BIVARIATE CHROMATIC POLYNOMIALS OF MIXED GRAPHS

Master's Thesis

handed in by

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Abstract

For an undirected graph $G = (V, E)$, the chromatic polynomial $\chi_G(x)$ counts the number of vertex colourings as a function of the number of colours. Stanley's reciprocity theorem connects the chromatic polynomial with the enumeration of acyclic orientations of the graph G. One way to prove the reciprocity result is via the decomposition of chromatic polynomials as the sum of order polynomials over all acyclic orientations of the graph. Beck, Bogart, and Pham proved the analogue of this reciprocity theorem for the strong chromatic polynomials $\eta_G(x)$ for a mixed graph $G = (V, E, A)$. Dohmen–Pönitz–Tittmann provided a new two-variable generalization of the chromatic polynomial of undirected graphs. We extend this bivariate chromatic polynomial to mixed graphs, provide deletion-contraction formulae and prove a theorem which enumerates the chromatic polynomial of mixed graphs via the decomposition into the sum of bivariate order polynomials. Using this decomposition, we also prove a reciprocity result for the bivariate chromatic polynomials of mixed graphs.

Contents

1. Introduction

This thesis introduces and studies a new two-variable chromatic polynomial for mixed graphs. The aim of this thesis is to provide background on the classical chromatic polynomial and its two generalizations, introduce the new bivariate chromatic polynomial which generalizes the three previously known chromatic polynomials and to provide detailed results of its enumeration and reciprocity property.

In Chapter 2 we provide necessary background on undirected graphs, mixed graphs and study some properties such as deletions and contractions of vertices, edges, and arcs. We also recall basic facts and terminology from the theory of posets and order polynomials. The bicoloured posets and a two-variable order polynomial associated to these structures is also discussed in this section. We also motivate the concept of combinatorial reciprocity in this chapter.

In Chapter 3, we discuss the history of graph colouring problems which led to the development of chromatic polynomials. We provide a detailed background on the classical chromatic polynomial, its order polynomial decomposition through acyclic orientations and Stanley's reciprocity result of chromatic polynomials. In this chapter, we also study an extension of the univariate chromatic polynomials to mixed graph structures. In the next section, the study of generalized chromatic polynomials first introduced by Dohmen-Pönitz-Tittmann [12] is discussed. This completes the background set-up of our work.

Our main results appear in Chapters 4 and 5. In Chapter 4, we introduce a new two-variable chromatic polynomial for mixed graph structures and provide recurrence formulae via deletion-contraction of edges and arcs. We also show its connections with the chromatic polynomials described in Chapter 3.

In Chapter 5, we prove the polynomiality of this newly defined bivariate chromatic colouring function by constructing a decomposition to bivariate order polynomials of the bicoloured posets on the vertices of the graph induced by flat constructions and acyclic orientations. This formula provides a way of enumerating the bivariate chromatic polynomial for any mixed graph and we also obtain previously known decomposition results for chromatic polynomials discussed in Chapter 3 as a corollary. This helps us prove a reciprocity result for the bivariate chromatic polynomials of mixed graphs.

In Chapter 6, we provide some ideas for further continuation of this work in connecting lattice point enumeration to the bivariate colouring. We also suggest an approach to study this polynomial from a discrete geometric perspective via marked posets and marked order polytopes.

2. Graphs and Posets

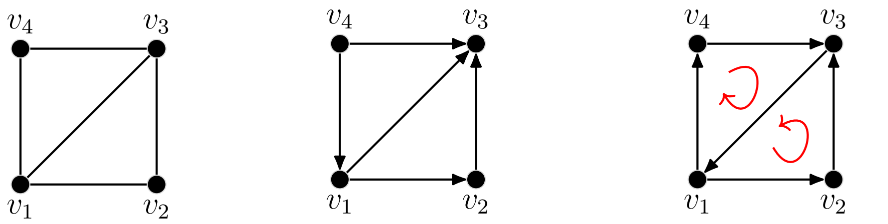
2.1. Graph Theory

A graph is an important structure in discrete mathematics. It consists of a set of objects in which some of the pairs of objects are related. An *undirected graph* G is a triple $G = (V, E, i)$ that consists of a set of *vertices* V, a set of (*undirected*) *edges* E and an incidence map i that associates an unordered pair of not necessarily distinct vertices to each edge of G. For an edge $e \in E$, if $i(e) = \{uv\}$, then we simply denote $e = uv$ and call the vertices u and v the end-vertices or incidence vertices of the edge e and the edge is said to be incident with these vertices. As the end-vertices of the edge are generally clear from the context, we will omit the incidence map and denote an undirected graph G as a pair $G = (V, E)$.

For a vertex v in the graph, $\delta_e(v)$ gives us the set of all edges incident with v. A graph G is said to be *finite* if the number of vertices and edges is finite. An edge with the same end-vertices is called a *loop*. A *simple graph* is a graph without multiple edges between any two vertices and without loops. Unless mentioned otherwise, we will assume that any undirected graph is simple and finite.

A *walk* w of length k is a sequence of k edges of the form $w = v_0v_1, \ldots, v_{k-1}v_k$. A walk with distinct vertices is called a *path* P and we denote it by the sequence of its vertices. A *cycle* is a walk that starts and ends at the same vertex with no vertex repetition.

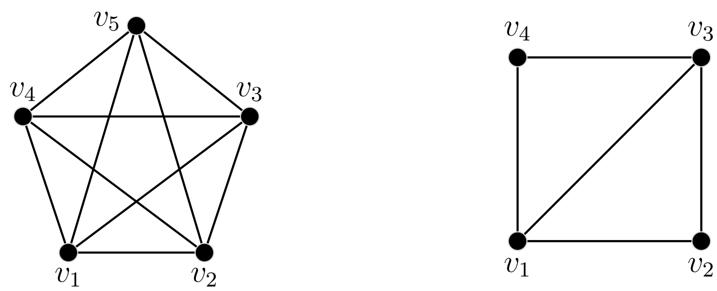
An *orientation* σ of an undirected graph G is an assignment of a direction to each edge of the graph. An oriented edge $e = uv \in E$ is either directed from u to v and denoted by $u \longrightarrow v$ or from v to u and denoted by $v \longrightarrow u$. Such an edge with a direction is called a *directed edge*. A *directed path* in an oriented graph is a sequence of distinct vertices $u_1, \ldots, u_r$ such that $u_i \longrightarrow u_{i+1}$ is a directed edge for each $i = 1, \ldots, r-1$. If in such a directed path there is an edge $u_r u_1$ such that the orientation of this edge is $u_r \longrightarrow u_1$, then this sequence of vertices is a *directed cycle*. An orientation σ of graph G is *acyclic* if it has no directed cycles. Figure 2.1 shows an undirected graph G on four vertices and examples of two of its orientations, an acyclic orientation σ_1 and a cyclic orientation σ_2.



G An orientation σ_1 An orientation σ_2

Figure 2.1.: A graph and its orientations. acyclic and cyclic.

A *complete graph* on n vertices, denoted K_n, is a graph in which every pair of distinct vertices is connected by a unique edge. A *planar graph* is a graph that can be drawn on a plane such that no straight edges of the graph cross each other. Figure 2.2 shows two examples of undirected graphs where 2.2a is a complete graph K_5 on five vertices which is not planar and graph 2.2b is an example of a planar graph.



(a) A complete graph K_5 (b) A planar graph

Figure 2.2.: Examples of undirected graphs.

A *mixed graph* $G = (V, E, A)$ is a graph on the vertex set V consisting of a set E of (undirected) edges and a set A of directed edges called *arcs*. We will denote the edges as $e = uv \in E$, and arcs as $a = \overrightarrow{uv} \in A$ for an arc directed from vertex u to vertex v. For example, Figure 2.3 illustrates a mixed graph $G = (V, E, A)$. For an arc

$a = \overrightarrow{uv}$, the vertex u is called the *tail* and the vertex v the *head* of the arc. Similar to an undirected graph, let $\delta_e(v)$ be the set of all incident edges to vertex v. For arcs in the mixed graph, we define $\delta_a(v) \coloneqq \{a \in A \mid \text{vertex } v \text{ is a tail or head of arc } a\}$ to be a set of arcs incident with the vertex v.

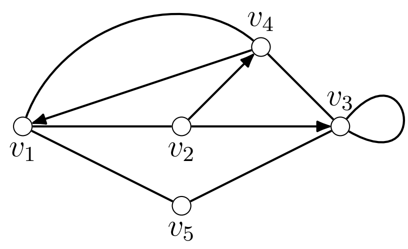


Figure 2.3.: A mixed graph $G = (V, E, A)$.

An orientation of a mixed graph G is obtained by orienting the edges, that is, assigning a direction to each edge $e \in E$. A mixed graph is *acyclic* if none of its possible orientations contains a directed cycle.

An undirected graph is a special case of a mixed graph where $A = \emptyset$, that is, there are no arcs in the graph. A *directed graph* $D = (V, A)$, sometimes called *digraph*, is another special case of a mixed graph in which there are no undirected edges. We can associate an underlying undirected graph G on the same vertex set for each mixed graph $G = (V, E, A)$ by replacing each arc with an edge. Similarly, a digraph can be viewed as an orientation of a mixed graph. Figure 2.4 shows the underlying undirected graph and an orientation of the mixed graph shown in Figure 2.3.

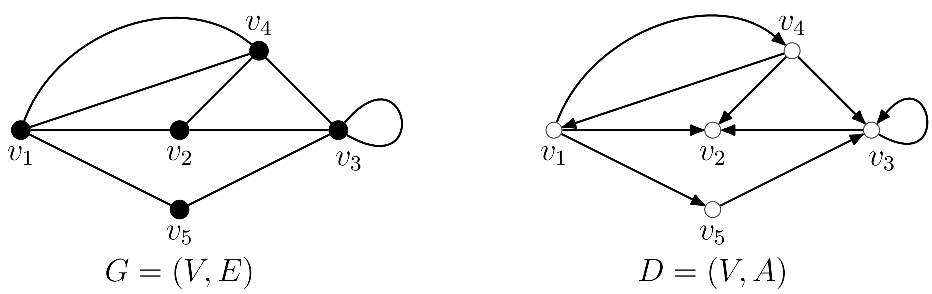


Figure 2.4.: The undirected graph G (left) and the directed graph D (right).

2.1.1. Deletion-Contraction

We introduce the concepts of deletion and contraction for a mixed graph. Let $G = (V, E, A)$ be a mixed graph.

Given an edge $e \in E$, we denote by $G - e$ the *edge deletion*, which is the subgraph of G obtained by removing e. Similarly, we define *arc deletion* $G - a$, where a is an arc. Moreover, if $v \in V$, the *vertex deletion* $G - v$ is the subgraph of G obtained by removing v and all edges and arcs incident to v. Figure 2.5 shows an example of vertex deletion in a mixed graph.

Formally,

- $G - e = (V, E - \{e\}, A)$;
- $G - a = (V, E, A - \{a\})$;
- $G - v = (V - \{v\}, E - \delta_e(v), A - \delta_a(v))$.

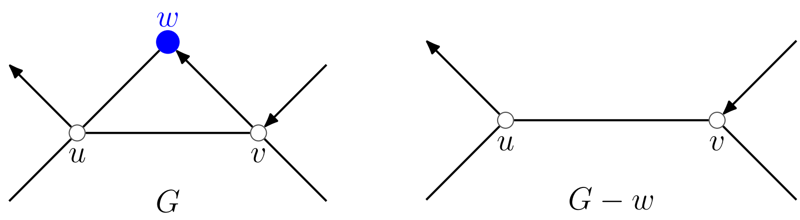


Figure 2.5.: Vertex deletion in a mixed graph.

An *edge contraction* in a graph G for an edge e is obtained removing e and identifying as equal the two vertices sharing this edge. It is denoted by G/e. Let v_e be the identified vertex.

Similarly, an *arc contraction* is the process of identifying the head and tail vertices of an arc. The graph resulting from contraction of an arc $a = \overrightarrow{uv}$ is $G/a = (V', E', A')$, where

$$
\begin{aligned}
V' &= V - \{u, v\} \cup \{v_a\} \\
E' &= \{ab \in E : \{a, b\} \cap \{u, v\} = \emptyset\} \cup \{v_a b \in E : ub \in E \text{ or } vb \in E\} \\
A' &= \left\{\overrightarrow{ab} \in A : \{a, b\} \cap \{u, v\} = \emptyset\right\} \bigcup \left\{\overrightarrow{v_a b} \in A : \overrightarrow{ub} \in A - a \text{ or } \overrightarrow{vb} \in A - a\right\} \\
&\quad \bigcup \left\{\overrightarrow{b v_a} \in A : \overrightarrow{bu} \in A - a \text{ or } \overrightarrow{bv} \in A - a\right\}.
\end{aligned}
$$

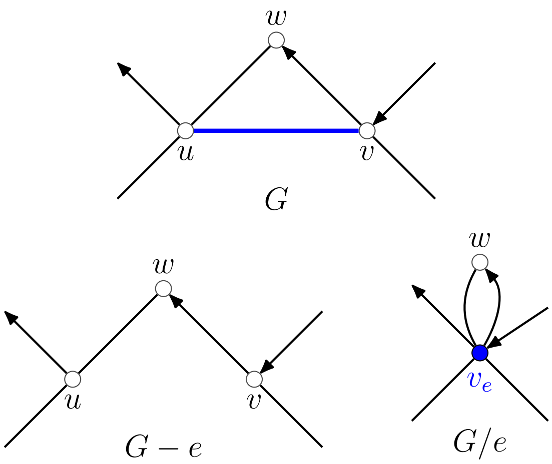


Figure 2.6.: Edge deletion-contraction in a mixed graph.

Figure 2.6 and Figure 2.7 show examples of edge deletion and edge contraction, and arc deletion and arc contraction in a mixed graph, respectively. Note that the contraction operation of edges and arcs can create multiple edges/arcs between two vertices of subgraph.

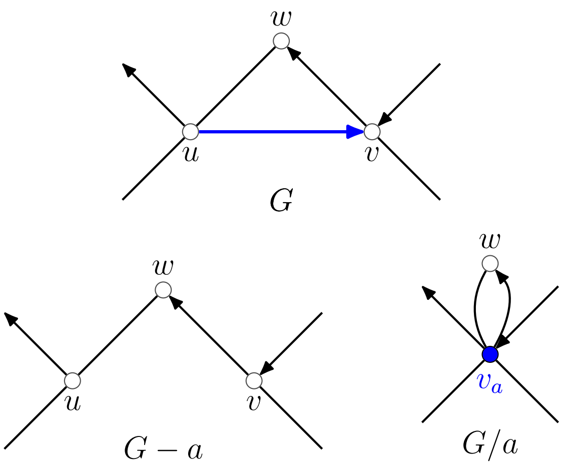


Figure 2.7.: Arc deletion-contraction in a mixed graph.

As seen above, undirected graphs and directed graphs are special cases of mixed graphs with $A = \emptyset$ and $E = \emptyset$ respectively. Hence the vertex, edge, arc deletion and contraction for undirected and directed graphs follow the same definitions as above with appropriate modifications.

2.2. Partially Ordered Sets

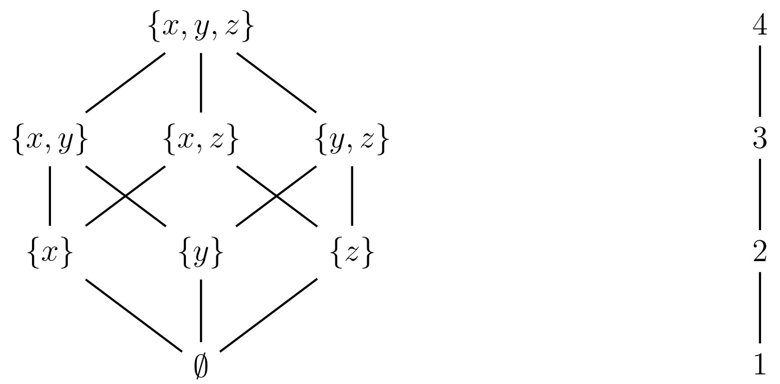
A *partially ordered set*, or simply called a *poset*, is a set P together with a binary relation $\preceq$ that is

- Reflexive: $a \preceq a$,
- Transitive: $a \preceq b, b \preceq c \Longrightarrow a \preceq c$,
- Antisymmetric: $a \preceq b, b \preceq a \Longrightarrow a = b$

for all $a, b, c \in P$. The poset $(P, \preceq)$ is *finite* if the set P is finite. We will consider only finite posets unless mentioned otherwise.

An element $a \in P$ is *covered* by an element b, if $a \preceq b$ and there is no element between a and b. The Hasse diagram of a poset P is a drawing of directed graph of cover relations in P as an (undirected) graph where the node a is drawn lower than the node b if $a \prec b$.

A poset in which each element is comparable to every other element is called a *chain*. That is, the poset $(P, \preceq)$ is a chain if we have either $a \preceq b$ or $b \preceq a$ for any two elements $a, b \in P$. The elements of a chain are *totally* or *linearly ordered*. An *antichain* is a poset in which no two distinct elements are comparable. Figure 2.8 shows examples of a poset and a chain as Hasse diagrams.



(a) A Hasse diagram of the power set of $\{x, y, z\}$ ordered by inclusion

(b) A chain poset on the set $[4] = \{1, 2, 3, 4\}$ by relation $\leq$

Figure 2.8.: Hasse Diagrams of some posets.

2.2.1. Order Polynomials

In 1970, Richard Stanley [24] introduced order polynomials while studying order preserving maps.

Definition 1. For posets P and P', a map $\phi : P \longrightarrow P'$ is (*weakly*) *order preserving* if for all $a, b \in P$

$$a \preceq_P b \Longrightarrow \phi(a) \preceq_{P'} \phi(b)$$

and *strictly order preserving* if

$$a \prec_P b \Longrightarrow \phi(a) \prec_{P'} \phi(b).$$

If the poset P' is a chain of length n, then it is isomorphic to the set $[n] \coloneqq \{1, 2, \dots, n\}$ with the natural order. We define the *order polynomial* as

$$\Omega_P(n) \coloneqq |\{\phi : P \longrightarrow [n] \text{ order preserving}\}|. \tag{2.1}$$

The *strict order polynomial* is defined as

$$\Omega_P^\circ(n) \coloneqq |\{\phi : P \longrightarrow [n] \text{ strictly order preserving}\}|. \tag{2.2}$$

Example 1. Consider a poset $P = [d]$, a chain of d elements.

- For computing $\Omega_P(n)$, we need to count maps ϕ for which the following condition is satisfied :

  $$1 \leq \phi(1) \leq \phi(2) \leq \cdots \leq \phi(d) \leq n.$$

  This gives us $\Omega_P(n) = \binom{n+d-1}{d}$.
- To compute the strict order polynomial, we have to count all possibilities that satisfy

  $$1 \leq \phi(1) < \phi(2) < \cdots < \phi(d) \leq n.$$

  This gives us $\Omega_P^\circ(n) = \binom{n}{d}$.

From the above definition, it is not clear that the functions $\Omega_P(n)$ and $\Omega_P^\circ(n)$ are polynomials but Example 1 provides some hints.

Proposition 2.1 ([7, Proposition 1.3.1])**.** *For a finite poset P, the functions $\Omega_P(n)$ and $\Omega_P^\circ(n)$ agree with polynomials of degree $|P|$ with rational coefficients.*

The idea of the proof of Proposition 2.1 is very similar to our proof of Theorem 3.2. This proposition and its proof also provide more information about the coefficients of the order polynomials. If we use the binomial basis $\left\{\binom{n}{r} : r = 0, 1, 2, \ldots\right\}$ instead of the monomial basis for order polynomials, then we get that the coefficients of these polynomials are integral.

A natural question to ask is how the two order polynomials (2.1 and 2.2) relate to each other. This was answered by Stanley as a reciprocity theorem of order polynomials.

Theorem 2.2 ([24, Theorem 3])**.** *For a finite poset P,*

$$\Omega_P(n) = (-1)^{|P|}\Omega_P^\circ(-n). \tag{2.3}$$

The order polynomials and this reciprocity result are closely related to chromatic polynomials. See Section 3.1.1.

The order polynomials are associated to the Ehrhart polynomials via order polytopes. This approach is useful in the geometric interpretation of combinatorial structures such as posets and the study of graph colourings via lattice point enumeration. This relation is discussed in Appendix A.

2.2.2. Bicoloured Posets

The following section is based entirely on the work done by Beck, Farahmand, Karunaratne and Ruiz in [9], where they introduce the bivariate order polynomial, provide the formulae for enumeration of these polynomials and prove reciprocity results.

Let $(P, \preceq)$ be a finite poset. This poset is called a *bicoloured poset* if P can be viewed as the disjoint union of sets C and S, called the *celeste* and *silver element sets*, respectively. We think of it as assigning one of the two colours to the elements of the poset. Any poset can be thought of as bicoloured poset with no celeste elements, that is, $C = \emptyset$.

Consider the poset illustrated in Figure 2.9. This is a poset on the set $\{v_1, v_2, v_3, v_4, v_5\}$ where the elements $\{v_3, v_4, v_5\}$ are celeste and the remaining are silver elements.

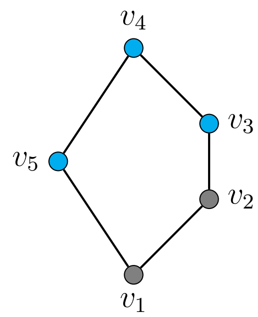


Figure 2.9.: A bicoloured poset.

The colour labelling of the elements of the bicoloured poset is captured in the order preserving maps by introducing another variable. The order preserving maps are extended to the bivariate scenario as follows.

Definition 2. The map $\varphi : P \longrightarrow [x]$ is called an *order preserving* (x, y)*-map* if

$$a \preceq b \implies \varphi(a) \leq \varphi(b) \quad \text{for all } a, b \in P \qquad \text{and} \qquad \varphi(c) \geq y \quad \text{for all } c \in C\,.$$

The function $\Omega_{P,C}(x, y)$ counts the number of order preserving (x, y)-maps.

Definition 3. The map $\varphi : P \longrightarrow [x]$ is a *strictly order preserving* (x, y)*-map* if

$$a \prec b \implies \varphi(a) < \varphi(b) \quad \text{for all } a, b \in P \qquad \text{and} \qquad \varphi(c) > y \quad \text{for all } c \in C\,.$$

The function $\Omega^{\circ}_{P,C}(x, y)$ counts the number of strictly order preserving (x, y)-maps.

It is not obvious that the counting functions $\Omega_{P,C}(x, y)$ and $\Omega^{\circ}_{P,C}(x, y)$ are polynomials but it was proved in [9, Lemma 5] via a decomposition formula in terms of certain permutation statistics for linear extensions of the poset.

We call the functions $\Omega^{\circ}_{P,C}(x, y)$ and $\Omega_{P,C}(x, y)$ *bivariate order polynomial* and *weak bivariate order polynomial*, respectively. From the definition, we observe:

Corollary 2.3. *For a bicoloured poset P,*

$$\Omega^{\circ}_{P,C}(x, 0) = \Omega^{\circ}_{P}(x)$$
$$\Omega_{P,C}(x, 1) = \Omega_{P}(x).$$

This corollary shows that the bivariate order polynomial is a natural extension of the univariate case discussed earlier in Section 2.2.1.

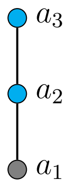


Figure 2.10.: A poset $P = \{a_1, a_2, a_3\}$ with $C = \{a_2, a_3\}$.

Example 2. Consider a poset $P = \{a_1, a_2, a_3\}$ with $a_1 \prec a_2 \prec a_3$ and $C = \{a_2, a_3\}$, as shown in Figure 2.10. To compute the bivariate order polynomial, we want to count all possible strictly order preserving (x, y)–maps $\phi : P \longrightarrow [x]$ such that $\phi(a_2) > y$ and $\phi(a_3) > y$.

The first case

$$1 \le \phi(a_1) \le y < \phi(a_2) < \phi(a_3) \le x$$

gives y choices for $\phi(a_1)$ and $\binom{x-y}{2}$ choices for choosing $\phi(a_2)$ and $\phi(a_3)$, whereas the second case

$$y < \phi(a_1) < \phi(a_2) < \phi(a_3) \le x$$

yields $\binom{x-y}{3}$ choices. Therefore,

$$\begin{aligned}\Omega^{\circ}_{P,C}(x, y) &= y\binom{x-y}{2} + \binom{x-y}{3} \\ &= \frac{1}{6}\left(x^3 - 3x^2 + 2x - 3xy^2 + 3xy - 2y\right).\end{aligned}$$

The bivariate order polynomial and weak bivariate order polynomial are related to each other via reciprocity, where we enumerate the bivariate order polynomial at negative integers for both variables.

Theorem 2.4 ([9, Theorem 1]). *For a bicoloured poset P with the set of celeste elements C,*

$$(-1)^{|P|}\,\Omega^{\circ}_{P,C}(-x, -y) \;=\; \Omega_{P,C}(x, y + 1)\,.$$

This reciprocity result plays an important role in the decomposition result for bivariate chromatic polynomials of undirected graphs discussed further in Section 3.3. It is also used in the reciprocity theorem of bivariate chromatic polynomials of mixed graphs proved in Section 5.2.

2.3. Combinatorial Reciprocity Theorems

A combinatorial reciprocity theorem is a kind of duality relating two combinatorial structures via their counting functions. This was first established by Richard Stanley in 1974 [26]. A combinatorial reciprocity result generally relates the counting function of one combinatorial structure to the counting function of another evaluated at negative integers. A comprehensive study of numerous reciprocity theorems in enumerative geometric combinatorics can be found in a book[1] by Beck and Sanyal [7].

In precise terms, there are two main requirements for univariate reciprocity results of some combinatorial class of objects, say $\mathcal{C}$, where the function $f(n)$ counts the elements of $\mathcal{C}$ of size $n \in \mathbb{N}$. The requirements are:

- the function $f(n)$ is some polynomial when restricted to $\mathbb{Z}_{>0}$;
- the evaluation of the function at negative integer, that is, $f(-n)$ is some integral value up to a sign for all $n \in \mathbb{Z}_{>0}$.

This yields exciting results in combinatorial enumeration for various functions and class of objects. More notably, it connects some classes of objects to discrete geometry through Ehrhart theory. This enables us to study many combinatorial objects using geometry. It also provides an alternative way of proving some results using the Ehrhart-Macdonald reciprocity. Two such functions that are at the centre of the study in this thesis are chromatic polynomials and order polynomials. Not only are they connected via acyclic orientations of graphs, but also are related to each other via lattice-point enumeration and Ehrhart theory of inside-out polytopes.

One of the earlier results of multidimensional reciprocity is by Beck [5] where Stanley's reciprocity result is generalized for a multidimensional version. The requirement for

[1] This book 'Combinatorial Reciprocity Theorems: An Invitation To Enumerative Geometric Combinatorics' has provided a solid background and reference for the reciprocity results in this thesis.

reciprocity results of two-variable counting functions is similar to the univariate scenario—at least for the two-variable counting functions mentioned in this thesis. The two-variable maps mentioned in this thesis are polynomials. These polynomials are evaluated where both variables are negative integers, and the value obtained is the value of a function at non-negative evaluation of both variables, up to a sign with some interesting combinatorial interpretation.

3. A Review of Chromatic Polynomials

3.1. Graph Colourings

We begin with the concept of vertex colouring of an undirected simple graph G.

Definition 4. A *k-colouring* of a graph G is a map $c : V \longrightarrow [k] := \{1, 2, \ldots, k\}$. A k-colouring c is called *proper* if the vertices sharing an edge have distinct colours, that is,

$$c(u) \neq c(v) \qquad \text{whenever} \qquad uv \in E.$$

The fame around the problem of finding proper graph colourings primarily dates back to a question asked in the year 1852 by Francis Guthrie [18]. Like many other long unresolved famous mathematical problems, this problem also has a fairly simple statement.

Theorem 3.1 ([2], [1])**.** *Every planar undirected graph has a proper* 4*-colouring.*

This theorem (earlier called *four colour conjecture*) was proved in 1976 by Kenneth Appel and Wolfgang Haken using computer assisted techniques. There had been many unsuccessful attempts at solving this problem in the past and despite their failure at proving the theorem, some of these attempts led to further advances in various aspects of graph theory. One such instance was the chromatic polynomials which were introduced by Birkhoff in 1912 [11]. Birkhoff introduced the *chromatic polynomial* $\chi_G(k)$ as a counting function of all proper k-colourings of graph G. Formally,

$$\chi_G(k) = |\{c : V \longrightarrow [k] \text{ proper colourings of graph } G\}|. \tag{3.1}$$

The following observation is due to Birkhoff (1912) and Whitney (1932).

Theorem 3.2 ([11], [30]). *For any undirected graph G, the function $\chi_G(k)$ is a polynomial in k.*

In more precise terms, for a loop-less undirected graph $G = (V, E)$, the chromatic function $\chi_G(k)$ agrees with a polynomial of degree $|V|$ with integer coefficients and for graphs containing loops, $\chi_G(k) = 0$. By slight abuse of notation we will call this chromatic function the (*classical*) *chromatic polynomial* of an undirected graph and denote it by $\chi_G(x)$ as a polynomial in the variable x. Some simple examples are computed below.

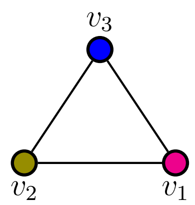


Figure 3.1.: An example of a graph colouring for graph K_3.

Example 3. For a graph $G = (V, E)$ with $V = \{u, v\}$ and $E = \{uv\}$, as shown in Figure 3.3a,

$$\chi_G(x) = x(x-1) = x^2 - x.$$

Example 4. For the graph K_3,

$$\chi_{K_3}(x) = x(x-1)(x-2) = x^3 - 3x^2 + 3x. \tag{3.2}$$

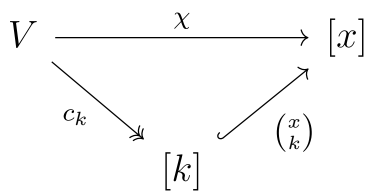
The most commonly discussed proof of the Theorem 3.2 is via deletion contraction formula given by the following remark.

Remark 1 ([29]). $\chi_G(x) = \chi_{G-e}(x) - \chi_{G/e}(x)$.

One elegant way to prove Theorem 3.2 is by factoring a chromatic counting function as a composition of two maps.

Proof of Theorem 3.2. (based on the proof of Proposition 1.3.1 in [7])

Let $G = (V, E)$ be a graph and let $\chi\colon V \longrightarrow [x]$ be the map counting all proper vertex colourings. For $k \leq n$, the map χ factors uniquely into a surjective map of proper vertex colourings using exactly k colours followed by an injective map distributing k colours into x possible colours. Counting these surjective maps gives some finite number c_k. Distributing these k colours into x possible colours is an injective map. This distribution can be done in $\binom{x}{k}$ ways.



This gives us the chromatic counting function as

$$\chi_G(x) = \sum_{k=1}^{|V|} c_k \binom{x}{k}$$

where c_k = the number of proper colourings with exactly k colours.

Hence $\chi_G(x)$ is a polynomial in x with degree at most $|V|$. □

3.1.1. Acyclic Orientations and Graph Colourings

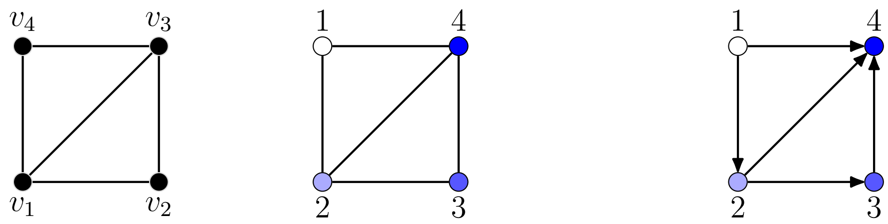
We have order polynomials of posets (Section 2.2.1) and chromatic polynomials of undirected graphs (Theorem 3.2). Stanley gave a connection between these two polynomials via acyclic orientations of an undirected graph [25]. This result by Stanley not only connects these two concepts but also provides a way towards a geometric interpretation of vertex colourings via lattice point enumeration and inside-out polytopes through the reciprocity result of chromatic polynomials[1].

[1]Details in Appendix A

Consider an undirected graph $G = (V, E)$ with $|V| = n$. Let $c : V \longrightarrow [k]$ be a proper colouring of the graph. We define an orientation of edges of the graph associated with this colouring by the colour gradient. For an edge $uv \in E$, we give a direction to the edge such that the direction is increasing in the colour labels of end vertices. That is, for $c(u) < c(v)$, we orient the edge as $u \longrightarrow v$. Since no two vertices connected by an edge have same colour label in proper colourings of the graph, we can have direction assignments for all edges of the graph to obtain an orientation, say ρ. We call this orientation ρ of graph G to be an *orientation induced by proper colouring* c of the graph G.

Remark 2 ([7, Proposition 1.1.3]). For an undirected graph G with proper colouring c, the induced orientation ρ is acyclic.

Figure 3.2 illustrates an example of proper graph colouring and the induced acyclic orientation.



(a) A graph G (b) A proper colouring (c) Induced acyclic orientation

Figure 3.2.: An example of a proper colouring and induced acyclic orientation.

The orientation ρ and a k-colouring c of G are *compatible* if for every directed edge $u \longrightarrow v$ in the oriented graph, we have $c(u) \leq c(v)$. The pair (ρ, c) are *strictly compatible* if for every directed edge $u \longrightarrow v$, we have $c(u) < c(v)$.

We can now state Stanley's combinatorial reciprocity result for chromatic polynomials.

Theorem 3.3 ([25]). *For an undirected graph $G = (V, E)$ and its chromatic polynomial $\chi_G(k)$,*

> $(-1)^{|V|}\chi_G(-k) =$ *the number of compatible pairs (ρ, c) where c is a k-colouring and ρ an acyclic orientation of G,*

$(-1)^{|V|}\chi_G(-1) =$ *the number of acyclic orientations of* G.

This theorem gives a combinatorial interpretation for the evaluation of chromatic polynomial at negative integers. It also connects to the geometric picture provided by Green's correspondence theorem [17] which connects the regions of a certain hyperplane arrangement associated to the graph and its acyclic orientations. It is further connected to the Ehrhart polynomials of inside-out polytopes as shown by Beck and Zaslavasky [8]. [2]

Recall that the proper vertex colourings induce acyclic orientations on the graph G (Remark 2). For any graph G with an acyclic orientation, we can construct a poset induced by the orientation and vice versa.

Construction:

- Poset from oriented graph:
  Let G be the graph with an induced acyclic orientation ρ. The poset P_G is a set of all vertices of the graph with binary relation defined by reachability between two vertices using the directed paths in the graph. That is, for elements $a, b \in P$, we say $a \preceq b$ if there is a directed path from vertex a to vertex b in the graph G with orientation ρ. This relation forms a partial order as it is

  Reflexive: A vertex is reachable from itself,
  Transitive: If there are directed paths from vertex u to v and from vertex v to w, then vertex w can be reached from vertex u,
  Antisymmetry: There is no simultaneous directed path from vertex a to b and from vertex b to a as it forms a directed cycle but the oriented graph is acyclic.

- Oriented graph from a poset:
  The Hasse diagram of a poset can be viewed as a directed graph with direction of edges along the upwards direction of cover relations. This gives us an acyclic oriented graph.

This gives a connection between a poset and acyclic orientations of an undirected graph. Combining these properties, we get Stanley's decomposition result of chromatic polynomials.

[2] Details in Appendix A.

Theorem 3.4 ([25]). *For a graph G,*

$$\chi_G(x) = \sum_{\substack{\text{acyclic} \\ \text{orientations } \rho}} \Omega^\circ_\rho(x).$$

This result for chromatic polynomials provides a new way of counting $\chi_G(x)$ by computing order polynomials of posets arising from acyclic orientations. Since we know from Proposition 2.1 that the function $\Omega^\circ_P(x)$ is a polynomial, this result also shows that $\chi_G(x)$ is indeed a polynomial. The Theorem 3.4 combined with the combinatorial reciprocity approach of both order polynomials and chromatic polynomials forms the basis of the work presented in Chapter 5 where these results are further generalized to the bivariate chromatic polynomials of mixed graphs.

3.2. Chromatic Polynomial of Mixed Graphs

Let $G = (V, E, A)$ be a simple mixed graph. A *proper k-colouring* of its vertices is an assignment of colour labels not exceeding an integer k such that the end-vertices of edges have distinct colours and the tail of any arc has a strictly smaller colour label than its head.

Definition 5. A proper k-colouring of a simple mixed graph $G = (V, E, A)$ is a map $c\colon V \longrightarrow [k]$ such that

- $c(u) \neq c(v)$ for all $e = uv \in E$, and
- $c(u) < c(v)$ for all $a = \overrightarrow{uv} \in A$.

Let $\eta_G(k)$ be the number of possible proper k-colourings of G. It was shown that the function $\eta_G(k)$ is indeed a polynomial in k of degree $|V|$ ([23], [10]). We call the function $\eta_G(k)$ the *strong chromatic polynomial* of a mixed graph G.

Mixed graph colourings have important applications in scheduling problems. One such example is of a job-shop scheduling with processing times subject to disjunctive and precedence constraints. Another example can be scheduling of lectures and exams where some lectures have to happen before exams with additional constraints on exam precedence. There is also a nice analogue between the study of Golomb rulers, its combinatorial reciprocity and mixed graph colourings $\eta_G(k)$ studied by Beck, Bogart and Pham in [6].

Similarly, a weaker version of this mixed graph chromatic polynomial was studied by Beck, Blado, Crawford, Jean-Louise and Young [10] via posets and order polynomials. The term 'weak' is used for the non-strict inequality constraint in the colouring c (where $c(u) \leq c(v)$ for all $a = \overrightarrow{uv} \in A$) of end-vertices of arcs of mixed graphs. Both these polynomials also have important applications to various types of scheduling problems (e.g. [16], [19], [23]).

Example 5. Consider an arc $\overrightarrow{uv}$ as in Figure 3.3b

$$\eta_G(x) = \binom{x}{2} = \frac{1}{2}\left(x^2 - x\right). \tag{3.3}$$

An overview of various chromatic polynomials

Graphs	**Polynomial**		
	Univariate		Bivariate
Undirected	$\chi_G(x)$	$\longrightarrow$	$P(G, x, y)$
	$\downarrow$		$\downarrow$
Mixed	$\eta_G(x)$	$\longrightarrow$	$\chi_G(x, y)$

3.3. Generalized Chromatic Polynomial

In 2003, Dohmen, Pönitz and Tittmann [12] introduced a generalization[3] of the chromatic polynomial by weakening the requirements of proper vertex colourings.

[3] A well known generalization of the chromatic polynomial is by Tutte and it generalizes $\chi_G(x)$ to the Tutte polynomial [28].

Definition 6 ([12])**.** For an undirected simple graph $G = (V, E)$, the generalized chromatic polynomial $P(G, x, y)$ is defined as the counting function of colourings $c : V \longrightarrow [x] = \{1, 2, \ldots, x\}$, $1 \leq y \leq x$, that satisfies for every edge $uv \in E$

$$c(u) \neq c(v) \qquad \text{or} \qquad c(u) > y.$$

This colouring criterion ensures that the end-vertices of an edge have distinct colours if the colour labelling is in the set $\{1, 2, \ldots, y\}$. For $y = x$, we have no feasible colouring with end-vertices of an edge having the same colour label and this polynomial reduces to the classical chromatic polynomial (Equation 3.1).

Corollary 3.5. *$P(G, x, x) = \chi_G(x)$.*

This bivariate chromatic polynomial helps in generalizing the independence polynomial as well as the matching polynomial. It is also closely related to Stanley's chromatic symmetric function. The readers may refer to Dohmen, Pönitz and Tittmann [12] for a detailed study of generalized chromatic polynomial. The polynomiality of the generalized chromatic polynomial is proved by expressing it in terms of classical chromatic polynomials of subgraphs of G. For a simple undirected graph G with $X \subseteq V$ a vertex subset, $G - X$ is the subgraph obtained by removing all vertices of X.

Theorem 3.6 ([12, Theorem 1])**.** *Let G be a graph. Then,*

$$P(G, x, y) = \sum_{X \subseteq V} (x - y)^{|X|} \chi_{G-X}(y).$$

The deletion-contraction formula for generalized chromatic polynomials is given by the following proposition.

Proposition 3.7 ([4], [20])**.** *Let $G = (V, E)$ be a simple undirected graph and $e \in E$ an edge.*

$$P(G, x, y) = P(G - e, x, y) - P(G/e, x, y) + (x - y)P(G/e - v_e, x, y).$$

Proof. ([20]) The number of proper colourings of the graph where the end-vertices of edge $e = uv \in E$ have distinct colour labellings, that is $c(u) \neq c(v)$, is counted by $P(G-e,x,y) - P(G/e,x,y)$. There are exactly $(x-y)P(G/e - v_e, x, y)$ ways of colouring vertices u and v such that they have same colour from $\{y+1,\dots,x\}$ colours. □

(a) Edge graph K_2 (b) Arc graph

Figure 3.3.: Simple graphs on 2 vertices.

Example 6. Consider the graph $G = P_2$ with $V = \{u, v\}, E = \{uv\}$ as in Figure 3.3a. The generalized chromatic polynomial for an undirected edge $e = uv$, is computed as follows:

$$P(P_2, x, y) = x(x-1) + (x-y) = x^2 - y. \tag{3.4}$$

Example 7. Consider the graph $G = K_3$ as shown in Figure 3.1. The generalized chromatic polynomial for K_3 is

$$P(K_3, x, y) = x^3 - 3xy + y. \tag{3.5}$$

A combinatorial reciprocity result for $P(G,x,y)$ is obtained from the reciprocity result of bivariate order polynomials (Theorem 2.4) and a decomposition formula [9, Lemma 8]. For a graph $G = (V,E)$, a *flat* H of the graph is obtained by a series of edge contractions. Let $V(H)$ be the vertices of H with $C(H)$ the set of vertices of H that resulted from contractions in G.

Theorem 3.8 ([9, Theorem 2])**.**

$$P(G,-x,-y) = \sum_{H \text{ flat of } G} (-1)^{|V(H)|} m_H(x,y) \tag{3.6}$$

where $m_H(x,y)$ is the number of pairs (σ, c) consisting of an acyclic orientation σ of H and a compatible colouring $c : V(H) \longrightarrow [x]$ such that $c(v) > y$ if $v \in C(H)$.

Here an orientation σ and a colouring c are compatible if $c(u) \leq c(v)$ for any edge oriented from u to v ($u \longrightarrow v$) in orientation σ of the graph.

4. A New Two-variable Chromatic Polynomial of Mixed Graphs

We introduce a new two-variable chromatic polynomial for a mixed graph.

Definition 7. For a mixed graph $G = (V, E, A)$, the *bivariate chromatic polynomial* $\chi_G(x, y)$ is defined as the counting function of colourings $c : V \longrightarrow [x] = \{1, 2, \ldots, x\}$, $1 \leq y \leq x$ that satisfies

i. for every undirected edge $uv \in E$,

$$c(u) \neq c(v) \text{ or } c(u) > y; \tag{4.1}$$

ii. for every directed arc $\overrightarrow{uv} \in A$,

$$c(u) < c(v) \text{ or } c(u) > y. \tag{4.2}$$

It is not obvious that this counting function of colourings is a polynomial in x and y for a mixed graph G. We prove the polynomiality in Theorem 5.1. It is important to note that while counting the possible colourings, the 'or' in this definition is an exclusive or in the condition 4.2.

For $y = x$, the bivariate chromatic polynomial reduces to the univariate (strong) chromatic polynomial of the mixed graph. If the graph is undirected, that is, for $A = \emptyset$, the polynomial $\chi_G(x, y)$ reduces to the generalized chromatic polynomial (Definition 6) of an undirected graph. This shows that the bivariate chromatic polynomial of mixed graph is a natural generalization of all three chromatic polynomials discussed in Chapter 3.

Corollary 4.1. *For a mixed graph G, $\chi_G(x, x) = \eta_G(x)$.*

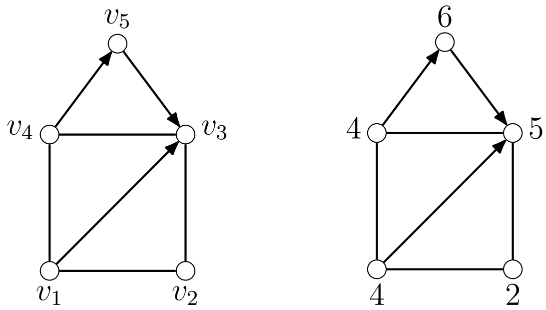


Figure 4.1.: A bivariate colouring of a mixed graph G with $\{1, 2, \ldots, 6\}$ colours with $y = 3$.

Corollary 4.2. *For a mixed graph G with $A = \emptyset$, $\chi_G(x, y) = P(G, x, y)$.*

Corollary 4.3. *For a mixed graph G with $A = \emptyset$, $\chi_G(x, x) = \chi_G(x)$.*

We compute some examples.

Example 8. Consider a graph G with $V = \{u, v\}, E = \{uv\}, A = \emptyset$ as shown in Figure 3.3a. We compute the bivariate chromatic polynomial as

$$\chi_G(x, y) = x^2 - y. \tag{4.3}$$

It is worth noting that this is exactly the bivariate chromatic polynomial of an edge as computed in Example 6. And for $y = x$, it reduces to the classical chromatic polynomial computed in Example 3.

Example 9. Consider the graph shown in Figure 3.3b, with $V = \{u, v\}, E = \emptyset$, $A = \{\overrightarrow{uv}\}$. To count all possible feasible colourings, we have the following cases:

1. $1 \leq c(u) < c(v) \leq x$
   The colour labels for vertices u and v can be chosen in exactly $\binom{x}{2}$ ways.
2. $c(u) > y$
   As we already counted the ways of choosing the colour labels which follow the condition $y < c(u) < c(v)$ in case 1, we need to consider the colouring possibilities satisfying the conditions $c(u) \leq c(v)$ and $c(u) > y$. This can be counted as follows:

a) $1 \leq c(v) \leq y < c(u) \leq x$
There are exactly $y(x-y)$ ways of choosing the colours.

b) $y < c(v) \leq c(u) \leq x$
The number of such possible colourings is given by $\binom{x-y+1}{2}$.

We get the bivariate chromatic polynomial

$$\chi_G(x,y) = \binom{x}{2} + y(x-y) + \binom{x-y+1}{2} = \frac{1}{2}\left(2x^2 - xy - y\right). \tag{4.4}$$

If we substitute $y = x$ in Equation (4.4), we get $\chi_G(x,x) = \frac{1}{2}\left(x^2 - x\right)$. This is the univariate chromatic polynomial of mixed graph as computed in Example 5.

4.1. Deletion-Contraction Computations

We provide deletion-contraction formulae for the bivariate chromatic polynomial. Let $G = (V, E, A)$ be a mixed graph. Let v_e denote the vertex obtained after the contraction of edge e.

Proposition 4.4. *If G is a mixed graph and $e \in E$ is an edge, then*

$$\chi_G(x,y) = \chi_{G-e}(x,y) - \chi_{G/e}(x,y) + (x-y)\chi_{(G/e)-v_e}(x,y).$$

Proof. As in the univariate case, $\chi_{G-e}(x,y) - \chi_{G/e}(x,y)$ gives the number of proper colourings of the graph G with $c(u) \neq c(v)$ for $uv = e \in E$. There are exactly $(x-y)\chi_{(G/e)-v_e}(x,y)$ ways of colouring vertices u and v such that they have same colour from $\{y+1, \dots, x\}$ colours. □

We provide a main theorem of this section, which gives an arc deletion-contraction computation for the bivariate chromatic polynomial of a mixed graph. It is important to note that this theorem does not provide the polynomiality of the function $\chi_G(x,y)$. For an arc $a = \overrightarrow{uv} \in A$, let $G_a := (V, E, A - \{\overrightarrow{uv}\} \cup \{\overrightarrow{vu}\})$, that is, the graph obtained by reversing the direction of arc a. Figure 4.2 illustrates an example. For an arc contraction at a, let v_a denote the vertex obtained after the contraction.

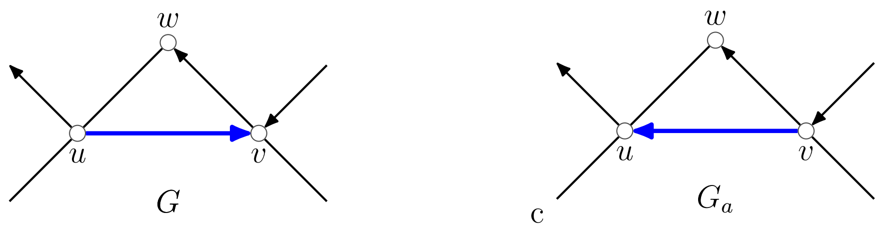


Figure 4.2.: The graph G and G_a for $a = \overrightarrow{uv}$.

Theorem 4.5. *If G is a mixed graph and $a \in A$ is an arc, then*

$$\chi_G(x,y) + \chi_{G_a}(x,y) = \chi_{G-a}(x,y) - \chi_{G/a}(x,y) + (x-y)(1-x+y)\chi_{(G/a)-v_a}(x,y) + (x-y)\left[\chi_{G-a-v}(x,y) + \chi_{G-a-u}(x,y)\right]. \tag{4.5}$$

Proof. Let $a = \overrightarrow{uv}$. Let C be the set of all bivariate colourings of G and let C_a be the set of all bivariate colourings of mixed graph G_a. By inclusion-exclusion,

$$|C| + |C_a| = |C \cup C_a| + |C \cap C_a|.$$

This implies
$$\chi_G(x,y) + \chi_{G_a}(x,y) = |C \cup C_a| + |C \cap C_a|. \tag{4.6}$$

For a colouring $c \in C \cup C_a$, we count the number of ways the following colouring conditions are satisfied: $c(u) < c(v)$ or $c(v) < c(u)$ or $c(u) > y$ or $c(v) > y$. This means, we have to count the number of ways of colouring vertices u and v such that they can have any colour labels from the set $\{1, 2, \dots, x\}$ except that the vertices can not have equal colours with labels in the set $\{1, 2, \dots, y\}$. This is exactly counted by $\chi_{G-a}(x,y) - \chi_{G/a}(x,y) + (x-y)\chi_{(G/a)-v_a}(x,y)$. This implies that

$$|C \cup C_a| = \chi_{G-a}(x,y) - \chi_{G/a}(x,y) + (x-y)\chi_{(G/a)-v_a}(x,y). \tag{4.7}$$

A colouring $c \in C \cap C_a$ if and only if the following conditions mentioned above are satisfied.

Case 1: $c(u) < c(v)$ and $c(u) > c(v)$
There does not exist a feasible colouring in $C \cap C_a$ that satisfies these conditions simultaneously.

Case 2: $y < c(v)$ with $c(u) \leq c(v)$ and $y < c(u)$ with $c(v) \leq c(u)$
This implies that we want to count the feasible colourings with the colouring condition that satisfies $y < c(u) = c(v)$. This is counted in exactly $(x-y)\chi_{(G/a)-v_a}(x,y)$ ways.

Case 3: $c(u) < c(v)$ and $y < c(v)$ with $c(u) \leq c(v)$

This implies that the colouring c must satisfy $y < c(v)$ with $c(u) < c(v)$. There are two possibilities:

- $y < c(u) < c(v) \leq x$
  The colours for vertices u, v can be chosen in $\binom{x-y}{2}$ ways. Thus the number of possible graph colourings are counted by $\binom{x-y}{2}\chi_{(G/a)-v_a}(x, y)$.
- $1 \leq c(u) \leq y < c(v) \leq x$
  There are $(x - y)$ ways to colour vertex v. To colour vertex u, the condition $1 \leq c(u) \leq y$ needs to be satisfied. This is equivalent to counting colourings where $c(u) \leq x$ and removing the possible colourings with $c(u) > y$.
  This gives us the total number of possible graph colourings as
  $(x - y)\left(\chi_{G-a-u}(x, y) - (x - y)\chi_{(G/a)-v_a}(x, y)\right)$.

Hence there are exactly
$\binom{x-y}{2}\chi_{(G/a)-v_a}(x, y) + (x - y)\left(\chi_{G-a-u}(x, y) - (x - y)\chi_{(G/a)-v_a}(x, y)\right)$ ways.

Case 4: $c(v) < c(u)$ and $y < c(u)$ with $c(v) \leq c(u)$

This implies that the colouring c must satisfy $y < c(u)$ with $c(v) < c(u)$. There are two possibilities:

- $y < c(v) < c(u) \leq x$
  The colour labels for vertices u and v can be chosen in $\binom{x-y}{2}$ ways. Thus the possible graph colourings are counted by $\binom{x-y}{2}\chi_{(G/a)-v_a}(x, y)$.
- $1 \leq c(v) \leq y < c(u) \leq x$
  There are $(x - y)$ ways to colour the vertex u. For colouring of vertex v, the condition $1 \leq c(v) \leq y$ needs to be satisfied. This is equivalent to counting colourings where $c(v) \leq x$ and removing the possible colourings with $c(v) > y$.
  This gives us the total number of possible graph colourings as
  $(x - y)\left(\chi_{G-a-v}(x, y) - (x - y)\chi_{(G/a)-v_a}(x, y)\right)$

Hence, there are exactly

$$\binom{x-y}{2}\chi_{(G/a)-v_a}(x, y) + (x - y)\left(\chi_{G-a-v}(x, y) - (x - y)\chi_{(G/a)-v_a}(x, y)\right)$$

ways.

Thus,

$$\begin{aligned}|C \cap C_a| = (x-y)\chi_{(G/a)-v_a}(x,y) + 2\binom{x-y}{2}\chi_{(G/a)-v_a}(x,y) \\ + (x-y)\left[\chi_{G-a-v}(x,y) - (x-y)\chi_{G/a-v}(x,y)\right] \\ + (x-y)\left[\chi_{G-a-u}(x,y) - (x-y)\chi_{G/a-u}(x,y)\right].\end{aligned} \tag{4.8}$$

From Equations (4.6), (4.7) and (4.8) we get

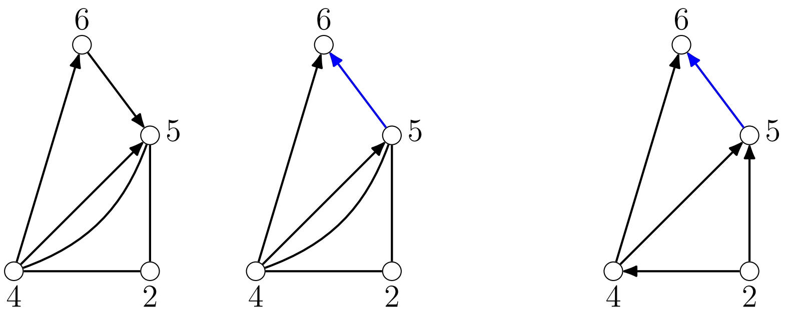
$$\begin{aligned}\chi_G(x,y) + \chi_{G_a}(x,y) &= \chi_{G-a}(x,y) - \chi_{G/a}(x,y) + 2(x-y)\chi_{(G/a)-v_a}(x,y) \\ &\quad + 2\binom{x-y}{2}\chi_{(G/a)-v_a}(x,y) \\ &\quad + (x-y)\left[\chi_{G-a-v}(x,y) - (x-y)\chi_{(G/a)-v_a}(x,y)\right] \\ &\quad + (x-y)\left[\chi_{G-a-u}(x,y) - (x-y)\chi_{(G/a)-v_a}(x,y)\right] \\ &= \chi_{G-a}(x,y) - \chi_{G/a}(x,y) + (x-y)(1-x+y)\chi_{(G/a)-v_a}(x,y) \\ &\quad + (x-y)\left[\chi_{G-a-v}(x,y) + \chi_{G-a-u}(x,y)\right].\end{aligned}$$

□

5. Order Polynomial Decomposition

For some subset of arcs B of a mixed graph G, the *redirection of arcs in G along the set B* is denoted by ${}_BG$, that is, a graph obtained by reversing the directions of the arcs in B.

For a mixed graph $G = (V, E, A)$, we recall that a *flat* of G is a mixed graph H that can be constructed from G by a series of contractions of edges and arcs. We denote the vertices, edges and arcs of the flat H by $V(H), E(H)$ and $A(H)$, respectively. The subset of vertices of H that results from contractions of G is denoted by $C(H)$. For some subset $B \subseteq A(H)$, the vertex set of tails of arcs in the set B is denoted by $C_B(H)$. The set $O(H, B)$ is the set of all acyclic orientations of the flat ${}_BH$.



(a) H (b) ${}_BH$ (c) An acyclic orientation of ${}_BH$

Figure 5.1.: A flat H of a graph G from Example 4.1 and ${}_BH$.

5.1. The Decomposition Result

In this section, we provide a formula for computing the bivariate chromatic colouring function by constructing a decomposition to bivariate order polynomials of the bicoloured posets on the vertices of the graph induced by flat constructions and acyclic orientations.

Theorem 5.1. *For a mixed graph G,*

$$\chi_G(x,y) = \sum_{H \text{ flat of } G} \sum_{B \subseteq A(H)} \sum_{\sigma \in O(H,B)} \Omega^{\circ}_{\sigma,\, C_B(H) \cup C(H)}(x,y).$$

Proof. Let $c : V \longrightarrow [x]$ be a colouring of the mixed graph G that satisfies the colouring conditions stated in (4.1) and (4.2). Note that the colours of the end-points of edges and arcs can be equal only if the colour labels are $> y$.

Let H be the flat of G obtained by contracting all edges and arcs whose end-points have the same colour. Thus the colours of vertices in $C(H)$ have colours $> y$. Let B be the subset of those arcs in H for which the colour gradient of end-vertices is decreasing along the direction of an arc, i.e., $B := \{a = \overrightarrow{uv} \in A(H) \mid c(u) > c(v)\}$. Hence, the colour labels of vertices in $C_B(H)$ are $> y$. Now consider the flat ${}_BH$ obtained by reversing the directions of the arcs in B. Here the direction of each arc follows the increase in colour gradient on end-vertices. We orient the edges of ${}_BH$ along the colour gradient, i.e., for the edge vw, we introduce the orientation $v \to w$ if and only if $c(v) < c(w)$.

No two vertices in the flat that are connected by an arc or a directed edge have identical colour labels. As the direction of arcs and edges follows strictly increasing colour gradients, the orientation is acyclic: Suppose $u \to v_1 \to v_2 \to \cdots \to v_k \to u$ was a directed cycle, then we would obtain a contradiction as $c(u) \not< c(u)$.

Regard this acyclic orientation as a binary relation $\preceq$ on $V(H)$ defined by $u \preceq v$ if $u \to v$. This gives us a bicoloured poset P where the vertices in $C_B(H) \cup C(H)$ are the celeste elements (with colour labelling $> y$). The colouring c is an order preserving (x,y)–map on the bicoloured poset P. The bivariate order polynomial $\Omega^{\circ}_{\sigma,\, C_B(H) \cup C(H)}(x,y)$ counts all such order preserving maps.

Conversely, given a flat H of G, a subset of arcs $B \subseteq A(H)$ and an acyclic orientation σ in $O(H, B)$, a strictly order preserving (x, y)–map counted by $\Omega^{\circ}_{\sigma, C_B(H) \cup C(H)}(x, y)$ can be extended to a colouring of G as follows. All the vertices of ${}_BH$ get colours such that the colour gradient follows the direction of edges and arcs given by the orientation σ (that is, the colour gradient increases in the direction of edges of ${}_BH$ and along the direction opposite to the arc direction of H for arcs in set B). The celeste elements of the bicoloured poset induced by orientation σ is given by the set $C_B(H) \cup C(H)$. Hence the vertices in set $C_B(H)$ get colour labels $> y$. The colouring is then extended to graph G such that the vertices of the graph G that result in contractions to form the flat H get equal colours $> y$. This gives a colouring of the mixed graph G.

Consider two distinct colourings c_1 and c_2 of G. We need to show that the corresponding order preserving maps, say ϕ_1 and ϕ_2, are also distinct.

Construct the flats H_1 and H_2 of the graph by contracting those edges and arcs that have end-vertices with equal colour labels with respect to colourings c_1 and c_2 respectively.

If $H_1 \neq H_2$, then the posets on the vertices of the flats will be different for each colouring. This will give us distinct order preserving (x, y)–maps.

Suppose H_1 equals H_2, that is, both the flats are identical, then define the sets $B_i = \{a = \overrightarrow{uv} \in A(H) \mid c_i(u) > c_i(v)\}$ for $i = 1, 2$. If these sets are distinct, that is, $B_1 \neq B_2$, then the vertex ordering in the corresponding posets will be distinct. This will give distinct order preserving (x, y)–maps for the colourings.

Now consider the case where $B_1 = B_2$, then the graphs ${}_{B_1}H_1$ and ${}_{B_2}H_2$ are identical. Orient the edges in these flats along the increase in colour gradient, i.e., for edge vw, we have the orientation $v \longrightarrow w$ if and only if $c_i(v) < c_i(w)$ for $i = 1, 2$ respectively. If there is some edge e in the flat H_i $(i = 1, 2)$, for which the direction in ${}_{B_2}H_2$ after orientation is reverse to the direction in ${}_{B_2}H_2$, then we have a different ordering on the vertices resulting in two distinct posets. As the order preserving (x, y)–maps are on these posets, the maps will be distinct. If the orientation for all edges is identical in ${}_{B_i}H_i$ for both $i = 1, 2$, then we have $H_1 = H_2, B_1 = B_2$ and orientation which gives us a bicoloured poset σ with celeste elements $C_{B_1}(H_1) \cup C(H_1)$. The strict bivariate order polynomial $\Omega^{\circ}_{\sigma, C_B(H) \cup C(H)}(x, y)$ counts all possible order preserving (x, y)-maps on this bicoloured poset exactly once. This completes the proof. $\square$

Since the bivariate order polynomial $\Omega^{\circ}_{\sigma, C_B(H)\cup C(H)}(x, y)$ is a polynomial and sum of polynomials is a polynomial, the Theorem 5.1 also proves that the function $\chi_G(x, y)$ is a polynomial in variables x and y for any mixed graph G.

Undirected graphs are mixed graphs where $A = \emptyset$. As a special case of Theorem 5.1, the following corollary recovers the decomposition result for bivariate chromatic polynomials of undirected graph first stated and proved in [9].

Corollary 5.2 ([9, Lemma 8])**.** *For an undirected graph $G = (V, E)$,*

$$\chi_G(x, y) = \sum_{H \text{ flat of } G} \sum_{\substack{\sigma \text{ acyclic} \\ \text{orientation of } H}} \Omega^{\circ}_{\sigma, C(H)}(x, y).$$

For a flat H of a directed graph G with $B \subseteq A(H)$, let $\sigma(H, B)$ be the poset on the vertex set of ${}_BH$.

Corollary 5.3. *For a directed graph G,*

$$\chi_G(x, y) = \sum_{H \text{ flat of } G} \sum_{B \subseteq A(H)} \Omega^{\circ}_{\sigma(H,B), C_B(H)\cup C(H)}(x, y).$$

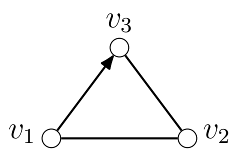


Figure 5.2.: A mixed graph on 3 vertices.

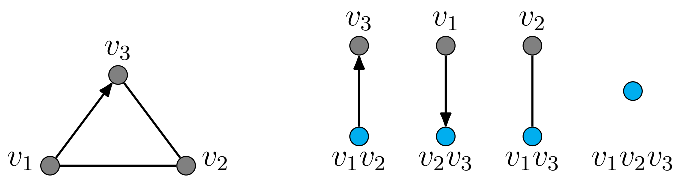


Figure 5.3.: Flats H_i of G.

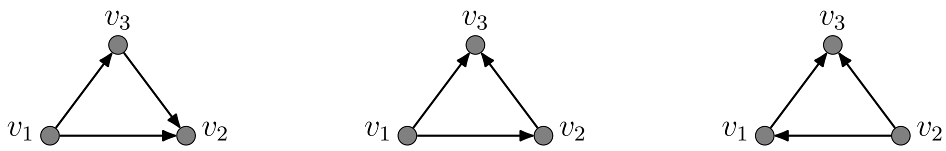


Figure 5.4.: Acyclic orientations of ${}_{\emptyset}H_1$.

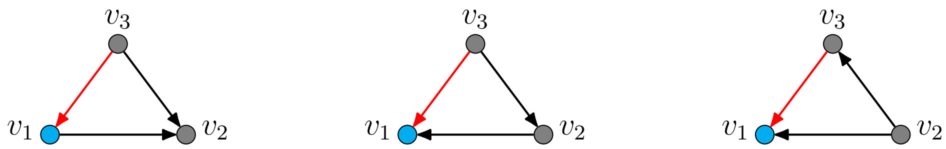


Figure 5.5.: Acyclic orientations of ${}_{B}H_1$ for $B = \{v_1v_3\}$.

Example 10. Consider the mixed graph G shown in Figure 5.2. The bivariate chromatic polynomial of G is computed using the flats and acyclic orientations constructed in the proof of Theorem 5.1 as shown in Figure 5.3.

As a next step, we look at each flat H_i $(i = 1, 2, 3)$ and each subset $B \in A(H_i)$ and then compute the bivariate order polynomials for each feasible acyclic orientation of ${}_{B}H_i$.

For $H_1 = G$, first consider $B = \emptyset$. In this case, we get in total 3 acyclic orientations of ${}_{\emptyset}H_1$ as shown in Figure 5.4. For each acyclic orientation, we get the bivariate order polynomial

$$\Omega^{\circ}_{\sigma,\emptyset}(x, y) = \binom{x}{3}.$$

Now we consider the flat ${}_{B}H_1$ where $B = \{\overrightarrow{v_1v_3}\}$. There are three acyclic orientations as shown in Figure 5.5 with vertex v_1 as celeste element. The corresponding bivariate order polynomial summed over all three acyclic orientations is

$$\sum_{\sigma \in O(H,B)} \Omega^{\circ}_{\sigma,\,\{v_1\}}(x, y) = 2(x - y)\binom{y}{2} + 3y\binom{x - y}{2} + 3\binom{x - y}{3}.$$

Now we consider the flats of G obtained by contraction of an edge or an arc in the graph as shown in Figure 5.6. We compute the bivariate order polynomial for each poset.

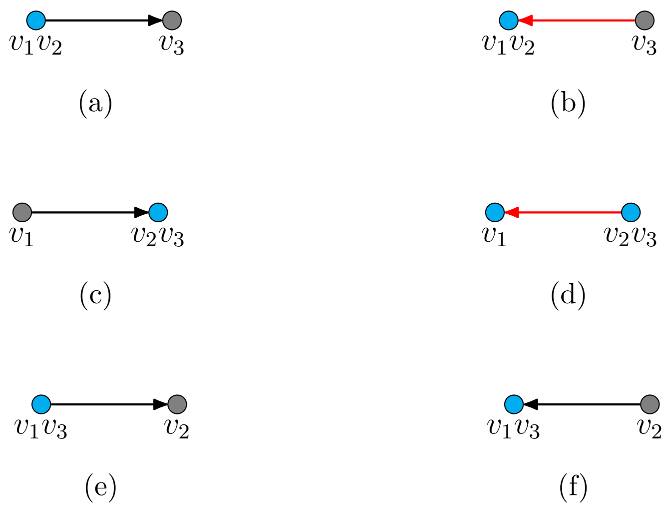


Figure 5.6.: Flats H_2 of G.

a) For the flat shown in Figure 5.6a, the bivariate order polynomial is

$$\Omega^{\circ}_{\sigma_1,\{v_1v_2\}}(x,y) = \binom{x-y}{2}.$$

b) For the flat shown in Figure 5.6b, the bivariate order polynomial is

$$\Omega^{\circ}_{\sigma_2,\{v_1v_2\}}(x,y) = y(x-y) + \binom{x-y}{2}.$$

c) For the flat shown in Figure 5.6c, the bivariate order polynomial is

$$\Omega^{\circ}_{\sigma_1,\{v_2v_3\}}(x,y) = y(x-y) + \binom{x-y}{2}.$$

d) For the flat shown in Figure 5.6d, the bivariate order polynomial is

$$\Omega^{\circ}_{\sigma_2,\{v_2v_3\}\cup\{v_1\}}(x,y) = \binom{x-y}{2}.$$

e) For the flat shown in Figure 5.6e, the bivariate order polynomial is

$$\Omega^{\circ}_{\sigma_1,\{v_1v_3\}}(x,y) = \binom{x-y}{2}.$$

f) For the flat shown in Figure 5.6f, the bivariate order polynomial is

$$\Omega^{\circ}_{\sigma_2,\{v_1v_3\}}(x,y) = y(x-y) + \binom{x-y}{2}.$$

For the flat H_3 of G where all edges and arcs are contracted, the bivariate order polynomial is $\Omega^{\circ}_{\sigma,\{v_1v_2v_3\}} = (x-y)$.

Hence,

$$\begin{aligned}\chi_G(x,y) &= 3\binom{x}{3} + 2(x-y)\binom{y}{2} + 3y\binom{x-y}{2} + 3\binom{x-y}{3} + 3y(x-y) + 6\binom{x-y}{2} \\ &= 2x^3 - 3x^2 - 2xy^2 - xy + x + y + y^2 + y^3.\end{aligned}$$

5.1.1. Alternative Formulation

For a mixed graph G, let G^u denote the underlying undirected graph.

Theorem 5.4. *For a mixed graph G,*

$$\chi_G(x,y) = \sum_{H \text{ flat of } G} \; \sum_{\substack{\sigma' \text{ acyclic} \\ \text{orientation of } H^u}} \Omega^{\circ}_{\sigma',\, C(H)\cup T(\sigma')}(x,y).$$

Proof. Let $c : V \longrightarrow [x]$ be a colouring of the mixed graph G that satisfies the colouring conditions stated in (4.1) and (4.2). Note that the colours of the end-points of edges and arcs can be equal only if their colour labels are $> y$. Let H be a flat of G obtained by contracting all edges and arcs whose end-points have the same colour. Thus the colours of vertices in $C(H)$ have colour labels $> y$. Consider H^u, the underlying undirected graph of flat H. We orient the edges of H^u along the colour gradient, that is, for the edge uv, we introduce the orientation $u \longrightarrow v$ if and only if $c(u) < c(v)$. Let σ' be such an orientation. No two vertices in H^u that are connected by an edge have identical colour labels. This gives us that the orientation σ' is acyclic. Let $T(\sigma')$ be the set of vertices defined as $T(\sigma') := \{v \in V(H) \mid \overrightarrow{vw} \in A(H) \text{ and } v \longleftarrow w \text{ in } \sigma' \text{ of } H^u\}$. As the colour gradient is decreasing along the direction of the arcs for arcs with tail vertices in the set $T(\sigma')$, we have $c(u) > y$ for each $u \in T(\sigma')$ from the colouring constraints. Now we regard the acyclic orientation σ' as a binary relation on the set $V(H^u)$ defined by $u \preceq v$ if $u \longrightarrow v$. This gives us a bicoloured poset P where the vertices in the set $C(H) \cup T(\sigma')$ are celeste elements. The colouring c is an order preserving (x,y)–map on the bicoloured poset P. The bivariate order polynomial $\Omega^{\circ}_{\sigma',\, C(H)\cup T(\sigma')}(x,y)$ counts all such order preserving maps.

Conversely, given a flat H of G, an acyclic orientation σ' of H^u and a set $T(\sigma')$, an order preserving (x, y)–map counted by $\Omega^{\circ}_{\sigma', C(H)\cup T(\sigma')}(x, y)$ can be extended to a colouring of G as follows. All the vertices of H^u (and in turn of H) get colours such that the colour gradient follows the acyclic orientation σ' (that is, the colour gradient increases in the direction of orientation of the edges of H^u). The celeste elements of the bicoloured poset induced by the orientation σ' is given by the set $C(H) \cup T(\sigma')$. Hence the vertices in the set $T(\sigma')$ get colours $> y$. The colouring is then extended to the graph G such that the vertices of the graph G that result in contractions to form the flat H get equal colours $> y$. This gives a colouring of the mixed graph G.

Consider two distinct colourings c_1 and c_2 of G. We need to show that the corresponding order preserving maps, say ϕ_1 and ϕ_2, are distinct. Construct the flats H_1 and H_2 of the graph by contracting those edges and arcs that have end-vertices with equal colour labels with respect to colourings c_1 and c_2 respectively. If $H_1 \neq H_2$, then the posets on the vertices of the underlying undirected graphs H_1^u and H_2^u will be different for each colouring. This will give us distinct order preserving $(x, y)-$ maps.

Suppose $H_1 = H_2$, that is, both flats are identical, then the underlying undirected graphs H_1^u and H_2^u will also be identical. Let σ_1' and σ_2' be corresponding acyclic orientations of H_1^u and H_2^u respectively.

Now define $T_i(\sigma_i') \coloneqq \{v \in V(H_i) \mid \overrightarrow{vw} \in A(H_i) \text{ and } v \longleftarrow w \text{ in } \sigma_i' \text{ of } H_i^u\}$ for $i = 1, 2$. If these sets are distinct, that is, if $T_1 \neq T_2$, then the celeste elements in the corresponding bicoloured posets will be distinct resulting in different vertex orderings which will give distinct order preserving (x, y)–maps for corresponding colourings.

If for the vertex sets, $T_1(\sigma_1') = T_2(\sigma_2')$ but the acyclic orientations σ_1' and σ_2' are distinct, then the posets induced by these acyclic orientations will be distinct resulting in distinct order preserving (x, y)–maps for corresponding graph colourings.

If the flats, the acyclic orientation and the celeste sets are identical, then the bicoloured posets corresponding to both colourings are the same. The bivariate order polynomial $\Omega^{\circ}_{\sigma', C(H)\cup T(\sigma')}(x, y)$ counts all possible order preserving (x, y)–maps on this bicoloured poset exactly once. Hence we are done. □

For an undirected graph G, we get that $H^u = H$ and $T(\sigma') = \emptyset$ and hence Corollary 5.2 also follows from Theorem 5.4.

5.2. Reciprocity Result

We now prove a reciprocity theorem for the bivariate chromatic polynomials of mixed graphs in which we evaluate this polynomial where both the variables are negative integers.

The orientation σ and the colouring $c : V \longrightarrow [x]$ of the mixed graph G that satisfies the colouring conditions stated in (4.1) and (4.2), are said to be *compatible* if $c(u) \leq c(v)$ for any edge/arc directed from u to v in the orientation σ.

For a flat H of a mixed graph G, let H^u be the underlying undirected graph with some acyclic orientation σ'. Let $T(\sigma')$ be the set of all tail vertices of arcs of H for which the orientation of an edge in σ' is opposite to the direction of the corresponding arc in H.

Theorem 5.5. *For a mixed graph G,*

$$\chi_G(-x,-y) = \sum_{H \text{ flat of } G} (-1)^{|V(H)|} \; m_{H^u}(x,y).$$

where $m_{H^u}(x,y)$ counts the number of compatible pairs (σ', c) consisting of acyclic orientations σ' of H^u and compatible colourings c with $c(v) > y$ if $v \in C(H) \cup T$.

Proof. By the reciprocity result of bivariate order polynomials (Theorem 2.4),

$$\begin{aligned}
\chi_G(-x,-y) &= \sum_{H \text{ flat of } G} \;\; \sum_{\substack{\sigma' \text{ acyclic} \\ \text{orientation of } H^u}} (-1)^{|V(H)|} \Omega_{\sigma', C(H)\cup T(\sigma')}(x, y+1) \\
&= \sum_{H \text{ flat of } G} (-1)^{|V(H)|} \sum_{\substack{\sigma' \text{ acyclic} \\ \text{orientation of } H^u}} \Omega_{\sigma', C(H)\cup T(\sigma')}(x, y+1) \\
&= \sum_{H \text{ flat of } G} (-1)^{|V(H)|} \; m_{H^u}(x,y).
\end{aligned}$$

Here, $\Omega_{\sigma', C(H)\cup T(\sigma')}(x, y+1)$ counts the number of order preserving maps $\varphi : \sigma' \longrightarrow [x]$ subject to the following conditions:

- For $u \in C(H) \cup T(\sigma')$, we have $\varphi(u) \geq y+1$;
- The map φ is compatible with σ'.

□

As undirected graphs are special cases of mixed graphs with $A = \emptyset$, we get

Corollary 5.6 ([9, Theorem 2]). *For an undirected graph $G = (V, E)$,*

$$P(G, -x, -y) = \sum_{H \text{ flat of } G} (-1)^{|V(H)|} m_H(x, y),$$

where $m_H(x, y)$ is the number of pairs (σ, c) consisting of an acylic orientation σ of H and a compatible colouring $c : V(H) \longrightarrow [x]$ such that $c(v) > y$ if $v \in C(H)$.

6. Future Directions

The classical chromatic polynomial $\chi_G(x)$ is related to the enumeration of interior lattice points in inside-out polytope[1] $(P, \mathcal{H})$. For a mixed graph $G = (V, E, A)$, consider some dilate of the polytope $P = [0,1]^{|V(G)|}$ and the hyperplane arrangement $\mathcal{H} = \{\{x_i = x_j\} \text{ for } ij \in E \cup A \text{ and } \{x_k = y\} \text{ for } 1 \leq k \leq n\} \subseteq \mathbb{R}^{|V(G)|}$. The bivariate chromatic polynomial $\chi_G(x, y)$ might also have a similar geometric interpretation in terms of lattice-point enumeration for an inside-out polytope like structure $(P, \mathcal{H})$. This might give us connections between the Ehrhart polynomial of some polytope and the bivariate chromatic polynomial of mixed graphs.

Another approach might be via marked posets. For a finite poset Π, let A be a subset of Π containing all minimal and maximal elements of Π. Let a vector $\lambda = (\lambda_a)_{a \in A} \in \mathbb{R}^A$ be a marking of the elements of A with real numbers. A triple (Π, A, λ) is called a *marked poset*. Bicoloured posets can be thought of as marked posets where the set $\{0, y, x\}$ provides markings. All celeste elements of the poset are bounded below by y and above by x, whereas all silver elements are bounded below by 0. All the elements of the bicoloured poset are bounded above by x.

For an order preserving map $\lambda : A \longrightarrow \mathbb{R}$, the *marked order polytope* is defined as

$$\mathcal{O}_{\Pi,A}(\lambda) := \left\{ x \in \mathbb{R}^{\Pi \setminus A} \;\middle|\; \begin{array}{l} x_p \leq x_q \text{ for } p < q \\ \lambda_a \leq x_p \text{ for } a < p \\ x_p \leq \lambda_a \text{ for } p < a \end{array} \right\} \tag{6.1}$$

where p and q represent elements of $\Pi \setminus A$, and a represents an element of A [3].

In this setting, from [21, Theorem 2.6] we know that for an integral-valued order preserving map $\lambda : A \longrightarrow \mathbb{Z}$,

$$\Theta_{(\Pi,A)}(\lambda) := |\mathcal{O}_{\Pi,A}(\lambda) \cap \mathbb{Z}^{|\Pi|}| \tag{6.2}$$

[1]Details in Appendix A

is a polynomial. This polynomial also relates to the Ehrhart polynomial of marked order polytopes. Jochemko-Sanyal [21] also showed that the polynomial $\Theta_{(\Pi,A)}(\lambda)$ is related to partial graph colourings.

As the bicoloured posets are a special case of marked posets, the bivariate order polynomial and bivariate chromatic polynomial of mixed graphs must be related to the polynomial $\Theta_{(\Pi,A)}(\lambda)$ and the Ehrhart polynomial of marked order polytopes. This approach might also provide a geometric interpretation and understanding of bivariate order and chromatic polynomials.

Appendix A.

Ehrhart Theory

A *polytope* P is the convex hull of a finite set of the form $P = \text{conv}(S) \subseteq \mathbb{R}^n$ for some finite set $S \subseteq \mathbb{R}^n$. In 1962, Ehrhart [14] showed that, for an n-dimensional lattice polytope P, the lattice point enumerator $E(kP) = \#(kP \cap \mathbb{Z}^n)$ is a polynomial k of degree n where the coefficients $E_i(P)$, $0 \leq i \leq n$, depend only on P. This polynomial is called the *Ehrhart polynomial* and can be written as $E(kP) = \sum_{i=0}^{n} E_i(P)k^i$.

The famous Ehrhart-Macdonald reciprocity theorem [13, 22] states that

$$E(\text{int}(kP)) = (-1)^n \sum_{i=0}^{n} E_i(P)(-k)^i \tag{A.1}$$

where int() denotes the interior. The leading coefficient $E_n(P)$ equals vol(P), where vol() denotes the n-dimensional Lebesgue measure on $\mathbb{R}^n$. Ehrhart [15] also showed that the second leading coefficient $E_{n-1}(P)$ admits a simple geometric interpretation in terms of the lattice surface area of P.

Order Polytopes[1]

In [27], Stanley introduced the order and chain polytopes built from a poset. Both polytopes have since been studied quite thoroughly due to their interesting combinatorial properties. We will discuss the order polytopes. The *order polytope* of a poset

[1] Reference: [7, Chapter 6]

$(\Pi, \preceq)$, denoted $\mathcal{O}_\Pi$, is

$$\mathcal{O}_\Pi = \{(x_1, \ldots, x_d) : 0 \leq x_i \leq x_j \leq 1 \text{ for all } i \preceq j\}.$$

Stanley showed that, given a poset Π with n elements, its order polytope $\mathcal{O}_\Pi$ has normalized volume equal to the number of linear extensions of Π. It can also be shown that the vertices of order polytopes have coordinates in $\{0, 1\}$. This gives that the order polytopes are contained in the cube $[0, 1]^d$. Combining the properties of order polytopes and Ehrhart's theorems, we get the following theorem.

Theorem A.1 ([7, Proposition 6.3.2])**.** *Let* Π *be a finite poset. Then*

$$\Omega_\Pi(n) = E_{\mathcal{O}_\Pi}(n - 1).$$

This theorem shows the geometric interpretation of order polynomials and also proves Proposition 2.1 about polynomiality of order polynomials. This theorem combined with Ehrhart-Macdonald Reciprocity (A.1) gives an alternative way to prove the univariate order polynomial reciprocity Theorem 2.2.

Lattice-Point Enumeration[2]

Recall that Theorem 3.4 provides a decomposition of classical chromatic polynomial into the order polynomials via acyclic orientations of a graph. It states that

$$\chi_G(x) = \sum_{\substack{\text{acyclic} \\ \text{orientations } \rho}} \Omega_P^\circ(x).$$

Combining this with Theorem A.1, we get

Proposition A.2.

$$\chi_G(x) = \sum_{\substack{\textit{acyclic} \\ \textit{orientations } \rho}} E_{\mathcal{O}_\Pi^\circ}(n + 1). \tag{A.2}$$

[2]Reference: [7, Section 7.1]

Geometrically, $E_{\mathcal{O}_\Pi^\circ}(n+1)$ counts all the lattice points in the interior of the $(n+1)$-th dilate of the order polytope. Since there is an acyclic orientation associated with every proper colouring of a graph $G = (V, E)$, $E_{\mathcal{O}_\Pi^\circ}(n+1)$ counts all proper n-colourings of G as interior lattice points of dilated order polytope. In other words, this counts all interior lattice points in the cube $(0, n+1)^{|V|}$ except for those points which have coordinates $x_i = x_j$ for $ij \in E$.

Inside-Out Polytopes[3]

Let P be a full-dimensional convex polytope in $\mathbb{Z}^d$ with some hyperplane arrangement $\mathcal{H} \in \mathbb{R}^d$. Consider the lattice point enumeration inside P such that the interior lattice points that lie on the hyperplanes in $\mathcal{H}$ are not counted. This can be viewed as the hyperplane arrangement forming an additional boundary inside the polytope P. This polytope and hyperplane arrangement pair is called *inside-out polytope* and is denoted by $(P, \mathcal{H})$. The lattice point enumeration can be similarly generalized for this set-up of inside-out polytope $(P, \mathcal{H})$:

$$I_{P,\mathcal{H}}(n) := |n(P \setminus \mathcal{H}) \cap \mathbb{Z}^d|.$$

That is, $I_{P,\mathcal{H}}(n)$ counts those points in $\mathbb{Z}^d$ that are in the polytope nP but off the hyperplanes in $\mathcal{H}$.

Consider an undirected graph $G = (V, E)$. Let $H_{ij} := \{x \in \mathbb{R}^{|V|} : x_i = x_j\}$. The graphical hyperplane arrangement is given by $\mathcal{H} := \{H_{ij} \colon \text{ for each } ij \in E\}$. Let $P = [0,1]^{|V|}$. A theorem by Greene [7, Lemma 7.2.4] states that the regions of graphical hyperplane arrangement $\mathcal{H}$ are in one-to-one correspondence with acyclic orientations of graph G. The chromatic polynomial $\chi_G(x)$ counts exactly those lattice points in the $(x+1)$-th dilate of P° which do not lie on any of the hyperplanes H_{ij} for each $ij \in E$. Hence,

Proposition A.3.

$$\chi_G(x) = I_{P^\circ,\mathcal{H}}(x+1).$$

This is another geometric interpretation for chromatic polynomial via inside-out polytopes. The reciprocity result can also be proved using this technique.

[3]Reference: [7, Section 7.3] and [8]

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