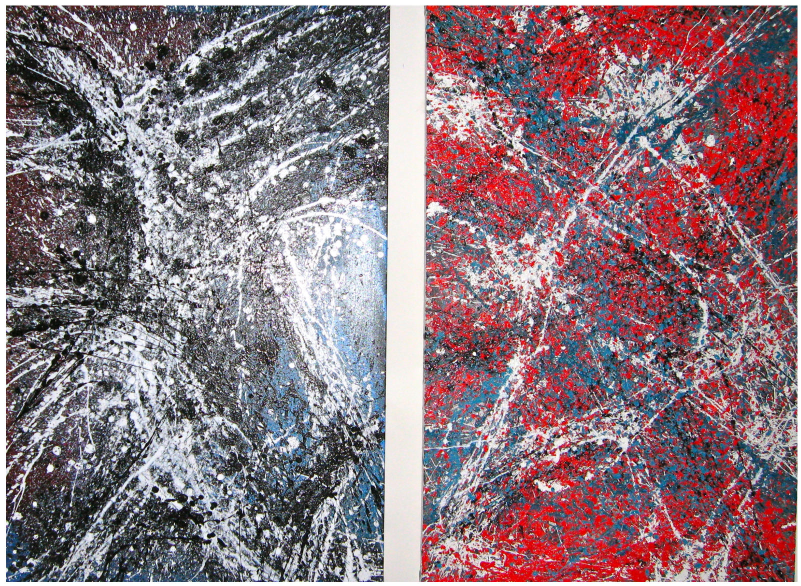
A MINIMAL-DISTANCE CHROMATIC POLYNOMIAL FOR SIGNED GRAPHS



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Master of Arts
In
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by

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CERTIFICATION OF APPROVAL

I certify that I have read *A MINIMAL-DISTANCE CHROMATIC POLYNOMIAL FOR SIGNED GRAPHS* by Nicholas E. Dowdall and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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A MINIMAL-DISTANCE CHROMATIC POLYNOMIAL FOR SIGNED GRAPHS

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In the early 20^{th} century the chromatic polynomial was introduced as a way to count the proper colorings of a graph. It was generalized to signed graphs, graphs consisting of an unsigned graph and a signing function that labels each edge with a positive or negative sign. In 2009 Babson and Beck developed a two-variable chromatic polynomial for unsigned graphs by requiring colors of adjacent nodes in a graph to be a minimal-color apart. We extend this idea to signed graphs for integral and modular coloring values, showing in both cases that the counting function is a piecewise-defined quasipolynomial of period 1 or 2. Furthermore, we establish a reciprocity relationship that mirrors Stanley's reciprocity theorem on the chromatic polynomial for an unsigned graph.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Chapter 1

Introduction

In 1912 George Birkhoff introduced a counting function for planar graphs in an effort to solve the four-color map problem [5]. In 1932 Whitney generalized Birkhoff's idea to the case of general graphs [9]. This famous function is referred to as the *chromatic polynomial* of a graph. Let $G = (V, E)$ be a graph with edge set E and node set V where we allow multiple edges, but typically not loops, so that E is a multi-set consisting of 2-element subsets of V. A *proper coloring* is one that colors adjacent nodes in G differently, thus a proper k-coloring of G is a vector $\mathbf{x} \in [k]^{|V|}$ with $x_i \neq x_j$ if $\{i, j\} \in E$. The chromatic polynomial $\chi_G(k)$ is now a classic counting function in graph theory which counts the number of proper colorings of a graph given a color set $[k] := \{1, 2, ..., k\}$ and a graph G.

After Whitney first established that $\chi_G(k)$ was indeed a polynomial in 1932, $\chi_G(k)$

has been the subject of much research. It has been generalized several times, in particular in 2009 by Babson and Beck to allow for the notion of minimal distance between colors of adjacent nodes [1]. That is, if two nodes of a graph are adjacent then they are required to be some minimal distance m apart in color. Thus, m becomes a second parameter of the coloring function. We say that a k-coloring $x \in [k]^{|V|}$ has *minimal distance* m if $|x_i - x_j| \geq m$ whenever $\{i, j\} \in E$. Babson and Beck showed in [1] that this new counting function is a piecewise-defined polynomial in k and m.

This paper generalizes the above ideas to signed graphs. A *signed graph* $\Sigma = (G, \sigma)$ consists of a graph $G = (V, E)$ (multiple edges and loops allowed) and a signature σ that labels each edge with $+$ or $-$. A *k-coloring* of a signed graph is a vector $\mathbf{x} \in [-k, k]^{|V|}$ where $[-k, k] := \{-k, -(k-1), ..., 0, ..., k-1, k\}$. We say $\mathbf{x}$ is *proper* if whenever there is an edge ij with sign ϵ then $x_i \neq \epsilon x_j$. The function

$$\chi_\Sigma(2k+1) := \text{the number of proper } k\text{-colorings of } \Sigma$$

was proved to be a polynomial in 1982 by Thomas Zaslavsky [10]. If ij is an edge of Σ with sign ϵ, an *m-minimal k-coloring* will satisfy $-m < x_i - \epsilon x_j < m$. The first main result of this paper is that the function that counts the number of k-colorings of a signed graph with minimal distance m is a piecewise-defined quasipolynomial (which we define in Equation (2.2)).

Theorem 1.1. *Let Σ be a signed graph and $c_\Sigma(m,k) := \#$ of k-colorings of Σ with minimal distance m. Then $c_\Sigma(m,k)$ is a piecewise-defined quasipolynomial in m and k of period 1 or 2.*

Letting $\mathbb{Z}_k$ denote a cyclic group of order k we can define a *modular* k-coloring with minimal distance m of a signed graph (see Equation (4.2)) and extend the above theorem.

Theorem 1.2. *Let Σ be a signed graph with $\eta_\Sigma(m,k) := \#$ of modular k-colorings of Σ with minimal distance m. Then $\eta_\Sigma(m,k)$ is a piecewise-defined quasipolynomial in m and k of period 1 or 2.*

Our second main result is that c_Σ and η_Σ satisfy a reciprocity relationship similar to Stanley's Theorem on the evaluation of chromatic polynomials at negative integers [8] and the Beck-Babson extension of Stanley's Theorem [1].

Theorem 1.3. *The piecewise-defined quasipolynomials $c_\Sigma(m,k)$ and $\eta_\Sigma(m,k)$ satisfy*

$$(-1)^{|V|}c_\Sigma(-m,-k) = c_\Sigma(m+1,k-1)$$

and

$$(-1)^{|V|}\eta_\Sigma(-m,-k) = \eta_\Sigma(m+1,k).$$

Chapter 2

Methodology

2.1 Geometry

The main idea is to change our problem of counting the ways to color a signed graph into the problem of counting the integral points contained within rational convex polytopes. Given the unsigned complete graph K_2, we can easily visualize this in two dimensions. The possible k-colorings are the integer lattice points in the positive orthant that are within the $k \times k$ square with the proper colors avoiding the $x_1 = x_2$ line. See Figure 2.1.

If we wish our graph coloring to have a minimal distance m, using the graph K_2 we see that proper colors now need to avoid a strip symmetric about the $x_1 = x_2$ line where any point within the strip has coordinates satisfying $|x_1 - x_2| < m$, see Figure 2.2.

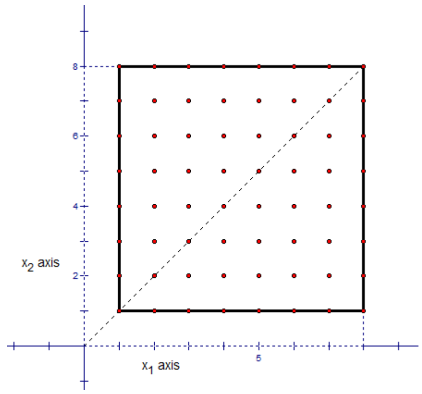

Figure 2.1: Proper k-colorings of $G = K_2$.

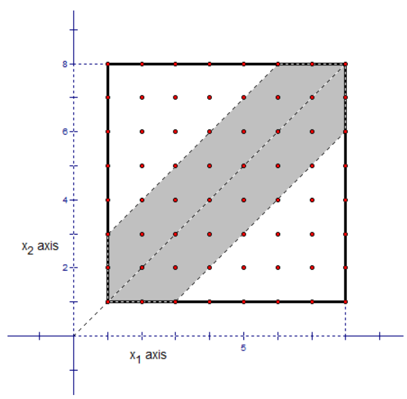

Figure 2.2: Minimal distance colorings ($G = K_2, m = 2$).

If $m = 1$ we recover the classical chromatic polynomial. The proper colorings of the signed graph $\pm K_2$, two adjacent nodes with a double edge one with $\epsilon = +$ and the other with $\epsilon = -$, can also be visualized in two dimensions. Now our proper colorings need to avoid the lines $x_1 = x_2$ and $x_1 = -x_2$, see Figure 2.3.

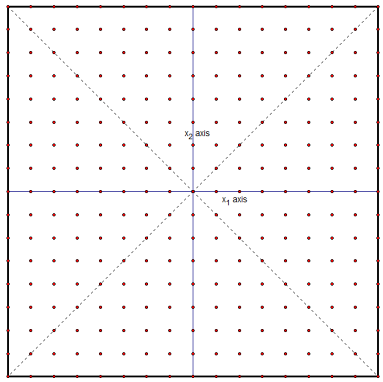

Figure 2.3: Proper k-colorings of the signed graph $\pm K_2$.

We wish need to extend these geometric ideas to that of m-minimal k-colorings for signed graphs. To accomplish this we will need some ideas from Ehrhart theory which were also used in [1] and [4].

2.2 Ehrhart Theory

Given any finite point set $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\} \subset \mathbb{R}^{\mathbf{d}}$ let the *convex polytope* $\mathcal{P} = \{\lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + ... + \lambda_n\mathbf{v_n} : \text{all } \lambda_k > 0 \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}$ be the smallest convex set containing these points. For convenience we say $\mathcal{P} = \text{conv}\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$. The *dimension* of a polytope $\mathcal{P}$ is the dimension of the affine space

$$\operatorname{span}\mathcal{P} := \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in \mathcal{P}, \lambda \in \mathbb{R}\}$$

spanned by $\mathcal{P}$. If $\mathcal{P}$ has dimension d we say $\mathcal{P}$ is a d-polytope. The hyperplane $h = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$ is a *supporting hyperplane* of $\mathcal{P}$ if $\mathcal{P} \subset \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \leq b\}$ or $\mathcal{P} \subset \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \geq b\}$, i.e., $\mathcal{P}$ lies entirely on one side of h. If h is a supporting hyperplane of $\mathcal{P}$, we say $\mathcal{P} \cap h$ is a *face* of $\mathcal{P}$. By tradition the $(d-1)$-dimensional faces are called *facets*, the 1-dimensional faces *edges*, and the 0-dimensional faces *vertices* of $\mathcal{P}$.

A *rational polytope* is one with vertices in $\mathbb{Q}^d$. An *integral polytope* is one with vertices in $\mathbb{Z}^d$. Ehrhart theory tells us that under certain conditions the function that counts integral lattice points inside a polytope is well behaved and takes on the form of a quasipolynomial.

A *quasipolynomial* is a function $q(t) = c_d(t)t^d + c_{d-1}(t)t^{d-1} + \cdots + c_0(t)$, where the $c_i(t)$ are periodic functions with integral period p. If $c_d(t)$ is not identically zero then we say that $q(t)$ has *degree* d. Equivalently, the function $Q : \mathbb{N} \mapsto \mathbb{N}$ is a quasipolynomial if there exists polynomials $q_0, q_1, ..., q_{k-1}$ such that $Q(n) = q_i(n)$ whenever $n \equiv i \bmod k$. The q_i are called the *constituents* of Q.

Proposition 2.1. *Let* $\{Q_1, Q_2, ..., Q_n\}$ *be quasipolynomials in the indeterminant* t *with periods* $k_1, k_2, ..., k_n$ *respectively. Then*

$$P = \sum_{i=1}^{n} Q_i$$

is a quasipolynomial with period $d \mid \mathrm{lcm}[k_1, k_2, ..., k_n]$. □

Remark. Notice that Proposition 2.1 implies if we sum a finite number of quasipolynomials all with the same **prime** period p, the resulting quasipolynomial will have period p or 1. We will use this fact in our results with $p = 2$.

Theorem 2.2. (**Ehrhart's Theorem on Rational Polytopes [3].**) *If* $\mathcal{P}$ *is a rational convex d-polytope, then the function that counts the integral points contained in integral dilates of* $\mathcal{P}$*,* $\mathcal{L}_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$*, is a quasipolynomial in* t *of degree* d*. Its period divides the least common multiple of of the denominators of the coordinates of the vertices of* $\mathcal{P}$*.*

Theorem 2.3. *If* $\mathcal{P}$ *is an integral convex d-polytope, then the constant term of the Ehrhart polynomial* $\mathcal{L}_{\mathcal{P}}$ *is* 1*. (For a proof see [3, pp.* $65-68$*] .)*

Example 2.1. *To see Ehrhart theory in action consider the triangular polytope in two dimensions* $\mathcal{P} = \triangle = \operatorname{conv}\{\mathbf{v_1} = (0,0), \mathbf{v_2} = (0,1), \mathbf{v_3} = (1,0)\}$*. Any dilation of* $\mathcal{P}$ *by the positive integer* t *is* $t\triangle = \operatorname{conv}\{t\mathbf{v_1} = (0,0), t\mathbf{v_2} = (0,t), t\mathbf{v_3} = (t,0)\}$*.* (*See Figure 2.4.*)

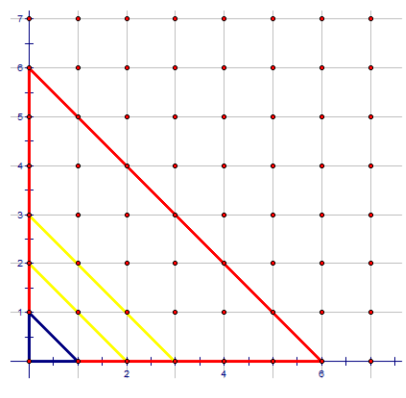

Figure 2.4: $\triangle = \operatorname{conv}\{t\mathbf{v_1} = (0,0), t\mathbf{v_2} = (0,t), t\mathbf{v_3} = (t,0)\}$, $t = 1, 2, 3, ..., 6$.

Ehrhart theory tells us that $\mathcal{L}_\triangle(t) = c_2(t)t^2 + c_1(t)t^1 + c_0(t)$*. Since the vertices of* $\triangle$ *are all integral, the* c_i *all have period* 1*, i.e., they are all constants. Thus we have* $\mathcal{L}_\triangle(t) = c_2t^2 + c_1t + c_0$*. By Theorem 2.3,* $\mathcal{L}_\triangle(0) = 1$*. Checking Figure 2.4 we see that* $\mathcal{L}_\triangle(1) = 3$ *and* $\mathcal{L}_\triangle(2) = 6$*. This gives*

$$\begin{aligned} c_0 &= 1 \\ c_2 + c_1 + 1 &= 3 \\ 4c_2 + 2c_1 + 1 &= 6. \end{aligned}$$

Solving this system yields $\mathcal{L}_\triangle(t) = \frac{1}{2}t^2 + \frac{3}{2}t + 1$.

Example 2.2. *Suppose our polytope is not integral. What does this do to the polynomiality of* $\mathcal{L}_\mathcal{P}$*? Let* $\mathcal{P} = \triangle = \text{conv}\{\mathbf{v_1} = (0,0), \mathbf{v_2} = (0, \frac{1}{2}), \mathbf{v_3} = (\frac{1}{2}, 0)\}$*. By Theorem 2.2* $\mathcal{L}_\triangle(t) = c_2(t)t^2 + c_1(t)t + c_0(t)$*. When* $t = 2k$*,* $\triangle$ *is integral and we recover the same geometric interpretation as in Figure 2.4. By Theorem 2.3* $\mathcal{L}_\triangle(0) = 1$*. Using interpolation once more we get* $\mathcal{L}_\triangle(2) = 3$ *and* $\mathcal{L}_\triangle(4) = 6$*. Thus for* $t = 0, 2, 4, 6, ...$ *we have* $\mathcal{L}_\triangle(t) = \frac{1}{8}t^2 + \frac{3}{4}t + 1$*. Theorem 2.2 tells us that when* t *is odd, our second constituent is of the form* $c_2t^2 + c_1t + c_0$ *where the* c_i *are constants. Relying on Figure 2.5 we get* $\mathcal{L}_\triangle(1) = 1$*,* $\mathcal{L}_\triangle(3) = 3$*,* $\mathcal{L}_\triangle(5) = 6$*. Hence* $c_2 + c_1 + c_0 = 1$*,* $9c_2 + 3c_1 + c_0 = 3$*, and* $25c_2 + 5c_1 + c_0 = 6$*. Solving this system yields* $c_2 = \frac{1}{8}, c_1 = \frac{1}{2}$ *and* $c_0 = \frac{3}{8}$*. Therefore we may write*

$$\mathcal{L}_{\triangle}(t) = \begin{cases} \frac{1}{8}t^2 + \frac{3}{4}t + 1 & \textit{if } t \equiv 0 \ \textit{mod } 2, \\ \frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8} & \textit{if } t \equiv 1 \ \textit{mod } 2. \end{cases}$$

$\square$

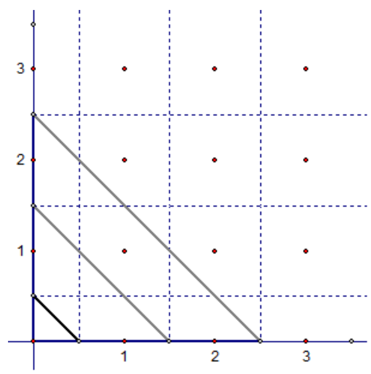

Figure 2.5: $\triangle = \text{conv}\{t\mathbf{v_1} = (0,0), t\mathbf{v_2} = (0, \frac{t}{2}), t\mathbf{v_3} = (\frac{t}{2}, 0)\}$, $t = 1, 3, 5$.

2.3 Signed Graphs

Returning to our underlying graph G, fix an orientation on G, called the *initial orientation*. This assigns to each edge $e \in E$ a *head* $h(e) \in V$ and a *tail* $t(e) \in V$.

Thus, $h(e)$ and $t(e)$ are the nodes incident to e.

The *adjacency matrix of a signed graph* [7] is defined as the matrix $A_\Sigma = (a_{ij})_{n\times m}$, $\Sigma = (G, \sigma)$, where $G = (V, E)$ with $V = \{v_1, v_2, ..., v_n\}$ and $E = \{e_1, e_2, ..., e_m\}$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if } \ h(e_j) = v_i, \\ -\epsilon & \text{if } \ t(e_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.3. *Consider the graph $\Sigma = -K_3$, the complete graph on three nodes and $\epsilon = -$ for each edge. We have $V = \{v_1, v_2, v_3\}$ and $E = \{e_1, e_2, e_3\}$. Fix an initial orientaion $v_1 \mapsto v_2 \mapsto v_3 \mapsto v_1$. The associated adjacency matrix is:*

$$A_\Sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The reader should take note that $|\det(A_\Sigma)| = 2$. *This shows that the adjacency matrix of a signed graph is in general not totally unimodular. In fact it is not difficult to show that the determinant of such a matrix can grow without bound. This means that we will be unable to use similar methods as in [1]. Therefore develop a new argument in Chapter 4.*

A *real affine hyperplane* of $\mathbb{R}^n$ is an $(n-1)$-dimensional affine subspace of the form $\{x \in \mathbb{R}^n : \; a_n x_n + a_{n-1}x_{n-1} + \cdots + a_1 x_1 = b\}$ with at least one of the $a_i \neq 0$. We will just say hyperplane when there is no confusion possible. A hyperplane therefore separates $\mathbb{R}^n$ into two affine half spaces described by $a_n x_n + a_{n-1}x_{n-1} + \cdots + a_1 x_1 < b$ and $a_n x_n + a_{n-1}x_{n-1} + \cdots + a_1 x_1 > b$.

A *hyperplane arrangement* $\mathcal{H}$ is a set of finitely many affine hyperplanes $h \in \mathbb{R}^n$. This arrangement divides the space into regions. An *open region* is a connected component of $\mathbb{R}^n \backslash \mathcal{H}$ and a *closed region* is the topological closure of an open region.

The *hyperplane arrangement of a signed graph* [4] Σ is

$$\begin{aligned}\mathcal{H}[\Sigma] \;\; = \;\; & \{h^{\epsilon}_{ij} : \Sigma \text{ has an edge } ij \text{ with sign } \epsilon\} \\ & \cup\{x_i = 0 : \Sigma \text{ has a half edge at node } v_i\} \\ & \cup\{0 = 0 : \Sigma \text{ has a loose edge}\}\end{aligned}$$

where the hyperplane $h_{ij}^{\epsilon} := (x_i = \epsilon x_j)$. A half edge is one which is incident to one vertex with multiplicity 1. A loose edge is one that is not incident to any vertices.

Since a k-coloring of Σ is a vector $\mathbf{x} \in [-k,k]^{|V|}$ and a proper k-coloring must avoid h_{ij}^{ϵ} for all $ij \in E$, a proper k-coloring of a signed graph is a vector $\mathbf{x} \in [-k,k]^{|V|} \backslash \mathcal{H}[\Sigma]$. Notice that $\mathcal{H}(\Sigma)$ is central and divides $[-k,k]^{|V|}$ into finitely many, say n, bounded regions in $\mathbb{R}^{|V|}$, i.e., polytopes. We have $[-k,k]^{|V|} \backslash \mathcal{H}[\Sigma] = \bigcup_{j=1}^{n} \mathcal{P}_j$, where the $\mathcal{P}_j$ are half open polytopes. Thus, a proper k-coloring of the signed graph Σ is an integral vector $\mathbf{x} \in \mathcal{P}_j$ for some j, while the lattice points on the boundaries that are on $\mathcal{H}(\Sigma)$ are the non-proper colorings of Σ.

In order to know the number of proper k-colorings of Σ we need only count the number of lattice points in $\bigcup_{j=1}^{n} \mathcal{P}_j$. Since the $\mathcal{P}_j$ are disjoint,

$$\#\left[\left(\bigcup_{j=1}^{n} \mathcal{P}_j\right) \cap \mathbb{Z}^{|V|}\right] = \sum_{j=1}^{n} \#\left(\mathcal{P}_j \cap \mathbb{Z}^{|V|}\right) \tag{2.1}$$

This was done by Beck and Zaslavsky in [4] by constructing the Ehrhart polynomial associated to each $\mathcal{P}_j$ and then summing the individual polynomials.

Now we wish to consider a minimal coloring distance between nodes of our signed graph Σ. If ij is an edge of Σ with sign ϵ, a k-coloring with minimal distance m will avoid $-m < x_i - \epsilon x_j < m$.

Example 2.4. *Let $\Sigma = \pm K_2$ be the graph on two nodes with a double edge, one with $\epsilon = +$ and the other with $\epsilon = -$ and set $m = 3$ and $k = 8$. Now our proper colorings need to avoid two strips symmetric about the lines $x_1 = x_2$ and $x_1 = -x_2$. From a hyperplane point of view we have an arrangement $\mathcal{H}(\Sigma) = \{(x_1 = x_2 + 3), (x_1 = x_2 - 3), (x_1 = -x_2 + 3), (x_1 = -x_2 - 3)\}$. See Figure 2.6.*

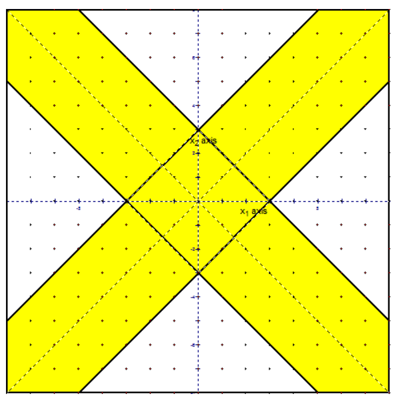

Figure 2.6: Minimal distance k-colorings of $\pm K_2$ with $m = 3, k = 8$.

Before we develop our two-variable counting function for signed graphs we need to lay some ground work using generating functions.

Chapter 3

Generating Functions

3.1 The Integer-Point Transform

A *pointed cone* $\mathcal{K}_{\mathbf{v}} \subseteq \mathbb{R}^d$ is a set of the form

$$\mathcal{K}_{\mathbf{v}} = \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + ... + \lambda_m \mathbf{w}_m : \lambda_i \geq 0\}$$

where $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m \in \mathbb{R}^n$ are such that there exists a hyperplane h for which $h \cap \mathcal{K}_{\mathbf{v}} = \{\mathbf{v}\}$; that is, $\mathcal{K}_{\mathbf{v}} \setminus \{\mathbf{v}\}$ lies strictly on one side of h. The vector $\mathbf{v}$ is called the *apex* of $\mathcal{K}_{\mathbf{v}}$, and the $\mathbf{w}_k$'s are the *generators* of $\mathcal{K}_{\mathbf{v}}$. The cone $\mathcal{K}_{\mathbf{v}}$ is *rational* if $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m \in \mathbb{Q}^d$. The *dimension* of $\mathcal{K}_{\mathbf{v}}$ is the dimension of the affine space spanned by $\mathcal{K}_{\mathbf{v}}$; if $\mathcal{K}_{\mathbf{v}}$ is of dimension d, we call it a d-cone. Notice that any cone with rational generators can be written as a cone with integral generators by clearing denominators.

Let $\mathcal{S}$ be a rational cone or polytope in $\mathbb{R}^d$. Define the multivariate generating function

$$\sigma_{\mathcal{S}}(\mathbf{z}) = \sigma_{\mathcal{S}}(z_1, z_2, \ldots, z_d) := \sum_{\mathbf{m}\,\in\,\mathcal{S}\cap\,\mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$$

with $\mathbf{z}^{\mathbf{m}} := z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$ for the integer vector $\mathbf{m} = (m_1, m_2, ..., m_d)$. We call $\sigma_{\mathcal{S}}$ the *integer-point transform* of $\mathcal{S}$ or simply the *generating function* of $\mathcal{S}$.

Example 3.1. *Let* $\mathcal{K} = [0, \infty)$ *be a* 1*-dimensional cone. Then*

$$\sigma_{\mathcal{K}}(z) = \sum_{m \in\, [0,\infty)\cap\,\mathbb{Z}} z^m = \sum_{m \geq 0} z^m = \frac{1}{1-z}.$$

Example 3.2. *Let* $\mathcal{K}_{(0,0)} := \{\lambda_1(1,1) + \lambda_2(-1,1), \quad \lambda_i \geq 0\}$*. Set* $\Pi = \{\lambda_1(1,1) + \lambda_2(-1,1), \quad 0 \leq \lambda_i < 1\} \subset \mathbb{R}^2$*. We refer to the half-open parallelogram* Π *as* the fundamental region *of* $\mathcal{K}_{(0,0)}$*. We wish to tile all of* $\mathcal{K}_{(0,0)}$ *with* Π*. First list all non-negative integer transforms of the vertices* $(1,1), (-1,1)$ *of* Π*, which are the* *generators of $\mathcal{K}_{(0,0)}$. That is,*

$$\begin{aligned}
\sum_{\mathbf{m}=j(1,1)+k(-1,1);j,k,\geq 0} \mathbf{z}^{\mathbf{m}} &= \sum_{j\geq 0}\sum_{k\geq 0} \mathbf{z}^{j(1,1)+k(-1,1)} = \sum_{j\geq 0}\sum_{k\geq 0} \mathbf{z}^{j(1,1)}\mathbf{z}^{k(-1,1)} \\
&= \sum_{j\geq 0} \mathbf{z}^{j(1,1)} \sum_{k\geq 0} \mathbf{z}^{k(-1,1)} = \sum_{j\geq 0} z_1^j z_2^j \sum_{k\geq 0} z_1^{-k} z_2^k \\
&= \frac{1}{(1-z_1z_2)(1-z^{-1}z_2)}.
\end{aligned}$$

Next, take all integer points in Π and add to them non-negative, linear, integer combinations of the generators $(1,1)$ and $(-1,1)$. Let $\mathcal{L}_{(m,n)} := \{(m,n)+j(1,1)+k(-1,1) : j,k \in \mathbb{Z}_{\geq 0}\}$. Thus, $\mathcal{K}_{(0,0)}$ is the disjoint union of the subsets of $\mathcal{L}_{(m,n)}$ as (m,n) ranges over each lattice point in $\Pi \cap \mathbb{Z}^2 = \{(0,0),(0,1)\}$. Hence,

$$\begin{aligned}
\sigma_{\mathcal{K}_{(0,0)}}(\mathbf{z}) &= \left((z_1z_2)^{(0,0)} + (z_1z_2)^{(0,1)}\right) \sum_{\mathbf{m}=j(1,1)+k(-1,1);j,k,\geq 0} \mathbf{z}^{\mathbf{m}} \\
&= \frac{1+z_2}{(1-z_1z_2)(1-z^{-1}z_2)} = \frac{\sigma_\Pi(\mathbf{z})}{(1-z_1z_2)(1-z_1^{-1}z_2)}.
\end{aligned}$$

$\square$

Let $\mathcal{S} = \mathbf{w} + \mathcal{T}$, $\mathbf{w}$ an integer vector. Then

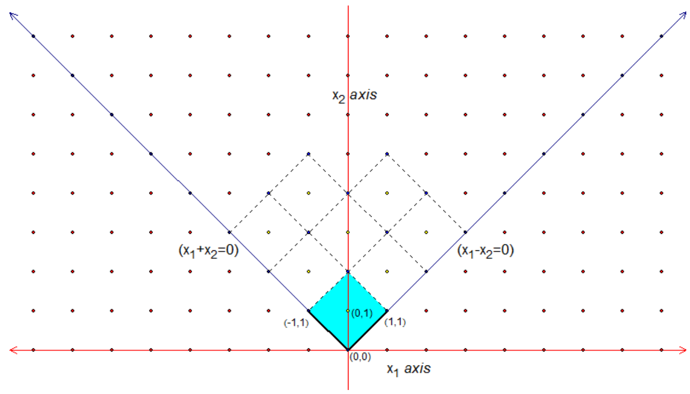

Figure 3.1: The simplicial cone $\mathcal{K}_{(0,0)}$ and its fundamental parallelogram.

$$\sigma_{\mathcal{S}}(\mathbf{z}) = \sum_{\mathbf{m} \in \mathcal{S} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{t} \in \mathcal{T} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{w}+\mathbf{t}} = \mathbf{z}^{\mathbf{w}} \sum_{\mathbf{t} \in \mathcal{T} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{t}} = \mathbf{z}^{\mathbf{w}} \sigma_{\mathcal{T}}(\mathbf{z}). \tag{3.1}$$

Theorem 3.1. *Let $\mathcal{K} := \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_d \mathbf{w}_d : \lambda_i \geq 0\}$ be a simplicial d-cone (i.e., the $\mathbf{w}_i$ are independent), with $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$. Then for $\mathbf{v} \in \mathbb{R}^d$, the integer-point transform $\sigma_{\mathbf{v}+\mathcal{K}}$ of the shifted cone $\mathbf{v} + \mathcal{K}$ is the rational function*

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2}) \cdots (1-\mathbf{z}^{\mathbf{w}_d})}$$

where Π is the half open parallelepiped

$$\Pi := \{\lambda_1 \mathbf{w}_2 + \lambda_2 \mathbf{w}_1 + \cdots + \lambda_d \mathbf{w}_d : \ 0 \leq \lambda_i < 1\}.$$

For a complete proof see [3, p.60] .

Example 3.3. *Let Let* $\mathcal{K}_{(1,1)} := \{(1,1) + \lambda_1(1,1) + \lambda_2(-1,1), \ \ \lambda_i \geq 0\}$. *Notice that* $\mathcal{K}_{(1,1)} = (1,1) + \mathcal{K}_{(0,0)}$. *For* t *an integer we have* $t\mathcal{K}_{(1,1)} = t(1,1) + \mathcal{K}_{(0,0)}$. *Therefore*

$$\begin{aligned}
\sigma_{t\mathcal{K}_{(1,1)}}(\mathbf{z}) &= \frac{\sigma_{t(1,1)+\Pi}(\mathbf{z})}{(1 - z_1 z_2)(1 - z_1^{-1} z_2)} = \frac{\mathbf{z}^{(t,t)} \sigma_{\Pi}(\mathbf{z})}{(1 - z_1 z_2)(1 - z_1^{-1} z_2)} \\
&= \frac{(z_1^t z_2^t)(1 + z_2)}{(1 - z_1 z_2)(1 - z_1^{-1} z_2)} \qquad \textit{(Refer to Figure 3.2.)}
\end{aligned}$$

□

If $\mathcal{K}$ is simplicial with $\mathbf{v}$ integral, Theorem 3.1 together with (3.1) says

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2}) \cdots (1 - \mathbf{z}^{\mathbf{w}_d})} = \frac{\mathbf{z}^{\mathbf{v}} \sigma_{\Pi}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2}) \cdots (1 - \mathbf{z}^{\mathbf{w}_d})}.$$

Moreover, since every pointed cone can be triangulated into simplicial cones, Theorem 3.1 also tells us for **any** pointed cone

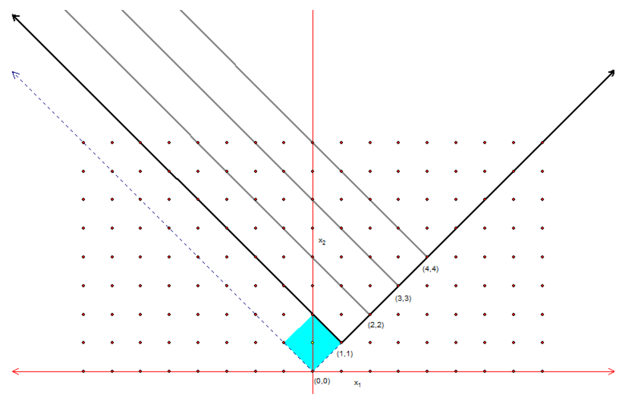

Figure 3.2: The simplicial cone $t\mathcal{K}_{(1,1)}$ and its dilates for $t = 1, 2, 3, 4$.

$$\mathcal{K}_{\mathbf{v}} = \{\mathbf{v} + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + ... + \lambda_m\mathbf{w}_m : \lambda_i \geq 0\}$$

with $\mathbf{v} \in \mathbb{R}^d, \mathbf{w}_i \in \mathbb{Z}^d$, $\sigma_{\mathcal{K}_{\mathbf{v}}}(z)$ is a rational function in the coordinates of $\mathbf{z}$.

3.2 Brion's Theorem

Let $\mathcal{P}$ be a rational convex polytope with vertices $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$. The **vertex cone** $\mathcal{K}_{\mathbf{v}_i}$ is the smallest pointed cone with apex $\mathbf{v}_i$ such that $\mathcal{P} \subset \mathcal{K}_{\mathbf{v}_i}$. The next beautiful and surprising theorem was discovered by Michel Brion in 1988. It tells us that to understand the generating function of a rational polytope, it is enough to understand the generating functions of its vertex cones.

Theorem 3.2. (Brion's theorem [6]) *Suppose $\mathcal{P}$ is a rational convex polytope. Then we have the following identity of rational functions:*

$$\sigma_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{v} \ \textit{a vertex of}\ \mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z}). \tag{3.2}$$

For a proof see [3, pp.157 – 160] .

Notice that since $\sigma_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$ by definition, if we evaluate $\sigma_{\mathcal{P}}$ at $\mathbf{z} = (1, 1, ..., 1) = \mathbf{1}$ we have

$$\sigma_{\mathcal{P}}(\mathbf{1}) = \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} \mathbf{1}^{\mathbf{m}}$$

which is the lattice-point count of $\mathcal{P}$. Now consider the generating function of $\mathcal{P}$ and all its dilates by the integer-valued parameter t:

$$\sigma_{t\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{m} \in t\mathcal{P} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}.$$

Evaluating at $\mathbf{z} = \mathbf{1}$ we get the lattice-point count of $t\mathcal{P}$. If we use Brion's theorem, expanding the right hand side and evaluating at $\mathbf{z} = \mathbf{1}$ would create singularities in the generating functions. These singularities can be dealt with in a simple way as is outlined in the proof of the next theorem. By definition the Ehrhart polynomial $L_{\mathcal{P}}(t) = \sigma_{t\mathcal{P}}(\mathbf{1})$. Thus, we can reprove Ehrhart's theorem on rational polytopes using Brion's theorem in straightforward manner. We outline a proof here, but for a complete proof see [3, p.161] .

Since every polytope can be triangulated, it is enough to only consider simplices. Let $\triangle$ be a rational d-simplex whose vertices have coordinates with denominator dividing p. Thus, $p\triangle$ is integral which implies $L_{\triangle}(pt) = L_{p\triangle}(t)$ is a polynomial in t. Let $0 \leq r < p$. Then Brion's theorem implies

$$
\begin{aligned}
L_{\triangle}(r + pt) &= \sum_{\mathbf{m} \in (r+tp)\triangle \cap \mathbb{Z}^d} 1 \\
&= \lim_{\mathbf{z} \mapsto \mathbf{1}} \sigma_{(r+tp)\triangle}(\mathbf{z}) \\
&= \lim_{\mathbf{z} \mapsto \mathbf{1}} \sum_{\mathbf{v} \text{ vertex of } \triangle} \sigma_{(r+pt)\mathcal{K}_{\mathbf{v}}}(\mathbf{z})
\end{aligned}
$$

Since $\triangle$ is a simplex, each of the vertex cones $\mathcal{K}_{\mathbf{v}}$ are simplicial. We have

$$
\begin{aligned}
\mathcal{K}_{\mathbf{v}} &= \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + ... + \lambda_d \mathbf{w}_d : \lambda_i \geq 0\} \text{ and} \\
(r+pt)\mathcal{K}_{\mathbf{v}} &= \{(r+pt)(\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + ... + \lambda_d \mathbf{w}_d) : \lambda_i \geq 0\} \\
&= \{(r+pt)\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + ... + \lambda_d \mathbf{w}_d : \lambda_i \geq 0\} \\
&= \{tp\mathbf{v} + r\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + ... + \lambda_d \mathbf{w}_d : \lambda_i \geq 0\} \\
&= tp\mathbf{v} + \{r\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + ... + \lambda_d \mathbf{w}_d : \lambda_i \geq 0\} \\
&= tp\mathbf{v} + r\mathcal{K}_{\mathbf{v}}
\end{aligned}
$$

where the $\mathbf{w}_i$ are the generators of $\mathcal{K}_{\mathbf{v}}$. Since $p\triangle$ is integral, $p\mathbf{v}$ is an integer vector. Hence

$$
\begin{aligned}
\sigma_{(r+tp)\mathcal{K}_{\mathbf{v}}}(\mathbf{z}) &= \mathbf{z}^{tp\mathbf{v}} \sigma_{r\mathcal{K}_{\mathbf{v}}}(\mathbf{z}) \quad \text{which implies} \\
L_{\triangle}(r+pt) &= \lim_{\mathbf{z} \mapsto \mathbf{1}} \sum_{\mathbf{v} \text{ vertex of } \triangle} \mathbf{z}^{tp\mathbf{v}} \sigma_{r\mathcal{K}_{\mathbf{v}}}(\mathbf{z}).
\end{aligned}
$$

Note that the rational functions $\sigma_{r\mathcal{K}_{\mathbf{v}}}(\mathbf{z})$ do not depend on t. Thus to compute $L_{\triangle}(r+pt)$ we use L'Hôsptial's rule (repetitively if needed) to remove the singularities of the rational functions $\sigma_{r\mathcal{K}_{\mathbf{v}}}$ as we take the limit for $\mathbf{z} \mapsto 1$. Since t only appears in the monomials $\mathbf{z}^{tp\mathbf{v}}$ we obtain linear factors of t each time we differentiate. Once we remove all of the singularities, when we evaluate the remaining rational function at $\mathbf{z} = 1$ we obtain a polynomial in t, one polynomial for each r. Thus, our counting function is a quasipolynomial in t. Therefore Brion's theorem implies Ehrhart's theorem for rational polytopes. $\square$

The polytopes that we will be interested in are described by two different types of hyperplanes. While one set of hyperplanes is controlled by the parameter k, the number of colors, the other set of hyperplanes is controlled by m, the minimal distance between colors. Because of this, a unique situation arises where there are different rates of dilation on different facets of our polytope. Thus, we will need a slight generalization of the above ideas.

3.3 Multivariate Dilation

Let $\mathcal{P}$ be any rational convex d-polytope, i.e., $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : M\mathbf{x} \leq \mathbf{c}\}$, with $M \in \mathbb{Z}^{n \times d}$ and $\mathbf{c} \in \mathbb{Z}^n$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be an integer valued vector. Let $\mathcal{P}_\gamma := \{\mathbf{x} \in \mathbb{R}^d : M\mathbf{x} \leq \mathbf{c} + \gamma\}$. For the moment, we will only be interested in those γ that preserve the combinatorial equivalence of $\mathcal{P}$. The face lattice of a polytope $\mathcal{P}$, denoted $\mathrm{FL}(\mathcal{P})$, is a poset ordered by inclusion of the faces of $\mathcal{P}$. We say $\mathcal{P}$ and $\mathcal{P}_\gamma$ are *combinatorially equivalent* if $\mathrm{FL}(\mathcal{P}) = \mathrm{FL}(\mathcal{P}_\gamma)$. Let $\mathbf{w}$ be a vertex of $\mathcal{P}_\gamma$. Since $\mathcal{P}_\gamma$ is d-dimensional, we only need d hyperplanes to describe $\mathbf{w}$. Thus, there exists a full dimensional, invertible, sub-matrix A of M such that

$A\mathbf{w} = \mathbf{b} + \gamma$, where after suitable renumbering $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$ with corresponding entries $\mathbf{b}$ from $\mathbf{c}$. Therefore, whenever we select an arbitrary vertex of $\mathcal{P}_\gamma$ it will suffice to refer to it as $\mathbf{v}_\gamma$. Since A is an invertible linear operator, we have $\mathbf{v}_\gamma = A^{-1}(\mathbf{b} + \gamma) = A^{-1}\mathbf{b} + A^{-1}\gamma = \mathbf{v} + A^{-1}\gamma$. By $\mathrm{FL}(\mathcal{P}) = \mathrm{FL}(\mathcal{P}_\gamma)$ we have $\mathbf{v}$ is a vertex of $\mathcal{P}$. For our purposes it will be more convenient to write

$$\mathbf{v}_\gamma = \mathbf{v} + \sum_{i=1}^{d} \gamma_i \mathbf{u}_i$$

where $\mathbf{u}_i$ is the i^{th} column of A^{-1}. Let $\gamma_j = m$ for all $j \in W$ and $\gamma_i = 0$ otherwise, where $W \subseteq \{1, 2, \ldots, (d-1), d\}$ and $m \in \mathbb{Z}$. That is, we want to take an arbitrary vertex $\mathbf{v}$ of $\mathcal{P}$ and move some subset of facets that $\mathbf{v}$ is on by the same integer value m. Thus,

$$\mathbf{v}_\gamma = \mathbf{v} + m \left(\sum_{i \in W} \mathbf{u}_i \right) \tag{3.3}$$

where $\mathbf{v}$ is a vertex of $\mathcal{P}$. Let $p_\mathbf{v}$ be the lcm of the denominators of the entries of $\mathbf{u}_i$ for all $i \in W$. Set

$$p_* = \operatorname{lcm}[p_{\mathbf{v}} : \ \mathbf{v} \text{ a vertex of } \mathcal{P}]. \tag{3.4}$$

Consider the rational vertex cone $\mathcal{K}_{\mathbf{v}_\gamma} = \{\mathbf{v}_\gamma + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_n\mathbf{w}_n : \lambda_i \geq 0\}$. Let $\mathbf{v}_\gamma$ be as in (3.3) where we write $m = r + lp_*$, $r = 0, 1, 2, \ldots, (p_* - 1)$. Here we are considering l to be the variable. Finally, set $\sum_{i \in W} \mathbf{u}_i = \mathbf{u}(\mathbf{v})$. Then

$$\begin{aligned}
\mathcal{K}_{\mathbf{v}_\gamma} &= \{\mathbf{v}_\gamma + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_n\mathbf{w}_n : \lambda_i \geq 0\} \\
&= \{(m\mathbf{u}(\mathbf{v}) + \mathbf{v}) + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_n\mathbf{w}_n : \lambda_i \geq 0\} \\
&= \{lp_*\mathbf{u}(\mathbf{v}) + (r\mathbf{u}(\mathbf{v}) + \mathbf{v}) + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_n\mathbf{w}_n : \lambda_i \geq 0\}.
\end{aligned}$$

Since $lp_*\mathbf{u}(\mathbf{v})$ is integral by definition of p_* and $(r\mathbf{u}(\mathbf{v}) + \mathbf{v})$ is just shifting $\mathcal{K}_{\mathbf{v}}$ by $r\mathbf{u}(\mathbf{v})$ and does not contain the variable l, we have

$$\sigma_{\mathcal{K}_{\mathbf{v}_\gamma}}(\mathbf{z}) = \mathbf{z}^{lp_*\mathbf{u}(\mathbf{v})}\sigma_{\mathcal{K}_{r\mathbf{u}(\mathbf{v})+\mathbf{v}}}(\mathbf{z}). \tag{3.5}$$

Theorem 3.3. *Let $\mathcal{P} = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$ and $\mathcal{P}_{m,W} = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b} + m\sum_{j \in W} \mathbf{e}_j\}$, $\mathbf{e}_j$ the j^{th} standard basis vector, be convex rational d-polytopes. Set $c_W(m) = \#(\mathbb{Z}^d \cap \mathcal{P}_{m,W})$ for all m such that $\mathcal{P}$ and $\mathcal{P}_{m,W}$ are combinatorially equivalent. Then $c_W(m)$*

is a quasipolynomial in m of period $q|p_*$, with p_* as defined in (3.4)*.*

Proof. By Brion's Theorem we have

$$\sigma_{\mathcal{P}_{m,W}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}_{m,W}} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z}).$$

Take any vertex $\mathbf{v}$ of $\mathcal{P}_{m,W}$ and compute A^{-1} and $\mathbf{u}(\mathbf{v})$ as before. By (3.3) and (3.5)

$$\begin{aligned}
\sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z}) &= \mathbf{z}^{lp_*\mathbf{u}(\mathbf{v})}\sigma_{\mathcal{K}_{r\mathbf{u}(\mathbf{v})+\mathbf{v}}}(\mathbf{z}) \quad \text{which implies} \\
c(r+lp_*) &= \lim_{\mathbf{z}\mapsto\mathbf{1}} \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}_{m,W}} \mathbf{z}^{lp_*\mathbf{u}(\mathbf{v})}\sigma_{\mathcal{K}_{r\mathbf{u}(\mathbf{v})+\mathbf{v}}}(\mathbf{z}).
\end{aligned}$$

The rational functions $\sigma_{\mathcal{K}_{\mathbf{ru}(\mathbf{v})+\mathbf{v}}}(\mathbf{z})$ do not depend on l. Thus to compute $c(r+lp_*)$ we use L'Hôsptial's rule (repetitively if needed) to remove the singularities of the rational functions $\sigma_{\mathcal{K}_{r\mathbf{u}(\mathbf{v})+\mathbf{v}}}$ as we take the limit for $\mathbf{z} \mapsto 1$. As we have already seen, once we remove all of the singularities, when we evaluate the remaining rational function at $\mathbf{z} = 1$ we obtain a polynomial in l. $\square$

Chapter 4

Quasipolynomiality of $c_\Sigma(m,k)$

Recall $c_\Sigma(m,k)$:=# *of k-colorings of a signed graph* Σ *with minimal coloring distance m.* We may assume for our purposes that Σ is connected. Also recall that a proper k-coloring of a signed graph Σ is an integral vector $\mathbf{x} \in \mathcal{P}_j$ for some j, where $\mathcal{P}_j$ is as in (2.1). We now shift every hyperplane $h^\epsilon_{ij} \in \mathcal{H}(\Sigma)$ by $\pm m$, m an integer. Thus, if $h^\epsilon_{ij} = x_i + \epsilon x_j = 0$ is a member of $\mathcal{H}(\Sigma)$ then the hyperplanes $x_i + \epsilon x_j = \pm m$ are a member of $\mathcal{H}_m(\Sigma)$, the *shifted hyperplane arrangement* of Σ. Now we wish to impose a minimal coloring distance between nodes of our signed graph Σ. An m-minimal k-coloring of Σ will avoid $-m < x_i - \epsilon x_j < m$, whenever $(x_i - \epsilon x_j = 0) \in \mathcal{H}(\Sigma)$, i.e., $ij \in E$ with sign ϵ where E is the edge set of Σ.

The m-minimal k-colorings are are contained in

$$[-k,k]^{|V|} \setminus \cup \{\mathbf{x} \in \mathbb{R}^{|V|} : -m < x_i - \epsilon x_j < m, ij \in E \text{ with sign } \epsilon\} = \bigcup_{j=1}^{n} \mathcal{M}_j$$

where $\mathcal{M}_j$ are pairwise disjoint polytopes. Referring to Figure 4.1, $\bigcup \mathcal{M}_j$ is the union of the white regions.

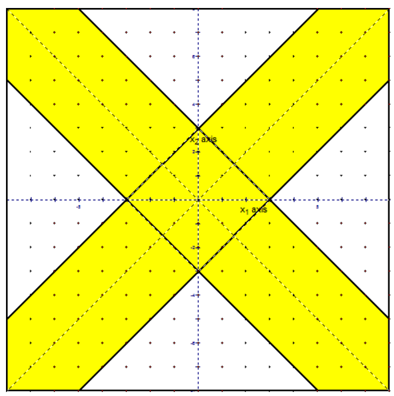

Figure 4.1: Minimal distance k-colorings of a signed graph with $m = 3, k = 8$.

$c_\Sigma(m,k) = \# \left[\left(\bigcup_{j=1}^{n} \mathcal{M}_j \right) \cap \mathbb{Z}^{|V|} \right] = \sum_{j=1}^{n} \# \left(\mathcal{M}_j \cap \mathbb{Z}^{|V|} \right)$ since the $\mathcal{M}_j$ are disjoint. What remains to be done is prove $c_\Sigma(m,k)$ is a piecewise-defined quasipolynomial. To do this we will need to determine the integrality of an arbitrary $\mathcal{M}_j$, but first we construct an example polynomial.

4.1 Computation of $c_{\pm K_2}(m, k)$

Recall that the sum of the first N positive odd integers is N^2. With this in mind we refer to the geometry in Figure 4.1. Each $\mathcal{M}_j$ for $j = 1, 2, 3, 4$ (white regions) contains our m-minimal k-colorings of $\pm K_2$. The $\mathcal{M}_j$ are isomorphic as lattice-polytopes, i.e., they each contain an equal number of lattice points. The lattice-point count for any $\mathcal{M}_j$ is the sum of odd integers. The exact number of odd integers being summed is dependent upon the relationship of m and k in so far as we sum $k - (m + 1)$ consecutive, odd positive integers. Also notice that if $m = k$ the lattice-point count is 1 while for all $m > k$ the lattice-point count is 0. Thus letting $N = (k - m + 1)$ we have

$$c_{\pm K_2}(m, k) = \begin{cases} 4(k - m + 1)^2 & \text{if } k \geq m, \\ 0 & \text{if } k < m. \end{cases}$$

Thus

$$
\begin{aligned}
(-1)^2 c_{\pm K_2}(-m,-k) &= 4(-k+m+1)^2 \\
&= 4(k-m-1)^2 \\
&= 4((k-1)-(m+1)+1)^2 \\
&= c_{\pm K_2}(m+1,k-1) \text{ as guaranteed by Theorem 1.3.}
\end{aligned}
$$

4.2 Half Integrality

Lemma 4.1. *The vertices of each $\mathcal{M}_j$ are half integral.*

Proof. Let Σ be a connected component on d nodes $\{w_1, w_2, ..., w_d\}$ of a signed graph. Let A_Σ be the $n \times d$ adjacency matrix of Σ. Let $B_\Sigma = A_\Sigma | I_{d\times d}$, where we stack the $d \times d$ identity matrix with A_Σ. Then $\mathcal{M}_j = \{\mathbf{x} \in \mathbb{R}^d : B_\Phi \mathbf{x} \leq \mathbf{c}\}$, $\Phi \subseteq \Sigma$ a sub-graph on d nodes. From the definition of $\mathcal{M}_j$ we know the entries of $\mathbf{c}$ are from $\{\pm m, \pm k\}$, however for the remainder of this proof we will only need to assume that the entries of $\mathbf{c}$ are integral. Let $\mathbf{v}$ be a vertex of $\mathcal{M}_j$. Then for some full dimensional sub-matrix $B_\tau \subseteq B_\Phi$ and some sub-graph $\tau \subseteq \Phi$, $\mathbf{v}$ satisfies $B_\tau \mathbf{v} = \mathbf{b}$, with $\mathbf{b}$ containing the associated entries of $\mathbf{c}$. Notice this implies that τ is still on d nodes, else $\mathbf{v}$ would not be a vertex of $\mathcal{M}_j$. τ must contain a cycle, since if not then τ is a connected graph without a cycle on d nodes, i.e., a tree graph on d nodes.

This in turn would imply B_τ has one more column than rows, hence $B_\tau \mathbf{v} = \mathbf{b}$ has infinitely many solutions, a contradiction.

Choose any adjacent nodes w_i, w_j of τ with edge sign ϵ. This yields the equation $v_i - \epsilon v_j = n$, $n \in \mathbb{Z}$, with v_i a coordinate of $\mathbf{v}$, the solution vector to $A_\tau \mathbf{v} = \mathbf{b}$. Then $v_j = \pm v_i \pm n$. Similarly, if w_r is adjacent to w_j then $\pm v_i \pm n - \epsilon v_r = \pm l$, $l \in \mathbb{Z}$, which implies $v_r = \pm v_i \pm t$, $t \in \mathbb{Z}$.

Let $\pi \subset \tau$ be any cycle of length k with nodes $\{w_1, w_2, \ldots, w_k\}$, and let $\{v_1, v_2, \ldots, v_k\}$ be the corresponding coordinates of $\mathbf{v}$ (after suitable reordering). Let w_i, w_{i+1} and w_1, w_k be adjacent. From the above we have $v_i = \pm v_1 \pm n_i$, n_i an integer, $i = 1, 2, ..., k$. Since w_1, w_k are adjacent we have:

$$
\begin{aligned}
v_1 \pm v_1 \pm (n_1 + n_k) &= \pm l, \quad l \text{ an integer, which implies} \\
v_1 \pm v_1 &= n, \quad n \text{ an integer}
\end{aligned}
$$

If $2v_1 = n$ then $v_1 = \frac{n}{2}$ so that v_1 is integral or half integral depending on the parity of n. If $0 = n$ then v_1 could have been any value, contradicting our assumption that $A_\tau \mathbf{v} = \mathbf{b}$ had a unique solution. Since τ is connected, there exists a path from $w_1 \in \pi$ to any other $w_i \in \tau$. Hence, $v_i = \pm v_1 \pm r n_1$ for some integer r for all v_i. Therefore, $v_i = \pm \frac{n}{2} + r n_1 = \frac{q}{2}$ for some integer q, v_i an entry of $\mathbf{v}$. Since $\mathbf{v}$ was an arbitrary vertex of $\mathcal{M}_j$, we see that every $\mathcal{M}_j$ is half integral as desired. $\square$

4.3 Proof of Theorem 1.1

Let $c_j(m,k) := \#\left(\mathcal{M}_j \cap \mathbb{Z}^{|V|}\right)$. Hence

$$c_\Sigma(m,k) = \sum_{j=1}^{n} \#\left(\mathcal{M}_j \cap \mathbb{Z}^{|V|}\right) = \sum_{j=1}^{n} c_j(m,k). \tag{4.1}$$

Lemma 4.2. *For a fixed k, $c_j(m,k)$ is a piecewise-defined quasipolynomial in m of period 1 or 2.*

Proof. If $m \geq k$ then $c_j(m,k) = 0$, thus we may assume $m < k$. By Lemma 4.1, the vertices of $\mathcal{M}_j$ are half integral, therefore $p_* = 1$ or 2 as defined in (3.4). By the definition of $\mathcal{M}_j$, there exists a finite number of combinatorial classes as m varies. For each class, Theorem 3.3 says that $c_j(m,k)$ is a quasipolynomial of period 1 or 2. By Proposition 2.1, the function obtained by summing each $c_j(m,k)$ over all combinatorial classes is a piecewise-defined quasipolynomial of period 1 or 2. $\square$

Notice that if we fix m and let k vary, using a similar proof we obtain a piecewise-defined quasipolynomial in k of period 1 or 2.

Proof of Theroem 1.1. By (4.1) and Lemma 4.2 the result follows. □

4.4 Modular Colorings

Let $\mathbb{Z}_k$ denote a cyclic group of order k and let $\varphi : \mathbb{Z}_k \mapsto \{0, 1, \ldots, k-1\}$ be the canonical map that lets us realize $n \in \mathbb{Z}_k$ as a nonnegative integer $\varphi(n)$. We say a modular k-coloring $\mathbf{x} \in \mathbb{Z}_k^V$ of a signed graph Σ has *minimal distance* m if

$$m \leq \varphi(x_i - \epsilon x_j) \leq k - m \tag{4.2}$$

whenever $ij \in E$ with sign ϵ. Thus, we may think of a modular k-coloring with minimal distance m as a lattice point $\mathbf{x}$ in $[0, k-1]^V \cap \mathbb{Z}^V$ whenever $\mathbf{x}$ satisfies (4.2).

Let $\eta_\Sigma(m, k)$ denote the number of modular k-colorings with minimal distance m of a signed graph Σ. If we follow a similar setup and proof as used in the case of $c_\Sigma(m, k)$ we see there is one main difference for $\eta_\Sigma(m, k)$. The colorings we are after that lie in $\mathbf{x} \in [0, k-1]^V \cap \mathbb{Z}^V$ are now contained in polytopes in the positive orthant. The proof of Theorem 1.1 relied most heavily on Lemma 4.1. Recall we proved $A_\tau \mathbf{v} = \mathbf{b}$ had half-integral solutions and our assumption on $\mathbf{b}$ was integrality of the entries. Hence, our proof is still valid in the modular case. Thus, substituting $[0, k-1]^V$ for $[-k, k]^V$ and $m \leq \varphi(x_i - \epsilon x_j) \leq k - m$ for $-m \leq x_i - \epsilon x_j \leq m$, each of the proofs for the integral case still hold in the modular case and Theorem 1.2 follows.

There is one subtlety that should be addressed. Geometrically we are now counting lattice points in polytopes in the positive orthant. We gain a few new combinatorial classes as there is now "wrapping" of the non-coloring regions created by modular arithmetic (see Figure 4.2). However, there are still finitely many combinatorial classes, therefore the new regions pose no new problems.

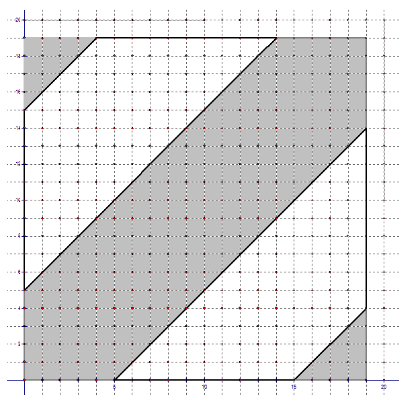

Figure 4.2: Modular coloring of the signed graph $+K_2$, $k = 20$ and $m = 5$

4.5 Computation of $\eta_{+K_2}(m, k)$

Recall that the sum of the first consecutive n positive integers is $\frac{n(n+1)}{2}$. With this is mind we refer to the geometry in Figure 4.2. Each $\mathcal{M}_j$, for $j = 1, 2$ contains our m-minimal modular k-colorings of $+K_2$. Each $\mathcal{M}_j$ contains a number of lattice points equal to the sum of $n = k - m$ consecutive positive integers minus the first $m - 1$ consecutive positive integers. Geometrically this is one of the large white triangles with a smaller grey triangle removed.

Thus, counting the lattice points in any white region gives

$$\begin{aligned}
\mathcal{M}_j \cap \mathbb{Z}^2 &= \frac{(k-m+1)(k-m)}{2} - \frac{m(m-1)}{2} \\
&= \frac{k(k-2m+1)}{2} \\
&= \frac{k(k-(2m-1))}{2}.
\end{aligned}$$

The last equality tells us that the lattice point count is zero if $k < 2m$, therefore

$$\eta_{+K_2}(m, k) = \begin{cases} k^2 - 2km + k & \text{if } k \geq 2m, \\ 0 & \text{if } k < 2m. \end{cases}$$

Thus

$$
\begin{aligned}
(-1)^2\eta_{+K_2}(-m,-k) &= k^2 - 2km - k \\
&= k^2 - 2km - 2k + k \\
&= k^2 - 2k(m+1) + k \\
&= \eta_{+K_2}(m+1,k) \quad \text{as guaranteed by Theorem 1.3.}
\end{aligned}
$$

Chapter 5

Reciprocity

A common theme in combinatorics is to evaluate counting functions, such as chromatic polynomials, at negative integers and show these evaluations have an interpretation as mathematical objects. Eugene Ehrhart first conjectured the following theorem which was proved about a decade later by I.G. Macdonald in 1971 [3].

Theorem 5.1. (Ehrhart–Macdonald). *Suppose $\mathcal{P}$ is a convex rational d-polytope with $\mathcal{P}^\circ$ its relative interior. Define $L_\mathcal{P}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^d)$. Then the evaluation of the quasipolynomial $L_\mathcal{P}$ at negative integers yields*

$$L_\mathcal{P}(-t) = (-1)^d L_{\mathcal{P}^\circ}(t).$$

We will need the following multi-variant version of Ehrhart reciprocity proved by Matthias Beck in 2002 [2].

Theorem 5.2. (Beck). *Let $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\}$ be a rational d-polytope. Let $L_A(\mathbf{b}) := \#\{\mathbf{x} \in \mathbb{Z}^d : A\mathbf{x} \leq \mathbf{b}\}$ denote the number of lattice points in $\mathcal{P}$. Then the evaluation of the piecewise-defined quasipolynomial L_A at $-\mathbf{b}$ yields*

$$(-1)^d L_A(-\mathbf{b}) = \# \left\{ \mathbf{x} \in \mathbb{Z}^{\mathbf{d}} : \mathbf{A}\mathbf{x} < \mathbf{b} \right\}.$$

Recall from the proof of Lemma 4.1 that $\mathcal{M}_j = \{\mathbf{x} \in \mathbb{R}^d : B_\Phi \mathbf{x} \leq \mathbf{c}\}$ is a half-integral d-polytope, where $\mathbf{c}$ has integral entries from $\{\pm m, \pm k\}$. By (4.1) we have

$$c_\Sigma(m, k) = \sum_{j=1}^{n} \# \left(\mathcal{M}_j \cap \mathbb{Z}^{|V|} \right) = \sum_{j=1}^{n} c_j(m, k).$$

Therefore

$$c_\Sigma(-m, -k) = \sum_{j=1}^{n} c_j(-m, -k),$$

and Theorem 5.2 tells us

$$\begin{aligned}(-1)^d c_j(-m,-k) &= \#\left\{\mathbf{x}\in\mathbb{Z}^d : B_\Phi\mathbf{x} < \mathbf{c}\right\}\\ &= \#\left(\mathcal{M}_j^\circ \cap \mathbb{Z}^d\right)\end{aligned}$$

where $\mathcal{M}_j^\circ$ is the relative interior of the polytope $\mathcal{M}_j$. Referring to Figure 5.1. each $\mathcal{M}_j^\circ$ is a grey shaded triangle.

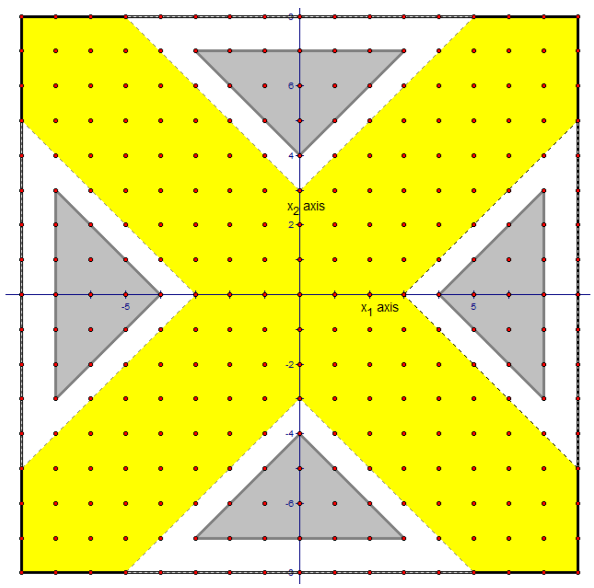

Figure 5.1: $\mathcal{M}_j^\circ$ interiors of $\pm K_2$ with $m = 3$ and $k = 8$.

The $\mathcal{M}_j^\circ$ are bounded by constraints of the form $\pm x_i < k$ and $\pm x_i \pm x_l > m$. Because these are integral conditions, the lattice points in $\left(\mathcal{M}_j^\circ \cap \mathbb{Z}^d\right)$ have constraints

$\pm x_i \leq k-1$ and $\pm x_i \pm x_l \geq m+1$. Therefore we have

$$(-1)^d c_j(-m,-k) = c_j(m+1,k-1).$$

This proves part one of Theorem 1.3.

We will need a generalization of Theorem 5.2 also due to Matthias Beck [2]. Collect a subset of the rows of the matrix A in the matrix A_1 and the corresponding entries in $\mathbf{b}$ in the vector $\mathbf{b}_1$. The remaining rows of A will be collected in the matrix A_2 and the corresponding entries of $\mathbf{b}$ in the vector $\mathbf{b}_2$. Let

$$L_{A_1,A_2}(\mathbf{b}) = \# \left\{ \mathbf{x} \in \mathbb{Z}^d : \begin{array}{ccc} A_1 & \mathbf{x} & \leq & \mathbf{b}_1 \\ A_2 & \mathbf{x} & < & \mathbf{b}_2 \end{array} \right\}.$$

If the union of the facets of $\mathcal{P}(\mathbf{b}) := \left\{ \mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b} \right\}$ whose normal vectors are the rows of A_1 is a manifold (possibly with boundary), then

$$(-1)^n L_{A_1,A_2}(-\mathbf{b}) = \# \left\{ \mathbf{x} \in \mathbb{Z}^d : \begin{array}{cccc} A_1 & \mathbf{x} & < & \mathbf{b}_1 \\ A_2 & \mathbf{x} & \leq & \mathbf{b}_2 \end{array} \right\}. \tag{5.1}$$

Remark. Theorem 5.2 follows as a special case with $A_1 = A$.

For the second part of Theorem 1.3 we use a geometric construction similar to that of $c_\Sigma(m,k)$. We would like to express $\eta_\Sigma(m,k) = \#\left(\cup \mathcal{M}_j \cap \mathbb{Z}^{|V|}\right)$ where the $\mathcal{M}_j$ are described by inequalities whose right-hand sides are now in $\{0, k, m, k-m\}$. Since we are working mod k, we are only using the colors from $\{0, 1, 2 \ldots, k-1\}$. This means the lattice points on the hyperplanes $x_i = k$ need to be excluded. Thus, the $\mathcal{M}_j$ are half open polytopes. Collect all the rows of B_Φ that correspond to hyperplanes of the form $x_i = k$ and place them into the matrix B_{Φ_1} and the corresponding entries of $\mathbf{c}$ in the vector $\mathbf{c}_1$. Collect the remaining rows of B_Φ and place them in the matrix B_{Φ_2} and the corresponding entries of $\mathbf{c}$ in the vector $\mathbf{c}_2$. We have

$$\begin{aligned} c_j(m,k) &= \#\left(\mathcal{M}_j \cap \mathbb{Z}^d\right) \\ &= \#\left\{ \mathbf{x} \in \mathbb{Z}^d : \begin{array}{ccc} B_{\Phi_1} & \mathbf{x} & < & \mathbf{c}_1 \\ B_{\Phi_2} & \mathbf{x} & \leq & \mathbf{c}_2 \end{array} \right\} \quad \text{and} \\ \eta_\Sigma(m,k) &= \sum_{j=1}^{n} c_j(m,k). \end{aligned}$$

By (5.1) we have

$$
\begin{aligned}
(-1)^d c_j(-m,-k) &= \# \left\{ \mathbf{x} \in \mathbb{Z}^d : \begin{array}{ccc} B_{\Phi_1} & \mathbf{x} & \leq & \mathbf{c}_1 \\ B_{\Phi_2} & \mathbf{x} & < & \mathbf{c}_2 \end{array} \right\} \\
&= \# \left(\mathcal{M}'_j \cap \mathbb{Z}^d \right).
\end{aligned}
$$

The rows of B_{Φ_2} correspond to hyperplanes of the form $\{\pm x_i = 0, \pm x_i \pm x_j = m, \pm x_i \pm x_j = k - m\}$. Notice that by changing the inequalities from weak to strong and strong to weak, we have altered $\mathcal{M}_j$ by closing all open facets and opening closed facets which we call $\mathcal{M}'_j$ above. Thus, our color set is now in $\{1, 2, \ldots, k\}$ (which still make up k colors) and our parameter m changes to $m+1$ (See Figure 5.2). This tells us that $\# \left(\mathcal{M}'_j \cap \mathbb{Z}^d\right) = c_j(m+1, k)$. Therefore

$$
\begin{aligned}
(-1)^d \eta_\Sigma(-m,-k) &= (-1)^d \sum_{j=1}^{n} c_j(-m,-k) \\
&= \sum_{j=1}^{n} c_j(m+1,k) \\
&= \eta_\Sigma(m+1,k),
\end{aligned}
$$

which proves part two of Theorem 1.3.

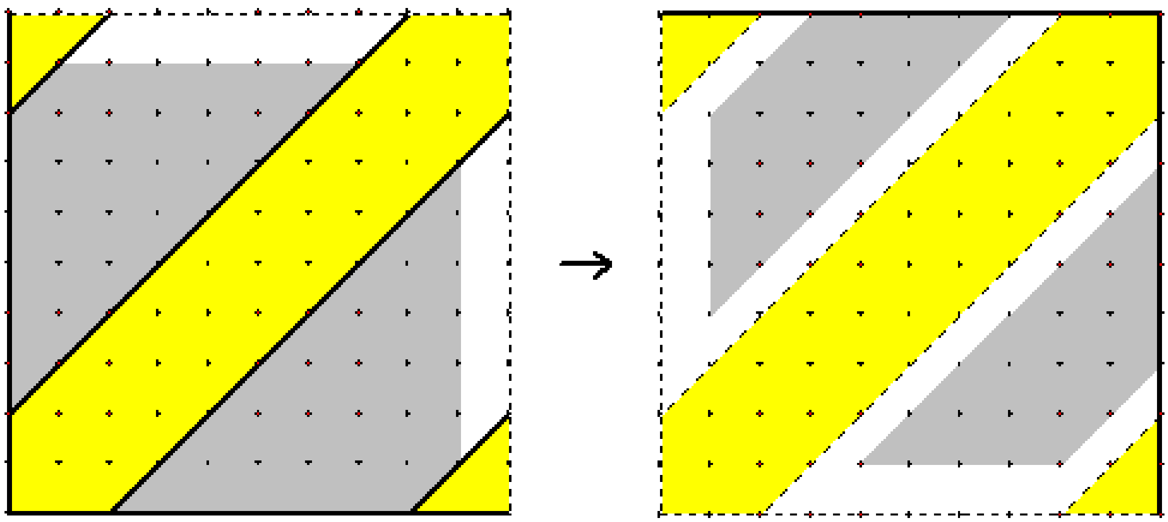

Figure 5.2: Modular Reciprocity of $+K_2$ with $m = 2$ and $k = 10$.

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