A LATTICE POINT ENUMERATION APPROACH TO PARTITION IDENTITIES

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *Integer Partitions* by George E. Andrews and Kimmo Eriksson and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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In this paper, we present a novel method to find generating functions of partition identities. Our method is based on integer-point enumeration in polyhedra. We show how lattice-point enumeration can be applied to partition identity theorems that were proved using MacMahon's Ω-operator, and establish the full generating functions of these theorems. In addition to introducing our new method, we establish connections between the different mathematic areas of Geometric Combinatorics and Number Theory.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Chapter 1

Introduction

1.1 Leibniz and Euler

According to [2], Leibniz (1646–1716) was the first person who was asking a question about the number of partitions of integers. Leibniz observed that there are three partitions of 3 (3, $2+1$, and $1+1+1$), five partitions of 4, seven partitions of 5, and eleven partitions of 6. These beginnings opened the field of partitions. On September 4, 1740, Naude (1684–1747) wrote Euler (1707–1783) to ask how many partitions there are of 50 into seven distinct parts. The correct answer is 522 [2]. However, it is not likely to be obtained by writing out all the ways of adding seven distinct positive integers to get 50. To solve this problem Euler introduced generating functions, arguably the most important innovation in the history of partitions. We can find the use of generating functions in the theory of partitions in [3, Ch. 13], [4, Chs. 1

and 2], [6, Ch. 5], and later in this paper.

1.2 Goal of This Paper

This paper presents a novel method for finding generating functions for various forms of partitions. In Chapter 2 we introduce the definitions of these objects and highlight the utility of generating functions. We also give some theorems of partition identities which Andrews et al proved in [5] and [7] using the Ω-operator [8].

Our method is based on integer-point enumeration in polyhedra [9]. Chapter 3 provides the mathematical background for this method. We start with the language of cones in term of the affine structure of $\mathbb{R}^d$ and integer-point transforms for rational cones. We introduce theorems and a lemma which help us to obtain the generating functions for the Ω theorems of Chapter 2.

The main results of this paper appear in Chapter 4. We will reprove and extend the theorems of Chapter 2 using our lattice-point enumeration approach. In particular, Chapter 4 provides the full generating functions of the theorems of Chapter 2.

The motivation for this paper is to shed new lights on known theorems. We hope that the method that we have used will establish further connections between geometric combinatorics and number theory.

Chapter 2

Partition Functions and Ω Theorems

2.1 Partition Functions

A **partition** of a positive integer n (or a partition of weight n) is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where the $\lambda_i's$ are non-negative integers such that $\sum_{i=1}^{k} \lambda_i = n$. The $\lambda_i's$ are the **parts** of the partition λ.

Example 2.1. The partitions of 5 are: (5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$, and $(1,1,1,1,1)$. In particular, the number of partitions of 5 is 7.

There are various forms of partitions. For example, we can have partitions of at most k parts, partitions into odd parts, partitions into distinct parts, and so on.

Example 2.2. The partitions of 5 into odd parts are: (5), $(3,1,1)$, and $(1,1,1,1,1)$. Thus, the number of partitions of 5 into odd parts is 3.

2.2 Generating Functions

Generating functions form a tool to deal with partitions. Let $\{a_k\}_{k=0}^{\infty}$ be an infinite sequence. The **generating function** of the sequence a_k is a function $F(x)$ expressed as the formal power series

$$F(x) = \sum_{k=0}^{\infty} a_k x^k \,.$$

Example 2.3. A key generating function is the one for the constant sequence 1, 1, 1, 1, 1, . . . , namely $F(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

Generating functions are very useful in that the degree of each monomial keeps track of the position in the sequence while the coefficient provides the actual value of the term. Then if we play by the rules of either formal or analytic power series, we may be able to derive results. The following is one of the examples of the power of generating functions provided in [9, p. 3].

Consider the classic example of the **Fibonacci sequence** f_k, named after Leonardo Pisano Fibonacci (1170–1250) and defined by the recursion

$$f_0 = 0,\ f_1 = 1,\ \text{ and } f_{k+2} = f_{k+1} + f_k \text{ for } k \geq 0\,.$$

This gives the sequence $\{f_k\}_{k=0}^{\infty} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$. Let

$$F(z) = \sum_{k\geq 0} f_k z^k \,.$$

We embed both sides of the recursion identity into their generating functions:

$$\sum_{k\geq 0} f_{k+2} z^k = \sum_{k\geq 0} (f_{k+1} + f_k) z^k = \sum_{k\geq 0} f_{k+1} z^k + \sum_{k\geq 0} f_k z^k \,. \tag{2.1}$$

Then the left-hand side of (2.1) is

$$\sum_{k\geq 0} f_{k+2} z^k = \frac{1}{z^2} \sum_{k\geq 0} f_{k+2} z^{k+2} = \frac{1}{z^2} \sum_{k\geq 2} f_k z^k = \frac{1}{z^2} (F(z) - z) \,,$$

and the right-hand side of (2.1) is

$$\sum_{k\geq 0} f_{k+1} z^k + \sum_{k\geq 0} f_k z^k = \frac{1}{z} F(z) + F(z) \,.$$

So (2.1) can be restated as

$$\frac{1}{z^2} (F(z) - z) = \frac{1}{z} F(z) + F(z) \,.$$

Solving for $F(z)$ we obtain

$$F(z) = \frac{z}{1 - z - z^2} \,.$$

$F(z)$ has the following partial fraction expansion

$$F(z) = \frac{z}{1 - z - z^2} = \frac{1/\sqrt{5}}{1 - \frac{1+\sqrt{5}}{2} z} - \frac{1/\sqrt{5}}{1 - \frac{1-\sqrt{5}}{2} z} \,.$$

Now, we use the well known geometric series from Example 2.3 with $x = \frac{1+\sqrt{5}}{2}z$ and $x = \frac{1-\sqrt{5}}{2}z$, respectively. We obtain

$$\begin{aligned} F(z) = \frac{z}{1-z-z^2} &= \frac{1}{\sqrt{5}} \sum_{k\geq 0} \left(\frac{1+\sqrt{5}}{2} z \right)^k - \frac{1}{\sqrt{5}} \sum_{k\geq 0} \left(\frac{1-\sqrt{5}}{2} z \right)^k \\ &= \sum_{k\geq 0} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right) . \end{aligned}$$

This provides the desired closed-form expression for the Fibonacci sequence

$$f_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^k - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^k .$$

The rational function obtained using the properties of geometric series is called a **rational generating function**. Often we will jump back and forth from generating functions to rational generating functions.

Furthermore, we can obtain interesting results about partitions by using generating functions. Throughout this paper we will see the utility of generating functions in establishing relations between various partition functions.

2.3 Ω Theorems

In this section, we introduce some partition theorems which Andrews et al established using the Ω-operator in [5] and [7]. First, we introduce some definitions.

2.3.1 k-Gon Partitions

Definition 2.1. [7, Definition 2] As the set of **non-degenerate k-gon partitions** into positive parts we define

$$\tau_k := \{(a_1, \ldots, a_k) \in \mathbb{Z}^k : 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \text{ and } a_1 + \cdots + a_{k-1} > a_k\} .$$

As the set of non-degenerate k-gon partitions of n into positive parts we define

$$\upsilon_k(n) := \{(a_1, \ldots, a_k) \in \tau_k : a_1 + \cdots + a_k = n\} .$$

The corresponding cardinality is denoted by

$$t_k(n) := |\upsilon_k(n)| .$$

The term "non-degenerate" refers to the restriction to strict inequality, i.e. to $a_1 + a_2 + \cdots + a_{k-1} > a_k$.

Definition 2.2. [7, Definition 3] For an integer $k \geq 0$, let

$$T_k(q) := \sum_{n \geq k} t_k(n) q^n ,$$

and

$$S_k(x_1, \ldots, x_k) := \sum_{(a_1,\ldots,a_k) \in \tau_k} x_1^{a_1} \ldots x_k^{a_k} .$$

Theorem 2.1. *[7, Theorem 1] Let $k \geq 3$ and $X_i = x_i \dots x_k$ for $1 \leq i \leq k$. Then*

$$\begin{aligned} S_k(x_1, \dots, x_k) = & \frac{X_1}{(1-X_1)(1-X_2)\cdots(1-X_k)} \\ & - \frac{X_1 X_k^{k-2}}{1-X_k} \frac{1}{(1-X_{k-1})(1-X_{k-2}X_k)(1-X_{k-3}X_k^2)\cdots(1-X_1X_k^{k-2})} \, . \end{aligned}$$

2.3.2 Partitions with Difference Conditions

Theorem 2.2. *[5, Theorem 3.1] Let $\triangle_m(n)$ denote the number of partitions of n into $2m+1$ nonnegative parts*

$$n = a_1 + a_2 + a_3 + \cdots + a_{2m+1} \, ,$$

where the parts are listed in non-increasing order and additionally

$$a_1 - a_2 - a_3 + a_4 \leq 0 \, ,$$

$$a_3 - a_4 - a_5 + a_6 \leq 0 \, ,$$

$$\vdots$$

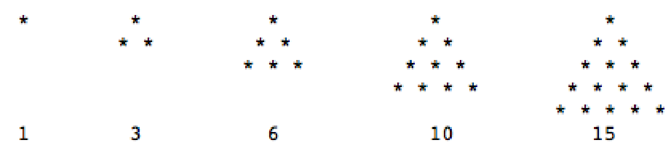
$$a_{2m-3} - a_{2m-2} - a_{2m-1} + a_{2m} \leq 0 \, ,$$

$$a_{2m-1} - a_{2m} - a_{2m+1} \leq 0 \, .$$

Then $\triangle_m(n)$ equals the number of partitions of n into parts that are either $\leq 2m$ and even or of the form $(j+1)(2m+1-j)$ with $0 \leq j \leq m$.

2.3.3 Partitions with Higher Order Difference Conditions

For the following theorem, we will need to define triangular numbers. A **triangular number** is the number of dots we need to make triangles. These are the first few triangular numbers. The following image is provided in [1].



The way to get these numbers without drawing pictures is to add up all the numbers that come before a certain number. For example, the tenth triangular number is $1+2+3+4+5+6+7+8+9+10$. By this method, a triangular number is, equivalently, the sum of the natural numbers from 1 to n, and the nth triangular number, T_n, is

$$T_n = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

Theorem 2.3. *[5, Theorem 4.1] Let $p_2(n)$ denote the number of partitions of n of the form $a_1 + a_2 + .. + a_s$ (s arbitrary) where the first differences are nonnegative, i.e., $a_i - a_{i+1} \geq 0$ for $1 \leq i \leq s-1$, and the the second differences are nonnegative, i.e., $a_i - 2a_{i+1} + a_{i+2} \geq 0$ for $1 \leq i \leq s-1$ (assuming $a_{s+1} = 0$). Then $p_2(n)$ equals* *the number of partitions of n into triangular numbers.*

2.3.4 Partitions With Mixed Difference Conditions

Theorem 2.4. *[5, Theorem 5.1] Let $p_{\pm}(m,n)$ denote the number of partitions of n of the form $a_1 + a_2 + ... + a_m$, wherein $a_i - a_{i+1} \geq 0$ for $1 \leq i \leq m-1$, while $a_i - 2a_{i+1} + a_{i+2} \leq 0$ for $1 \leq i \leq m-1$ (with $a_{m+1} = 0$). Then $p_{\pm}(m,n)$ equals the number of partitions of n of the form $(m-j)(m+j+1)/2$ where $0 \leq j \leq m$.*

Note that Theorems 2.1–2.4 constituted the main content of [5] and [7]. In Chapter 4, we will prove and extend the above theorems using lattice-point enumeration.

Chapter 3

Polyhedra

3.1 The Language of Cones

Let $\mathbf{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$. Then a **hyperplane** is a set of the form $\{\mathbf{x} \in \mathbb{R}^d : a_1x_1+a_2x_2+\cdots+a_dx_d = b\}$ and a **halfspace** is a set of the form $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} \leq b\}$.

A **pointed cone** $K \subseteq \mathbb{R}^d$ is a set of the form

$$K = \{\mathbf{v} + \lambda_1\mathbf{w_1} + \lambda_2\mathbf{w_2} + \cdots + \lambda_m\mathbf{w_m} : \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\},$$

where $\mathbf{v}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_m} \in \mathbb{R}^d$ are such that there exists a hyperplane H for which $H \cap K = \{\mathbf{v}\}$. The vector $\mathbf{v}$ is called the **apex** of K and each $\mathbf{w_i}$ is called a **generator**. K is said to be **rational** if all of its generators and apex are rational. The **dimension** of K is the dimension of the affine space spanned by K; if K is of dimension d, we call it a d-cone. A d-cone K is said to be **simplicial** if it has exactly d linearly independent generators.

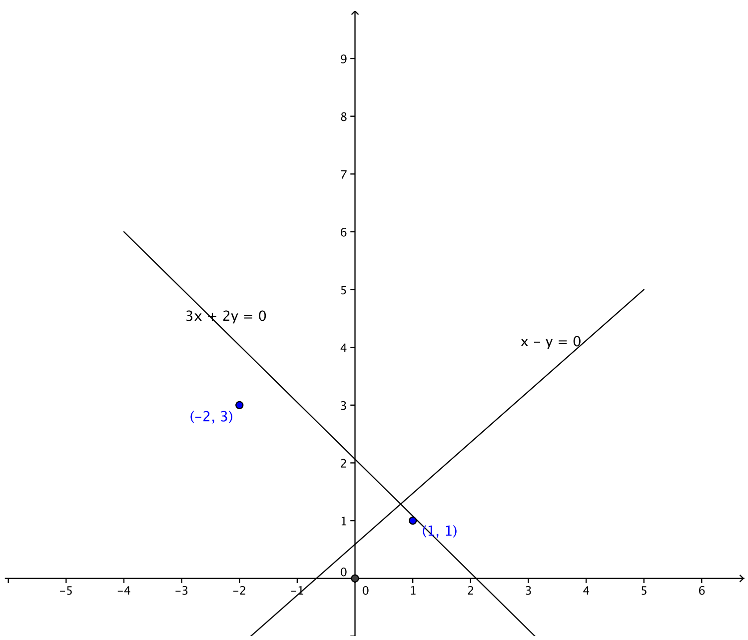


Figure 3.1: The simplicial cone $K = \{(0,0) + \lambda_1(-2,3) + \lambda_2(1,1) : \lambda_1, \lambda_2 \geq 0\}$.

Figure 3.1 shows the hyperplanes $3x + 2y = 0$, $x - y = 0$ and the cone K is created by the intersection of two halfspaces $3x + 2y \geq 0$ and $x - y \leq 0$.

We say that the hyperplane $H = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} = b\}$ is a **supporting hyperplane** of the pointed d-cone K if K lies entirely on one side of H, that means,

$$K \subset \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} \leq b\} \text{ or } K \subset \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} \geq b\}.$$

A **face** of K is a set of the form $K \cap H$, where H is a supporting hyperplane of K. The $(d-1)$-dimensional faces are called **facets** and the 1-dimensional faces are called **edges** and the apex of K is its unique 0-dimensional face.

3.2 Integer-Point Transforms for Rational Cones

For a cone $K \subset \mathbb{R}^n$, let

$$\sigma_K(\mathbf{z}) = \sigma_K(z_1, z_2, \ldots, z_d) = \sum_{\mathbf{m} \in K \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}},$$

with the usual monomial notation $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$. The generating function σ_K lists all integer points in K in a special form: not as a list of vectors, but as a formal sum of monomials. For example, the integer point $(3,4)$ would be listed as the monomial $z_1^3 z_2^4$. We call σ_K the **integer-point transform** of K; the function σ_K also goes by the name moment generating function or simply generating function of K.

Now we are ready to state the following theorem, which helps us to find the generating function of a simplicial cone.

Theorem 3.1. *[9, Theorem 3.5] Let K be an n-dimensional, rational, simplicial cone with generators, $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n} \in \mathbb{Z}^n$. Then*

$$\sigma_K(\mathbf{z}) = \frac{\sigma_{\pi_K}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{w_1}})(1 - \mathbf{z}^{\mathbf{w_2}}) \cdots (1 - \mathbf{z}^{\mathbf{w_n}})}.$$

where π_K *is the half-open parrallelepiped*

$$\pi_K := \{\lambda_1 \mathbf{w_1} + \lambda_2 \mathbf{w_2} + \cdots + \lambda_n \mathbf{w_n} : 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n < 1\}.$$

Example 3.1. Find $S(z_1, z_2) = \sum_{2m_1 \geq m_2, 2m_2 \geq m_1} z_1^{m_1} z_2^{m_2}$, where the sum ranges over all pairs (m_1, m_2) of non-negative integers satisfying the indicated inequalities.

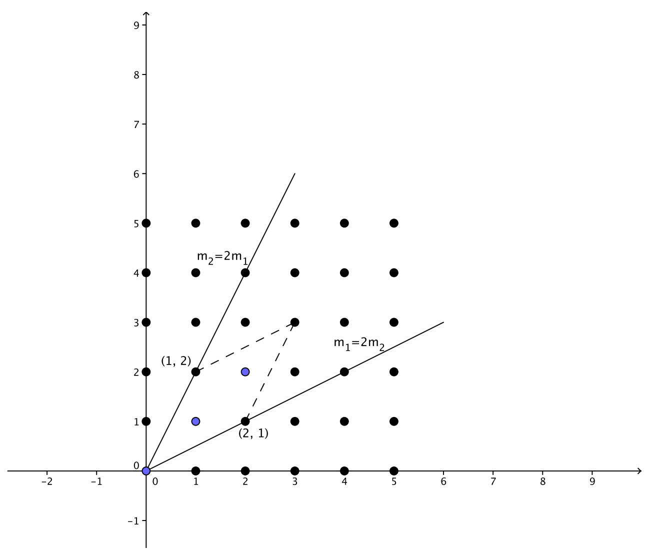


Figure 3.2: The cone K and its fundamental parallelogram.

Proof. We see that all the integer points (m_1, m_2) satisfying the indicated inequalities lie on the intersection of the halfspaces $m_1 \leq 2m_2$ and $m_2 \leq 2m_1$, as shown in Figure 3.2. This intersection creates the two-dimensional simplicial cone K with generators $(2,1)$ and $(1,2)$. Hence,

$$K = \{\lambda_1(1,2) + \lambda_2(2,1) : \lambda_1, \lambda_2 \geq 0\}$$

and

$$\pi_K = \{\lambda_1(1,2) + \lambda_2(2,1) : 1 > \lambda_1, \lambda_2 \geq 0\}.$$

Applying Theorem 3.1, we obtain

$$\sigma_K(z_1, z_2) = \frac{\sigma_{\pi_K}(z_1, z_2)}{(1 - z_1 z_2^2)(1 - z_1^2 z_2)}.$$

Figure 3.2 shows $\pi_K \cap \mathbb{Z}^2 = \{(0,0), (1,1), (2,2)\}$. This implies $\pi_K = 1 + z_1 z_2 + z_1^2 z_2^2$. Therefore,

$$S(z_1, z_2) = \sigma_K(z_1, z_2) = \frac{1 + z_1 z_2 + z_1^2 z_2^2}{(1 - z_1^2 z_2)(1 - z_1 z_2^2)}.$$

$\square$

The following lemma is a well-known result in lattice-point enumeration. This version was formulated in [10] for easy application to partition and composition enumeration problems.

Lemma 3.2. *Let $C = [c_{i,j}]$ be an $n \times n$ matrix of integers such that $C^{-1} = B = [b_{i,j}]$ exists and $b_{i,j}$ are all nonnegative integers. Let $e_1, \ldots, e_n$ be nonnegative integers.*

For each $1 \le i \le n$, *let* c_i *be the constraint*

$$c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \cdots + c_{i,n}\lambda_n \ge e_i.$$

Let S_C *be the set of nonnegative integer sequences* $\lambda = (\lambda_1, \lambda_2 \ldots, \lambda_n)$ *satisfying the constraints* c_i *for all* i, $1 \le i \le n$. *Then the generating function for* S_C *is:*

$$F_C(x_1, x_2, \ldots, x_n) = \sum_{\lambda \in S_C} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} = \frac{\prod_{j=1}^{n} (x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})^{e_j}}{\prod_{j=1}^{n} (1 - x_1^{b_{1,j}} x_2^{b_{2,j}} \cdots x_n^{b_{n,j}})}.$$

Chapter 4

Geometric Proofs of Ω Theorems

This chapter contains proofs of the Ω theorems that are listed in Chapter 2, using lattice-point enumeration. Before proving the Ω theorems, we start with an elementary problem.

4.1 Integer-Sided Triangles of Perimeter n

The following has been posed as a problem and solved using the Ω-operator in [7, Problem 1]. We want to show that the problem can be solved using lattice-point enumeration.

Problem: Let $t_3(n)$ be the number of non-congruent triangles whose sides have integer length and whose perimeter is n. For instance, $t_3(9) = 3$, corresponding to $3+3+3, 2+3+4, 1+4+4$. Find $\sum_{n \geq 3} t_3(n) q^n$.

The corresponding generating function is

$$T_3(q) := \sum_{n \geq 3} t_3(n) q^n = \sum\nolimits^* q^{a_1+a_2+a_3},$$

where $\sum^*$ is the restricted summation over all positive integer triples (a_1, a_2, a_3) satisfying $a_1 \leq a_2 \leq a_3 \leq 1$ and $a_1 + a_2 > a_3$. In other words, we want to find all partitions of n of the form $a_1 + a_2 + a_3$, where $1 \leq a_1 \leq a_2 \leq a_3$ and $a_1 + a_2 > a_3$.

Proof. Figure 4.1 shows all integer points (a_1, a_2, a_3) that satisfy the conditions above lie in the cone

$$K = \{(\lambda_1(0,1,1) + \lambda_2(1,1,1) + \lambda_3(1,1,2) : \lambda_1, \lambda_3 \geq 0 \text{ and } \lambda_2 > 0\},$$

which is the intersection of three halfspaces: $a_1 \leq a_2$, $a_2 \leq a_3$ and $a_1 + a_2 > a_3$. This implies

$$\pi_K = \{(\lambda_1(0,1,1) + \lambda_2(1,1,1) + \lambda_3(1,1,2) : 1 > \lambda_1, \lambda_3 \geq 0 \text{ and } 1 \geq \lambda_2 > 0\},$$

and $\pi_K \cap \mathbb{Z}^3 = \{(1,1,1)\}$. Using Theorem 3.1, we obtain

$$\sigma_K(x_1, x_2, x_3) = \frac{x_1 x_2 x_3}{(1 - x_1 x_2)(1 - x_1 x_2 x_2)(1 - x_1^2 x_2 x_3)}.$$

Note that $\sigma_K(x_1, x_2, x_3) = S_3(x_1, x_2, x_3)$ in the language of Theorem 2.1.

Since $n = a_1 + a_2 + a_3$, we let $q = x_1 = x_2 = x_3$. Hence, the generating function of K:

$$\sigma_K(q, q, q) = T_3(q) = \sum_{n \geq 3} t_3(n) q^n = \sum\nolimits^{*} q^{x_1 + x_2 + x_3} = \frac{q^3}{(1 - q^2)(1 - q^3)(1 - q^4)}\,. \tag{4.1}$$

□

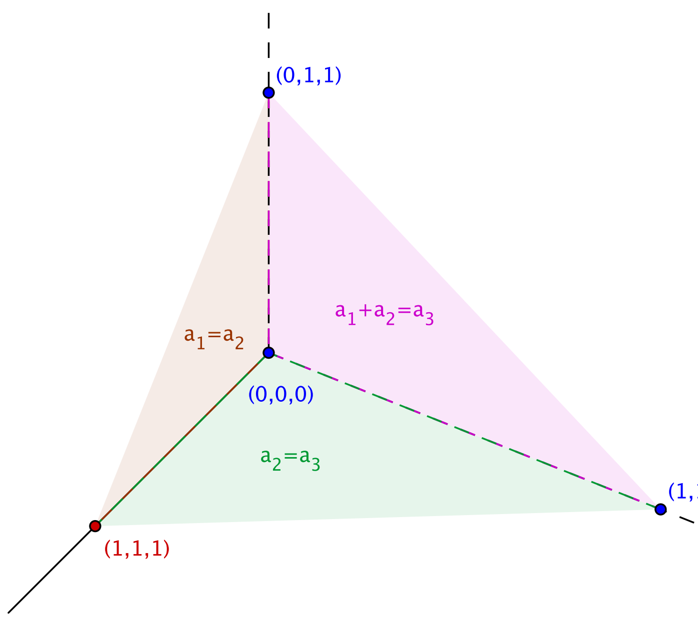


Figure 4.1: The cone $K = \{(\lambda_1(0,1,1) + \lambda_2(1,1,1) + \lambda_3(1,1,2) : \lambda_1, \lambda_3 \geq 0 \text{ and } \lambda_2 > 0\}$.

Now we consider Theorem 2.1, the generalization of the triangle problem to k-gons, where $k \geq 3$, which was proved using the Ω-operator in [7, Theorem 1]. In the following section, with the lattice-point enumeration method in hand, we are able to reprove this main result for k-gon partitions.

4.2 Geometric Proof of Theorem 2.1

In Section 4.1, we computed the generating functions $T_3(q) = \sum_{n\geq 3} t_3(n)q^n$ and $S_3(x_1, x_2, x_3)$. Our goal in this section is to compute the generating function

$$S_k(x_1, x_2, \ldots, x_k) = \sum_{(a_1,a_2,\ldots,a_k)\in\tau_k} x_1^{a_1} \cdots x_k^{a_k} \,,$$

where

$$\tau_k = \{(a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k : a_k \geq a_{k-1} \geq \cdots \geq a_1 > 0 \text{ and } a_1+a_2+\cdots+a_{k-1} > a_k\} \,.$$

Proof of Theorem 2.1. Let

$$K := \{(a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k : a_k \geq a_{k-1} \geq \cdots \geq a_1 > 0\} \,.$$

and

$$P := \{(a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k : a_k \geq a_{k-1} \geq \cdots \geq a_1 > 0 \text{ and } a_1+a_2+\cdots+a_{k-1} \leq a_k\} \,.$$

We see that $\tau_k = K \setminus P$. The constraints of K are given by the system

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \\ a_k \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{4.2}$$

Now let C be the matrix on the left side of (4.2). Then $\det(C) = 1$, which implies that C is invertible, and

$$C^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We see that the integer entries of C^{-1} are all nonnegative, and Lemma 3.2 gives the generating function of K :

$$\sigma_K(x_1, x_2, \dots, x_k) = \frac{X_1}{(1 - X_1)(1 - X_2) \cdots (1 - X_k)}.$$

Recall that $X_i = x_i \cdots x_k$ for $1 \leq i \leq k$.

The constraints of P are given by the system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{k-1} \\ a_k \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} . \tag{4.3}$$

Since $x_{k-1} \geq x_{k-2} \geq \cdots \geq x_1 \geq 1$ and $x_k \geq x_1 + x_2 + \cdots + x_{k-1}$, this implies $x_k \geq x_{k-1}$. Therefore, we do not need the condition $x_k \geq x_{k-1}$.

Now, let D be the matrix on the left side of (4.3). Then $\det(D) = 1$, and

$$D^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 0 \\ k & k-1 & k-2 & k-3 & \dots & 3 & 2 & 1 & 1 \end{bmatrix} .$$

Again, we see that the integer entries of D^{-1} are all nonnegative, and Lemma 3.2

gives the generating function of P :

$$\sigma_P(x_1, x_2, \ldots, x_k) = \frac{X_1 X_k^{k-2}}{(1 - X_k)(1 - X_{k-1})(1 - X_{k-2}X_k)(1 - X_{k-3}X_k^2) \cdots (1 - X_1 X_k^{k-2})}.$$

We have $\tau_k = K \setminus P$, and so

$$\begin{aligned} S_k(x_1, x_2, \ldots, x_k) &= \sigma_K(x_1, x_2, \ldots, x_k) - \sigma_P(x_1, x_2, \ldots, x_k) \\ &= \frac{X_1}{(1 - X_1)(1 - X_2) \cdots (1 - X_k)} \\ &\quad - \frac{X_1 X_k^{k-2}}{(1 - X_k)(1 - X_{k-1})(1 - X_{k-2}X_k)(1 - X_{k-3}X_k^2) \cdots (1 - X_1 X_k^{k-2})}. \end{aligned}$$

□

To show that our method can be applied for other cases of partition identities, we will reprove Theorem 2.2 using our method in the following section.

4.3 Geometric Proof of Theorem 2.2

In this section, our goal simplifies to computing the generating function of $\triangle_m(n)$:

$$\sigma_K(q) := \sum_{n \geq 0} \triangle_m(n) q^n = \sum_{(a_1, a_2, \ldots, a_{2m+1}) \in K} q^{a_1 + a_2 + \cdots + a_{2m+1}},$$

where

$$K := \{(a_1, a_2, \ldots, a_{2m+1}) \in \mathbb{Z}_{2m+1} : a_1 \geq a_2 \geq \cdots \geq 0 \text{ and}$$

$$a_1 - a_2 - a_3 + a_4 \leq 0\,,$$

$$a_3 - a_4 - a_5 + a_6 \leq 0\,,$$

$$\vdots$$

$$a_{2m-3} - a_{2m-2} - a_{2m-1} + a_{2m} \leq 0\,,$$

$$a_{2m-1} - a_{2m} - a_{2m+1} \leq 0\}\,.$$

Proof of Theorem 2.2. We see that $a_1 - a_2 - a_3 + a_4 \leq 0 \Rightarrow a_1 + a_4 \leq a_2 + a_3$. This implies $a_3 \geq a_4$ because $a_1 \geq a_2$. Similarly, $a_{2m-3} - a_{2m-2} - a_{2m-1} + a_{2m} \leq 0 \Rightarrow a_{2m-3} + a_{2m} \leq a_{2m-2} + a_{2m-1}$. This implies $a_{2m-1} \geq a_{2m}$, and $a_{2m-1} - a_{2m} - a_{2m+1} \leq 0 \Rightarrow a_{2m-1} \leq a_{2m} + a_{2m+1}$. This condition guarantees that $a_{2m+1} \geq 0$ because

$a_{2m-1} \geq a_{2m}$. Therefore, the constraints of K are given by the system

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_{2m} \\ a_{2m+1} \end{bmatrix} \geq 0\,. \tag{4.4}$$

Let A be the matrix on the left side of (4.4). Then

$$\det A = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Adding all odd-numbered rows together we get

$$\det A = \begin{vmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & \vdots & & & & & & \vdots \\
\vdots & & & & & & \vdots & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1
\end{vmatrix}.$$

Thus, $|\det A| = 1$, and

$$A^{-1} = \begin{bmatrix} m+1 & 1 & 1 & \dots & 1 & m & m-1 & \dots & 2 & 1 \\ m & 1 & 1 & \dots & 1 & m & m-1 & \dots & 2 & 1 \\ m & 0 & 1 & \dots & 1 & m & m-1 & \dots & 2 & 1 \\ m-1 & 0 & 1 & \dots & 1 & m-1 & m-1 & \dots & 2 & 1 \\ m-1 & 0 & 0 & \dots & 1 & m-1 & m-1 & \dots & 2 & 1 \\ \vdots & & & \dots & \vdots & & & \dots & & \vdots \\ 1 & 0 & 0 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

The integer entries of A^{-1} are all nonnegative, and Lemma 3.2 gives the generating function of K as

$\sigma_K(x_1, x_2, \dots, x_{2m+1}) =$

$$\frac{1}{(1 - x_1^{m+1} x_2^m \cdots x_{2m+1})(1 - x_1 x_2) \cdots (1 - x_1 x_2 \cdots x_{2m+1})}.$$

Now let $q = x_1 = x_2 = \cdots = x_{2m+1}$, we get $\sigma_K(q, q, \cdots, q)$ as

$$\frac{1}{(1-q^2)\cdots(1-q^{2m})(1-q^{2m+1})(1-q^{4m})(1-q^{3(2m-1)})\cdots(1-q^{m(m+2)})(1-q^{(m+1)(m+1)})}.$$

Hence, the generating function we set out to find is

$$\sigma_K(q) = \prod_{j=1}^{m} \frac{1}{(1-q^{2j})} \prod_{i=0}^{m} \frac{1}{(1-q^{(i+1)(2m+1-i)})}\,,$$

and this is precisely the generating function for the partitions described in Theorem 2.2. $\square$

Note that we actually derived the full generating function of Theorem 2.2.

Theorem 4.1. *For an integer $m \geq 0$, let*

$$\sigma(x_1, x_2, \ldots, x_{2m+1}) := \sum{}^{*} x_1^{a_1} x_2^{a_2} \cdots x_{2m+1}^{a_{2m+1}}\,,$$

where $\sum^*$ is the restricted summation over all nonnegative integers $(a_1, a_2, \ldots, a_{2m+1})$ satisfying

$$a_1 \geq a_2 \geq \cdots \geq 0 \text{ and}$$

$$a_1 - a_2 - a_3 + a_4 \leq 0\,,$$

$$a_3 - a_4 - a_5 + a_6 \leq 0\,,$$

$$\vdots$$

$$a_{2m-3} - a_{2m-2} - a_{2m-1} + a_{2m} \leq 0\,,$$

$$a_{2m-1} - a_{2m} - a_{2m+1} \leq 0\,.$$

Let $X_i = x_1 \cdots x_i$ *for* $1 \leq i \leq 2m+1$. *Then*

$$\sigma(x_1, x_2, \cdots, x_{2m+1}) = \frac{1}{(1-X_2)\cdots(1-X_{2m})(1-X_1X_3X_5\cdots X_{2m+1})(1-X_3X_5\cdots X_{2m+1})\cdots(1-X_{2m+1})}.$$

Once we have seen the results of Theorems 2.2 and 4.1, it is natural to consider a variety of partition identities related to further difference conditions.

4.4 Geometric Proofs of Theorem 2.3

In this section, we give a novel proof of Theorem 2.3. This theorem requires us to prove that $p_2(n)$ equals the number of partitions of n into triangular numbers.

Proof of Theorem 2.3. The generating function of $p_2(n)$ is

$$\sum_{n\geq 0} p_2(n)q^n = \sum_{(a_1,a_2,\ldots,a_s)\in K} q^{a_1+a_2+\cdots+a_s},$$

where

$$K := \left\{ (a_1, a_2, \ldots, a_s) \in \mathbb{Z}^s \; : \; \begin{array}{c} a_1 \geq a_2 \geq \cdots \geq a_s \geq 0 \text{ and } a_i - 2a_{i+1} + a_{i+2} \geq 0 \\ \text{for } 1 \leq i \leq s-1 \text{ and } a_{s+1} = 0 \end{array} \right\}.$$

Claim: The conditions $a_i - 2a_{i+1} + a_{i+2} \geq 0$ for $1 \leq i \leq s-1$ and $a_{s+1} = 0$ guarantee that $a_i \geq a_{i+1}$ for $1 \leq i \leq s-1$.

Proof of Claim: We will use induction on s.

Base case: If $s = 1$ then $a_1 \geq 0$.

If $s = 2$, we have $a_1 - 2a_2 \geq 0$, this implies $a_1 \geq 2a_2$. Thus $a_1 \geq a_2$.

Induction step: Assume the claim is true for $s-1$, i.e, if $a_i - 2a_{i+1} + a_{i+2} \geq 0$ for $1 \leq i \leq s-2$, then $a_1 \geq a_2 \geq \cdots \geq a_{s-1}$. We want to prove that it is also true for s. The condition $a_i - 2a_{i+1} + a_{i+2} \geq 0$ implies $a_{s-1} \geq 2a_s$. Therefore, $a_{s-1} \geq a_s$. Next, $a_{s-2} - 2a_{s-1} + a_s \geq 0 \Rightarrow a_{s-2} + a_s \geq 2a_{s-1}$. We have shown that $a_{s-1} \geq a_s$. Thus $a_{s-2} \geq a_{s-1}$. Now we use the induction step for $(a_1, a_2, \ldots, a_{s-1})$, and the claim is proven.

Thus, the conditions for K can be simplified:

$$K := \left\{ (a_1, a_2, \ldots, a_s) \in \mathbb{Z}^s \ : \begin{array}{c} a_s \geq 0 \text{ and } a_i - 2a_{i+1} + a_{i+2} \geq 0 \\ \text{for } 1 \leq i \leq s-1 \text{ and } a_{s+1} = 0 \end{array} \right\}.$$

Therefore, the constraints of K are given by the system

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ \vdots \\ a_{s-2} \\ a_{s-1} \\ a_s \end{bmatrix} \geq 0\,. \tag{4.5}$$

Let A be the matrix on the left side of (4.5). Then $\det(A) = 1$, and

$$A^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & s-3 & s-2 & s-1 & s \\ 0 & 1 & 2 & 3 & \dots & s-4 & s-3 & s-2 & s-1 \\ 0 & 0 & 1 & 2 & \dots & s-5 & s-4 & s-3 & s-2 \\ \vdots & & & & \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The integer entries of A^{-1} are all nonnegative. Thus, Lemma 3.2 gives the generating function of K:

$$\sigma_K(x_1, x_2, \ldots, x_s) = \frac{1}{(1-x_1)(1-x_1^2x_2)\cdots(1-x_1^sx_2^{s-1}\cdots x_s)}$$
$$= \prod_{i=1}^{s} \frac{1}{(1-x_1^ix_2^{i-1}\ldots x_i)} .$$

Since $n = a_1+a_2+\cdots+a_s$, we let $q = x_1 = x_2 = \cdots = x_s$. The generating function of $p_2(n)$ is

$$\sigma_K(q) = \prod_{i=1}^{s} \frac{1}{(1-q^iq^{i-1}\ldots q)} = \prod_{i=1}^{s} \frac{1}{1-q^{\binom{i+1}{2}}} ,$$

which is the generating function for partitions into triangular numbers. Thus, $p_2(n)$ equals the number of partitions of n into triangular numbers. $\square$

Note that we actually derived the full generating function of Theorem 2.3.

Theorem 4.2. *For an integer $s \geq 1$, let*

$$\sigma(x_1, x_2, \ldots, x_s) := \sum\nolimits^* x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_s^{a_s} ,$$

where $\sum^*$ is the restricted summation over all nonnegative integers $(a_1, a_2, \cdots, a_s)$ satisfying $a_s \geq 0$ and $a_i - 2a_{i+1} + a_{i+2} \geq 0$ for $1 \leq i \leq s-1$ and $a_{s+1} = 0$. Then

$$\sigma(x_1, x_2, \ldots, x_s) = \prod_{i=1}^{s} \frac{1}{(1-x_1^ix_2^{i-1}\cdots x_i)} .$$

Once one becomes aware of the discoveries that the geometry of lattice-point enumeration yields almost painlessly, it is possible to produce the next result.

4.5 Geometric Proof Of Theorem 2.4

In this section, we give a geometric proof that $p_{\pm}(m,n)$ equals the number of partitions of n of the form $(m-j)(m+j+1)/2$ where $0 \leq j \leq m$.

Proof of Theorem 2.4. The generating function of $p_{\pm}(m,n)$ is

$$\sum_{n\geq 0} p_{\pm}(m,n)q^n = \sum_{(a_1,a_2,\ldots,a_m)\in K} q^{a_1+a_2+\cdots+a_m},$$

where

$$K := \left\{ (a_1,a_2,\ldots,a_m) \in \mathbb{Z}^m \;:\; \begin{array}{c} a_1 \geq a_2 \geq \cdots \geq a_m \geq 0 \text{ and } a_i - 2a_{i+1} + a_{i+2} \leq 0 \\ \text{for } 1 \leq i \leq m-1 \text{ and } a_{m+1} = 0 \end{array} \right\}.$$

Claim: If $a_1 \geq a_2$ and $a_i - 2a_{i+1} + a_{i+2} \leq 0$, then $a_i \geq a_{i+1}$ and $a_m \geq 0$ for $1 \leq i \leq m-1$.

Proof of Claim: We will use induction on m. Base case: If $m = 2$, we have $a_1 \geq a_2$ and $a_1 - 2a_2 \leq 0 \Rightarrow a_1 \geq a_2$ and $a_1 \leq 2a_2$. Thus $a_2 \geq 0$.

If $m = 3$, then we have $a_1 \geq a_2$ and $a_1 - 2a_2 + a_3 \leq 0 \Rightarrow a_1 \geq a_2$ and $a_1 + a_3 \leq 2a_2$. Thus, a_2 must be greater or equal to a_3. Next, $a_2 - 2a_3 \leq 0 \Rightarrow a_2 \leq 2a_3$; however, $a_2 \geq a_3$. Therefore, $a_3 \geq 0$.

Induction step: Assume the claim is true for $m-1$. We need to show it is also true for m. From the induction step , we get $a_i \geq a_{i+1}$ for $1 \leq i \leq m-2$. Thus, $a_{m-2} - 2a_{m-1} + a_m \leq 0 \Leftrightarrow a_{m-2} + a_m \leq 2a_{m-1}$ implies that $a_{m-1} \geq a_m$ because $a_{m-2} \geq a_{m-1}$. In addition, $a_{m-1} - 2a_m \leq 0 \Leftrightarrow a_{m-1} \leq 2a_m$ and $a_{m-1} \geq a_m$ imply that $a_m \geq 0$. Therefore, the claim is proven.

Thus, the conditions for K can be simplified:

$$K := \left\{ (a_1, a_2, \ldots, a_s) \in \mathbb{Z}^s \ : \begin{array}{c} a_1 \geq a_2 \text{ and } a_i - 2a_{i+1} + a_{i+2} \leq 0 \\ \text{for } 1 \leq i \leq m-1 \text{ and } a_{m+1} = 0 \end{array} \right\}.$$

Therefore, the constraints of K are given by the system

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ \vdots \\ a_{m-2} \\ a_{m-1} \\ a_m \end{bmatrix} \geq 0\,. \quad (4.6)$$

Now let C be the matrix on the left side of (4.6). Then

$$\det C = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & 2 \end{vmatrix}.$$

Adding the i^{th} row to the $(i+1)^{th}$ row for $1 \leq i \leq m$, we obtain

$$\det C = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Thus $\det C = 1$, and

$$C^{-1} = \begin{bmatrix} m & m-1 & m-2 & m-3 & \dots & 4 & 3 & 2 & 1 \\ m-1 & m-1 & m-2 & m-3 & \dots & 4 & 3 & 2 & 1 \\ m-2 & m-2 & m-2 & m-3 & \dots & 4 & 3 & 2 & 1 \\ \vdots & & & & \vdots & & & & \vdots \\ 3 & 3 & 3 & 3 & \dots & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & \dots & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The integer entries of C^{-1} are all nonnegative. Thus, Lemma 3.2 gives the generating function of K:

$$\sigma_K(x_1, x_2, \dots, x_m) =$$

$$\frac{1}{(1 - x_1^m x_2^{m-1} \cdots x_{m-1}^2 x_m)(1 - x_1^{m-1} x_2^{m-1} \cdots x_{m-1}^2 x_m) \cdots (1 - x_1 x_2 \cdots x_{m-1} x_m)}.$$

Since $n = a_1 + a_2 + \cdots + a_m$, we let $q = x_1 = x_2 = \cdots = x_m$ and obtain

$$\sigma_k(q) = \frac{1}{(1 - q^{m+(m-1)+\cdots+1})(1 - q^{(m-1)+(m-1)+\cdots+1}) \cdots (1 - q^{m+(m-1)})(1 - q^m)}.$$

Thus, the generating function we wanted to find is

$$\sigma_K(q) = \prod_{j=1}^{m} \frac{1}{(1 - q^{j(2m-j+1)/2})},$$

and this is precisely the generating function for the partitions described in Theoreom 2.4. $\square$

Note that in our proof we actually derived the full generating function of Theorem 2.4.

Theorem 4.3. *For an integer $m \geq 2$, let*

$$\sigma(x_1, x_2, \ldots, x_m) := \sum\nolimits^{*} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m},$$

where $\sum^*$ is the restricted summation over all nonnegative integers $(a_1, a_2, \ldots, a_m)$ satisfying $a_1 \geq a_2$ and $a_i - 2a_{i+1} + a_{i+2} \leq 0$ for $1 \leq i \leq m-1$ and $a_{m+1} = 0$. Let $X_j = x_1 \cdots x_j$ for $1 \leq j \leq m$. Then

$$\sigma(x_1, x_2, \ldots, x_m) = \frac{1}{(1 - X_1 X_2 \cdots X_m)(1 - X_2 \cdots X_m) \cdots (1 - X_{m-1} X_m)(1 - X_m)}.$$

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