BIVARIATE ORDER POLYNOMIALS

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Contents

Chapter 1

Introduction

"If you want to go fast, go alone. If you want to go far, bring others along."

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The motivation for this thesis is based on two ideas. The first idea comes from the paper *A chromatic-like polynomial for ordered sets* by Richard Stanley where he provides a method to decompose the chromatic polynomial into order polynomials [8]. Stanley also provides a way to compute the order polynomial of a poset P using a general formula involving descents. Ultimately, Stanley provides a reciprocity theorem establishing a relationship between strict and weak order polynomials [9]. In *A new two-variable generalization of the chromatic polynomial*, Klaus Dohmnen, André Pönitz, and Peter Tittmann generalized the chromatic polynomial to two variables [5].

We investigate a way to decompose the bivariate chromatic polynomial into bivariate order polynomials. We introduce and study bivariate order polynomials and provide a way to compute them given any poset. We discover a decomposition analogous to Stanley's and establish a reciprocity theorem for bivariate order polynomials.

In Chapter 2 we introduce basic concepts in graph theory and hyperplane arrangements. We define what a graph is, special types of graphs, and deletion and contraction of edges. We define hyperplane arrangements, particularly the graphical arrangement that comes from a graph.

In Chapter 3 we begin by defining the chromatic polynomial of a graph. We provide a method to compute the chromatic polynomial using deletion and contraction. We then define partially ordered sets and the relationship between acyclic oriented graphs and posets. We investigate order polynomials, their reciprocity theorem, and the decomposition of the chromatic polynomial using order polynomials.

Our main results in the two variable scenario occurs in Chapter 4. We begin by defining the bivariate chromatic polynomial and a method for its computation. In Proposition 4.3 we explicitly give a way to compute the order polynomial of a bicolored chain. Theorem 4.4 generalizes a formula to compute the order polynomial of a linear extension that comes from a bicolored poset. That leads us to Theorem 4.5 which generalizes a way to compute the bivariate strict order polynomial of a bicolored poset. In Theorem 4.7 we then present a decomposition of the bivariate chromatic polynomial using bivariate order polynomials. Finally, we introduce marked posets, their connection to bicolored posets and future work,

Chapter 2

Background

"Injustice anywhere is a threat to justice everywhere."

- Martin Luther King, Jr.

In this chapter we introduce basic concepts in graph theory and discrete geometry. In Section 2.1 we familiarize ourselves with basic concepts and examples in graph theory. We start discussing hyperplane arrangements in Section 2.2 and their connection with graph theory. Furthermore, in Section 2.3 we briefly examine basic Ehrhart Theory properties and theorems.

2.1 Graph Theory

We dive into fundamental concepts of graph theory. Our goal is to gather some necessary tools to understand graphs and their properties.

2.1.1 Basic Properties

A **graph** G is a pair $G = (V, E)$ consisting of a set V and a multiset $E \subseteq \{\{i, j\} : i, j \in V\}$. The set V consists of elements that are **vertices** or **nodes** of G. The set E contains the **edges** of G. An edge $e \in E$ is of the form $\{i, j\}$ but for the sake of simplicity we will denote e as ij.

A vertex j is **incident** with an edge e if $j \in e$. The two vertices incident to e are its **endpoints**. We say that two vertices i and j are **adjacent** if there is an edge $ij \in E$. **Multiple edges** are two or more edges incident to the same vertices.

A **simple** graph is a graph with no multiple edges. We will assume that any graph is simple and finite. If we face the situation of having multiple edges we will simply replace them with a single edge.

A **walk of length** m is a sequence of m edges of the form

$$v_0v_1, v_1v_2, \dots, v_{m-1}v_m.$$

A walk with distinct vertices is called a **path** P. We refer to a path by the sequence of its vertices, $P = abc \dots jk$ and we say that P is a path from a to k. A **cycle** is a walk that

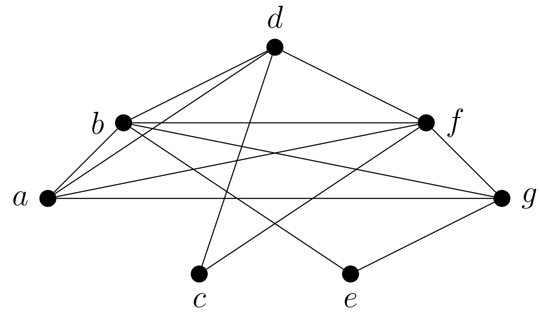

Figure 2.1: A simple graph.

starts and ends at the same vertex and no other vertex is repeated. A graph is **connected** when there is a path between every pair of vertices. If the graph is not connected then we say that it is **disconnected**. A subgraph $H = (\hat{V}, \hat{E})$ of G is a graph whose vertices are a subset of V and whose edges are a subset of E and $ij \in \hat{E} \to i, j \in \hat{V}$.

We provide classical examples of graphs.

Example 2.1. A **null graph** is a graph having no edges.

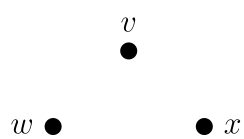

Figure 2.2: A null graph with 3 vertices.

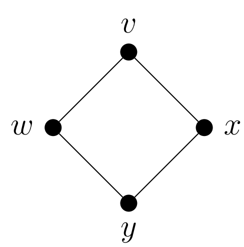

Figure 2.3: A cycle graph on 4 vertices.

Example 2.2. A **complete graph** is a graph in which each pair of vertices is joined by an edge.

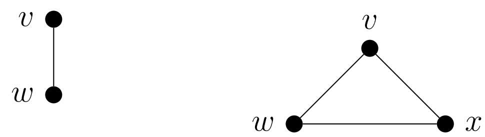

Figure 2.4: Example of complete graphs with 2 and 3 vertices, K_2 and K_3.

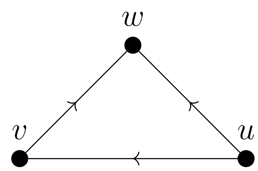

Figure 2.5: An acyclic orientation of K_3.

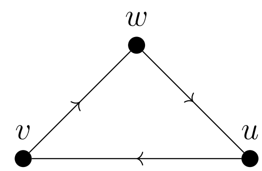

Figure 2.6: A nonacyclic orientation of K_3.

Figure 2.4 shows complete graphs. An **orientation** of G is an assignment of a direction to each edge ij denoted by $i \to j$ or $j \to i$. An orientation of G is **acyclic** if it has no coherently directed cycles. In Figure 2.5 we have an acyclic orientation of K_3 and Figure 2.6 a nonacyclic orientation of K_3.

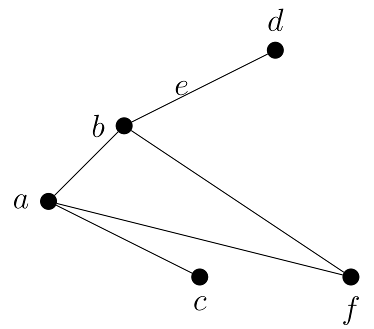

Figure 2.7: G with edge e.

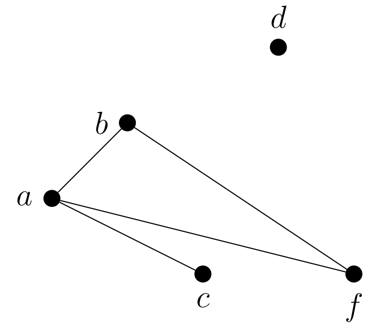

Figure 2.8: Resulting graph $G - e$.

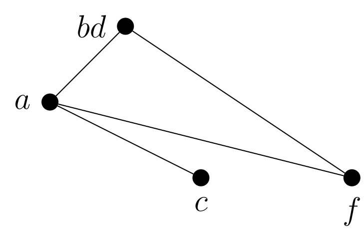

Figure 2.9: Resulting graph G/e.

2.1.2 Deletion - Contraction

Given a graph G and $e \in E$ we say that $G - e$ is the subgraph of G that results in deleting edge e.

Moreover, G/e is the subgraph of G that results in contracting edge $e = ij$. A **contraction** of e means that we identify the endpoints of e, into a new vertex which then becomes adjacent to every vertex formerly adjacent to i or j. Note that loops that result from a contraction is removed. More precisely, $G/e = (V', E')$ where $V' := V - \{i, j\} \cup \{v_e\}$ and $E' := \{ab \in E : \{a, b\} \cap \{i, j\} = \emptyset\} \cup \{v_e b : ib \in E - \{e\} \text{ or } jb \in E - \{e\}\}$.

Figure 2.7 we have G with an edge $e = bd$ and Figure 2.8 is the resulting graph when we delete e. In Figure 2.9 we denote the same e but instead we contract the edge and we obtain the resulting graph G/e.

We can also consider contracting $S \subseteq E$ of G by contracting the edges of S one at a time; the result does not depend on the order of the contractions. Note that this subset S must be connected component of G. We will denote a contraction in the same way we did earlier as G/S. Also, it's possible that two different subsets S_1 and S_2 of E can result in $G/S_1 = G/S_2$. Figure 2.10 and Figure 2.11 present the same resulting graphs even when the subsets of vertices are different.

In the next section we provide insight into a geometric approach to graph theory.

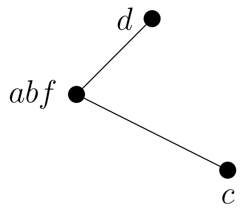

Figure 2.10: Resulting graph $G/\{a,b\} \cup \{a,f\}$.

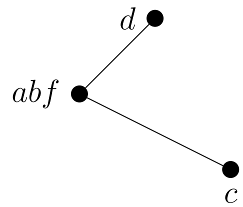

Figure 2.11: Resulting graph $G/\{a,b\} \cup \{a,f\} \cup \{b,f\}$.

2.2 Hyperplane Arrangements

This section presents background information on the geometric approach of our work. An (affine) **hyperplane** is a set of the form

$$H := \{x \in \mathbb{R}^d : a \cdot x = b\}$$

where a is a fixed nonzero vector in $\mathbb{R}^d$, b is a displacement in $\mathbb{R}$ and $a \cdot x$ is the standard dot product. An **arrangement** $\mathcal{H}$ is a finite set of hyperplanes. This arrangement divides the space into regions. A **region** of $\mathcal{H}$ is a maximal connected component of $\mathbb{R}^d - \cup_{A \in \mathcal{H}} A$.

Example 2.3. The d-dimensional **real braid arrangement** is the arrangement $\mathcal{H} = \{x_j = x_k : 1 \leq j < k \leq d\}$ with $\binom{d}{2}$ hyperplanes.

Let us assume that G is a simple graph with edges E and $d = |V|$. The **graphical arrangement** $\mathcal{A}_G$ in $\mathbb{R}^d$ is the arrangement

$$\{x_i = x_j : ij \in E\}.$$

The graphical arrangement is a subarrangement of the braid arrangement introduced in Example 2.3. If $G = K_d$, the complete graph on d vertices, then $\mathcal{A}_{K_d}$ is the fulll braid arrangement.

Moreover, each region of the hyperplane arrangement $\mathcal{A}_G$ corresponds to an acyclic orientation of G: Each region of the $\mathcal{A}_G$ is determined by some inequalities $x_i < x_j$ or $x_i > x_j$ where i and j come from edge ij. Naturally, $x_i < x_j$ implies that we can orient the edge ij from i to j and $x_i > x_j$ implies that we can orient the edge ij from j to i. We can see in Example 2.12 the regions of $\mathcal{A}_{K_3}$ and the acyclic orientations.

This can be generalized by the following proposition.

Proposition 2.1 ([7])**.** *The regions of $\mathcal{A}_G$ are in one-to-one correspondence with the acyclic orientations of G.*

We encourage the reader to read [12] for more interesting results.

The intersections of hyperplanes in $\mathcal{H}$ are **flats**. Particularly, we can think of a flat in the following different ways. First, all possible intersections of the arrangement are the possible flats. Because the graphical arrangement comes from a graph G these hyperplane intersections come from the possible contractions of G.

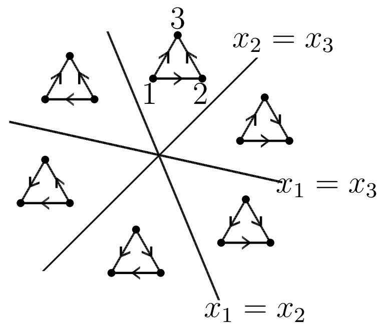

Figure 2.12: The regions of $\mathcal{A}_{K_3}$ and their corresponding acyclic orientations [3].

Proposition 2.2 ([3])**.** *The set of contractions of G are in bijection with the set of flats of $\mathcal{A}_G$.*

Secondly, we can also think of a flat through transitivity. Suppose that we have the following hyperplanes $x_1 = x_2$, $x_2 = x_3$, $\ldots$, $x_{k-1} = x_k$. Then that means that it must be true that $x_1 = x_k$ since the equality is transitive. Transitivity gives us the tools to be able to decompose the bivariate chromatic polynomial in Chapter 4.

2.3 Ehrhart Theory

A **convex polytope** $\mathcal{P}$ is the convex hull of finitely many points in $\mathbb{R}^d$; in other words, given a finite set $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$, the polytope $\mathcal{P}$ is the smallest convex set containing those points. We say $\mathcal{P}^\circ$ is the interior of $\mathcal{P}$. A **dilated polytope** $k\mathcal{P}$ is

$$k\mathcal{P} = \{(kx_1, kx_2, \ldots, kx_d) : (x_1, x_2, \ldots, x_d) \in \mathcal{P}\}$$

for some $k \in \mathbb{Z}_{>0}$ and polytope $\mathcal{P}$. If $\mathcal{P} = \text{conv}\{V\}$ where $V \subset \mathbb{Z}^d$ then $\mathcal{P}$ is called a **lattice polytope**.

Lattice polytopes are used in Ehrhart theory to study the number of lattice points in integral dilates of $\mathcal{P}$. Let $\mathcal{P}$ be a d-dimensional lattice polytope in $\mathbb{R}^d$. The lattice point enumerator for the kth dilate is

$$E_{\mathcal{P}}(k) := \#(k\mathcal{P} \cap \mathbb{Z}^d)$$

for $k \in \mathbb{Z}_{>0}$. The $E_{\mathcal{P}}(k)$ is a polynomial in k of degree d with rational coefficients known as the **Ehrhart polynomial**.

Theorem 2.3 ([6])**.** *Let $\mathcal{P}$ be a convex polytope. $E_{\mathcal{P}^\circ}(k)$ and $E_{\mathcal{P}}(k)$ are polynomials in k with rational coefficients.*

Theorem 2.4 (Ehrhart–Macdonald Reciprocity, [6])**.** *For any d-dimensional lattice polytope $\mathcal{P}$,*

$$E_{\mathcal{P}^\circ}(k) = (-1)^d E_{\mathcal{P}}(-k).$$

Chapter 3

Classical Scenario

"They tried to bury us; They didn't know we were seeds."

-

This chapter will begin by presenting the chromatic polynomial and order polynomial in one variable. We explore the idea that the chromatic polynomial is decomposed into order polynomials by way of relationship between acyclic oriented graphs and posets.

3.1 The Chromatic Polynomial

In 1912, George David Birkhoff [4], in an attempt to prove the four coloring theorem, discovered the chromatic polynomial. The four color theorem was originally conjectured in 1852 and thus proving it would have been a revolutionary event.

An n-**coloring** of a graph G is a map

$$\phi : V \to \{1, 2, \ldots, n\}$$

where $\{1, 2, \ldots, n\}$ can be thought of as a set of colors.

A **proper** n-coloring of G is a map

$$\phi : V \to \{1, 2, \ldots, n\}$$

where $\phi(i) \neq \phi(j)$ whenever i and j are adjacent.

The **chromatic polynomial** $P(G; n)$ counts the number of proper graph colorings as a function of n colors.

Proposition 3.1 (Birkhoff, [4]). *The chromatic polynomial $P(G; n)$ of G is a polynomial in n with integral coefficients.*

We can compute the chromatic polynomial of G using the method of *deletion and contraction.* This is a recursive method that enables us to delete and contract an edge and compute the chromatic polynomials of smaller subgraphs.

Proposition 3.2. *Let G be a graph and e an edge of G. $P(G; n) = P(G - e; n) - P(G/e; n)$.*

Example 3.1. Let us count the number of proper n-colorings of K_3 found in Figure 2.4. We will use Proposition 3.2 to compute the chromatic polynomial. We will let $e = vx$, that means that we will delete the edge e and then consider the case where we contract e.

First, we will compute $P(G - e; n)$. For the single vertex v we have n possible colors, for node w, we have $n - 1$ possible colors, since we want the proper colorings for node x we will have $n - 1$. Therefore, by the product property the number of possible proper n-colorings is $P(G - e; n) = n(n - 1)^2$. Now, we compute $P(G/e; n)$ using the same approach and we have $P(G/e; n) = n(n - 1)$. Therefore,

$$P(G; n) = n(n - 1)^2 - n(n - 1) = n^3 - 3n^2 + 2n.$$

Naturally, we can then question if there are alternative approaches to computing the chromatic polynomial. In the next section we will dive into a way to decompose the chromatic polynomial using order polynomials.

3.2 Ordered Structures

In 1970, Richard Stanley introduced a notion of counting order preserving maps and its connection with chromatic polynomials [8].

A **partially ordered set** or **poset** is a set P with a binary relation $R \subseteq P \times P$ satisfying three conditions: reflexivity, antisymmetry and transitivity, in other words, a set that has a partial order. For the following three conditions we will assume that $a, b, c \in P$. Reflexivity means that if $a \in P$ then $a \preceq a$. Antisymmetry means that if $a \preceq b$ and $b \preceq a$ then $a = b$. Transitivity means that if $a \preceq b$ and $b \preceq c$ then $a \preceq c$.

Example 3.2. Let $P = \mathbb{Z}_{\geq 1}$ where the relation is divisibility. This is an infinite poset.

Example 3.3. Let $P = B_n$, the Boolean poset on n elements be the set of subsets of the set $\{1, 2, \ldots, n\}$ where the relation is containment. This is a finite poset.

In Section 2.1.1 we introduced acyclic oriented graphs. We will utilize that concept to give rise to the structure of a poset.

Given any graph G we can count the number of possible orientations G can have. In particular, there are $(2)^{|E|}$ ways to orient a graph since we have two choices per edge. Yet, this is not really interesting. Stanley discovered that taking the chromatic polynomial and evaluating it at -1 gives us the number of acyclic orientations up to a sign.

Theorem 3.3 (Stanley, [10])**.** *If G is a graph with k vertices, then $(-1)^k P(G; -1)$ is equal to the number of acyclic orientations of G.*

Example 3.4. Consider K_3 in Figure 2.4. The chromatic polynomial of K_3 is

$$P(G; n) = n^3 - 3n^2 + 2n.$$

By Theorem 3.3 to find the number of acyclic orientations of K_3, we compute

$$(-1)^3 P(G; -1) = (-1)^3(-1 - 3 - 2) = 6,$$

and indeed K_3 has 6 acyclic orientations. Moreover we encountered these in Figure 2.12.

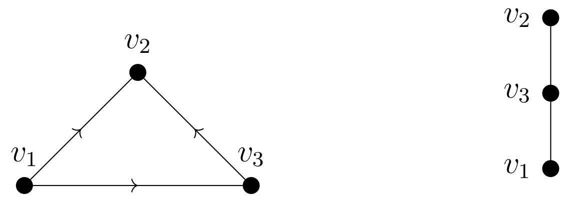

Figure 3.1: An acyclic orientation of K_3 and its corresponding poset.

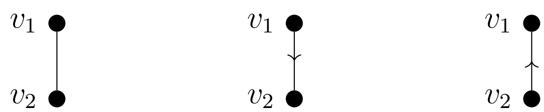

Figure 3.2: K_2 and its acyclic orientations, σ_1 and σ_2.

There is a natural way to go from acyclic oriented graphs to posets. Suppose we have an acyclic oriented graph G_σ. The elements of the associated poset P are vertices of G_σ. For a, $b \in P$, define $a \preceq b$ if and only if there is a coherently oriented path from i to j in G_σ.

Example 3.5. In Figure 3.1, we see one of the acyclic orientations of K_3 and the poset that is obtained from following the process described above.

Let P be a finite poset and Γ_n be a chain of length n. A chain is a totally ordered set, and in particular we can think of it as the set $\{1, 2, \ldots, n\}$ with the natural order. A map $\phi : P \to \Gamma_n$ is **order preserving** if $x \le y$ implies that $\phi(x) \le \phi(y)$. Let $\Omega_P(n)$ count the number of such order preserving maps. A map $\phi : P \to \Gamma_n$ is **strictly order preserving** if $x < y$ implies that $\phi(x) < \phi(y)$. $\Omega^\circ_P(n)$ counts the number of such strictly order preserving maps.

Theorem 3.4 (Stanley, [8])**.** *Let P be a partially ordered set. $\Omega_P(n)$ and $\Omega^\circ_P(n)$ are polynomials in n with rational coefficients.*

Theorem 3.5 (Stanley, [8])**.** *Let P be a finite poset. Then,*

$$\Omega(n) = (-1)^{|P|}\Omega^\circ(-n).$$

There is a strong relation between strictly order preserving maps and proper colorings of a graph. Let us begin by counting the number of ways to color K_2. We can start with a vertex and we have n ways to color it and for the second vertex we have $n - 1$ choices. And so

$$P(K_2, n) = n(n-1) = n^2 - n.$$

In Figure 3.2 we see that we have two acyclic orientations for K_2, σ_1 and σ_2. We can create a poset out of each acyclic orientation. We can find an order polynomial that counts the number of strictly order preserving maps from P to $[n]$.

Figure 3.3 provides us with the corresponding poset of σ_1 and σ_2. Note that the Hasse diagram of both resulting posets are structurally the same. Hence both order polynomials

Figure 3.3: Acyclic orientations σ_1 and σ_2 and their corresponding poset.

must be the same. Let v_1 and v_2 be elements in P_1 and $v_2 > v_1$. Then to count the number of ways to map it to Γ_n we know we must preserve the order:

$$n \geq \phi(v_1) > \phi(v_2) > 0,$$

we have $\binom{n}{2}$ ways to do this. So

$$\Omega^\circ_{P_1}(n) = \binom{n}{2}, \qquad \Omega^\circ_{P_2}(n) = \binom{n}{2},$$

and adding the two order polynomials gives us the chromatic polynomial:

$$\Omega^\circ_{P_1}(n) + \Omega^\circ_{P_2}(n) = n^2 - n = P(K_2, n)$$

The following theorem generalizes this process of decomposing the chromatic polynomial.

Theorem 3.6 (Stanley, [10]). *Let G be a graph. Then*

$$P(G; n) = \sum_{\sigma} \Omega^\circ_\sigma(n),$$

where the sum is over all possible acyclic orientations σ of G.

Given a poset P, Stanley introduced two geometric interpretations of P. These are known as an order polytope and a chain polytope. We focus on the order polytope but note these two polytopes share many interesting results [11].

Let $f : P \to \mathbb{R}$ be a function, the **order polytope** $\mathcal{O}(P)$ of the poset P is the subset of $\mathbb{R}^P$ defined by the conditions

$$0 \leq f(x) \leq 1 \ \forall x \in P$$

and

$$f(x) \leq f(y) \text{ if } x \leq y \in P.$$

The following theorem gives a relation between the Ehrhart polynomial of an order polytope and the order polynomial of a poset. In other words, the Ehrhart polynomial of an order polytope is an order polynomial in disguise.

Theorem 3.7 (Stanley, [11]). $E_{\mathcal{O}(P)}(n) = \Omega_P(n+1)$.

In the following chapter we will extend many of these results to two variables.

Chapter 4

The Bivariate Scenario

"Pies para que los quiero, si tengo alas para volar."

- Frida Kahlo

In this chapter we provide a setting parallel to Chapter 3 but in two variables. We begin by introducing the bivariate chromatic polynomial and some of its properties. We then present a method to decompose the bivariate chromatic polynomial using bivariate order polynomials of marked posets.

4.1 The Bivariate Chromatic Polynomial

In 2003, Dohmen, Pönitz, and Tittmann [5] generalized the chromatic polynomial to two variables x and y. A **generalized proper (x, y)-coloring** of G is a map

$$\phi : V \to \{1, 2, \dots, y, y + 1, \dots, x\}$$

where $\phi(v_1) \neq \phi(v_2)$ whenever v_1 and v_2 are adjacent and $\phi(v_1), \phi(v_2) \leq y$. Consequently, we can color adjacent vertices with the same color only if the color is greater than y. The **bivariate chromatic polynomial** $P(G; x, y)$ with $y \leq x$ counts the number of generalized proper (x, y)-colorings.

If we consider this generalized chromatic polynomial, the one-variable chromatic polynomial is a special case. Namely, for $x = y$, $P(G; y, y)$ is the usual chromatic polynomial $P(G; y)$ that we defined in Section 3.1. Another special case is $y = 0$; then $P(G; x, 0) = x^n$ where n is the number of vertices of G. In addition to generalizing the chromatic polynomial, the bivariate chromatic polynomial generalizes the independence polynomial and the matching polynomial. The bivariate chromatic polynomial is independent of the Tutte polynomial. The reader is encouraged to visit [5] for further details about these polynomials.

Naturally, we would like to know how to compute this bivariate chromatic polynomial. Using deletion and contraction will not be possible in this scenario since the colors greater than y can be improper. Theorem 4.1 gives us tools to compute the bivariate chromatic polynomial.

Theorem 4.1 (Dohmen, Pönitz, Tittmann, [5])**.** *Let G be a graph and let $G - X$ be the subgraph obtained from G by removing all vertices of $X \subseteq V$ and their incident edges. Then*

$$P(G; x, y) = \sum_{X \subseteq V} (x - y)^{|X|} P(G - X; y).$$

By Theorem 4.1, we find all possible subgraphs of G and use the above formula to compute the bivariate chromatic polynomial. In addition, we can deduce the following result.

Corollary 4.2 (Dohmen, Pönitz, Tittmann, [5])**.** *Let G be a finite graph and $y \leq x$. $P(G; x, y)$ is a polynomial in x and y with integral coefficients.*

Example 4.1. For $G = K_3$, we begin by finding all possible $X \subseteq V$.

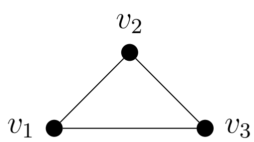

Figure 4.1: All possible three-element subsets of V.

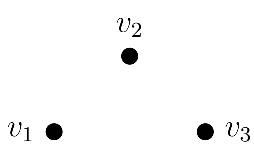

Figure 4.2: All possible one-element subsets of V.

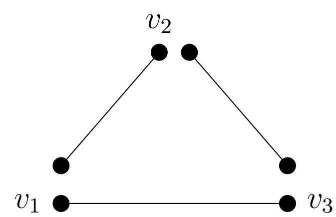

Figure 4.3: All possible two-element subsets of V.

By Theorem 4.1,

$$P(K_3; x, y) = (x-y)^0(y^3-3y^2+2y)+3(x-y)^1(y^2-y)+3(x-y)^2(y)+(x-y)^3(1) = x^3-3xy+2y.$$

We would like to find a way to decompose the bivariate chromatic polynomial by analogy with Theorem 3.6. The following section will provide us with an alternative way to visualize the chromatic polynomial using ordered structures.

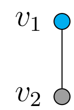

Figure 4.4: Bicolored poset with one celeste element.

4.2 Ordered Structures

4.2.1 Bicolored Posets

A **bicolored poset** is a poset with two types of elements: celeste and silver elements. Let $P = C \uplus S$ be a bicolored poset where we think of C as the set of celeste elements and S the set of silver elements. A map $\phi : P \to [x]$ is an **order preserving (x,y)-map** if $p_i \preceq p_j$ with $p_i, p_j \in P$ implies that $\phi(p_i) \leq \phi(p_j)$ and if $c \in C$ implies that $\phi(c) \geq y$. Let

$$\Omega_{P,C}(x, y) := \text{ \# of order preserving } (x, y)\text{-maps} \; .$$

A map $\phi : P \to [x]$ is a **strictly order preserving** (x, y)**-map** if $p_i \prec p_j$ implies that $\phi(p_i) < \phi(p_j)$ and if $c \in C$ implies that $\phi(c) > y$. Let

$$\Omega^{\circ}_{P,C}(x, y) := \text{ \# of strictly order preserving } (x, y)\text{-maps} \; .$$

Note that it must be true that $y \leq x$. It is not obvious that these are polynomials in x and y. Yet, intuitively as the one variable case, we will prove that $\Omega_{P,C}(x, y)$ and $\Omega^{\circ}_{P,C}(x, y)$ are polynomials in x and y.

Example 4.2. We will begin by providing a way to obtain the order polynomial for the bicolored poset P in Figure 4.4. From the Hasse diagram we obtain that $v_1 > v_2$.

We want to find all possible strictly order preserving (x, y)-maps from P to $[x]$. We know that $[x]$ is a chain of x elements $\{1, 2, \ldots, y, y + 1, \ldots, x\}$ and the order of P must be preserved. We have the following two cases. Either

$$0 < \phi(v_2) \leq y < \phi(v_1) \leq x; \tag{4.1}$$

since v_1 is a celeste element, $\phi(v_1)$ must be greater than y by definition. This gives us $x - y$ choices for $\phi(v_1)$ and y choices for $\phi(v_2)$. In our second case we have

$$0 < y < \phi(v_2) < \phi(v_1) \leq x \tag{4.2}$$

which we have $\binom{x-y}{2}$ possible ways for. The sum of (4.1) and (4.2) will give us the bivariate order polynomial of P:

$$\Omega^{\circ}_{P,C}(x, y) = (x - y)y + \binom{x - y}{2} = \frac{1}{2}(x^2 - x - y^2 - y).$$

The following proposition will generalize this way for us to compute the bivariate order polynomial of a bicolored chain.

Proposition 4.3. *If $P = \{v_n \prec v_{n-1} \prec \cdots \prec v_{k+1} \prec v_k \prec \cdots \prec v_2 \prec v_1\}$ is a chain of length n and v_{k+1} is the minimal celeste element in C then*

$$\Omega^{\circ}_{P,C}(x,y) = \sum_{a=0}^{k} \binom{x-y}{n-a}\binom{y}{a}.$$

We denote this polynomial as $f_{n,k}(x,y)$.

Proof. We wish to count the number of strictly order preserving maps from $\phi : P \to [x]$ such that $\phi(v_{k+1}) > y$. This gives rise to a set of inequalities for ϕ; we will have $n-k$ cases. In the first case we have the following inequalities for ϕ:

$$0 < \phi(v_n) < \cdots < \phi(v_{k+2}) \leq y < \phi(v_{k+1}) < \cdots < \phi(v_1) \leq x.$$

We have $\binom{x-y}{n-k}\binom{y}{k}$ possibilities. In the second case, $\phi(v_{k+2})$ is greater than y, so ϕ

$$0 < \phi(v_n) < \cdots \leq y < \phi(v_{k+2}) < \phi(v_{k+1}) < \cdots < \phi(v_1) \leq x.$$

The number of possible ways to do this is $\binom{x-y}{n-k+1}\binom{y}{k-1}$. For the third case we have $\phi(v_{k+3}) > y$ and we repeat the process as above. In the last case we have $\phi(v_n) > y$ and the linear inequalities are

$$0 < y < \phi(v_n) < \cdots < \phi(v_{k+2}) < \phi(v_{k+1}) < \cdots < \phi(v_1) \leq x.$$

There are $\binom{x-y}{n}$ possible ways. Therefore, $\Omega^{\circ}_{P,C}(x,y) = \sum_{a=0}^{k} \binom{x-y}{n-a}\binom{y}{a}$. $\square$

Now we want to compute the bivariate order polynomial when P is not a chain. Before we can do so let us take a step back and revisit some important tools introduced by Stanley [13].

Assume $S = \{1, 2, \ldots, n-1, n\}$, we say that a **permutation** w of S is a linear order $w_1, w_2, \ldots, w_n$ of the elements S. We can consider w as a word $w_1 w_2 \cdots w_n$; this word corresponds to the bijection $w : S \to S$ given by $w(i) = w_i$. There are various statistics associated with permutations of S.

Let $w = w_1 w_2 \cdots w_n$ and $1 \leq i \leq n-1$ then i is a **descent** of w if $w_i > w_{i+1}$. Similarly, we say that i is an **ascent** if $w_i < w_{i+1}$. The **descent set** of w is defined as

$$D(w) = \{i : w_i > w_{i+1}\} \subseteq [n-1].$$

We also say that $\mathrm{des}(w) = |D(w)|$, the number of descents in the permutation w. Similarly, the **ascent set** of w is defined as

$$A(w) = \{i : w_i < w_{i+1}\} \subseteq [n-1].$$

We also say that $\mathrm{asc}(w) = |A(w)|$, the number of ascents in the permutation w. In particular, $\mathrm{des}(w) + \mathrm{asc}(w) = n - 1$. The reader is encouraged to visit [13] and [9] for further details.

Example 4.3. Let $S = \{1, 2, 3\}$ then all possible permutations of S are $\{123, 132, 213, 231, 312, 321\}$. Consider $w = 132$; we see that $w_2 > w_3$ thus 2 is a descent of w.

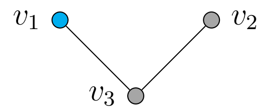

Figure 4.5: Bicolored poset with celeste element v_1.

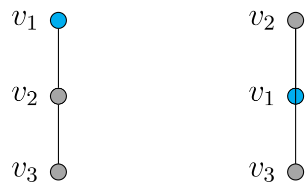

Figure 4.6: Linear extensions of P.

Let P be a bicolored poset with cardinality n. Let $\omega : P \to \{v_1, \ldots, v_n\}$ be a bijection called a **labeling** of P. If for $a \prec b$ in P we have $\omega(a) = v_j$ and $\omega(b) = v_k$ where $j \geq k$, we call this a **natural reverse labeling**. A **linear extension** L is a chain that **refines** P, i.e., if $a \preceq_P b$ then $a \preceq_L b$. Given P with a fixed natural reverse labeling we find all possible linear extensions the following way:

- if $a \preceq_P b$ then $a \preceq_L b$
- if a and b are incomparable elements in P then we have two cases: $a \preceq b$ or $b \preceq a$.

We let $\mathcal{L}(P)$ denote the set of linear extensions of P. For any given linear extension L we associate a word, w, by reading off the indices of the labeling from bottom to top. If $v_i \prec v_j$ implies that $a_{w(i)} \prec a_{w(j)}$ then we say that w **respects** P.

In Figure 4.5 we have a bicolored poset with a natural reverse labeling. We see in Figure 4.6 that P gives rise to two linear extensions $L(321), L(312)$ which correspond to the words 321 with $v_3 < v_2 < v_1$ and 312 with $v_3 < v_1 < v_2$. We say an order preserving (x, y)-map is of **type** w if $\phi(v_{w_i}) \leq \phi(v_{w_{i+1}})$ for every ascent i of w and $\phi(v_{w_i}) < \phi(v_{w_{i+1}})$ for every descent i of w. Let $\hat{\Omega}_{L(w),C}(x, y)$ be the number of (x, y)-maps of type w.

Theorem 4.4. *Let $L(w) = \{v_{w_1} \prec v_{w_2} \prec \cdots \prec v_{w_k} \prec v_{w_{k+1}} \prec \cdots \prec v_{w_n}\}$ be the linear extension of P that corresponds to the permutation $w = w_1w_2 \cdots w_kw_{k+1} \cdots w_n$. Let $v_{w_{k+1}}$ be the minimal celeste element in $L(w)$ and let $\tilde{w} = w_1w_2 \cdots w_{k+1}$. Then*

$$\hat{\Omega}_{L(w),C}(x, y) = \sum_{a=0}^{k} \binom{x - y + \operatorname{asc}(w) - \operatorname{asc}(\tilde{w})}{n - a} \binom{y + \operatorname{asc}(\tilde{w})}{a}.$$

Proof. Let $L(w)$ be the linear extension that corresponds to $w = w_1w_2 \cdots w_kw_{k+1} \cdots w_n$. We want to count the number of (x, y)-maps of type w. We represent ϕ as a string of inequalities where $\phi(v_{w_i}) \leq \phi(v_{w_{i+1}})$ if i is an ascent of w and $\phi(v_{w_i}) < \phi(v_{w_{i+1}})$ if i is a descent of w . We get rid of weak inequalities by creating a bijection $\phi \mapsto \bar{\phi}$. We define the bijection as follows:

$$\bar{\phi}(v_{w_1}) = \phi(v_{w_1})$$

$$\bar{\phi}(v_{w_2}) = \phi(v_{w_2}) + \operatorname{asc}(w_1 w_2)$$

$$\vdots$$

$$\bar{\phi}(v_{w_k}) = \phi(v_{w_k}) + \operatorname{asc}(w_1 w_2 \cdots w_k)$$

$$\bar{\phi}(v_{w_{k+1}}) = \phi(v_{w_{k+1}}) + \operatorname{asc}(w_1 w_2 \cdots w_k w_{k+1})$$

$$\vdots$$

$$\bar{\phi}(v_{w_n}) = \phi(v_{w_n}) + \operatorname{asc}(w_1 w_2 \cdots w_k w_{k+1} \cdots w_n).$$

The difference between $\operatorname{asc}(w_1 w_2 \cdots w_{i-1} w_i)$ and $\operatorname{asc}(w_1 w_2 \cdots w_i w_{i+1})$ will be either 0 or 1 depending on whether i is an ascent or not. Therefore, $\bar{\phi}(v_{w_1})$ to $\bar{\phi}(v_{w_n})$ is now strictly increasing. Thus we can now represent $\bar{\phi}$ as the following string of strict inequalities:

$$0 < \bar{\phi}(v_{w_1}) < \bar{\phi}(v_{w_2}) < \bar{\phi}(v_{w_3}) < \cdots < \bar{\phi}(v_{w_{k+1}}) < \cdots < \bar{\phi}(v_{w_n}) \leq x + \operatorname{asc}(w)$$

and $y + \operatorname{asc}(\tilde{w}) < \bar{\phi}(v_{w_k})$ By Proposition 4.3,

$$\hat{\Omega}_{L(w),C}(x,y) = \sum_{a=0}^{k} \binom{x - y + \operatorname{asc}(w) - \operatorname{asc}(\tilde{w})}{n-a} \binom{y + \operatorname{asc}(\tilde{w})}{a}$$

as we wished to prove. □

Now we are ready to compute the bivariate order polynomial of P in Figure 4.5. We have the following two cases.

- The linear extension that comes from 321 can be represented as

  $$L(321) = \{v_3 \prec v_2 \prec v_1\}.$$

  We can represent an (x,y)-map ϕ of type 321 as the string of inequalities

  $$0 < \phi(v_3) < \phi(v_2) < \phi(v_1) \leq x$$

  and $y < \phi(v_1)$. By Proposition 4.3,

  $$\hat{\Omega}_{L(321),C}(x,y) = \sum_{a=0}^{2} \binom{x-y}{3-a} \binom{y}{a} = \frac{1}{6}(x^3 - y^3 - 3x^2 + 3y^2 + 2x - 2y). \qquad (4.3)$$

- The linear extension that comes from 312 can be represented as

  $$L(312) = \{v_3 \prec v_1 \prec v_2\}.$$

  We can represent the (x,y)-map ϕ as the string of inequalities

  $$0 < \phi(v_3) < \phi(v_1) \leq \phi(v_2) \leq x$$

since we have a descent at 2 and $y < \phi(v_1)$. In order for us to be able to compute the strict order polynomial we must shift to get rid of the weak inequalities by adding 1. This then becomes

$$0 < \phi(v_3) < \phi(v_1) < \phi(v_2) + 1 \le x + 1$$

and $y < \phi(v_1)$. Through the bijection $\phi \mapsto \bar{\phi}$ with $\bar{\phi}(v_1) = \phi(v_1)$, $\bar{\phi}(v_2) = \phi(v_2) + 1$ and $\bar{\phi}(v_3) = \phi(v_3)$, this becomes

$$0 < \bar{\phi}(v_3) < \bar{\phi}(v_1) < \bar{\phi}(v_2) \le x + 1$$

with $y < \bar{\phi}(v_1)$. By Proposition 4.4 the order polynomial is

$$\hat{\Omega}_{L(312),C}(x, y) = \sum_{a=0}^{1} \binom{x - y + 1}{3 - a} \binom{y}{a} = \frac{1}{6}(x^3 - 3xy^2 + 2y^3 + 3xy - 3y^2 - x + y). \quad (4.4)$$

The sum of (4.3) and (4.4) gives us the strict order polynomial of P:

$$\Omega^{\circ}_{P,C}(x, y) = \frac{1}{6}(2x^3 - 3xy^2 + y^3 - 3x^2 + 3xy + x - y).$$

Theorem 4.5. *Let P be a poset with a fixed natural reverse labeling. Then*

$$\Omega^{\circ}_{P,C}(x, y) = \sum_{L(w)\in\mathcal{L}(P)} \hat{\Omega}_{L(w),C}(x, y)$$

where the sum is over all possible linear extensions $L(w)$ that respect P.

Proof. We will show there is a bijection between strictly order preserving maps ϕ of P and pairs of linear extension $L(w)$ and an order preserving map of type $L(w)$. That is, given a fixed P and a strictly order preserving map ϕ there is a corresponding linear extension $L(w)$ and a strictly order preserving (x, y)-map.

Let ϕ be an order preserving map of P to the chain $[x]$. Take all images of ϕ and order them in ascending order. We have $v_{w_i} \prec v_{w_{i+1}}$ whenever $\phi(v_{w_i}) = \phi(v_{w_{i+1}})$. In addition, it must be true that $\phi(v_{w_i}) < \phi(v_{w_{i+1}})$ when $w_i < w_{i+1}$. We then obtain the linear extension $L(w) = \{v_{w_1} \prec v_{w_2} \prec \cdots \prec v_{w_k} \prec v_{w_{k+1}} \prec \cdots \prec v_{w_n}\}$ that corresponds to the word $w = w_1 w_2 \cdots w_k w_{k+1} \cdots w_n$.

The inverse map is even simpler. Recall P is fixed and so take $\phi : L(w) \mapsto [x]$ and make ϕ our strictly order preserving map of $P \mapsto [x]$.

Therefore,

$$\Omega^{\circ}_{P,C}(x, y) = \sum_{L(w)\in\mathcal{L}(P)} \hat{\Omega}_{L(w),C}(x, y).$$

□

Corollary 4.6. *Let P be a bicolored poset and C is the set of celeste colors. Then $\Omega_{P,C}(x, y)$ and $\Omega^{\circ}_{P,C}(x, y)$ are polynomials in x and y.*

This follows from Theorem 4.5.

The main motivation for this line of work was to find a way to decompose the bivariate chromatic polynomial into bivariate order polynomials. We begin by stating the theorem and providing an example about its decomposition.

Theorem 4.7. *Let G be a finite graph and $y \leq x$ be positive integers. Assume P is the acyclic orientation of H regarded as a poset and C is the set of celeste elements that come from the contracted vertices of H.*

$$P(G; x, y) = \sum_{\substack{\text{contractions} \\ H \text{ of } G}} \sum_{\substack{\text{acyclic} \\ \text{orientations} \\ \sigma \text{ of } H}} \Omega^{\circ}_{P(\sigma),C(H)}(x, y)$$

Before proving Theorem 4.7 we illustrate it with an example. We begin by decomposing K_3 into its possible contractions. We denote v_{ij} as the resulting vertex after contracting edge v_iv_j. Figure 4.7 shows all possible ways to contract K_3.

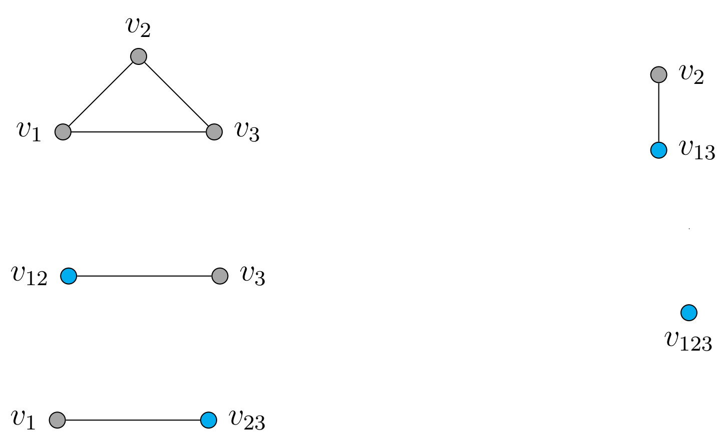

Figure 4.7: Contractions of K_3.

We give a set of steps to decompose the bivariate chromatic polynomial of G:

- Find all possible subgraphs of H of G that result from contractions;
- Color the contracted vertices celeste and the others silver;
- Find all possible acyclic orientations of H;
- Regard each acyclic orientation as a bicolored posets;
- Compute the bivariate order polynomial of each poset.

Let us note that when we do not contract anything then we have the one variable case of the chromatic polynomial. Below we will compute the bivariate order polynomial of each P that comes from an acyclic oriented subgraph. We then add up all order polynomials and obtain the bivariate chromatic polynomial.

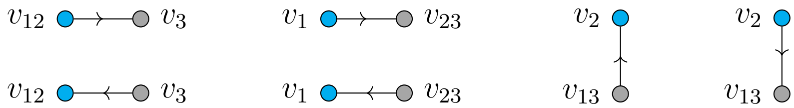

Figure 4.8: Acyclic orientations of contractions of K_3.

- <u>No contractions:</u> We have the one variable case, $P(K_3; x) = x^3 - 3x^2 + 2x$.
- <u>Contract 1 edge:</u> We have two acyclic orientations that lead to two posets, P_1 and P_2.

$$\Omega^\circ_{P_1,v_{12}}(x,y) = \binom{x-y}{2} + (x-y)y \qquad\qquad \Omega^\circ_{P_2,v_{12}}(x,y) = \binom{x-y}{2}$$

- <u>Contract 2 edges:</u> $\Omega^\circ_{P,v_{123}}(x,y) = x - y$ Thus,

$$P(K_3; x, y) = x^3 - 3xy + y = P(K_3, x) + 3\Omega^\circ_{P_1,v_{12}}(x,y) + 3\Omega^\circ_{P_2,v_{12}}(x,y) + \Omega^\circ_{P,v_{123}}(x,y)$$

Proof of Theorem 4.7. Let m be a general proper (x,y)-coloring of G. Let H be the graph obtained by contracting every edge whose endpoints have the same color. The coloring m induces an acyclic orientation σ of H. We regard σ as a poset P and the contracted vertices of H as the celeste elements C of P. The (x,y)-coloring m can be thought of as an order preserving (x,y)-map on P, C.

Conversely, let H be a contracted subgraph of G with σ acyclic orientation and ϕ a strictly order preserving map. Since G is fixed, the contracted components will correspond to the connected vertices that have the same color. The order preserving (x,y)-map has an order and so this will become the generalized proper coloring of G.

Therefore,

$$P(G; x, y) = \sum_{\substack{\text{contractions} \\ H \text{ of } G}} \sum_{\substack{\text{acyclic} \\ \text{orientations} \\ \sigma \text{ of } H}} \Omega^\circ_{P(\sigma),C(H)}(x,y).$$

□

We conclude our line of work by providing the reader with an interesting connection and an open problem.

Chapter 5

Future Work

"You must be the change you wish to see in the world."

- Mahatma Gandhi

The introduction of bicolored posets gave rise to an interesting connection to marked posets. Formally, we provide the definition introduced in [1] by Ardila, Bliem and Salazar. Let P be a poset, and $A \subseteq P$ which contains all maximal and minimal elements, and $\lambda = (\lambda_a)_{a \in A}$ is a vector in $\mathbb{R}^A$. This vector represents the marking of elements of A with real numbers. We say that (P, A, λ) is a **marked poset**. Bicolored posets can be considered a special marked poset with three positive numbers as our markings $\{0, y, x\}$. This means that all silver elements will be bounded below by 0, all celeste elements will be bounded below by y and all the elements will be bounded above by x. The Hasse diagram of the bicolored poset P that we saw in Figure 4.4 represented as a marked poset is illustrated in Figure 5.1.

The reason why this connection is quite interesting is because having marked posets enables us to obtain a marked polytope. Given a marked polytope we can obtain the Ehrhart polynomial as it was argued in [1]. Our intuition and [2] leads us to believe that reciprocity theorem for bivariate order polynomials of bicolored posets is true.

Conjecture 5.1. *Let P be a bicolored poset and C be the set of celeste elements. Then*

$$\Omega^{\circ}_{P,C}(x, y) = (-1)^{|P|}\Omega_{P,C}(-x, -y).$$

A classical approach to proving the reciprocity theorem for order polynomials in one variable was through Ehrhart theory as we know that an order polynomial is an Ehrhart polynomial in disguise. Nonetheless, more work needs to be done in order to prove the above conjecture and it is left as an open problem.

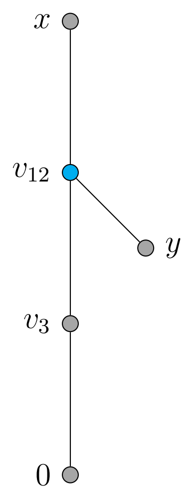

Figure 5.1: Marked Poset with minimal and maximal elements.

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