Eulerian Polynomials for Bidirected Graphs

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by

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Certification of Approval

I certify that I have read Eulerian Polynomials for Bidirected Graphs by Panya Sukphranee and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirement for the degree Masters of Arts at San Francisco State University.

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Abstract

The descent and inversion statistics are two well-studied statistics in combinatorics that are defined for the permutation group S_n. P.A. MacMahon introduced the major index $\operatorname{maj}\pi = \sum_{i=1}^{n-1} i\chi(\pi_i > \pi_{i+1})$ on S_n (more generally on words of a totally ordered alphabet), and proved that the inversion and major index statistics of S_n are equidistributed. Foata and Zeilberger (1995) generalized these statistics to directed graphs. In 2023, Celano et al studied the evaluations of the Eulerian polynomials (for digraphs) at -1 and found that the inversion statistic can be viewed as the descent statistic on a complete graph. Therefore, the Eulerian and Mahonian polynomials are captured as just Eulerian polynomials on the path and the complete graph, respectively. We generalize the descent and inversion statistics on signed permutations to bidirected graphs, and extend properties of Eulerian polynomials on digraphs to bidirected graphs.

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Introduction

We generalize Eulerian polynomials to the structure of bidirected graphs (Definition 3 below). We present basic properties and results of this generalization. Previous work by [2, 3] generalized the Eulerian polynomials to digraphs and we capture these as a special case.

The Eulerian polynomial is a well-studied topic in combinatorics that encodes information about the descent permutation statistic, and a historical motivation for studying them is presented in Chapter 2. Permutation statistics refer to numerical quantities of interest associated with permutations. We are interested in two permutation statistics: the descent statistic and inversion statistic. There are two versions of each statistic: type-A (Definition 7) corresponding to unsigned permutations, and type-B (Definition 1) corresponding to signed permutations.

We introduce the matrix representation of a bidirected graph in Section 4 and the signed graph structure (Definition 2). Signed graphs and bidirected graphs have different origins and were constructed independently. However, every bidirected graph is associated with a signed graph which can be viewed as its underlying structure. Signed graphs have a one-to-many relationship to bidirected graphs. The bidirected graph is our main object of study and we define the descent permutation statistic for it (Definiton 4) along with its matrix representation (Corollary 1.0.1).

We generalize the Eulerian polynomial to bidirected graphs (Definition 6) and present properties relating the Eulerian polynomials to the graph structure (Theorem 1.0.2). For example, given a bidirected graph, we present an expression for its Eulerian polynomial in terms of its disjoint-subgraph's Eulerian polynomials. We prove that the Eulerian polynomial is preserved under a switching operation (Theorem 1.0.4). We prove that the absolute evaluation at -1 is preserved for all orientations of the same signed graph.

In Chapter 7, we derive the classical type-B generating function for an Eulerian polynomial and identify it with the Eulerian polynomial for the path graph.

Chapter 1

Main Results

Signed Permutations and the Hyperoctahedral Group B_n We define signed permutations as permutations on $[-n,n] \setminus \{0\} := \{\pm 1, \ldots, \pm n\}$, writing them as a set of pairs $\{(\pi, \varepsilon) : \pi \in S_n, \varepsilon \in \{\pm 1\}^n\}$ with

$$(\pi, \varepsilon)k = \varepsilon_k \cdot \pi(k) =: \varepsilon_k \cdot \pi_k$$

[1, pg. 928]. Each pair (π, ε) is characterized by its mapping of elements $k \in [n]$, $(\pi, \varepsilon) : [n] \to [-n,n] \setminus \{0\}$. Then $(\pi, \varepsilon)(-k)$ is defined to be $-(\pi, \varepsilon)k$ so that $(\pi, \varepsilon)(-k) := -\pi_k \varepsilon_k$. Therefore, the mapping $(\pi, \varepsilon) : [-n,n] \setminus \{0\} \to [-n,n] \setminus \{0\}$ becomes a bijection. Under composition, this set becomes a group known as the hyperoctahedral group B_n. Signed permutations are also known as type-B permutations, and we define its descent set as the classical type-B descent.

Definition 1. *The classical type-B descent set is defined to be* $\mathrm{Des_B}(\pi, \varepsilon) = \{i \in [0, n-1] : \varepsilon_i \pi_i > \varepsilon_{i+1} \pi_{i+1}\}$, *where* $\pi_0 \varepsilon_0 := 0$.

(Note that if $\varepsilon_1 \pi_1 < 0$, then we have a descent in the zeroth position).

Graph Structures Next, we introduce signed graphs and bidirected graphs [8]. The bidirected graph is our main object of study, where we extend the signed permutations to act on its vertices. A signed graph comes about as an underlying layer that can be interpreted as classifying bidirected graphs.

Definition 2. *A signed graph is a graph (V,E) with a map $\gamma : E \to \{\pm\}$. The map γ makes each edge into what we can call a (+) positive edge or (-) negative edge. We can denote a signed graph by (V,E,γ).*

Definition 3. *A bidirected graph is a graph $D = (V,E)$ along with a mapping $\tau : V \times E \to \{0, \pm 1\}$, such that*

$$\tau(v,e) = \begin{cases} 0 & \text{if } v \notin e, \\ \\ \pm 1 & \text{if } v \in e. \end{cases}$$

τ is interpreted as assigning a direction to e at v (if e contains v). $\tau(v,e) = -1$ is interpreted as assigning a direction away from v, and $\tau(v,e) = +1$ as assigning a direction towards v. We denote a bidirected graph by $\tilde{D} = (V,\tilde{E}) = ((V,E),\tau) = (V,E,\tau)$.

Although these structures are constructed independently from a graph, there is a relationship between them: a bidirected graph is equivalent to an oriented signed graph [9, pg. 6]. This relationship is captured in the following diagram:

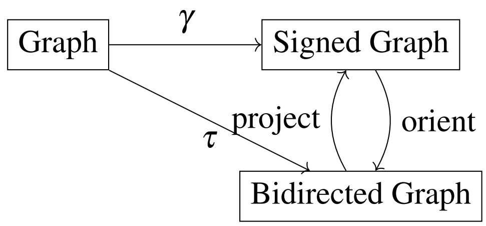


Figure 1.1: Relationship between graphs

The possible orientations of a signed graph are determined by the signs of its edges described in Section 3.3. Conversely, a bidirected graph corresponds to some signed graph. We distinguish between the terms biorient and orient as acting on graphs and signed graphs, respectively [7, 9, 10]. We define a projection mapping from the bidirected edge its underlying signed edge also in Section 3.3.

Matrix Representation In Section 4, we define an *appended incidence matrix* for a bidirected graph $\tilde{D}$. This allows us to have a matrix representation for computation, and we denote this by $\Sigma_{\tilde{D}}$. The matrix is defined to be 0-indexed. For $i, j \geq 0$, its entries are given by

$$(\Sigma_{\tilde{D}})_{ij} = \tau(i, e_j)$$

(note that we need to enumerate the edges before having a matrix representation). We define the signed permutation's matrix representation as a column vector and denote this by σ. Its entries are given by $\sigma_i = \pi_i \varepsilon_i$.

The Descent Set With the signed permutation group B_n generalized to act on the vertices of a graph, the classical type-B descent is generalized to the context of a bidirected graph, denoted by

$\mathrm{Des}_{\tilde{D}}(\pi,\varepsilon)$. A descent between two vertices is determined by the bidirected edge connecting them.

Definition 4. *The descent set of a signed permutation with respect to a given bidirected graph $\tilde{D}$ is defined to be*

$$\mathrm{Des}_{\tilde{D}}(\pi,\varepsilon) = \left\{ e \in \tilde{E} : \varepsilon_i \pi_i \cdot \tau(i,e) + \varepsilon_j \pi_j \cdot \tau(j,e) < 0 \text{ and } i,j \in e \right\}.$$

Theorem 1.0.1. *The matrix representation of a descent set of a signed permutation with is given by*

$$\mathrm{Des}_{\tilde{D}}(\pi,\varepsilon) = \left\{ i : \left(\sigma^T \cdot \Sigma_{\tilde{D}}\right)_i < 0 \right\} = \left\{ i : \sum_k \sigma_k \left(\Sigma_{\tilde{D}}\right)_{ki} < 0 \right\} = \left\{ i : \sum_k \varepsilon_k \pi_k \left(\Sigma_{\tilde{D}}\right)_{ki} < 0 \right\}.$$

Definition 5. $\mathrm{des}_{\tilde{D}}(\pi,\varepsilon) = \#\mathrm{Des}_{\tilde{D}}(\pi,\varepsilon)$.

The Eulerian Polynomial The generalizations of B_n and descent lay the foundation for introducing the Eulerian polynomial for bidirected graphs.

Definition 6. *The Eulerian polynomial for the bidirected graph $\tilde{D}$ is* $A_{\tilde{D}}(t) = \sum_{(\pi,\varepsilon)\in B_n} t^{\mathrm{des}_{\tilde{D}}(\pi,\varepsilon)}$.

This is our object of interest, and we will refer to it as "the Eulerian polynomial", while refering to the classical type-B Eulerian polynomial explicitly as "the classical Eulerian polynomial" (Equation 2.2). We can compare this to the Eulerian polynomial generalized to digraphs (Equation 2) which are defined for the permutation group S_n [2, 3].

Our definition captures the classical Eulerian polynomial as a special case when $\tilde{D}$ is a path graph (Section 5.0.1). Moreover, the classical type-B inversion (Equation 11) is a captured as a special case when $\tilde{D}$ is a complete graph (Lemma 5.0.1).

Basic Properties of the Eulerian Polynomial We observe some basic properties of the Eulerian polynomial and prove these in Section 6.

Theorem 1.0.2. *Let $\tilde{D} = (D, \tau)$ be a bidirected graph with n vertices and m edges. We use the notation $(\tau(e,i) \cdot i, \tau(e,j) \cdot j)$ for a bidirected edge.*

1. *The polynomial $A_{\tilde{D}}(t)$ is palindromic with center $\frac{m}{2}$.*

2. *Suppose $(-i, j), (i, -j) \in \tilde{E}$. Let $\tilde{E}' = \tilde{E} - \{(-i, j), (i, -j)\}$, and $\tilde{D}' = \left(V, \tilde{E}'\right)$ be the bidirected graph with edges $(-i, j)$ and $(i, -j)$ removed. Then*
$$A_{\tilde{D}}(t) = t \cdot A_{\tilde{D}'}(t).$$

3. *Similarly, if $(i, j), (-i, -j) \in \tilde{E}$, let $\tilde{D}'$ be the bidirected graph $\tilde{D}$ with edges $(i, j), (-i, -j)$ removed. Then*
$$A_{\tilde{D}}(t) = t \cdot A_{\tilde{D}'}(t).$$

4. *If $\tilde{D} = \bigsqcup_{i=1}^{r} \tilde{D}_i$ is a disjoint union of bidirected graphs of orders $n_1, n_2, \ldots, n_r$, then*
$$A_{\tilde{D}}(t) = \binom{n}{n_1, \ldots, n_r} \cdot \prod_{i=1}^{r} A_{\tilde{D}_i}(t).$$

These properties are extensions of Proposition 2.1(a)-(c) of [2] for digraphs (Section 2). Our analog to Proposition 2.1(b) of [2] is true only if the pairs of edges had the same underlying signed edge (Section 3.3).

Changing the Biorientation of an Edge Changing the biorientation of a bidirected edge generally changes the graph's associated Eulerian polynomial. But if we do so while maintaining the underlying signed graph, this is called a *reorientation* of the edge [7] and we have the following invariance:

Theorem 1.0.3. *If $\tilde{D} = (V, \tilde{E})$ and $\tilde{D}' = (V, \tilde{E}')$ are orientations of the same signed graph $D = (V, E)$, then $|A_{\tilde{D}}(-1)| = |A_{\tilde{D}'}(-1)|$.*

The absolute value of the evaluation of the Eulerian polynomial at -1 is preserved under another operation of *switching* at a vertex.

Switching We introduce the operation of switching a given bidirected graph at a vertex v' in Section 6.1. The switching operation modifies the edges connected to v' by reversing the signs of all edges adjacent to v'. Consider the bidirected graph $((D, E), \tau)$ undergoing a switch at v', the resulting graph is denoted $((D, E), \tau')$ and τ' is given by

$$\tau'(v, e) = \begin{cases} -\tau(v, e) & \text{if } v = v' \\ \tau(v, e) & \text{if } v \neq v'. \end{cases} \tag{1.1}$$

In terms of the incidence matrix, the entries of the row corresponding to v' changes signs (see Example 8). We found that switching leaves the Eulerian polynomial preserved.

Theorem 1.0.4. *Let D and D' be bidirected graphs before and after performing a switching operation. The Eulerian polynomial is preserved under switching: $A_{\tilde{D}}(t) = A_{\tilde{D}'}(t)$.*

Corollary 1.0.5. *Performing a switch keeps $|A_{\tilde{D}}(-1)|$ preserved.*

Finally, in Section 7, we present an exponential generating function for the classical type-B Eulerian polynomial. This also serves as a generating function for the path graph's Eulerian polynomial. The exponential generating function for the classical type-A Eulerian polynomial is [6]

$$\sum_{n\geq 0} A_n(q)\frac{x^n}{n!} = \frac{q-1}{q-e^{(q-1)x}}.$$

Theorem 1.0.6. *Let $b_n(q)$ be the classical type-B Eulerian polynomial. Then*

$$\sum_{n\geq 0} b_n(q)\frac{x^n}{n!} = \frac{1-q}{e^{-(1-q)x}-qe^{(1-q)x}}.$$

Corollary 1.0.7. $\sum\limits_{n\geq 0} b_n(q)\frac{x^n}{n!} = \frac{1-q}{e^{-(1-q)x}-qe^{(1-q)x}}$ *is the exponential generating function of the path graph Eulerian polynomial.*

The evaluation at -1 for the exponential generating function has a closed form identity.

Corollary 1.0.8.

$$\sum_{n\geq 0} b_n(-1)\frac{x^n}{n!} = \frac{2}{e^{-2x}+e^{2x}} = \operatorname{sech}(2x).$$

Chapter 2

Historical Background

Definition 7. *The classical type-A descent set is defined to be* $\mathrm{Des_A}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$, *where* $\pi \in S_n$.

The corresponding descent statistic is $\mathrm{des_A}(\pi) = \#\mathrm{Des_A}(\pi) = \#\{i \in [n-1] : \pi_i > \pi_{i+1}\}$.

Definition 8. *The classical type-A Eulerian polynomial is defined as* $\sum_{\sigma \in S_n} t^{\mathrm{des_A}\,\sigma}$.

The type-A Eulerian polynomial is at the center of finding a closed-form identity to the alternating series and its reciprocal analog. In *Remarques sur un beau rapport entre les series des puissances tant directes que reciproques* in Memoires de l'Academie des Sciences de Berlin 17 (1768), Leonard Euler develops a relationship between the alternating series

$$\zeta(s) = 1^s - 2^s + 3^s - \dots \quad \text{and} \quad \varphi(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \dots$$

as $\varphi(s) = (1 - 2^{1-s})\zeta(s)$. The terms of $\zeta(s)$ are embedded as coefficients in the power series expansions

$$1 - s + s^2 - s^3 + \ldots = \frac{1}{1+s}$$
$$1 - 2s + 3s^2 - 4s^3 + \ldots = \frac{1}{(1+s)^2}$$
$$1 - 2^2 s + 3^2 s^2 - 4^2 s^3 + \ldots = \frac{1-s}{(1+s)^3}$$
$$1 - 2^3 s + 3^3 s^2 - 4^3 s^3 + \ldots = \frac{1-4s+s^2}{(1+s)^4}$$
$$1 - 2^4 s + 3^4 s^2 - 4^4 s^3 + \ldots = \frac{1-11s+11s^2-s^3}{(1+s)^5}$$
$$\vdots$$

The numerator has a pattern that is not obvious, and are called the classical Eulerian polynomials.

The expansions above are a few expansions of the identity [1] with $t = -s$

$$\sum_{k\geq 0}(k+1)^n t^k = \frac{\sum_{\pi\in S_n} t^{\mathrm{des_A}(\pi)}}{(1-t)^{n+1}}. \tag{2.1}$$

We have a corresponding definition for the (classical) type-B Eulerian polynomial

Definition 9. *The classical type-B Eulerian polynomial is defined as* $\sum_{(\pi,\varepsilon)\in B_n} t^{\mathrm{des_B}(\pi,\varepsilon)}$.

The corresponding identity for the type-B Eulerian polynomial can be derived from Theorem 6.6 of [1, pg. 948]

$$\sum_{m\geq 0}(2m+1)^n q^m = \frac{\sum_{(\pi,\varepsilon)\in B_n} t^{\mathrm{des_B}(\pi,\varepsilon)}}{(1-q)^{n+1}}. \tag{2.2}$$

Another well-studied statistic that comes up when dealing with permutations is the inversion statistic.

Definition 10. *The classical inversion statistic for a type-A permutation is defined to be*

$$\mathrm{Inv_A}(\pi) = \left\{(i,j) : \pi_i > \pi_j \text{ and } i < j\right\}.$$

Definition 11. *The classical inversion statistic for a type-B permutation is defined to be*

$$\mathrm{Inv_B}(\pi,\varepsilon) = \left\{(i,j) : \varepsilon_i\pi_i > \varepsilon_j\pi_j \text{ and } i < j\right\}.$$

Graph Generalization In 1995, Foata and Zeilberger [3] characterized direct graphs as possessing the Mahonian property of the inversion statistic and major index statistic having the same distribution. MacMahon introduced the major index statistic and further generalized both to arbitrary words (with repeats) [5, pg 135 of Vol 1]. To introduce these statistics, let X be a total ordered alphabet and a $w = x_1x_2\ldots x_m$, then

$$\operatorname{maj} w = \sum_{i=1}^{m-1} i\chi(x_i > x_{i+1}) \tag{2.3}$$

$$\operatorname{inv} w = \sum_{1\le i<j\le m} \chi(x_i > x_j), \tag{2.4}$$

and the rearrangement class of w is the class containing all permutations of the letters of w. MacMahon proved that for each $k \in \mathbb{Z}$, the number of words such that $\operatorname{maj} w = k$ is the same as the number of words such that $\operatorname{inv} w = k$. I.e., maj and inv have the same distribution over the rearrangement class. In terms of the generating functions, inv and maj have the same generating function : $\sum_w q^{\operatorname{maj} w} = \sum_w q^{\operatorname{inv} w}$. Foata and Zeilberger extended these statistics to directed graphs [3, pg. 81] (let U be a directed graph):

$$\mathrm{maj}_{\mathrm{U}}^{,}\, w = \sum_{i=1}^{m-1} i\chi((x_i, x_{i+1}) \in U) \tag{2.5}$$

$$\mathrm{inv}_{\mathrm{U}}^{,}\, w = \sum_{1 \le i < j \le m} \chi((x_i, x_j) \in U). \tag{2.6}$$

Note that a directed graph U has the structure of a relation. They proved that (2.5) and (2.6) are equidistributed if and only if the relation U is bipartitional [3, pg 82]. [3] notes five types of bipartite relationships and it seems that these corresponds to edges of a directed graph, where type (1) at the bottom of [3, pg. 1] corresponds to directed graphs with regular edges, e.g., no loops.

The generating functions for the descent and inversion statistic are called the Eulerian and Mahonian polynomials (respectively) and given by:

$$A_n(t) = \sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)} \text{ and } M_n(t) = \sum_{\sigma \in S_n} t^{\mathrm{inv}(\sigma)}.$$

The generalized descent set and descent statistic for a digraph (V, E) are given by

$$\mathrm{Des}_{\mathrm{D}}(\pi) = \left\{(i, j) \in E : \pi_i > \pi_j\right\} \text{ and } \mathrm{des}_{\mathrm{D}}(\pi) = \#\mathrm{Des}_{\mathrm{D}}(\pi),$$

respectively. In 2023, [2] presented the Eulerian polynomial for digraphs:

$$A_D(t) = \sum_{\sigma \in G_D} t^{\mathrm{des}_{\mathrm{D}}(\sigma)},$$

where $D = (V, E)$ and σ is a permutation on the vertices. They found that the descent statistic is captured as the (digraph) descent statistic of the path of $[n]$: $\mathrm{des}(\sigma) = \mathrm{des}_{P_n}(\sigma)$, and that the inversion statistic is captured as the (digraph) descent statistic of the complete graph of $[n]$:

$\mathrm{des}(\sigma) = \mathrm{des}_{K_n}(\sigma)$ [2, pg. 2 Remark 1.1]. It follows that the Eulerian and Mahonian polynomials were captured as the (diagraph) Eulerian polynomial for P_n and K_n, respectively:

$$A_n(t) = A_{P_n}(t) \text{ and } M_n(t) = A_{K_n}(t).$$

We extend these results of the to bidirected graphs.

Chapter 3

Graphs

3.1 Signed Graphs

Signed graphs are graphs with edges labeled either $(+)$ or $(-)$. They originated in social psychology, being first introduced by Harary [4] in 1953. Psychologists employed square matrices with entries $\{-1,0,1\}$ to represent individuals' feelings towards on another to be negative, indifferent, and positive, respectively. Symmetric matrices were then able to be represented by signed graphs. Since their introduction, signed graphs have arisen in numerous areas in pure and applied mathematics [8].

Example 1. *(Fig 3.1)* $V = \{1,2,3,4\}$, $E = \{\{1,2\},\{1,3\},\{1,4\}\}$, $\gamma(\{1,2\}) = +$, $\gamma(\{1,3\}) = -$, $\gamma(\{1,4\}) = -$.

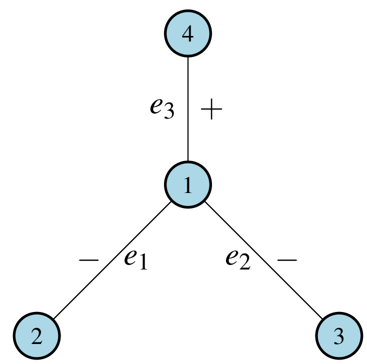


Figure 3.1: A signed graph with four vertices.

3.2 Bidirected Graphs

A graph is turned into a bidirected graph via a bidirection mapping we denote by $\tau : E \times V \to \{0, \pm 1\}$. Each edge is independently oriented, and we adopt the interpretation of -1 representing an end pointing away from its incident vertex and $+1$ as pointing towards its incident vertex.

Example 2. *(Fig. 3.2)* $V = \{1,2\}$, $E = \{e_1 = \{1,2\}\}$, $\tau(e_1,1) = -1,\ \tau(e_1,2) = +1$.

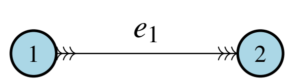


Figure 3.2: Example of a bidirected graph with two nodes.

Example 3. *(Fig. 3.3)* $V = \{1,2,3,4\}$, $E = \{e_1, e_2, e_3\}$, *where* $e_1 = \{1,2\}, e_2 = \{1,3\}, e_3 = \{1,4\}$. *The mapping* τ *is given by*

$$\tau(e_1,1) = +1, \tau(e_1,2) = +1, \tau(e_2,1) = -1, \tau(e_2,3) = +1, \tau(e_3,1) = -1, \tau(e_3,4) = +1.$$

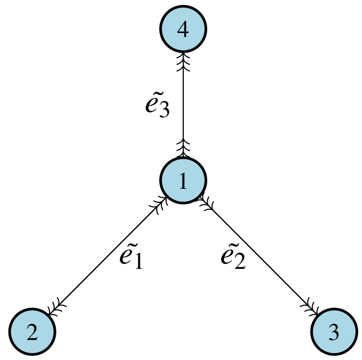


Figure 3.3: Example of a bidirected graph with four nodes.

Example 4. *We denote the bidirected graph Figure 4.1 by* $\tilde{D} = (D, \tau) = ((V, E), \tau)$*, where the underlying graph is given by* $V = \{1, 2, 3, 4\}$ *and* $E = \{e_1, e_2, e_3, e_4, e_5\}$*. The map* τ *is given by* $\tau(e_1, 1) = -1,\ \tau(e_1, 2) = -1,\ \tau(e_2, 1) = -1,\ \tau(e_2, 3) = +1,\ \tau(e_3, 1) = -1,\ \tau(e_3, 4) = +1,$ $\tau(e_4, 2) = -1,\ \tau(e_4, 4) = +1,\ \tau(e_5, 3) = +1,\ \tau(e_5, 4) = +1.$

3.3 Relationships between a Signed Graph and a Bidirected Graph

Signed edges can be oriented as depicted in Figure 3.4. Negative edges are mapped to bidirected edges that have both ends being $+1$ or both -1, and positive edges are mapped to bidirected edges with each end being opposite signs [8].

Given a bidirected graph (V, E, τ), if a sign mapping on (V, E) satisfies (Eqn 2.2 of [10])

$$\gamma(e) = -\tau(e, i) \cdot \tau(e, j)\ \forall\ e = \{i, j\} \in E, \tag{3.1}$$

then (V,E,γ) is said to be its underlying signed graph. Conversely, we call (V,E,τ) an orientation of (V,E,γ) if Eqn 3.1 is satisfied.

Denote the bidirected graph (V,E,τ) as $(V,\tilde{E})$ so that $\tilde{E}$ is a set of bidirected edges, we can define a projection mapping $\text{proj} : \tilde{E} \to \{\pm\}$ with the right side of Eqn 3.1 as

$$\text{proj}(\tilde{e}) = \text{proj}(\{e,\tau\}) = -\tau(e,i)\cdot\tau(e,j).$$

$$+ \mapsto \{\ \ll\!\!-\!\!\!-\!\!\ll\ ,\ \gg\!\!-\!\!\!-\!\!\gg\ \},$$
$$- \mapsto \{\ \gg\!\!-\!\!\!-\!\!\ll\ ,\ \ll\!\!-\!\!\!-\!\!\gg\ \}.$$

Figure 3.4: Orienting Signed Edges

Example 5. *The projection of $\tilde{e_3}$ from Figure 3.3 is* $\text{proj}(\tilde{e_3}) = -$.

Chapter 4

Matrix Representations

The Appended Incidence Matrix Let $\tilde{D} = \left(V, \tilde{E}\right)$ be a bidirected graph with n vertices. We identify the set of vertices with the set $[n]$ and write $V = [n]$. Then $\tilde{D}$ can be represented by a matrix $(\Sigma_{\tilde{D}})$ of dimension $(n+1) \times (\left|\tilde{E}\right| + 1)$ called an appended incidence matrix.

Definition 12. *The appended incidence matrix $\Sigma_{\tilde{D}}$ for a bidirected graph $\tilde{D}$ is defined via*

$$(\Sigma_{\tilde{D}})_{ij} = \begin{cases} 0 & \textit{if } i \neq 1, j = 0, \\ +1 & \textit{if } i = 1, j = 0, \\ \tau(e_j, i) & \textit{otherwise}. \end{cases}$$

We index the matrix from 0 and set the top row to 0 to be consistent with the convention $\varepsilon_0 \pi_0 = 0$. We associate columns 1 to $|E|$ of the matrix with the edges of a graph in lexicographic order, and associate rows 1 to n of the matrix with the vertices. The bottom right $n \times \left|\tilde{E}\right|$ submatrix has entries determined by τ:

$$\Sigma_{\tilde{D}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \tau(e_1,1) & & \dots & & \tau(e_{|E|},1) \\ 0 & \vdots & \ddots & & & \vdots \\ 0 & \vdots & & \ddots & & \vdots \\ 0 & \tau(e_1,n) & & \dots & & \tau(e_{|E|},n) \end{bmatrix}.$$

Example 6. *The appended incidence matrix for* $\tilde{D}$ *in Figure* 4.1 *is*

$$\Sigma_{\tilde{D}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & +1 & -1 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & 0 & +1 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & +1 & +1 \end{bmatrix}.$$

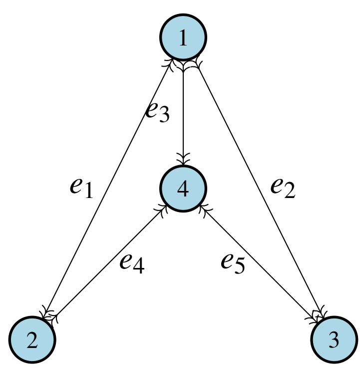


Figure 4.1: Bidirected graph $\tilde{D}$.

The Signed Permutation Vector A signed permutation $(\varepsilon, \pi) \in B_n$ is represented by a column vector σ with

$$\sigma_i = \varepsilon_i \pi_i,$$

where the first index is $\sigma_0 = \varepsilon_0 \pi_0 := 0$, yielding

$$\sigma = \begin{bmatrix} 0 \\ \varepsilon_1 \pi_1 \\ \vdots \\ \varepsilon_n \pi_n \end{bmatrix}.$$

Chapter 5

The Descent and Inversion for Bidirected Graphs

The Classical Descent Statistic Consider the bidirected path graph, $P_n = (V, \tilde{E})$, illustrated in Figure 5.1. This consists of n vertices that we identify with $[n]$, the $n-1$ underlying edges are $E = \{e_i\} = \{\{i, i+1\} : i \in [n-1]\}$, and the associated mapping τ is

$$\tau(e_i, j) = \begin{cases} 0 & \text{if } e_i \notin E, \\ -1 & \text{if } e_i \in E, j = \min(e_i), \\ +1 & \text{if } e_i \in E, j = \max(e_i). \end{cases}$$

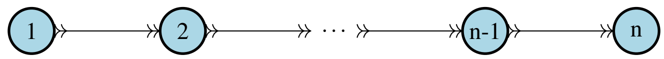


Figure 5.1: The path P_n of length n.

The resulting appended incidence matrix is

$$\Sigma_{\tilde{P}_n} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 0 & +1 & -1 & & 0 \\ 0 & 0 & +1 & \ddots & \vdots \\ \vdots & & \vdots & & -1 \\ 0 & 0 & 0 & 0 & +1 \end{bmatrix}.$$

If we multiply the vector representation σ of some $(\varepsilon,\pi) \in B_n$ with the path's incidence matrix, we obtain a vector

$$\sigma^T \cdot \Sigma_{\tilde{P}_n},$$

with entries in the form of $-\varepsilon_i\pi_i + \varepsilon_{i+1}\pi_{i+1}$. Negative entries then imply that $\varepsilon_{i+1}\pi_{i+1} < \varepsilon_i\pi_i$. Therefore, we can identify the classical type-B descent set $\mathrm{Des}_{\mathrm{B}}(\pi,\varepsilon)$ of section 1 with the product $\sigma^T \cdot \Sigma_{P_n}$ associated with the path. This discussion motivates and yields the following result:

Lemma 5.0.1. *For* $(\pi,\varepsilon) \in B_n$*,*

$$\mathrm{Des}_{\tilde{P}_n}(\pi,\varepsilon) = \left\{ i \in [0,n-1] : \left(\sigma^T \cdot \Sigma_{\tilde{P}_n}\right)_i < 0 \right\} = \mathrm{Des}(\pi,\varepsilon).$$

The Classical Inversion Statistic Consider the bidirected complete graph $K_n = (([n], E), \tau)$ with n vertices illustrated in Figure 5.2. The edges are given by

$$E = \{\{i, j\} \,|\, 1 \leq i < j \leq n\}.$$

There is an edge connecting every pair of vertices, and the edge at the smaller vertex is directed away from that vertex, and the edge at the larger vertex is directed towards that vertex. The associated mapping τ is given by

$$\tau(e,k) = \begin{cases} -1 & k = \min(e), \\ +1 & k = \max(e), \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

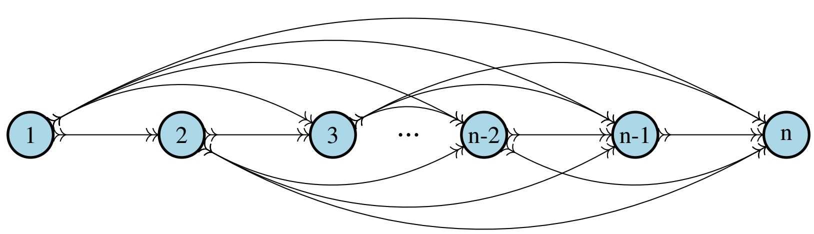


Figure 5.2: The bidirected complete graph K_n with n vertices.

Example 7. *The appended incidence matrix of K_4 illustrated in Figure* 5.3 *is given by*

$$\Sigma_{\tilde{K}_4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

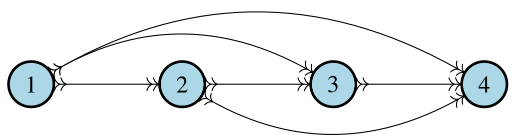


Figure 5.3: The complete graph of length 4.

Lemma 5.0.2. *The descent set for the complete graph is equal to the classical (type-B) inversion set, i.e.,* $\mathrm{Des}_{\tilde{\mathrm{K}}_\mathrm{n}}(\pi, \varepsilon) = \mathrm{Inv}_\mathrm{B}(\pi, \varepsilon)$.

Proof.

$$\begin{aligned}\mathrm{Des}_{\tilde{\mathrm{K}}_\mathrm{n}}(\pi,\varepsilon) &= \left\{k : \left(\sigma^T \cdot \Sigma_{\tilde{K}_n}\right)_k < 0\right\} = \left\{k : \sum_m \sigma_m \cdot \left(\Sigma_{\tilde{K}_n}\right)_{mk} < 0\right\} \\ &= \left\{(i,j) : \sigma_i \cdot (-1) + \sigma_j \cdot (+1) < 0 \text{ where } i < j\right\} = \left\{(i,j) : \varepsilon_i\pi_i > \varepsilon_j\pi_j \text{ where } i < j\right\} \\ &= \mathrm{Inv}_\mathrm{B}(\pi,\varepsilon).\end{aligned}$$

□

Chapter 6

The Eulerian Polynomial of a Bidirected Graph

Structural Properties

Proof of Thm 1.0.2.

1. We identify $(\pi, \varepsilon) \in B_n$ with a column vector σ (as in Section 3) and identify $(\pi, -\varepsilon)$ with $-\sigma$. Now consider an entry i in the matrix product $\sigma^T \cdot \Sigma_{\tilde{D}}$ that counts as a descent in $\mathrm{Des}(\pi, \varepsilon)$,

$$\left(\sigma^T \cdot \Sigma_{\tilde{D}}\right)_i < 0.$$

This is equivalent to the expression

$$\tau_i \varepsilon_i \pi_i + \tau_j \varepsilon_j \pi_j < 0 \iff \tau_i(-\varepsilon_i)\pi_i + \tau_j(-\varepsilon_j)\pi_j > 0.$$

Then

$$k = \operatorname{des}(\pi,\varepsilon) = \#\mathrm{Des}\left\{i : \left(\sigma^T \cdot \Sigma_{\tilde{D}}\right)_i < 0\right\} = \#\mathrm{Des}\left\{i : \left(-\sigma^T \cdot \Sigma_{\tilde{D}}\right)_i > 0\right\}$$
$$= m - \#\mathrm{Des}\left\{i : \left(-\sigma^T \cdot \Sigma_{\tilde{D}}\right)_i < 0\right\} = m - \operatorname{des}(\pi,-\varepsilon),$$

so that

$$\operatorname{des}(\pi,-\varepsilon) = m-k.$$

This implies that, in $A_{\tilde{D}}(t)$, the coefficient of t^k is the same as the coefficient of t^{m-k}.

2. For all $(\pi,\varepsilon) \in B_n$, either

$$\varepsilon_i\pi_i < \varepsilon_j\pi_j \text{ or } \varepsilon_i\pi_i > \varepsilon_j\pi_j.$$

i.e.,

$$\varepsilon_i\pi_i - \varepsilon_j\pi_j < 0 \text{ or } -\varepsilon_i\pi_i + \varepsilon_j\pi_j < 0.$$

This implies that exactly one of $(i,-j)$ and $(-i,j)$ is a descent. With both edges removed, it follows that for all $(\pi,\varepsilon) \in B_n$,

$$\operatorname{des}_{\tilde{D}'}(\pi,\varepsilon) = \operatorname{des}_{\tilde{D}}(\pi,\varepsilon) - 1.$$

Then

$$A_{\tilde{D}'}(t) = \sum_{(\pi,\varepsilon)\in B_n} t^{\operatorname{des}_{\tilde{D}'}(\pi,\varepsilon)} = \sum_{(\pi,\varepsilon)\in B_n} t^{\operatorname{des}_{\tilde{D}}(\pi,\varepsilon)-1} = t^{-1}A_{\tilde{D}}(t).$$

3. For all $(\pi,\varepsilon) \in B_n$, either

$$\varepsilon_i\pi_i + \varepsilon_j\pi_j < 0 \text{ or } \varepsilon_i\pi_i + \varepsilon_j\pi_j > 0,$$

i.e.,

$$\varepsilon_i\pi_i + \varepsilon_j\pi_j < 0 \text{ or } -\varepsilon_i\pi_i - \varepsilon_j\pi_j < 0.$$

This implies that either (i,j) or $(-i,-j)$ is a descent. The proof is the same as in part (2).

4. Let V and V_i denote the set of vertices of $\tilde{D}$ and $\tilde{D}_i$, respectively. Enumerate the vertices of $\tilde{D}$ by $[n]$ in a way that

$$V_1 = \{1,\ldots,n_1\}, V_2 = \{n_1+1,\ldots,n_2\},\ldots,V_r = \{n_{r-1}+1,\ldots,n_r\}.$$

We can express a signed permutation on V in terms of V_i by first choosing an ordered set partition $(P_1,P_2,\ldots,P_r)$ of $[n]$ of order $(n_1,n_2,\ldots,n_r)$ and defining a map

$$\sigma_i : V_i \to P_i \times \{\pm 1\}^{n_i},$$

where σ_i assigns to a vertex in V_i a unique number in P_i. Then for $\sigma \in B_n$,

$$\sigma(V) = \bigcup_{i=1}^{r} \sigma_i(V_i)$$

so that

$$\operatorname{des}_{\tilde{D}}(\sigma) = \sum_{i=1}^{r} \operatorname{des}_{\tilde{D}_i}(\sigma_i).$$

Since there are $\binom{n}{n_1,n_2,\ldots,n_r}$ ways to partition $[n]$ into type $(n_1,n_2,\ldots,n_r)$ partitions,

$$A_{\tilde{D}}(t) = \sum_{\sigma\in B_n} t^{\operatorname{des}_{\tilde{D}}(\sigma)} = \sum_{(P_1,P_2,\ldots,P_r)} \prod_{i=1}^{r} \left(\sum_{\sigma_i \in B_{n_i}} t^{\operatorname{des}_{\tilde{D}_i}(\sigma_i)} \right) = \binom{n}{n_1,n_2,\ldots,n_r} \prod_{i=1}^{r} A_{\tilde{D}_i}(t).$$

□

Preservation of the Absolute Evaluation at -1 We provide a proof for Proposition 1.0.3, which states that the absolute evaluation at -1 is the same for bidirected graphs with the same underlying signed graph.

Proof. It suffices to consider the case where $\tilde{E}$ and $\tilde{E}'$ differ by only one oriented edge. Let $\tilde{D}$ and $\tilde{D}'$ be bidirected graphs with their edge sets differing only in the k^{th} edge, $\tilde{E} = \{..,e_k,...\}$ and $\tilde{E}' = \left\{...,e'_k,...\right\}$, where e_k is negated to become e'_k.

We want to see what happens to the parity of the descent number of each signed permutation when negating e_k. We organize the signed permutations into two subsets: the set of all permutations with a descent at e_k and the set of all permutations without a descent at e_k.

Let $\eta = (\pi, \varepsilon)$ be a permutation with a descent at e_k, and $\eta' = (\pi, -\varepsilon)$ (which does not have a descent at e_k). It follows that η does not have a descent at the inverted edge e_k, therefore

$$\mathrm{des}_{\tilde{D}'}(\eta) = \mathrm{des}_{\tilde{D}}(\eta) - 1.$$

Similarly, η' has a descent at the inverted edge e_k and therefore

$$\mathrm{des}_{\tilde{D}'}(\eta') = \mathrm{des}_{\tilde{D}}(\eta') + 1.$$

We see that the parity of a signed-permutation's descent number changes when inverting the single edge: $\mathrm{des}_{\tilde{D}}(\sigma)$ is even if and only if $\mathrm{des}_{\tilde{D}'}(\sigma)$ is odd.

$$
\begin{aligned}
|A_{\tilde{D}}(-1)| &= \left| \left(\sum_{\mathrm{des}_{\tilde{D}}(\sigma)\ odd} (-1) \right) + \left(\sum_{\mathrm{des}_{\tilde{D}}(\sigma)\ even} (1) \right) \right| \\
&= \left| \left(\sum_{\mathrm{des}_{\tilde{D}'}(\sigma)\ even} (1) \right) + \left(\sum_{\mathrm{des}_{\tilde{D}'}(\sigma)\ odd} (-1) \right) \right| \\
&= \left| A_{\tilde{D}'}(-1) \right|
\end{aligned}
$$

Finally, two orientations of a the same signed graph differ by successions of inverting single edges and so$|A_{\tilde{D}}(-1)| = |A_{\tilde{D}'}(-1)|$ holds for all orientations of the same underlying signed graph. □

6.1 Switching at a Vertex

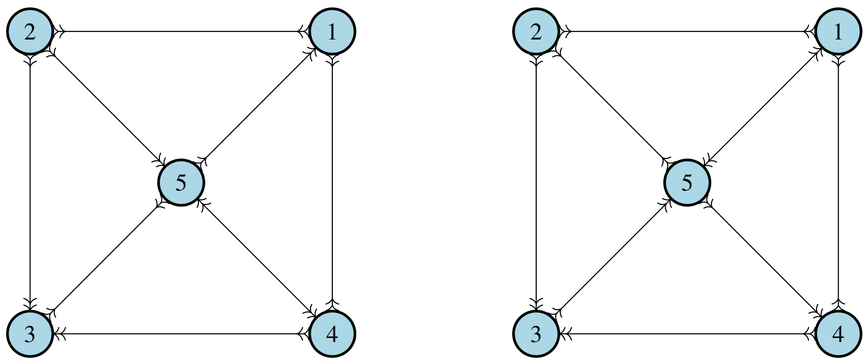
The switching operation is performed on a vertex of a bidirected graph by changing the direction of an edge incident at that vertex. We define switching on a bidirected graph by extending this notion on signed graphs described on pg. 51 of [11]. Consider a bidirected graph (V,E,τ) and let $\nu : V \to \{\pm\}$ be a sign function that identifies the vertices to be switched:

$$
\nu(i) = \begin{cases} - & \text{switch at } i \\ + & \text{do not switch at } i. \end{cases}
$$

Then switching means forming the bidirected graph (V,E,τ^{ν}), where τ^{ν} is defined by

$$
\tau^{\nu}(e,i) = \nu(i)\tau(e,i).
$$

It follows that row i of the incidence matrix is negated by switching. We show in Example 8 and example of switching at a vertex and it's resulting incidence matrix.



(a) Bidirected graph $\tilde{D}$. (b) Bidirected graph $\tilde{D}$.

Figure 6.1: Swtiching at a vertex

Example 8. *A switch is performed on vertex 5 on the graph represented by, shown in Figure 6.1a. The graph after the switch is represented by and shown in Figure 6.1b. The switch changes the sign of all entries in the 5^{th} row.*

$$\Sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\Sigma' = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \end{bmatrix}$$

Preservation under Switching We provide a proof which states that a switch operation at a vertex of a bidirected graph leaves its corresponding Eulerian polynomial preserved.

Proof of Proposition 1.0.4. Let $\tilde{D} = (V, \tilde{E})$ be a bidirected graph with $\{e_\alpha\}$ being the set of all edges of $\tilde{E}$ connected to vertex k. Let $\eta = (\pi, \varepsilon) \in B_n$ and $\eta' = (\pi, (\ldots, -\varepsilon_k, \ldots)) \in B_n$ (η and η' differ only by the k^{th} sign). Using the notation of τ from Eqn 5.1, we write $e_\alpha = (\tau(e_\alpha, \alpha) \cdot \alpha, \tau(e_\alpha, k) \cdot k)$ and note that the condition for e_α being in the descent set $\mathrm{Des}_{\tilde{D}}(\eta)$ is

$$\tau(e_\alpha, \alpha) \cdot \varepsilon_\alpha \cdot \pi_\alpha + \tau(e_\alpha, k) \cdot \varepsilon_k \cdot \pi_k < 0.$$

Now, let $\tilde{D}'$ be $\tilde{D}$ switched at k. Note $\tau'(e_\alpha, k) = -\tau(e_\alpha, k)$, and the condition for e_α being in the descent set $\mathrm{Des}_{\tilde{D}'}(\eta')$ is

$$\tau'(e_\alpha, \alpha) \cdot \varepsilon'_\alpha \cdot \pi'_\alpha + \tau'(e_\alpha, k) \cdot \varepsilon'_k \cdot \pi'_k < 0.$$

But

$$\tau'(e_\alpha, \alpha) \cdot \varepsilon'_\alpha \cdot \pi'_\alpha + \tau'(e_\alpha, k) \cdot \varepsilon'_k \cdot \pi'_k < 0 \iff$$

$$\tau(e_\alpha, \alpha) \cdot \varepsilon'_\alpha \cdot \pi'_\alpha - \tau(e_\alpha, k) \cdot \varepsilon'_k \cdot \pi'_k < 0 \iff$$

$$\tau(e_\alpha, \alpha) \cdot \varepsilon_\alpha \cdot \pi_\alpha - \tau(e_\alpha, k) \cdot (-\varepsilon_k) \cdot \pi_k < 0 \iff$$

$$\tau(e_\alpha, \alpha) \cdot \varepsilon_\alpha \cdot \pi_\alpha + \tau(e_\alpha, k) \cdot \varepsilon_k \cdot \pi_k < 0.$$

This shows us that

$$\mathrm{Des}_{\tilde{D}}(\eta) = \mathrm{Des}_{\tilde{D}'}(\eta'), \tag{6.1}$$

and $t^{\mathrm{des}_{\tilde{D}}(\eta)} = t^{\mathrm{des}_{\tilde{D}'}(\eta')}$. Since we are summing over all signed permutations,

$$A_{\tilde{D}}(t) = \sum_{\eta \in B_n} t^{\mathrm{des}_{\tilde{D}}(\eta)} = \sum_{\eta' \in B_n} t^{\mathrm{des}_{\tilde{D}}(\eta')} = \sum_{\eta' \in B_n} t^{\mathrm{des}_{\tilde{D}'}(\eta')} = A_{\tilde{D}'}(t).$$

□

Chapter 7

An Exponential Generating Function for Hyperoctahedral Eulerian Polynomial

We present a proof of Theorem 1.0.6, the generating function for the classical Eulerian polynomial of the hyperoctahedral group.

Proof of Thm 1.0.6. Recall that the Eulerian polynomial is and defined to be

$$b_n(q) = \sum_{(\pi,\varepsilon)\in B_n} q^{\mathrm{des}(\pi,\varepsilon)}.$$

We use it to expand the identity

$$\frac{b_n(q)}{(1-q)^{n+1}} = \sum_{m\geq 0}(2m+1)^n q^m.$$

This yields the exponential generating function

$$\frac{1}{1-q}\sum_{n\geq 0} b_n(q)\frac{x^n}{n!} = \sum_{n\geq 0}\sum_{m\geq 0}(2m+1)^n(1-q)^n\frac{x^n q^m}{n!}.$$

The right-hand side becomes

$$\sum_{m\geq 0} q^m \sum_{n\geq 0} \frac{[(2m+1)(1-q)x]^n}{n!} = \sum_{m\geq 0} q^m e^{(2m+1)(1-q)x} = \sum_{m\geq 0} q^m e^{2m(1-q)x} e^{(1-q)x}$$
$$= e^{(1-q)x} \sum_{m\geq 0} \left[q \cdot e^{2(1-q)x}\right]^m = e^{(1-q)x} \cdot \frac{1}{1-qe^{2(1-q)x}}.$$

Thus,

$$\sum_{n\geq 0} b_n(q) \frac{x^n}{n!} = \frac{(1-q)e^{(1-q)x}}{1-qe^{2(1-q)x}} = \frac{1-q}{e^{-(1-q)x} - qe^{(1-q)x}}.$$

□

Recall that from Section 5.0.1, the path graph descent set is the same set as the classical type-B descent set. This implies that $B_n(q) = \sum_{(\pi,\varepsilon)\in B_n} q^{\mathrm{des}_{\tilde{P}_n}(\pi,\varepsilon)}$, and therefore, $\frac{1-q}{e^{-(1-q)x}-qe^{(1-q)x}}$ is the exponential generating function for $A_{P_n}(t)$.

Example 9. *We compare Corollary 1.0.8 with $n = 4$ against $A_{\tilde{P}_4}(-1)$ to illustrate a case of the generating function agreeing with the evaluation of the Eulerian polynomial at -1. We have that*

$$A_{\tilde{P}_4}(t) = 1 + 76t + 230t^2 + 76t^3 + t^4,$$

so that $A_{\tilde{P}_4}(-1) = 80$. On the other hand,

$$B_4(-1) = \mathrm{sech}^{(4)}(2x)|_{x=0} = 16\,\mathrm{sech}(2x)(5\,\mathrm{sech}^4(2x) - 18\,\mathrm{sech}^2(2x)\tanh^2(2x) + \tanh^4(2x))|_{x=0} = 80.$$

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