - r_j\}y_j}}{e^{y_j} - 1} \\ &= \sum_{k=1}^{n-1} \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n \frac{e^{\{r_k - r_i\}y_i}}{e^{y_i} - 1} \cdot \frac{e^{y_k} - 1}{e^{y_k} - 1} + \sum_H \prod_{j=1}^{n-1} \frac{e^{\{r_n - r_j\}y_j}}{e^{y_j} - 1} \cdot \frac{e^{y_n} - 1}{e^{y_n} - 1} \\ &= \frac{\sum_{k=1}^{n-1} \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n e^{\{r_k - r_i\}y_i} \cdot e^{y_k} - \sum_{k=1}^{n-1} \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n e^{\{r_k - r_i\}y_i} \\ &\quad + \sum_H \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j} \cdot e^{y_n} - \sum_H \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j}}{\prod_{f=1}^n (e^{y_f} - 1)}. \end{aligned}$$

We will drop the denominator and continue to focus only on the numerator:

$$\begin{aligned}
& \sum_{k=1}^{n-1} \sum_H e^{\{r_k-r_n\}y_n+y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k-r_i\}y_i} - \sum_{k=1}^{n-1} \sum_H e^{\{r_k-r_n\}y_n} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k-r_i\}y_i} \\
& + \sum_H \prod_{j=1}^{n-1} e^{\{r_n-r_j\}y_j} \cdot e^{y_n} - \sum_H \prod_{j=1}^{n-1} e^{\{r_n-r_j\}y_j} \\
& = \sum_{k=1}^{n-1} \sum_H e^{\{r_k-r_n\}(-y_1-\dots-y_{n-1})+y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k-r_i\}y_i} \\
& - \sum_{k=1}^{n-1} \sum_H e^{\{r_k-r_n\}(-y_1-\dots-y_{n-1})} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k-r_i\}y_i} \\
& + \sum_H \prod_{j=1}^{n-1} e^{\{r_n-r_j\}y_j} \cdot e^{(-y_1-\dots-y_{n-1})} - \sum_H \prod_{j=1}^{n-1} e^{\{r_n-r_j\}y_j}.
\end{aligned}$$

Recall that $1 - \{r_i - r_j\} = \{-r_j - r_i\}$ since $r_i - r_j \notin \mathbb{Z}$. Thus the numerator equals

$$\begin{aligned}
& \sum_{k=1}^{n-1} \sum_H e^{\{r_n-r_k\}y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k-r_i\}-\{r_k-r_n\}y_i} \\
& - \sum_{k=1}^{n-1} \sum_H e^{-\{r_k-r_n\}y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k-r_i\}-\{r_k-r_n\}y_i} \\
& + \sum_H \prod_{j=1}^{n-1} e^{-\{r_j-r_n\}y_j} - \sum_H \prod_{j=1}^{n-1} e^{\{r_n-r_j\}y_j}.
\end{aligned} \tag{3.4}$$

In order to prove Theorem 3.1, we will show that the exponents of opposite signed terms in the numerator can be paired and so the sum vanishes. Examining (3.4),

we can see that only three types of exponents appear, $\{r_n - r_i\}$, $-\{r_i - r_n\}$, and $\{r_i - r_j\} - \{r_i - r_n\}$. A critical characteristic of the latter difference is that, by applying Lemma 3.2 or Lemma 3.3, $\{r_j - r_i\} - \{r_j - r_n\} = \{r_n - r_i\}$ or $-\{r_i - r_n\}$, respectively. This means the exponents can be condensed to just the first two forms, $\{r_n - r_i\}$ and $-\{r_i - r_n\}$. Moreover, we can represent these numbers by their sign since the sign of $\{r_j - r_i\} - \{r_j - r_n\}$ determines if it is equal to $\{r_n - r_i\}$ or $-\{r_i - r_n\}$. This lends us to the following representations:

$$\{r_n - r_i\} = +, \quad -\{r_i - r_n\} = - \quad \text{and} \quad C_{ij} = \text{sign}(\{r_j - r_i\} - \{r_j - r_n\}).$$

Thus every term of $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ can be represented by a sign vector, which we will state explicitly.

The exponent corresponding to $k = 1$ in

$$\sum_{k=1}^{n-1} \sum_H e^{\{r_n - r_k\}y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{(\{r_k - r_i\} - \{r_k - r_n\})y_i}$$

is

$$\{r_n - r_1\}y_1 + (\{r_1 - r_2\} - \{r_1 - r_n\})y_2 + \cdots + (\{r_1 - r_{n-1}\} - \{r_1 - r_n\})y_{n-1}$$

and is represented as the sign vector

$$(+, C_{12}, \dots, C_{1,n-1}).$$

Similarly, the $(k = 1)$ -exponent in

$$-\sum_{k=1}^{n-1} \sum_H e^{-\{r_k - r_n\}y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k - r_i\} - \{r_k - r_n\}y_i}$$

is

$$-\{r_1 - r_n\}y_1 + (\{r_1 - r_2\} - \{r_1 - r_n\})y_2 + \dots + (\{r_1 - r_{n-1}\} - \{r_1 - r_n\})y_{n-1},$$

and the sign vector representation is

$$(-, C_{12}, \dots, C_{1,n-1}).$$

Finally, the terms

$$\sum_H \prod_{j=1}^{n-1} e^{-\{r_j - r_n\}y_j} \text{ and } -\sum_H \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j}$$

are represented as the respective sign vectors

$$(-, -, \dots, -) \text{ and } (+, +, \dots, +).$$

Note that if the sign of, say, $\{r_1 - r_n\}$ and $\{r_4 - r_n\}$ are the same, it is not necessarily true that $\{r_1 - r_n\} = \{r_4 - r_n\}$. We will address this further in the following argument.

To prove that the terms of $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ do in fact cancel, we proceed to construct two matrices M_n and M'_n that consist of the sign vector representations of the terms of $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ and will show that $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} = 0$ equates to proving $M_n = M'_n$ after row swapping.

Let M_n be the matrix of all sign vectors representing the exponents of the positive terms of $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ and let M'_n be the matrix of all sign vectors representing the exponents of the negative terms. This means the sign vector that represents every exponent from the positive terms $\sum_{k=1}^{n-1} \sum_H e^{\{r_n - r_k\}y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k - r_i\} - \{r_k - r_n\}y_i}$ for each k is placed in the k th row of matrix M_n .

Similarly, the sign vector representation for every exponent from the negative terms $-\sum_{k=1}^{n-1} \sum_H e^{-\{r_k - r_n\}y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k - r_i\} - \{r_k - r_n\}y_i}$ for each k is placed in the k th

row of the matrix M'_n .

Finally, place the sign vector representing $\sum_H \prod_{j=1}^{n-1} e^{-\{r_j - r_n\}y_j}$ in the last row of M_n and the sign vector representing $-\sum_H \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j}$ in the last row of M'_n .

Notice that the placement of entry C_{ki} depends on the indices k and i of each term in $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ and given the symmetry among the terms, C_{ki} lives in the same row and column in both matrices. This latter fact implies that if $\text{sign}(\{r_n - r_i\}) = \text{sign}(\{r_n - r_j\})$ then $\{r_n - r_i\} = \{r_n - r_j\}$ only when $i = j$. Thus we have constructed the following matrices,

$$M_n = \begin{pmatrix} + & C_{12} & C_{13} & \cdots & C_{1,n-1} \\ C_{21} & + & C_{23} & \cdots & C_{2,n-1} \\ C_{31} & C_{32} & + & \cdots & C_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1,1} & C_{n-1,2} & C_{n-1,3} & \cdots & + \\ - & - & \cdots & - & - \end{pmatrix}$$

and

$$M'_n = \begin{pmatrix} - & C_{12} & C_{13} & \cdots & C_{1,n-1} \\ C_{21} & - & C_{23} & \cdots & C_{2,n-1} \\ C_{31} & C_{32} & - & \cdots & C_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1,1} & C_{n-1,2} & C_{n-1,3} & \cdots & - \\ + & + & \cdots & + & + \end{pmatrix}.$$

Last, we will show that $M_n = M'_n$ after row swapping implies $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ vanishes. Assume $M_n = M'_n$ up to row swapping. Then for each sign row vector

$$(C_{s1}, \dots, C_{s,s-1}, +, C_{s,s+1}, \dots, C_{s,n-1}) \in M_n$$

there exists

$$(C_{t1}, \dots, C_{t,t-1}, -, C_{t,t+1}, \dots, C_{t,n-1}) \in M'_n$$

such that

$$(C_{s1}, \dots, C_{s,s-1}, +, C_{s,s+1}, \dots, C_{s,n-1}) = (C_{t1}, \dots, C_{t,t-1}, -, C_{t,t+1}, \dots, C_{t,n-1}).$$

Also, we have for some row f in M_n

$$(C_{f1}, \dots, C_{f,f-1}, +, C_{f,f+1}, \dots, C_{f,n-1}) = (+, \dots, +)$$

and for some row g in M'_n

$$(-, \dots, -) = (C_{g1}, \dots, C_{g,g-1}, +, C_{g,g+1}, \dots, C_{g,n-1}).$$

For each identity, we will show that the sign row vectors correspond to canceling terms of $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$. Beginning with $(C_{s1}, \dots, C_{s,s-1}, +, C_{s,s+1}, \dots, C_{s,n-1}) \in M_n$, this corresponds to the exponent

$$\begin{aligned} & (\{r_s - r_1\} - \{r_s - r_n\}) y_1 + \dots + (\{r_s - r_{s-1}\} - \{r_s - r_n\}) y_{s-1} + \{r_n - r_s\} y_s \\ & + (\{r_s - r_{s+1}\} - \{r_s - r_n\}) y_{s+1} + \dots + (\{r_s - r_{n-1}\} - \{r_s - r_n\}) y_{n-1}, \end{aligned}$$

and more specifically, represents the positive term

$$e^{\left(\begin{aligned} & (\{r_s - r_1\} - \{r_s - r_n\}) y_1 + \dots + (\{r_s - r_{s-1}\} - \{r_s - r_n\}) y_{s-1} + \{r_n - r_s\} y_s \\ & + (\{r_s - r_{s+1}\} - \{r_s - r_n\}) y_{s+1} + \dots + (\{r_s - r_{n-1}\} - \{r_s - r_n\}) y_{n-1} \end{aligned} \right)}.$$

Similarly, $(C_{t1}, C_{t2}, \dots, C_{t,t-1}, -, C_{t,t+1}, \dots, C_{t,n-1}) \in M'_n$ represents the exponent

$$\begin{aligned} & (\{r_t - r_1\} - \{r_t - r_n\}) y_1 + \dots + (\{r_t - r_{t-1}\} - \{r_t - r_n\}) y_{t-1} - \{r_t - r_t\} y_t \\ & + (\{r_t - r_{t+1}\} - \{r_t - r_n\}) y_{t+1} + \dots + (\{r_t - r_{n-1}\} - \{r_t - r_n\}) y_{n-1}, \end{aligned}$$

and so also the negative term

$$-e \left(\begin{aligned} & (\{r_t - r_1\} - \{r_t - r_n\}) y_1 + \dots + (\{r_t - r_{t-1}\} - \{r_t - r_n\}) y_{t-1} - \{r_t - r_t\} y_t \\ & + (\{r_t - r_{t+1}\} - \{r_t - r_n\}) y_{t+1} + \dots + (\{r_t - r_{n-1}\} - \{r_t - r_n\}) y_{n-1} \end{aligned} \right).$$

Then

$$(C_{s1}, \dots, C_{s,s-1}, +, C_{s,s+1}, \dots, C_{s,n-1}) = (C_{t1}, \dots, C_{t,t-1}, -, C_{t,t+1}, \dots, C_{t,n-1}),$$

implies

$$\begin{aligned}
& \text{sign}(\{r_s - r_1\} - \{r_s - r_n\})y_1 + \cdots + \text{sign}(\{r_s - r_{s-1}\} - \{r_s - r_n\})y_{s-1} \\
& \quad + \{r_n - r_s\}y_s + \text{sign}(\{r_s - r_{s+1}\} - \{r_s - r_n\})y_{s+1} \\
& \quad + \cdots + \text{sign}(\{r_s - r_{n-1}\} - \{r_s - r_n\})y_{n-1} \\
& = \text{sign}(\{r_t - r_1\} - \{r_t - r_n\})y_1 + \cdots + \text{sign}(\{r_t - r_{t-1}\} - \{r_t - r_n\})y_{t-1} \\
& \quad - \{r_t - r_t\}y_t + \text{sign}(\{r_t - r_{t+1}\} - \{r_t - r_n\})y_{t+1} \\
& \quad + \cdots + \text{sign}(\{r_t - r_{n-1}\} - \{r_t - r_n\})y_{n-1}.
\end{aligned}$$

If $\text{sign}(\{r_s - r_i\} - \{r_s - r_n\}) = \text{sign}(\{r_t - r_i\} - \{r_t - r_n\})$ then, by Lemmas 3.2 and 3.3, $(\{r_s - r_i\} - \{r_s - r_n\}) = (\{r_t - r_i\} - \{r_t - r_n\})$. Thus,

$$\begin{aligned}
& (\{r_s - r_1\} - \{r_s - r_n\})y_1 + \cdots + (\{r_s - r_{s-1}\} - \{r_s - r_n\})y_{s-1} + \{r_n - r_s\}y_s \\
& \quad + (\{r_s - r_{s+1}\} - \{r_s - r_n\})y_{s+1} + \cdots + (\{r_s - r_{n-1}\} - \{r_s - r_n\})y_{n-1} \\
& = (\{r_t - r_1\} - \{r_t - r_n\})y_1 + \cdots + (\{r_t - r_{t-1}\} - \{r_t - r_n\})y_{t-1} - \{r_t - r_t\}y_t \\
& \quad + (\{r_t - r_{t+1}\} - \{r_t - r_n\})y_{t+1} + \cdots + (\{r_t - r_{n-1}\} - \{r_t - r_n\})y_{n-1}
\end{aligned}$$

and so

$$\begin{aligned}
& e \left(\begin{aligned} & (\{r_s - r_1\} - \{r_s - r_n\}) y_1 + \cdots + (\{r_s - r_{s-1}\} - \{r_s - r_n\}) y_{s-1} + \{r_n - r_s\} y_s \\ & + (\{r_s - r_{s+1}\} - \{r_s - r_n\}) y_{s+1} + \cdots + (\{r_s - r_{n-1}\} - \{r_s - r_n\}) y_{n-1} \end{aligned} \right) \\
& - e \left(\begin{aligned} & (\{r_t - r_1\} - \{r_t - r_n\}) y_1 + \cdots + (\{r_t - r_{t-1}\} - \{r_t - r_n\}) y_{t-1} - \{r_t - r_t\} y_t \\ & + (\{r_t - r_{t+1}\} - \{r_t - r_n\}) y_{t+1} + \cdots + (\{r_t - r_{n-1}\} - \{r_t - r_n\}) y_{n-1} \end{aligned} \right) \\
& = 0.
\end{aligned}$$

As for the identities

$$(C_{f1}, C_{f2}, \dots, C_{f,f-1}, +, C_{f,f+1}, \dots, C_{f,n-1}) = (+, +, \dots, +)$$

and

$$(-, -, \dots, -) = (C_{g1}, C_{g2}, \dots, C_{g,g-1}, +, C_{g,g+1}, \dots, C_{g,n-1}),$$

a similar argument follows.

$(C_{f1}, C_{f2}, \dots, C_{f,f-1}, +, C_{f,f+1}, \dots, C_{f,n-1})$ represents the exponent

$$\begin{aligned}
& (\{r_f - r_1\} - \{r_f - r_n\}) y_1 + \cdots + (\{r_f - r_{f-1}\} - \{r_f - r_n\}) y_{f-1} + \{r_n - r_f\} y_f \\
& + (\{r_f - r_{f+1}\} - \{r_f - r_n\}) y_{f+1} + \cdots + (\{r_f - r_{n-1}\} - \{r_f - r_n\}) y_{n-1}
\end{aligned}$$

and thus the term

$$e^{\left((\{r_f - r_1\} - \{r_f - r_n\}) y_1 + \cdots + (\{r_f - r_{f-1}\} - \{r_f - r_n\}) y_{f-1} + \{r_n - r_f\} y_f \right. \\ \left. + (\{r_f - r_{f+1}\} - \{r_f - r_n\}) y_{f+1} + \cdots + (\{r_f - r_{n-1}\} - \{r_f - r_n\}) y_{n-1} \right)}.$$

The identity

$$(C_{f1}, C_{f2}, \dots, C_{f,f-1}, +, C_{f,f+1}, \dots, C_{f,n-1}) = (+, +, \dots, +)$$

means $\{r_f - r_i\} - \{r_f - r_n\} = \{r_n - r_i\}$. Thus

$$e^{\left((\{r_f - r_1\} - \{r_f - r_n\}) y_1 + \cdots + (\{r_f - r_{f-1}\} - \{r_f - r_n\}) y_{f-1} + \{r_n - r_f\} y_f \right. \\ \left. + (\{r_f - r_{f+1}\} - \{r_f - r_n\}) y_{f+1} + \cdots + (\{r_f - r_{n-1}\} - \{r_f - r_n\}) y_{n-1} \right)} \\ = e^{\left(\{r_n - r_1\} y_1 + \cdots + \{r_n - r_{f-1}\} y_{f-1} - \{r_f - r_n\} y_f \right. \\ \left. + \{r_n - r_{f+1}\} y_{f+1} + \cdots + \{r_n - r_{n-1}\} y_{n-1} \right)} \\ = \sum_H \prod_{i=1}^{n-1} e^{\{r_n - r_i\} y_i}.$$

The exponent of the negative term $-\sum_H \prod_{j=1}^{n-1} e^{\{r_n - r_j\} y_j}$ is represented by the sign vector $(+, +, \dots, +)$ which cancels with the above.

$(C_{g1}, C_{g2}, \dots, C_{g,g-1}, +, C_{g,g+1}, \dots, C_{g,n-1})$ represents the exponent

$$\begin{aligned} & (\{r_g - r_1\} - \{r_g - r_n\}) y_1 + \dots + (\{r_g - r_{g-1}\} - \{r_g - r_n\}) y_{g-1} - \{r_g - r_n\} y_g \\ & + (\{r_g - r_{g+1}\} - \{r_g - r_n\}) y_{g+1} + \dots + (\{r_g - r_{n-1}\} - \{r_g - r_n\}) y_{n-1} \end{aligned}$$

and thus the negative term

$$-e^{\left(\begin{aligned} & (\{r_g - r_1\} - \{r_g - r_n\}) y_1 + \dots + (\{r_g - r_{g-1}\} - \{r_g - r_n\}) y_{g-1} - \{r_g - r_n\} y_g \\ & + (\{r_g - r_{g+1}\} - \{r_g - r_n\}) y_{g+1} + \dots + (\{r_g - r_{n-1}\} - \{r_g - r_n\}) y_{n-1} \end{aligned} \right)}.$$

The identity

$$(-, -, \dots, -) = (C_{g1}, C_{g2}, \dots, C_{g,g-1}, +, C_{g,g+1}, \dots, C_{g,n-1})$$

means $\{r_g - r_i\} - \{r_g - r_n\} = -\{r_i - r_n\}$. Thus

$$\begin{aligned} & -e^{\left(\begin{aligned} & (\{r_g - r_1\} - \{r_g - r_n\}) y_1 + \dots + (\{r_g - r_{g-1}\} - \{r_g - r_n\}) y_{g-1} - \{r_g - r_n\} y_g \\ & + (\{r_g - r_{g+1}\} - \{r_g - r_n\}) y_{g+1} + \dots + (\{r_g - r_{n-1}\} - \{r_g - r_n\}) y_{n-1} \end{aligned} \right)} \\ & = -e^{\left(\begin{aligned} & -\{r_1 - r_n\} y_1 - \dots - \{r_{g-1} - r_n\} y_{g-1} - \{r_g - r_n\} y_g \\ & -\{r_{g+1} - r_n\} y_{g+1} - \dots - \{r_{n-1} - r_n\} y_{n-1} \end{aligned} \right)} \\ & = - \sum_H \prod_{i=1}^{n-1} e^{-\{r_i - r_n\} y_i}. \end{aligned}$$

The exponent of the positive term $\sum_H \prod_{j=1}^{n-1} e^{-\{r_j - r_n\}y_j}$ is represented by the sign vector $(-, -, \dots, -)$, which cancels with the above.

Thus the matching of rows of the matrices M_n and M'_n correspond to the matching of like terms of opposite signs and we have $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} = 0$. We have just revealed the potential of our main result,

$$M_n = M'_n, \text{ up to row swapping, implies } \sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} = 0.$$

We must now show that under certain conditions, we can indeed find $M_n = M'_n$ up to row swapping. By proving the following properties of the matrices M_n and M'_n , we are able to quickly show $M_n = M'_n$ up to row swapping and thus our reciprocity theorem follows.

Lemma 3.4. *M_n and M'_n are of the form such that M_n has all + entries on the diagonal and the last row has all - entries, and M'_n has all - entries on the diagonal and the last row has all + entries.*

Proof. As was explained above, the diagonal entries of M_n (excluding the last row

of $-$'s) represent the terms in the exponent of the sum

$$\sum_{k=1}^{n-1} \sum_H e^{\{r_n - r_k\}y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k - r_i\} - \{r_k - r_n\}y_i},$$

when $k = i$, the exponent is $\{r_n - r_k\} = +$. Likewise, the diagonal entries of M'_n (excluding the last row of $+$'s) represent the terms in the exponent of the sum

$$\sum_{k=1}^{n-1} \sum_H e^{-\{r_k - r_n\}y_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} e^{\{r_k - r_i\} - \{r_k - r_n\}y_i},$$

when $k = i$, this exponent is $-\{r_k - r_n\} = -$. □

Lemma 3.5. *As above, let $C_{ij} = \text{sign}(\{r_i - r_j\} - \{r_i - r_n\})$ where $i, j, n \in \mathbb{Z}_{>0}$ and $1 \leq i < j \leq n - 1$. Then $C_{ij} = +$ if and only if $C_{ji} = -$.*

Proof. Assume $C_{ij} = +$. Then $\{r_i - r_j\} - \{r_i - r_n\} \geq 0$ and by Lemma 3.2 $\{r_i - r_j\} - \{r_i - r_n\} = \{r_n - r_j\}$. We want to show $C_{ji} = -$ which is to show $\{r_j - r_i\} - \{r_j - r_n\} < 0$.

$$\begin{aligned} \{r_j - r_i\} - \{r_j - r_n\} &= \{r_j - r_i\} - 1 + 1 - \{r_j - r_n\} \\ &= -\{r_i - r_j\} + \{r_n - r_j\} \\ &= -\{r_i - r_n\} \\ &\Leftrightarrow C_{ji} = -. \end{aligned}$$

Assume $C_{ij} = -$. Then $\{r_i - r_j\} - \{r_i - r_n\} < 0$ and by Lemma 3.3 $\{r_i - r_j\} - \{r_i - r_n\} = -\{r_j - r_n\}$. We want to show $C_{ji} = +$ which is to show $\{r_j - r_i\} - \{r_j - r_n\} > 0$.

$$\begin{aligned}
\{r_j - r_i\} - \{r_j - r_n\} &= \{r_j - r_i\} + \{r_i - r_j\} - \{r_i - r_n\} \\
&= 1 - \{r_i - r_n\} \\
&= \{r_n - r_i\} \\
&\Leftrightarrow C_{ji} = +,
\end{aligned}$$

recalling that $\{r_j - r_i\} + \{r_i - r_j\} = 1$. □

By Lemma 3.5, it follows that the sign of one difference of fractional parts $\{r_i - r_j\} - \{r_i - r_n\}$ is dependent on the sign of another difference of fractional parts $\{r_j - r_i\} - \{r_j - r_n\}$. The lemma implies C_{ij} determines C_{ji} . Thus, we will rename $C_{ji} = -C_{ij}$, meaning if $C_{ij} = +$ then $C_{ji} = -$ and if $C_{ij} = -$ then $C_{ji} = +$. Thus, we can update the sign vector matrices utilizing this new information as

$$M_n = \begin{pmatrix} + & C_{12} & C_{13} & \cdots & C_{1,n-1} \\ -C_{12} & + & C_{23} & \cdots & C_{2,n-1} \\ -C_{13} & -C_{23} & + & \cdots & C_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -C_{1,n-1} & -C_{2,n-1} & -C_{3,n-1} & \cdots & + \\ - & - & \cdots & - & - \end{pmatrix}$$

and

$$M'_n = \begin{pmatrix} - & C_{12} & C_{13} & \cdots & C_{1,n-1} \\ -C_{12} & - & C_{23} & \cdots & C_{2,n-1} \\ -C_{13} & -C_{23} & - & \cdots & C_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -C_{1,n-1} & -C_{2,n-1} & -C_{3,n-1} & \cdots & - \\ + & + & \cdots & + & + \end{pmatrix},$$

and we can state the following property.

Lemma 3.6. M_n and M'_n exhibit antisymmetry about the diagonal.

Lemma 3.7. As above, let $C_{ij} = \text{sign}(\{r_i - r_j\} - \{r_i - r_n\})$ where $i, j, n \in \mathbb{Z}_{>0}$ and $1 \leq i < j \leq n - 1$.

If $C_{ij} = +$ and $C_{ik} = -$ then $C_{jk} = -$.

Proof. Assume $C_{ij} = +$ and $C_{ik} = -$. Then $\{r_i - r_j\} - \{r_i - r_n\} > 0$ and $\{r_i - r_k\} - \{r_i - r_n\} < 0$, and by Lemmas 3.2 and 3.3

$$\{r_i - r_j\} - \{r_i - r_n\} = \{r_n - r_j\} \quad (3.5)$$

and

$$\{r_i - r_k\} - \{r_i - r_n\} = -\{r_k - r_i\}. \quad (3.6)$$

Then the difference (3.5)-(3.6) is positive and we get

$$\{r_i - r_j\} - \{r_i - r_k\} = \{r_n - r_j\} + \{r_k - r_i\}.$$

The final identity is positive, which means the left hand side is positive. Then by Lemma 3.2

$$\{r_i - r_j\} - \{r_i - r_k\} = \{r_k - r_j\},$$

and we have

$$\{r_k - r_j\} = \{r_n - r_j\} + \{r_k - r_i\}.$$

We want to show $C_{jk} = -$ which reduces to showing $\{r_j - r_k\} - \{r_j - r_n\} < 0$.

$$\begin{aligned}
\{r_j - r_k\} - \{r_j - r_n\} &= \{r_j - r_k\} - 1 + 1 - \{r_j - r_n\} \\
&= -\{r_k - r_j\} + \{r_n - r_j\} \\
&= -\{r_k - r_i\} \\
&\Rightarrow C_{jk} = -.
\end{aligned}$$

□

Lemma 3.8. *There exists a unique row with k + 's, for each $0 \leq k \leq n - 1$, in the matrix M_n .*

Proof. We begin by showing that every row of the matrix M_n is unique. Assume on the contrary that row m and row l of M_n are equal. Then we can view the rows as follows:

$$\begin{array}{rcccccccc}
\text{row } m: & -C_{1m} & -C_{2m} & \cdots & + & \cdots & C_{ml} & \cdots & C_{m,n-1} \\
\text{row } l: & -C_{1l} & -C_{2l} & \cdots & -C_{ml} & \cdots & + & \cdots & C_{l,n-1}
\end{array}$$

Then $C_{ml} = +$ and $-C_{ml} = +$, but by Lemma 3.5, $C_{ml} = +$ implies $-C_{ml} = -$, a contradiction. Therefore, the rows of the matrix M_n are unique.

Next, we will show that no two rows contain the same number of + 's. Assume on the contrary that row m and row l of M_n contain exactly i + 's, are not equal

(and thus $1 \leq i < n - 1$) and look the same as above. We will examine the elements C_{ml} of row m and $-C_{ml}$ of row l . Let $C_{ml} = +$. Since the m th row does not contain only +’s, there exists a $-$ in column, say, w . Then by Lemma 3.7, the entry C_{wl} is $-$. So, for every $-$ in row m , Lemma 3.7 can be applied to show there is a $-$ in the same column entry of row l . But row l also contains a $-$ on the diagonal since it is a row of the matrix M'_n . Thus, row l contains $i - 1$ many +’s, a contradiction. The same argument holds for the entry $-C_{ml} = +$ of row l . Thus, no two rows can have an equal number of +’s.

We’ve shown no two rows contain the same number of +’s and that every row is unique. Thus, for each $0 \leq k \leq n - 1$, there exists a unique row with k +’s. \square

Lemma 3.9. *There exists a unique row with k +’s, for each $0 \leq k \leq n - 1$, in the matrix M'_n .*

Proof. M_n and M'_n share all entries C_{ij} for $1 \leq i < j \leq n - 1$. Then the k +’s in the k th row of M_n determine that the k th row of M'_n contains $k - 1$ +’s since the + diagonal entry of M'_n becomes a $-$. Therefore, M'_n exhibits the same uniqueness property of M_n . \square

We can now prove our main Theorem 3.1.

Proof. Our goal is to show $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$. Assume $\frac{x_u - h_u}{a_u} - \frac{x_v - h_v}{a_v} \notin \mathbb{Z}$ whenever

$1 \leq u < v \leq n$ and $h_u, h_v \in \mathbb{Z}$ which allows us to implement the following matrix

argument. We've seen that $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ gives rise to the following matrices

$$M_n = \begin{pmatrix} + & C_{12} & C_{13} & \cdots & C_{1,n-1} \\ -C_{12} & + & C_{23} & \cdots & C_{2,n-1} \\ -C_{13} & -C_{23} & + & \cdots & C_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -C_{1,n} & -C_{2,n-1} & -C_{3,n-1} & \cdots & + \\ - & - & \cdots & - & - \end{pmatrix}$$

and

$$M'_n = \begin{pmatrix} - & C_{12} & C_{13} & \cdots & C_{1,n-1} \\ -C_{12} & - & C_{23} & \cdots & C_{2,n-1} \\ -C_{13} & -C_{23} & - & \cdots & C_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -C_{1,n-1} & -C_{2,n-1} & -C_{3,n-1} & \cdots & - \\ + & + & \cdots & + & + \end{pmatrix},$$

where M_n represents the exponents of all positive terms of $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ and M'_n represents all negative terms. As was shown above, to prove that the matrices M_n and M'_n contain the same rows is equivalent to showing every positive term of $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ has a canceling negative term. Thus our problem reduces to showing $M_n = M'_n$ after row swapping.

We've proved in Lemma 3.4 that M_n has +'s on the diagonal and in row $n + 1$, M'_n has -'s on the diagonal and in row $n + 1$, Lemma 3.6 shows both matrices exhibit antisymmetry, and Lemma 3.7 tells us that two opposite signed entries in a row off the diagonal imply the sign of another entry in the matrix is a -. These three lemmas lead to conclude Lemma 3.8 which states that for both matrices M_n and M'_n there exists a unique row of k +'s for every $1 \leq k \leq n - 1$. We can use row swapping to place the unique rows of k +'s in the same row of matrix M_n as they appear in matrix M'_n such that $M_n = M'_n$, which, by our previous argument,

implies $\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} = 0$. □

Chapter 4

A New Proof of Hall–Wilson–Zagier’s Reciprocity Theorem

The original proof of Hall, Wilson and Zagier’s reciprocity theorem for the generalized Dedekind–Rademacher sum ultimately uses cotangent identities to show

$$\begin{aligned} & \Omega \begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} + \Omega \begin{pmatrix} b & c & a \\ y & z & x \\ Y & Z & X \end{pmatrix} + \Omega \begin{pmatrix} c & a & b \\ z & x & y \\ Z & X & Y \end{pmatrix} \\ &= \begin{cases} -1/4 & \text{if } (x, y, z) \in (a, b, c)\mathbb{R} + \mathbb{Z}^3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here we give an example of applying the approach used to prove Theorem 3.1, simplifying the proof of Theorem 2.4.

New proof of Theorem 2.4. Let $(a_1, a_2, a_3) \in \mathbb{Z}_{>0}$ and $(x_1, x_2, x_3) \in \mathbb{R}$. For non-zero variables y_1, y_2, y_3 such that $y_1 + y_2 + y_3 = 0$ we want to show that

$$\begin{aligned} \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \end{pmatrix} \\ = \begin{cases} -1/4 & \text{if } (x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Recall identity (3.3),

$$\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} = \sum_{k=1}^n \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n \beta(r_k - r_i, y_i),$$

where

$$\beta(\alpha, Y) = \begin{cases} \frac{1}{2} \cdot \frac{e^Y + 1}{e^Y - 1} & \text{for } \alpha \in \mathbb{Z}, \\ \frac{e^{\{\alpha\}Y}}{e^Y - 1} & \text{for } \alpha \notin \mathbb{Z}, \end{cases}$$

and $\sum_H = \sum_{h_1 \bmod a_1} \sum_{h_2 \bmod a_2} \cdots \sum_{h_n \bmod a_n}$. Then we have

$$\begin{aligned} \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \end{pmatrix} \\ = \sum_{k=1}^3 \sum_H \prod_{\substack{i=1 \\ i \neq k}}^3 \beta(r_k - r_i, y_i). \end{aligned}$$

We must examine the following cases.

- (i) $(x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3$;
- (ii) $(x_i, x_j) \in (a_i, a_j)\mathbb{R} + \mathbb{Z}^2$ for some $1 \leq i < j \leq 3$ but not (i);
- (iii) None of the above.

We will begin with case (i). Let $(x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3$. Then $x_i = \lambda a_i + z_i$ for each i where $\lambda \in \mathbb{R}$ and $z_i \in \mathbb{Z}$. Thus

$$\begin{aligned} \frac{h_i + x_i}{a_i} - \frac{h_j + x_j}{a_j} &= \frac{h_i + \lambda a_i + z_i}{a_i} - \frac{h_j + \lambda a_j + z_j}{a_j} \\ &= \frac{h_i + z_i}{a_i} - \frac{h_j + z_j}{a_j}. \end{aligned} \tag{4.1}$$

Since $h_i + z_i$ only permutes the modular values of h_i , we can make a change in

indices (letting $h_i - z_i = \bar{h}_i$) so that (4.1) becomes

$$\frac{\bar{h}_i}{a_i} - \frac{\bar{h}_j}{a_j}$$

and

$$\begin{aligned} \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \end{pmatrix} \\ = \sum_{k=1}^3 \sum_H \prod_{\substack{i=1 \\ i \neq k}}^3 \beta(r_k - r_i, y_i) \\ = \sum_{k=1}^3 \sum_{\bar{H}} \prod_{\substack{i=1 \\ i \neq k}}^3 \beta\left(\frac{\bar{h}_k}{a_k} - \frac{\bar{h}_i}{a_i}, y_i\right) \\ = \sum_{k=1}^3 \sum_H \prod_{\substack{i=1 \\ i \neq k}}^3 \beta\left(\frac{h_k}{a_k} - \frac{h_i}{a_i}, y_i\right), \end{aligned}$$

since we sum h_i over a complete residue system mod a_i .

Since $(a_i, a_j) = 1$, then $\frac{h_i}{a_i} - \frac{h_j}{a_j} \in \mathbb{Z}$ occurs only when $h_i = h_j = 0$. Thus, we can split up the sum $\sum_{k=1}^3 \sum_H \prod_{\substack{i=1 \\ i \neq k}}^3 \beta\left(\frac{h_k}{a_k} - \frac{h_i}{a_i}, y_i\right)$ into two parts, a term when all

$h_i = 0$ and a sum over the remaining $h_i \in \mathbb{Z}$ where $(h_1, h_2, h_3) \neq (0, 0, 0)$

$$\begin{aligned} & \sum_{k=1}^3 \sum_H \prod_{\substack{i=1 \\ i \neq k}}^3 \beta \left(\frac{h_k}{a_i} - \frac{h_i}{a_i}, y_i \right) \\ &= \beta(0, y_2) \beta(0, y_3) + \beta(0, y_1) \beta(0, y_3) + \beta(0, y_1) \beta(0, y_2) \\ &+ \sum_{H \setminus \{0,0,0\}} \left(\begin{aligned} & \beta \left(\frac{h_1}{a_1} - \frac{h_2}{a_2}, y_2 \right) \beta \left(\frac{h_1}{a_1} - \frac{h_3}{a_3}, y_3 \right) \\ & + \beta \left(\frac{h_2}{a_2} - \frac{h_1}{a_1}, y_1 \right) \beta \left(\frac{h_2}{a_2} - \frac{h_3}{a_3}, y_3 \right) \\ & + \beta \left(\frac{h_3}{a_3} - \frac{h_1}{a_1}, y_1 \right) \beta \left(\frac{h_3}{a_3} - \frac{h_2}{a_2}, y_2 \right) \end{aligned} \right). \end{aligned}$$

Let us first address the term when all $h_i = 0$. We use the following cotangent identities

$$\cot(y) = i \frac{e^{2\pi y} + 1}{e^{2\pi y} - 1}, \quad (4.2)$$

$$\cot(\alpha) + \cot(\beta) = \frac{\cot(\alpha) \cot(\beta) - 1}{\cot(\alpha + \beta)}. \quad (4.3)$$

Combined with the definition of $\cot y$, $\beta(0, y)$ becomes

$$\beta(0, y) = \frac{1}{i} \cot \frac{y}{2i}. \quad (4.4)$$

Let $y_k^* = \frac{y_k}{2^i}$. Then

$$\begin{aligned}
& \beta(0, y_2) \beta(0, y_3) + \beta(0, y_1) \beta(0, y_3) + \beta(0, y_1) \beta(0, y_2) \\
&= \frac{1}{4} \frac{e^{y_2} + 1}{e^{y_2} - 1} \frac{e^{y_3} + 1}{e^{y_3} - 1} + \frac{1}{4} \frac{e^{y_1} + 1}{e^{y_1} - 1} \frac{e^{y_3} + 1}{e^{y_3} - 1} + \frac{1}{4} \frac{e^{y_1} + 1}{e^{y_1} - 1} \frac{e^{y_2} + 1}{e^{y_2} - 1} \\
&= -\frac{1}{4} (\cot y_2^* \cot y_3^* + \cot y_1^* \cot y_3^* + \cot y_1^* \cot y_2^*) \\
&= -\frac{1}{4} (\cot y_2^* (\cot y_3^* + \cot y_1^*) + \cot y_1^* \cot y_3^*) \\
&= -\frac{1}{4} \left(\cot y_2^* \left(\frac{\cot y_1^* \cot y_3^* - 1}{\cot y_1^* + y_3^*} \right) + \cot y_1^* \cot y_3^* \right).
\end{aligned}$$

By assumption, $y_1 + y_2 + y_3 = 0$, so it follows that $y_1^* + y_3^* = -y_2^*$ and

$$\cot(y_1^* + y_3^*) = \cot(-y_2^*) = -\cot y_2^*.$$

Thus,

$$\begin{aligned}
& -\frac{1}{4} \left(\cot y_2^* \left(\frac{\cot y_1^* \cot y_3^* - 1}{\cot y_1^* + y_3^*} \right) + \cot y_1^* \cot y_3^* \right) \\
&= -\frac{1}{4} \left(\cot y_2^* \left(\frac{\cot y_1^* \cot y_3^* - 1}{-\cot y_2^*} \right) + \cot y_1^* \cot y_3^* \right) \\
&= -\frac{1}{4} ((-\cot y_1^* \cot y_3^* + 1) + \cot y_1^* \cot y_3^*) \\
&= -\frac{1}{4}.
\end{aligned}$$

It is left to show that the summand over $H \setminus \{0, 0, 0\}$ vanish. But we've seen this

sum before. It is just

$$\sum_{k=1}^3 \sum_{H \setminus \{0,0,0\}} \prod_{\substack{i=1 \\ i \neq k}}^3 \beta(r_k - r_i, y_i)$$

when $(x_1, x_2, x_3) = \vec{0}$. Since $r_k - r_i \notin \mathbb{Z}$, we can apply the same matrix argument from the proof of Theorem 3.1 and the terms cancel.

Assume case (ii). Let $(x_i, x_j) \in (a_i, a_j)\mathbb{R} + \mathbb{Z}^2$ for some $1 \leq i < j \leq 3$ but not (i). Without loss of generality, we assume $(x_1, x_2) \in (a_1, a_2)\mathbb{R} + \mathbb{Z}^2$. Then, as before, $x_1 = \lambda a_1 + z_1$ and $x_2 = \lambda a_2 + z_2$ for $\lambda \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{Z}$ so that

$$\begin{aligned} \frac{h_1 + x_1}{a_1} - \frac{h_2 + x_2}{a_2} &= \frac{h_1 + \lambda a_1 + z_1}{a_1} - \frac{h_2 + \lambda a_2 + z_2}{a_2} \\ &= \frac{h_1 + z_1}{a_1} - \frac{h_2 + z_2}{a_2}. \end{aligned}$$

Since z_1 and z_2 permute the summands over h_1 and h_2 , we introduce a change of variables and let $\bar{h}_1 = h_1 + z_1$ and $\bar{h}_2 = h_2 + z_2$. We can rewrite the differences involving r_3 as

$$\begin{aligned} \frac{h_i + x_i}{a_i} - \frac{h_3 + x_3}{a_3} &= \left(\frac{\bar{h}_i}{a_i} + \lambda - r_3 \right) \\ &= \left(\frac{\bar{h}_i}{a_i} - \frac{h_3 + x_3 - \lambda a_3}{a_3} \right) \\ &= \left(\frac{\bar{h}_i}{a_i} - \tilde{r}_3 \right). \end{aligned}$$

Then

$$\begin{aligned}
& \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \end{pmatrix} \\
&= \sum_{k=1}^3 \sum_H \prod_{\substack{i=1 \\ i \neq k}}^3 \beta(r_k - r_i, y_i) \\
&= \sum_{\bar{h}_1 \bmod a_1} \sum_{\bar{h}_2 \bmod a_2} \sum_{h_3 \bmod a_3} \left(\begin{aligned} & \beta\left(\frac{\bar{h}_1}{a_1} - \frac{\bar{h}_2}{a_2}, y_2\right) \beta\left(\frac{\bar{h}_1}{a_1} - \tilde{r}_3, y_3\right) \\ & + \beta\left(\frac{\bar{h}_2}{a_2} - \frac{h_1}{a_1}, y_1\right) \beta\left(\frac{\bar{h}_2}{a_2} - \tilde{r}_3, y_3\right) \\ & + \beta\left(\tilde{r}_3 - \frac{\bar{h}_1}{a_1} - \lambda, y_1\right) \beta\left(\tilde{r}_3 - \frac{\bar{h}_2}{a_2}, y_2\right) \end{aligned} \right) \\
&= \sum_H \left(\begin{aligned} & \beta\left(\frac{h_1}{a_1} - \frac{h_2}{a_2}, y_2\right) \beta\left(\frac{h_1}{a_1} - \tilde{r}_3, y_3\right) \\ & + \beta\left(\frac{h_2}{a_2} - \frac{h_1}{a_1}, y_1\right) \beta\left(\frac{h_2}{a_2} - \tilde{r}_3, y_3\right) \\ & + \beta\left(\tilde{r}_3 - \frac{h_1}{a_1}, y_1\right) \beta\left(\tilde{r}_3 - \frac{h_2}{a_2}, y_2\right) \end{aligned} \right).
\end{aligned}$$

We will again split up the final expression into two parts, one part includes all terms where $h_1 = h_2 = 0$ and the other part are all summands where h_1 and h_2 are not

both zero

$$\begin{aligned}
& \sum_H \left(\begin{array}{l} \beta \left(\frac{h_1}{a_1} - \frac{h_2}{a_2}, y_2 \right) \beta \left(\frac{h_1}{a_1} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\frac{h_2}{a_2} - \frac{h_1}{a_1}, y_1 \right) \beta \left(\frac{h_2}{a_2} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\tilde{r}_3 - \frac{h_1}{a_1}, y_1 \right) \beta \left(\tilde{r}_3 - \frac{h_2}{a_2}, y_2 \right) \end{array} \right) \\
&= \sum_{h_3 \bmod a_3} \left(\begin{array}{l} \beta(0, y_2) \beta(-\tilde{r}_3, y_3) \\ + \beta(0, y_1) \beta(-\tilde{r}_3, y_3) \\ + \beta(\tilde{r}_3, y_1) \beta(\tilde{r}_3, y_2) \end{array} \right) \\
&+ \sum_{\substack{H \\ (h_1, h_2) \neq (0,0)}} \left(\begin{array}{l} \beta \left(\frac{h_1}{a_1} - \frac{h_2}{a_2}, y_2 \right) \beta \left(\frac{h_1}{a_1} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\frac{h_2}{a_2} - \frac{h_1}{a_1}, y_1 \right) \beta \left(\frac{h_2}{a_2} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\tilde{r}_3 - \frac{h_1}{a_1}, y_1 \right) \beta \left(\tilde{r}_3 - \frac{h_2}{a_2}, y_2 \right) \end{array} \right).
\end{aligned}$$

First, we show

$$\sum_H \left(\begin{array}{l} \beta \left(\frac{h_1}{a_1} - \frac{h_2}{a_2}, y_2 \right) \beta \left(\frac{h_1}{a_1} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\frac{h_2}{a_2} - \frac{h_1}{a_1}, y_1 \right) \beta \left(\frac{h_2}{a_2} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\tilde{r}_3 - \frac{h_1}{a_1}, y_1 \right) \beta \left(\tilde{r}_3 - \frac{h_2}{a_2}, y_2 \right) \end{array} \right) = 0.$$

$$\begin{aligned}
& \sum_H \left(\begin{array}{l} \beta \left(\frac{h_1}{a_1} - \frac{h_2}{a_2}, y_2 \right) \beta \left(\frac{h_1}{a_1} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\frac{h_2}{a_2} - \frac{h_1}{a_1}, y_1 \right) \beta \left(\frac{h_2}{a_2} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\tilde{r}_3 - \frac{h_1}{a_1}, y_1 \right) \beta \left(\tilde{r}_3 - \frac{h_2}{a_2}, y_2 \right) \end{array} \right) \\
&= \sum_{h_3 \bmod a_3} \left(\frac{1}{2} \frac{e^{y_2} + 1}{e^{y_2} - 1} \cdot \frac{e^{\{-\tilde{r}_3\}y_3}}{e^{y_3} - 1} + \frac{1}{2} \frac{e^{y_1} + 1}{e^{y_1} - 1} \cdot \frac{e^{\{-\tilde{r}_3\}y_3}}{e^{y_3} - 1} + \frac{e^{\{\tilde{r}_3\}y_1}}{e^{y_1} - 1} \cdot \frac{e^{\{\tilde{r}_3\}y_2}}{e^{y_2} - 1} \right).
\end{aligned}$$

After multiplying each term by the common denominator and combining terms, we get the numerator

$$\sum_{h_3 \bmod a_3} \left(\begin{array}{l} e^{y_1+y_2+\{-\tilde{r}_3\}y_3} - e^{y_2+\{-\tilde{r}_3\}y_3} + e^{y_1+\{-\tilde{r}_3\}y_3} - e^{\{-\tilde{r}_3\}y_3} \\ + e^{y_1+y_2+\{-\tilde{r}_3\}y_3} - e^{y_1+\{-\tilde{r}_3\}y_3} + e^{y_2+\{-\tilde{r}_3\}y_3} - e^{\{-\tilde{r}_3\}y_3} \\ + 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2+y_3} - 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} \end{array} \right)$$

(writing all exponents in terms of y_1 and y_2)

$$= \sum_{h_3 \bmod a_3} \left(\begin{array}{l} e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} - e^{-\{-\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} + e^{\{\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} \\ - e^{-\{-\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} + e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} - e^{\{\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} \\ + e^{-\{-\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} - e^{-\{-\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} \\ + 2e^{-\{-\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} - 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} \end{array} \right) = 0.$$

We must now show

$$\sum_{\substack{H \\ (h_1, h_2) \neq (0, 0)}} \left(\begin{array}{l} \beta \left(\frac{h_1}{a_1} - \frac{h_2}{a_2}, y_2 \right) \beta \left(\frac{h_1}{a_1} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\frac{h_2}{a_2} - \frac{h_1}{a_1}, y_1 \right) \beta \left(\frac{h_2}{a_2} - \tilde{r}_3, y_3 \right) \\ + \beta \left(\tilde{r}_3 - \frac{h_1}{a_1}, y_1 \right) \beta \left(\tilde{r}_3 - \frac{h_2}{a_2}, y_2 \right) \end{array} \right) = 0.$$

Since $\pm \left(\frac{h_1}{a_1} - \frac{h_2}{a_2} \right), \pm \left(\frac{h_i}{a_i} - \tilde{r}_3 \right) \notin \mathbb{Z}$, we see that this is just another special case of our proof of Theorem 3.1 when $(x_1, x_2, x_3) = (0, 0, x_3 - \lambda a_3)$. Thus, these terms vanish.

Last, we apply our Theorem 3.1 to the final case. □

Chapter 5

The 4-variable reciprocity theorem

We've shown that the generic case of Hall–Wilson–Zagier's reciprocity theorem can be generalized to the n -variable case using a simple combinatorial argument. The remaining conditions left to deal with in the n -variable case cannot be addressed so easily, as shown by Bayad and Raouj [3] who use involved number theory to prove a reciprocity theorem for the multiple Dedekind–Rademacher sum. For $n = 4$, most of the remaining conditions can be dealt with and the following theorem is revealed.

Theorem 5.1. *Let $a_1, a_2, a_3, a_4 \in \mathbb{Z}$, $x_1, x_2, x_3, x_4 \in \mathbb{R}$, and y_1, y_2, y_3, y_4 be four nonzero variables such that $y_1 + y_2 + y_3 + y_4 = 0$. Then*

$$\sum_{k=1}^4 \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} = -\frac{1}{8i} \left(\cot \left(\frac{y_1}{2i} \right) + \cot \left(\frac{y_2}{2i} \right) + \cot \left(\frac{y_3}{2i} \right) + \cot \left(\frac{y_4}{2i} \right) \right)$$

if $(x_1, x_2, x_3, x_4) \in (a_1, a_2, a_3, a_4) \mathbb{R} + \mathbb{Z}^4$. The sum vanishes for all other cases, except possibly when

$$(x_i, x_j, x_k) \in (a_i, a_j, a_k) \mathbb{R} + \mathbb{Z}^3$$

and

$$\left(\frac{h_i + x_i}{a_i} - \frac{h_l + x_l}{a_l}, \frac{h_j + x_j}{a_j} - \frac{h_l + x_l}{a_l}, \frac{h_k + x_k}{a_k} - \frac{h_l + x_l}{a_l} \right) \notin \mathbb{Z}^3$$

for all $h_i, h_j, h_k, h_l \in \mathbb{Z}$, where $1 \leq i < j < k < l \leq 4$.

In the unknown case we believe the sum has the potential to simplify. After the proof of Theorem 5.1 we will show the challenge the unknown case poses in proving the sum has a closed formula and ultimately the difficulty in generalizing all cases to n -variables.

Proof. First, recall identity (3.3),

$$\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} = \sum_{k=1}^n \sum_H \prod_{\substack{i=1 \\ i \neq k}}^n \beta(r_k - r_i, y_i),$$

where

$$\beta(\alpha, Y) = \begin{cases} \frac{1}{2} \cdot \frac{e^Y + 1}{e^Y - 1} & \text{for } \alpha \in \mathbb{Z}, \\ \frac{e^{\{\alpha\}Y}}{e^Y - 1} & \text{for } \alpha \notin \mathbb{Z} \end{cases}$$

and $\sum_H = \sum_{h_1 \bmod a_1} \sum_{h_2 \bmod a_2} \cdots \sum_{h_n \bmod a_n}$.

Our approach to proving Theorem 5.1 will mimic that of our new proof of Theorem 2.4 outlined previously. Thus, we must check the following cases

- (i) $(x_1, x_2, x_3, x_4) \in (a_1, a_2, a_3, a_4) \mathbb{R} + \mathbb{Z}^4$;
- (ii) $(x_i, x_j, x_k) \in (a_i, a_j, a_k) \mathbb{R} + \mathbb{Z}^3$ for some $1 \leq i < j < k \leq 4$ but not (i);
- (iii) $(x_i, x_j) \in (a_i, a_j) \mathbb{R} + \mathbb{Z}^2$ for some $1 \leq i < j \leq 4$ but not (i) and not (ii);
- (iv) None of the above.

Cases (i), (iii), and (iv) are covered in the statement of Theorem 5.1 and will be proven here. Case (ii) will be discussed after the proof.

First, we see that case (iv) follows with Theorem 3.1.

Assume case (i). Let $(x_1, x_2, x_3, x_4) \in (a_1, a_2, a_3, a_4) \mathbb{R} + \mathbb{Z}^4$. As in our simplified proof of Theorem 2.4, our concern is if $\frac{h_i}{a_i} - \frac{h_j}{a_j} \in \mathbb{Z}$. Since this occurs only when $h_i = h_j = 0$ after introducing a change in variables, we can split the sum

$\sum_{k=1}^4 \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ into the terms where $h_i = 0$ for all $i = 1, 2, 3, 4$ and the remaining terms when not all $h_i = 0$

$$\begin{aligned} \sum_{k=1}^4 \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} &= \sum_{k=1}^4 \sum_H \prod_{\substack{i=1 \\ i \neq k}}^4 \beta(r_k - r_i, y_i) \\ &= \beta(0, y_2)\beta(0, y_3)\beta(0, y_4) + \beta(0, y_1)\beta(0, y_3)\beta(0, y_4) + \beta(0, y_1)\beta(0, y_2)\beta(0, y_4) \\ &\quad + \beta(0, y_1)\beta(0, y_2)\beta(0, y_3) \\ &\quad + \sum_{k=1}^4 \sum_{H \setminus \{0,0,0,0\}} \prod_{\substack{i=1 \\ i \neq k}}^4 \beta\left(\frac{h_k}{a_k} - \frac{h_i}{a_i}, y_i\right). \end{aligned}$$

We know that since in the last sum $\frac{h_k}{a_k} - \frac{h_i}{a_i} \notin \mathbb{Z}$, we can apply the idea of the proof of Theorem 3.1 and

$$\sum_{k=1}^4 \sum_{H \setminus \{0,0,0,0\}} \prod_{\substack{i=1 \\ i \neq k}}^4 \beta\left(\frac{h_k}{a_k} - \frac{h_i}{a_i}, y_i\right) = 0.$$

The interesting part consists of the remaining terms. We will now show that the remaining terms in fact simplify to a compact finite sum.

Let $y_k^* = \frac{y_k}{2i}$. Then

$$\begin{aligned}
& \sum_{k=1}^4 \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} \\
&= \beta(0, y_2)\beta(0, y_3)\beta(0, y_4) + \beta(0, y_1)\beta(0, y_3)\beta(0, y_4) + \beta(0, y_1)\beta(0, y_2)\beta(0, y_4) \\
&\quad + \beta(0, y_1)\beta(0, y_2)\beta(0, y_3) \\
&= -\frac{1}{8i}(\cot(y_2^*)\cot(y_3^*)\cot(y_4^*) + \cot(y_1^*)\cot(y_3^*)\cot(y_4^*) + \cot(y_1^*)\cot(y_2^*)\cot(y_4^*) \\
&\quad + \cot(y_1^*)\cot(y_2^*)\cot(y_3^*)) \\
&= -\frac{1}{8i}(\cot(y_3^*)\cot(y_4^*)(\cot(y_2^*) + \cot(y_1^*)) + \cot(y_1^*)\cot(y_2^*)(\cot(y_3^*) + \cot(y_4^*)))
\end{aligned}$$

(applying the cotangent identity to the sums $\cot(y_1^*) + \cot(y_2^*)$ and $\cot(y_3^*) + \cot(y_4^*)$)

$$= -\frac{1}{8i} \left(\cot(y_3^*)\cot(y_4^*) \left(\frac{\cot(y_2^*)\cot(y_1^*) - 1}{\cot(y_1^* + y_2^*)} \right) + \cot(y_1^*)\cot(y_2^*) \left(\frac{\cot(y_3^*)\cot(y_4^*) - 1}{\cot(y_3^* + y_4^*)} \right) \right)$$

(since $y_1 + y_2 + y_3 + y_4 = 0$, then $\cot(y_3^* + y_4^*) = \cot(-y_1^* - y_2^*) = -\cot(y_1^* + y_2^*)$)

$$\begin{aligned}
&= -\frac{1}{8i} \cdot \frac{\begin{pmatrix} \cot(y_3^*) \cot(y_4^*) (\cot(y_2^*) \cot(y_1^*) - 1) \\ -\cot(y_1^*) \cot(y_2^*) (\cot(y_3^*) \cot(y_4^*) - 1) \end{pmatrix}}{\cot(y_1^* + y_2^*)} \\
&= -\frac{1}{8i} (\cot(y_1^*) \cot(y_2^*) - \cot(y_3^*) \cot(y_4^*))
\end{aligned}$$

(solving the cotangent identity for $\cot(\alpha)\cot(\beta)$, we apply this identity to both cotangent products in the sum above)

$$\begin{aligned}
&= -\frac{1}{8i} \cdot \frac{\begin{pmatrix} \cot(y_1^* + y_2^*) (\cot(y_2^*) + \cot(y_1^*)) + 1 \\ -\cot(y_3^* + y_4^*) (\cot(y_3^*) + \cot(y_4^*)) - 1 \end{pmatrix}}{\cot(y_1^* + y_2^*)} \\
&= -\frac{1}{8i} \cdot \frac{\begin{pmatrix} \cot(y_1^* + y_2^*) (\cot(y_2^*) + \cot(y_1^*)) + 1 \\ +\cot(y_1^* + y_2^*) (\cot(y_3^*) + \cot(y_4^*)) - 1 \end{pmatrix}}{\cot(y_1^* + y_2^*)} \\
&= -\frac{1}{8i} \left(\cot(y_2^*) + \cot(y_1^*) + \frac{1}{\cot(y_1^* + y_2^*)} + \cot(y_3^*) + \cot(y_4^*) - \frac{1}{\cot(y_1^* + y_2^*)} \right) \\
&= -\frac{1}{8i} (\cot(y_1^*) + \cot(y_2^*) + \cot(y_3^*) + \cot(y_4^*)).
\end{aligned}$$

Next, we assume case (iii). Let $(x_i, x_j) \in (a_i, a_j)\mathbb{R} + \mathbb{Z}^2$ for some $1 \leq i < j \leq 4$ and not case (i) and not case (ii). Without loss of generality, assume $(x_1, x_2) \in$

$(a_1, a_2)\mathbb{R} + \mathbb{Z}^2$. Then, again, we can divide the sum $\sum_{k=1}^4 \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix}$ into terms

where $h_1 = h_2 = 0$ and the remaining terms when not both h_1 and h_2 are zero

$$\begin{aligned} \sum_{k=1}^4 \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} &= \sum_{h_3 \bmod a_3} \sum_{h_4 \bmod a_4} \begin{pmatrix} \beta(0, y_2) \beta(-\tilde{r}_3, y_3) \beta(-\tilde{r}_4, y_4) \\ +\beta(0, y_1) \beta(-\tilde{r}_3, y_3) \beta(-\tilde{r}_4, y_4) \\ +\beta(\tilde{r}_3, y_1) \beta(\tilde{r}_3, y_2) \beta(\tilde{r}_3 - \tilde{r}_4, y_4) \\ +\beta(\tilde{r}_4, y_1) \beta(\tilde{r}_4, y_2) \beta(\tilde{r}_4 - \tilde{r}_3, y_3) \end{pmatrix} \\ &+ \sum_{\substack{H \\ (h_1, h_2) \neq (0,0)}} \begin{pmatrix} \beta\left(\frac{h_1}{a_1} - \frac{h_2}{a_2}, y_2\right) \beta\left(\frac{h_1}{a_1} - \tilde{r}_3, y_3\right) \beta\left(\frac{h_1}{a_1} - \tilde{r}_4, y_4\right) \\ +\beta\left(\frac{h_2}{a_2} - \frac{h_1}{a_1}, y_1\right) \beta\left(\frac{h_2}{a_2} - \tilde{r}_3, y_3\right) \beta\left(\frac{h_2}{a_2} - \tilde{r}_4, y_4\right) \\ +\beta\left(\tilde{r}_3 - \frac{h_1}{a_1}, y_1\right) \beta\left(\tilde{r}_3 - \frac{h_2}{a_2}, y_2\right) \beta(\tilde{r}_3 - \tilde{r}_4, y_4) \\ +\beta\left(\tilde{r}_4 - \frac{h_1}{a_1}, y_1\right) \beta\left(\tilde{r}_4 - \frac{h_2}{a_2}, y_2\right) \beta(\tilde{r}_4 - \tilde{r}_3, y_3) \end{pmatrix}. \end{aligned}$$

Again, we see that all beta functions are evaluated at non-integer values and so the latter term vanishes by the proof method of Theorem 3.1. We now employ a similar matrix argument used in the proof of Theorem 3.1 to prove the first summand equals zero.

$$\begin{aligned}
& \sum_{h_3 \bmod a_3} \sum_{h_4 \bmod a_4} \left(\begin{array}{l} \beta(0, y_2) \beta(-\tilde{r}_3, y_3) \beta(-\tilde{r}_4, y_4) \\ + \beta(0, y_1) \beta(-\tilde{r}_3, y_3) \beta(-\tilde{r}_4, y_4) \\ + \beta(\tilde{r}_3, y_1) \beta(\tilde{r}_3, y_2) \beta(\tilde{r}_3 - \tilde{r}_4, y_4) \\ + \beta(\tilde{r}_4, y_1) \beta(\tilde{r}_4, y_2) \beta(\tilde{r}_4 - \tilde{r}_3, y_3) \end{array} \right) \\
&= \sum_{h_3 \bmod a_3} \sum_{h_4 \bmod a_4} \left(\begin{array}{l} \frac{1}{2} \cdot \frac{e^{y_2+1}}{e^{y_2-1}} \cdot \frac{e^{\{-\tilde{r}_3\}y_3}}{e^{y_3-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \\ + \frac{1}{2} \cdot \frac{e^{y_1+1}}{e^{y_1-1}} \cdot \frac{e^{\{-\tilde{r}_3\}y_3}}{e^{y_3-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \\ + \frac{e^{\{\tilde{r}_3\}y_1}}{e^{y_1-1}} \cdot \frac{e^{\{\tilde{r}_3\}y_2}}{e^{y_2-1}} \cdot \frac{e^{\{\tilde{r}_3-\tilde{r}_4\}y_4}}{e^{y_4-1}} \\ + \frac{e^{\{\tilde{r}_4\}y_1}}{e^{y_1-1}} \cdot \frac{e^{\{\tilde{r}_4\}y_2}}{e^{y_2-1}} \cdot \frac{e^{\{\tilde{r}_4-\tilde{r}_3\}y_3}}{e^{y_3-1}} \end{array} \right) \\
&= \sum_{h_3 \bmod a_3} \sum_{h_4 \bmod a_4} \left(\begin{array}{l} \frac{1}{2} \cdot \frac{e^{y_2+1}}{e^{y_2-1}} \cdot \frac{e^{\{-\tilde{r}_3\}y_3}}{e^{y_3-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \cdot \left(\frac{e^{y_1-1}}{e^{y_1-1}}\right) \\ + \frac{1}{2} \cdot \frac{e_1^y+1}{e_1^y-1} \cdot \frac{e^{\{-\tilde{r}_3\}y_3}}{e^{y_3-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \cdot \left(\frac{e^{y_2-1}}{e^{y_2-1}}\right) \\ + \frac{e^{\{\tilde{r}_3\}y_1}}{e^{y_1-1}} \cdot \frac{e^{\{\tilde{r}_3\}y_2}}{e^{y_2-1}} \cdot \frac{e^{\{\tilde{r}_3-\tilde{r}_4\}y_4}}{e^{y_4-1}} \cdot \left(\frac{2}{2} \left(\frac{e^{y_3-1}}{e^{y_3-1}}\right)\right) \\ + \frac{e^{\{\tilde{r}_4\}y_1}}{e^{y_1-1}} \cdot \frac{e^{\{\tilde{r}_4\}y_2}}{e^{y_2-1}} \cdot \frac{e^{\{\tilde{r}_4-\tilde{r}_3\}y_3}}{e^{y_3-1}} \cdot \left(\frac{2}{2} \left(\frac{e^{y_4-1}}{e^{y_4-1}}\right)\right) \end{array} \right) \\
& \quad e^{y_1+y_2+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} - e^{y_2+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} + e^{y_1+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} \\
& \quad - e^{\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} + e^{y_1+y_2+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} - e^{y_1+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} \\
& \quad + e^{y_2+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} - e^{\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} \\
& \quad + 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2+y_3+\{\tilde{r}_3-\tilde{r}_4\}y_4} - 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2+\{\tilde{r}_3-\tilde{r}_4\}y_4} \\
& \quad + 2e^{\{\tilde{r}_4\}y_1+\{\tilde{r}_4\}y_2+\{\tilde{r}_4-\tilde{r}_3\}y_3+y_4} - 2e^{\{\tilde{r}_4\}y_1+\{\tilde{r}_4\}y_2+\{\tilde{r}_4-\tilde{r}_3\}y_3} \\
&= \sum_{h_3 \bmod a_3} \sum_{h_4 \bmod a_4} \frac{\phantom{e^{y_1+y_2+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} - e^{y_2+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} + e^{y_1+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} - e^{\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} + e^{y_1+y_2+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} - e^{y_1+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} + e^{y_2+\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} - e^{\{-\tilde{r}_3\}y_3+\{-\tilde{r}_4\}y_4} + 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2+y_3+\{\tilde{r}_3-\tilde{r}_4\}y_4} - 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2+\{\tilde{r}_3-\tilde{r}_4\}y_4} + 2e^{\{\tilde{r}_4\}y_1+\{\tilde{r}_4\}y_2+\{\tilde{r}_4-\tilde{r}_3\}y_3+y_4} - 2e^{\{\tilde{r}_4\}y_1+\{\tilde{r}_4\}y_2+\{\tilde{r}_4-\tilde{r}_3\}y_3}}{2(e^{y_1}-1)(e^{y_2}-1)(e^{y_3}-1)(e^{y_4}-1)}.
\end{aligned}$$

We write the final equality in terms of y_1, y_2 and y_3 and get the numerator

$$\begin{aligned}
& e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + (\{-\tilde{r}_3\} - \{-\tilde{r}_4\})y_3} - e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + (\{-\tilde{r}_3\} - \{-\tilde{r}_4\})y_3} \\
& + e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + (\{-\tilde{r}_3\} - \{-\tilde{r}_4\})y_3} - e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + (\{-\tilde{r}_3\} - \{-\tilde{r}_4\})y_3} \\
& + e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + (\{-\tilde{r}_3\} - \{-\tilde{r}_4\})y_3} - e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + (\{-\tilde{r}_3\} - \{-\tilde{r}_4\})y_3} \\
& + e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + (\{-\tilde{r}_3\} - \{-\tilde{r}_4\})y_3} - e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + (\{-\tilde{r}_3\} - \{-\tilde{r}_4\})y_3} \\
& + 2e^{(\{\tilde{r}_3\} - \{\tilde{r}_3 - \tilde{r}_4\})y_1 + (\{\tilde{r}_3\} - \{\tilde{r}_3 - \tilde{r}_4\})y_2 + \{\tilde{r}_4 - \tilde{r}_3\}y_3} \\
& - 2e^{(\{\tilde{r}_3\} - \{\tilde{r}_3 - \tilde{r}_4\})y_1 + (\{\tilde{r}_3\} - \{\tilde{r}_3 - \tilde{r}_4\})y_2 - \{\tilde{r}_3 - \tilde{r}_4\}y_4} \\
& + 2e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{\tilde{r}_3 - \tilde{r}_4\}y_3} - 2e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4 - \tilde{r}_3\}y_3}.
\end{aligned}$$

Let every term in an exponent be represented by its sign. Then

$$\begin{aligned}
& \text{sign}\{\tilde{r}_4\} = +, \quad \text{sign}(-\{-\tilde{r}_4\}) = -, \\
& \text{sign}(\{-\tilde{r}_3\} - \{-\tilde{r}_4\}) = C_{03}, \quad \text{sign}(\{\tilde{r}_3\} - \{\tilde{r}_3 - \tilde{r}_4\}) = C_{30},
\end{aligned}$$

and let

$$-\{r_3 - r_4\} = - \quad \{r_4 - r_3\} = +.$$

We can construct the following matrices

$$M_4 = \begin{pmatrix} + & + & C_{03} \\ + & - & C_{03} \\ + & + & C_{03} \\ - & + & C_{03} \\ C_{30} & C_{30} & + \\ C_{30} & C_{30} & + \\ - & - & - \\ - & - & - \end{pmatrix} \quad \text{and} \quad M'_4 = \begin{pmatrix} - & + & C_{03} \\ - & - & C_{03} \\ + & - & C_{03} \\ - & - & C_{03} \\ C_{30} & C_{30} & - \\ C_{30} & C_{30} & - \\ + & + & + \\ + & + & + \end{pmatrix},$$

where M_4 consists of all vectors representing the exponents of the positive terms in the numerator and M'_4 consists of all vectors representing the exponents of the negative terms in the numerator. We can apply Lemmas 3.2 and 3.3 to the differences

$$\{-\tilde{r}_3\} - \{-\tilde{r}_4\} \quad \text{and} \quad \{\tilde{r}_3\} - \{\tilde{r}_3 - \tilde{r}_4\},$$

where $a = 0$ and $b = 0$, respectively. Then we can make a statement about the entries C_{03} and C_{30} .

Lemma 5.2. $C_{03} = +$ if and only if $C_{30} = -$.

Proof. Assume $C_{30} = +$. This means the difference $\{-\tilde{r}_3\} - \{-\tilde{r}_4\}$ is positive and,

by Lemma 3.2,

$$\{-\tilde{r}_3\} - \{-\tilde{r}_4\} = \{\tilde{r}_4 - \tilde{r}_3\}.$$

Then

$$\begin{aligned} \{\tilde{r}_3\} - \{\tilde{r}_3 - \tilde{r}_4\} &= \{\tilde{r}_3\} - 1 + 1 - \{\tilde{r}_3 - \tilde{r}_4\} \\ &= -\{-\tilde{r}_3\} + \{\tilde{r}_4 - \tilde{r}_3\} \\ &= -\{-\tilde{r}_4\}. \end{aligned}$$

Thus, $\{\tilde{r}_3\} - \{\tilde{r}_3 - \tilde{r}_4\}$ is negative and so $C_{30} = -$. □

Thus, $C_{30} = -C_{03}$ and the matrices become

$$M_4 = \begin{pmatrix} + & + & C_{03} \\ + & - & C_{03} \\ + & + & C_{03} \\ - & + & C_{03} \\ -C_{03} & -C_{03} & + \\ -C_{30} & -C_{30} & + \\ - & - & - \\ - & - & - \end{pmatrix}, \quad M'_4 = \begin{pmatrix} - & + & C_{03} \\ - & - & C_{03} \\ + & - & C_{03} \\ - & - & C_{03} \\ -C_{03} & -C_{03} & - \\ -C_{03} & -C_{03} & - \\ + & + & + \\ + & + & + \end{pmatrix}.$$

It is left to show that $M_4 = M'_4$ after row swapping.

The matrices M_4 and M'_4 exhibit some of the same properties as the matrices constructed in Chapter 3. Just as in Chapter 3, an entry in the i -th column and j -th row represents the term in an exponent of the summand evaluated at indices i and j . This guarantees that row swapping is allowed and thus comparable entries will always be found in the same row and column. Furthermore, the argument in Chapter 3 that shows matrix equality means opposite signed terms cancel holds for these matrices since the method of representation is the same.

Assume $C_{03} = +$ and the matrices are equal up to row swapping. The same is true for $C_{03} = -$. This finishes the proof of Theorem 5.1. \square

Finally, we shall discuss case (ii) and why we cannot find a closed formula in this case. Assume $(x_i, x_j, x_k) \in (a_i, a_j, a_k)\mathbb{R} + \mathbb{Z}^3$ for some $1 \leq i < j < k \leq 4$ and not case (i). Without loss of generality, let $(x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3$. Then as before, we divide the sum into the terms when $h_1 = h_2 = h_3 = 0$ and the remaining

terms when not all h_1, h_2 and h_3 are zero

$$\sum_{k=1}^4 \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & y_k \end{pmatrix} = \sum_{h_4 \bmod a_4} \begin{pmatrix} \beta(0, y_2) \beta(0, y_3) \beta(-\tilde{r}_4, y_4) \\ +\beta(0, y_1) \beta(0, y_3) \beta(-\tilde{r}_4, y_4) \\ +\beta(0, y_1) \beta(0, y_2) \beta(-\tilde{r}_4, y_4) \\ +\beta(\tilde{r}_4, y_1) \beta(\tilde{r}_4, y_2) \beta(\tilde{r}_4 - \tilde{r}_3, y_3) \end{pmatrix} \\ + \sum_{\substack{H \\ (h_1, h_2, h_3) \neq (0,0,0)}} \begin{pmatrix} \beta\left(\frac{h_1}{a_1} - \frac{h_2}{a_2}, y_2\right) \beta\left(\frac{h_1}{a_1} - \frac{h_3}{a_3}, y_3\right) \beta\left(\frac{h_1}{a_1} - \tilde{r}_4, y_4\right) \\ +\beta\left(\frac{h_2}{a_2} - \frac{h_1}{a_1}, y_1\right) \beta\left(\frac{h_2}{a_2} - \frac{h_3}{a_3}, y_3\right) \beta\left(\frac{h_2}{a_2} - \tilde{r}_4, y_4\right) \\ +\beta\left(\frac{h_3}{a_3} - \frac{h_1}{a_1}, y_1\right) \beta\left(\frac{h_3}{a_3} - \frac{h_2}{a_2}, y_2\right) \beta\left(\frac{h_3}{a_3} - \tilde{r}_4, y_4\right) \\ +\beta\left(\tilde{r}_4 - \frac{h_1}{a_1}, y_1\right) \beta\left(\tilde{r}_4 - \frac{h_2}{a_2}, y_2\right) \beta\left(\tilde{r}_4 - \frac{h_3}{a_3}, y_3\right) \end{pmatrix}.$$

The ideas of the proof of Theorem 3.1 can be applied to the latter summand since all β -functions are evaluated at non-integer values. Let us investigate what

goes awry in the first summand.

$$\begin{aligned}
& \sum_{h_4 \bmod a_4} \left(\begin{array}{l} \beta(0, y_2) \beta(0, y_3) \beta(-\tilde{r}_4, y_4) \\ +\beta(0, y_1) \beta(0, y_3) \beta(-\tilde{r}_4, y_4) \\ +\beta(0, y_1) \beta(0, y_2) \beta(-\tilde{r}_4, y_4) \\ +\beta(\tilde{r}_4, y_1) \beta(\tilde{r}_4, y_2) \beta(\tilde{r}_4 - \tilde{r}_3, y_3) \end{array} \right) \\
&= \sum_{h_4 \bmod a_4} \left(\begin{array}{l} \frac{1}{2} \cdot \frac{e^{y_2+1}}{e^{y_2-1}} \cdot \frac{1}{2} \cdot \frac{e^{y_3+1}}{e^{y_3-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \\ +\frac{1}{2} \cdot \frac{e^{y_1+1}}{e^{y_1-1}} \cdot \frac{1}{2} \cdot \frac{e^{y_3+1}}{e^{y_3-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \\ +\frac{1}{2} \cdot \frac{e^{y_1+1}}{e^{y_1-1}} \cdot \frac{1}{2} \cdot \frac{e^{y_2+1}}{e^{y_2-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \\ +\frac{e^{\{\tilde{r}_4\}y_1}}{e^{y_1-1}} \cdot \frac{e^{\{\tilde{r}_4\}y_2}}{e^{y_2-1}} \cdot \frac{e^{\{\tilde{r}_4\}y_3}}{e^{y_3-1}} \end{array} \right) \\
&= \sum_{h_4 \bmod a_4} \left(\begin{array}{l} \frac{1}{2} \cdot \frac{e^{y_2+1}}{e^{y_2-1}} \cdot \frac{1}{2} \cdot \frac{e^{y_3+1}}{e^{y_3-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \cdot \left(\frac{e^{y_1-1}}{e^{y_1-1}} \right) \\ +\frac{1}{2} \cdot \frac{e^{y_1+1}}{e^{y_1-1}} \cdot \frac{1}{2} \cdot \frac{e^{y_3+1}}{e^{y_3-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \cdot \left(\frac{e^{y_2-1}}{e^{y_2-1}} \right) \\ +\frac{1}{2} \cdot \frac{e^{y_1+1}}{e^{y_1-1}} \cdot \frac{1}{2} \cdot \frac{e^{y_2+1}}{e^{y_2-1}} \cdot \frac{e^{\{-\tilde{r}_4\}y_4}}{e^{y_4-1}} \cdot \left(\frac{e^{y_3-1}}{e^{y_3-1}} \right) \\ +\frac{e^{\{\tilde{r}_4\}y_1}}{e^{y_1-1}} \cdot \frac{e^{\{\tilde{r}_4\}y_2}}{e^{y_2-1}} \cdot \frac{e^{\{\tilde{r}_4\}y_3}}{e^{y_3-1}} \cdot \left(\frac{4}{4} \left(\frac{e^{y_4-1}}{e^{y_4-1}} \right) \right) \end{array} \right).
\end{aligned}$$

Expanding terms we get the numerator

$$\sum_{h_4 \bmod a_4} \left(\begin{aligned} & e^{y_1+y_2+y_3+\{-\tilde{r}_4\}y_4} - e^{y_2+y_3+\{-\tilde{r}_4\}y_4} \\ & + e^{y_1+y_2+\{-\tilde{r}_4\}y_4} - e^{y_1+\{-\tilde{r}_4\}y_4} \\ & + e^{y_1+y_3+\{-\tilde{r}_4\}y_4} - e^{y_3+\{-\tilde{r}_4\}y_4} \\ & + e^{y_1+\{-\tilde{r}_4\}y_4} - e^{\{-\tilde{r}_4\}y_4} \\ & e^{y_1+y_2+y_3+\{-\tilde{r}_4\}y_4} - e^{y_1+y_3+\{-\tilde{r}_4\}y_4} \\ & + e^{y_1+y_2+\{-\tilde{r}_4\}y_4} - e^{y_1+\{-\tilde{r}_4\}y_4} \\ & + e^{y_2+y_3+\{-\tilde{r}_4\}y_4} - e^{y_3+\{-\tilde{r}_4\}y_4} \\ & + e^{y_2+\{-\tilde{r}_4\}y_4} - e^{\{-\tilde{r}_4\}y_4} \\ & e^{y_1+y_2+y_3+\{-\tilde{r}_4\}y_4} - e^{y_1+y_2+\{-\tilde{r}_4\}y_4} \\ & + e^{y_1+y_3+\{-\tilde{r}_4\}y_4} - e^{y_1+\{-\tilde{r}_4\}y_4} \\ & + e^{y_2+y_3+\{-\tilde{r}_4\}y_4} - e^{y_2+\{-\tilde{r}_4\}y_4} \\ & + e^{y_3+\{-\tilde{r}_4\}y_4} - e^{\{-\tilde{r}_4\}y_4} \\ & + 4e^{\{\tilde{r}_4\}y_1+\{\tilde{r}_4\}y_2+\{\tilde{r}_4\}y_3+y_4} - 4e^{\{\tilde{r}_4\}y_1+\{\tilde{r}_4\}y_2+\{\tilde{r}_4\}y_3} \end{aligned} \right)$$

(writing the sum in terms of y_1, y_2 and y_3)

$$= \sum_{h_4 \bmod a_4} \left(\begin{array}{l} e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} \\ + e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} - e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ + e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} \\ + e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} \\ + e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} - e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ + e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} \\ + e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ + e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ + e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ + e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ + e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ + 4e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} - 4e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} \end{array} \right).$$

After canceling like terms, we are left with

$$= \sum_{h_4 \bmod a_4} \left(\begin{array}{l} e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} \\ + e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ + 2e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - 2e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \end{array} \right).$$

As with all the previous summands, this final sum exhibits an intriguing sym-

metry among the exponents. And though it is not yet clear how, it is quite possible that these few terms combine to equal something simple. The unknown fate of this extraneous case gives light to the level of complexity that builds when dealing with higher dimensional Dedekind–Bernoulli sums. Although our work and Bayad and Raouj’s work shows the extreme cases of Hall, Wilson and Zagier’s reciprocity law can be generalized, the 4-variable reciprocity Theorem shows us the more variables involved the more challenging computations become and generalizations of the intermediate cases are not strait forward. We leave this open question to the reader.

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