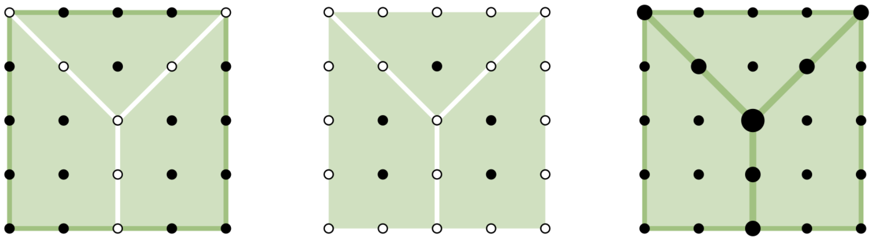

Extensions of Ehrhart theory and applications to combinatorial structures

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Summary

This dissertation presents recent contributions to Ehrhart theory and its applications in Combinatorics. It investigates the enumeration and structure of integer points subject to linear inequalities from a geometric perspective.

We give an introduction in Chapter 1 and background on polyhedral geometry and combinatorial structures used in this work in Chapter 2.

In Chapter 3, published in [Reh22], we use Ehrhart polynomials to count combinatorial and geometric data in generalized permutahedra and hypergraphs. Generalized permutahedra are a class of polytopes with many interesting combinatorial subclasses. We introduce pruned inside-out polytopes, a generalization of inside-out polytopes introduced by Beck–Zaslavsky [BZ06b], which have many applications such as recovering the famous reciprocity result for graph colorings by Stanley. We study the integer point count of pruned inside-out polytopes by applying classical Ehrhart polynomials and Ehrhart–Macdonald reciprocity. This yields a geometric perspective on and a generalization of a combinatorial reciprocity theorem for generalized permutahedra by Aguiar–Ardila [AA23], Billera–Jia–Reiner [BJR09], and Karaboghossian [Kar22]. Applying this reciprocity theorem to hypergraphic polytopes allows us to give a geometric proof of a combinatorial reciprocity theorem for hypergraph colorings by Aval–Karaboghossian–Tanasa [AKT20]. Aside from the reciprocity for generalized permutahedra, this proof relies only on elementary geometric and combinatorial properties of hypergraphs and their associated polytopes.

In Chapter 4, which is joint work with Eleon Bach and Matthias Beck [BBR24], we investigate the coefficients of the Ehrhart polynomial for special classes of zonotopes associated with signed graphs. There is a well-established dictionary between zonotopes, hyperplane arrangements, and (oriented) matroids. Arguably one of the most famous examples is the class of graphical zonotopes, also called acyclotopes, which encode subzonotopes of the type-A root polytope, the permutahedron. Stanley [Sta91] gave a general interpretation of the coefficients of the Ehrhart polynomial (integer-point counting function for a polytope) of a zonotope via linearly independent subsets of its generators. Applying this to the graphical case shows that Ehrhart coefficients count (labeled) forests of the graph of fixed sizes. Our first goal is to extend and popularize this story to other root systems, which on the combinatorial side is encoded by signed graphs analogously to the work by Greene–Zaslavsky [GZ83]. We compute the Ehrhart polynomial of the acyclotope in the signed case, and we give a matroid-dual construction. This gives rise to tocyclotopes and we compute their Ehrhart polynomials. Applying the same duality construction to a general integral matrix leads to a lattice Gale zonotope, whose face structure was studied by McMullen [McM71]. We describe the Ehrhart polynomials of lattice Gale zonotopes in terms of the given matrix.

Chapter 5 is joint work with Matthias Beck and Sophia Elia [BER23]. Here, we extend Ehrhart theory to consider rational dilates of polytopes. The Ehrhart quasipolynomial of a rational polytope P encodes fundamental arithmetic data of P, namely,

the number of integer lattice points in positive integral dilates of P. The enumerative theory of lattice points in rational (equivalently, real) dilates of rational polytopes is much younger, starting with work by Linke [Lin11], Baldoni–Berline–Köppe–Vergne [Bal+13], and Stapledon [Sta17]. We introduce a generating-function *ansatz* for rational Ehrhart quasipolynomials, which unifies several known results in classical and rational Ehrhart theory. In particular, we define γ-rational Gorenstein polytopes, which extend the classical notion to the rational setting and encompass the generalized reflexive polytopes studied by Fiset–Kasprzyk [FK08] and Kasprzyk–Nill [KN12].

In Chapter 6, which is joint work with Alexander E. Black and Raman Sanyal [ASR] we study poset permutahedra, an interesting class of polytopes arising as monotone path polytopes of order polytopes. Poset permutahedra are an amalgamation of order polytopes and permutahedra. We show that poset permutahedra give a unifying perspective on several recent classes of polytopes that occurred, for example, in connection with colorful subdivisions of polygons and Hessenberg varieties. As with order polytopes, the geometry and the combinatorics of poset permutahedra can be completely described in terms of the underlying poset. As applications of our results, we give a combinatorial description of the h-vectors of the partitioned permutahedra of Horiguchi–Masuda–Shareshian–Song [Hor+24b] and poset generalizations of Landau's score sequences of tournaments. To prove our results, we show that poset permutahedra arise from order polytopes via the fiber polytope construction of Billera–Sturmfels [BS92].

Zusammenfassung

Diese Dissertation präsentiert neue Beiträge zur Ehrhart-Theorie und deren Anwendung in der Kombinatorik. Sie untersucht die Struktur und Aufzählung von Gitterpunkten gegeben durch lineare Ungleichungen aus einer geometrischen Perspektive.

In Kapitel 1 befindet sich eine Einleitung und in Kapitel 2 erläutern wir das nötige Vorwissen aus der polyedrischen Geometrie und der Kombinatorik, das in dieser Arbeit verwendet wird.

In Kapitel 3, das in [Reh22] veröffentlicht wurde, nutzen wir Ehrhart-Polynome um kombinatorische und geometrische Daten von Verallgemeinerten Permutaedern und Hypergraphen zu zählen. Verallgemeinerte Permutaeder sind eine Klasse von Polytopen mit vielen interessanten kombinatorischen Unterklassen. Wir führen ausgeästete innen-außen Polytope (engl. "pruned inside-out polytopes") ein. Wir studieren die Gitterpunktzählfunktion von ausgeästete innen-außen Polytopen, in dem wir klassische Ehrhart-Polynome und Ehrhart–Macdonald-Reziprozität anwenden. Das ermöglicht nicht nur eine geometrische Perspektive auf ein kombinatorisches Reziprozitätsresultat für verallgemeinerte Permutaeder von Aguiar–Ardila [AA23], Billera–Jia–Reiner [BJR09], und Karaboghossian [Kar22], sonder auch dessen Verallgmeinerung.

In Kapitel 4, welches gemeinsame Arbeit mit Eleon Bach und Matthias Beck [BBR24] ist, untersuchen wir die Koeffizienten des Ehrhart-Polynoms für spezielle Klassen von Zonotopen, die zu vorzeichenbehafteten Graphen (engl. "signed graphs") gehören. Wir führen eine Dualitäts-Konstruktion für Gitter-Zonotope ein. Das führt zu der Definition von Tozyklotopen und wir berechnen auch deren Ehrhart-Polynom. Wenden wir dieselbe Dualitäts-Konstruktion auf allgemeine ganzzahlige Matrizen an, so erhalten wir die Definition von Gitter-Gale-Zonotopen, deren Seitenstruktur bereits von McMullen [McM71] untersucht wurde. Wir beschreiben deren Ehrhart-Polynome in Abhängigkeit von der gegebenen Matrix.

Kapitel 5 ist gemeinsame Arbeit mit Matthias Beck und Sophia Elia [BER23]. Wir erweitern die klassische Ehrhart-Theorie um rational Streckungen von Polytopen zu betrachten. Die abzählende Theorie für Gitterpunkte in rationalen (und reellen) Streckungen von rationalen Polytopen begann mit Arbeiten von Linke [Lin11], Baldoni–Berline–Köppe–Vergne [Bal+13], und Stapledon [Sta17]. Wir führen einen Ansatz über erzeugende Funktionen für rational Ehrhart-Quasipolynome ein, dieser vereinigt mehrere bekannte Resultate in klassischer und rationaler Ehrhart-Theorie.

In Kapitel 6, das gemeinsame Arbeit mit Alexander E. Black und Raman Sanyal ist, studieren wir Ordnungs-Permutaeder (engl. "poset permutahedra"). Diese bilden eine spannende Klasse von Polytopen, unter anderem da sie eine Fusion von Ordnungspolytopen und Permutaedern sind und als Monotone-Pfade-Polytope entstehen. Ordnungs-Permutaeder geben eine vereinheitlichende Perspektive auf mehrere neue Klassen von Polytopen und deren Gitterpunkte ergeben eine Verallgemeinerung von Punkte-Folgen von Turniergraphen mit Hilfe von Halbordnungen.

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1 Anmerkung von Marian: Was heißt hier höchstens halb so witzig, du hast ziemlich oft geschmunzelt!

Contents

1

Introduction

One common theme in this thesis is the study of the combinatorics, enumeration and structure of integer points subject to linear inequalities from a geometric perspective. The core of this is Ehrhart theory, which studies integer point enumeration in polyhedra. Integer point structures have applications in and connections to various mathematical areas such as Algebraic Combinatorics, Commutative Algebra, Representation Theory, Algebraic and Toric Geometry, as well as real world applications.

A **polytope** $\mathsf{Q} \subseteq \mathbb{R}^d$ is a bounded intersection of finitely many affine half-spaces. We can consider integer points in a polytope as integral solutions to a certain set of linear inequalities. Ehrhart theory is the study of the number of integer points in polytopes and their dilates. More concretely, let $\mathsf{Q} \subseteq \mathbb{R}^d$ be a rational polytope, i.e., one with rational coordinates in the vertices. The **denominator** of Q is the smallest integer k such that $k\mathsf{Q}$ has only integral coordinates in the vertices. For positive integers $n \in \mathbb{Z}_{>0}$ we define the **Ehrhart counting function**

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}; n) := |n\mathsf{Q} \cap \mathbb{Z}^d| .$$

Ehrhart [Ehr62] showed that for a rational polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ and $n \in \mathbb{Z}_{>0}$ the Ehrhart counting function $\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}; n)$ agrees with a quasipolynomial of degree equal to the dimension of Q and period dividing the denominator of Q. We define the **Ehrhart series** of a rational polytope as the generating series of its Ehrhart quasipolynomial:

$$\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P}; t) \ := \ 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}; n)\, t^n$$

The Ehrhart series can be written as a rational function of the form

$$\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P}; t) = \ \frac{\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P}; t)}{(1 - t^k)^{d+1}} .$$

Stanley [Sta80] showed that $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P}; t) \in \mathbb{Z}_{\geq 0}[t]$ has non-negative coefficients.

Typical research questions in Ehrhart theory include:

(i) What is the relation between special structures on the coefficients of the Ehrhart (quasi-)polynomials or $\mathrm{h}^*_{\mathbb{Z}}$-coefficients and properties of the corresponding polytopes?

(ii) Is there combinatorial meaning for coefficients in Ehrhart polynomials for special classes of polytopes?

(iii) Can Ehrhart polynomials be used to count integer valued functions related to combinatorial objects such as graphs, hypergraphs and matroids?

After introducing some background on Polyhedral Geometry, Ehrhart theory, and special classes of polytopes, often associated with combinatorial structures such as graphs, matroids, and hypergraphs in Chapter 2, we will address these research questions. We will now outline the following chapters and our main results.

Chapter 3: Pruned inside-out plytopes

For lattice polytopes, we know that the Ehrhart counting function $\mathrm{ehr}_{\mathbb{Z}}$ *agrees with a polynomial* for positive integers. This means that there exists a polynomial that we can evaluate at positive integers and the result is the value of the Ehrhart counting function. Even though we can plug in negative integers into that polynomial, a priori, there is no reason to believe that this would yield meaningful results. Eugéne Ehrhart conjectured (and proved for various special cases) that the following holds for rational polytopes $\mathsf{Q} \subseteq \mathbb{R}^d$:

$$(-1)^{\dim \mathsf{Q}} \mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}; -n) = \mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}^\circ; n) := \# \left(\mathbb{Z}^d \cap n\mathsf{Q}^\circ\right) \qquad \text{for } n \in \mathbb{Z}_{>0},$$

where Q° is the (relative) interior of the polytope Q. This was proved by Ian G. Macdonald [Mac71]. There are numerous instances of **combinatorial reciprocity results**, i.e., counting functions that agree with a polynomial for positive integers and where evaluating the same polynomial at negative integers agrees with a different but related counting function, see, e.g., [BS18].

Beck and Zaslavsky [BZ06b] developed the notion of inside-out polytopes, that is, polytopes dissected by hyperplanes. Counting integer points in a polytope but off certain hyperplanes turns out to be a useful tool to derive (quasi-)polynomiality results and reciprocity laws for counting functions coming from graph colorings and signed graph colorings, composition of integers, nowhere-zero flows on graphs and signed graphs, antimagic labellings, as well as magic, semimagic, and magic latin squares [BZ06a; BZ06c; BZ10].

In Chapter 3, based on [Reh22], we generalize this idea: A **pruned inside out polytope** $\mathsf{Q} \setminus \bigcup \mathcal{N}^{\mathrm{co}\,1}$ consists of the points that lie inside a polytope Q but not in the codimension-one cones $\mathcal{N}^{\mathrm{co}\,1}$ of a complete polyhedral fan $\mathcal{N}$, see Figure 3.2(a) below. We think of the codimension-one cones $\mathcal{N}^{\mathrm{co}\,1}$ defining a pruned inside-out polytope as *pruned* hyperplanes, hence the name. For a positive integer $n \in \mathbb{Z}_{>0}$, we define the **inner pruned Ehrhart function** as

$$\mathrm{in}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n) := \#\left(\mathbb{Z}^d \cap n \cdot \left(\mathsf{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1}\right)\right)\right).$$

Furthermore, we define a second counting function for pruned inside-out polytopes, the **cumulative pruned Ehrhart function** $\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n)$, for a positive integer $n \in \mathbb{Z}_{>0}$ as

$$\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n) := \sum_{\mathbf{y} \in \mathbb{Z}^d} \mathbb{1}_{n\mathsf{Q}}(\mathbf{y}) \cdot \# \left(\text{closed full-dim. cones in } \mathcal{N} \text{ containing } \mathbf{y}\right).$$

Theorem 3.5. *Let* $\mathsf{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1}\right) \subseteq \mathbb{R}^d$ *be a rational pruned inside-out polytope. Then the inner pruned Ehrhart function* $\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(n)$ *and the cumulative pruned*

Ehrhart function $\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n)$ *agree with quasipolynomials of degree* d *in* $n \in \mathbb{Z}_{>0}$ *and are related by reciprocity:*

$$(-1)^d \,\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(-n) = \mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n).$$

Applying Theorem 3.5 to dilations of the unit cube $[0,1]^d$ and the normal fan of a generalized permutahedron (see Figure 3.3) results in the following combinatorial reciprocity theorem.

Theorem 3.10. *For a generalized permutahedron* $\mathsf{P} \subseteq \mathbb{R}^d$ *and* $k = 0,\dots,d-1$,

$$\chi_{d,k}(\mathsf{P})(m) := \#\Big\{\mathbf{y} \in [m]^d \;:\; \mathbf{y}\text{-}\textit{maximum face } \mathsf{P}^{\mathbf{y}} \textit{ is a } k\text{-}\textit{face}\Big\}$$

agrees with a polynomial of degree $d-k$, *and*

$$(-1)^{d-k}\chi_{d,k}(\mathsf{P})(-m) = \sum_{\mathbf{y}\in[m]^d} \#\left(k\text{-}\textit{faces of } \mathsf{P}^{\mathbf{y}}\right).$$

Theorem 3.10 can be applied to subclasses of generalized permutahedra (see [AA23, Section 18]) to retrieve already known combinatorial reciprocity theorems for, e.g., matroid polynomials [BJR09], Bergmann polynomials of matroids, and Stanley's famous reciprocity theorem for graph colorings [Sta73]. We show how a combinatorial reciprocity theorem for hypergraph colorings in [AKT20] is a consequence of the combinatorial reciprocity for generalized permutahedra. The main tool is a vertex description of hypergraphic polytopes in terms of acyclic orientations of hypergraphs.

While special cases of Theorem 3.10 were previously obtained in [BJR09; AA23; Kar22] using Hopf algebraic methods, our geometric approach offers more flexibility in terms of weighted versions and easy generalizations, e.g., for deformed permutahedra in types B, C, or D [Ard+20].

Chapter 4: Acyclotopes and Tocyclotopes

Not only do the integer points in polyhedral complexes carry combinatorial data, in some cases the coefficients of Ehrhart polynomials do so too. Few general properties of Ehrhart coefficients are known: the leading coefficient is the volume, the second coefficient encodes facet volumina, and the constant term is the Euler characteristic.

For zonotopes, i.e., projections of cubes, we know a bit more: A matrix $\mathbf{A} = [\mathbf{a}_1,\dots,\mathbf{a}_m] \in \mathbb{R}^{d\times m}$ generates the **zonotope** $\mathsf{Z}(\mathbf{A}) := \mathbf{A}\,[0,1]^m$. Stanley [Sta91] showed that if $\mathbf{a}_1,\dots,\mathbf{a}_m \in \mathbb{Z}^d$, then the Ehrhart polynomial of $\mathsf{Z}(\mathbf{A})$ is

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{A}); n) = \sum_{\mathbf{F}} g(\mathbf{F})\, n^{|\mathbf{F}|},$$

where $\mathbf{F}$ ranges over all linearly independent subsets of $\{\mathbf{a}_1,\dots,\mathbf{a}_m\}$ and $g(\mathbf{F})$ is the greatest common divisor of all maximal minors of the matrix whose columns are the elements of $\mathbf{F}$.

There is a well-developed dictionary between three combinatorial objects: the zonotope $\mathsf{Z}(\mathbf{A})$ generated by $\mathbf{A}$, the (central) **hyperplane arrangement** $\mathcal{H}(\mathbf{A})$ with normal vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$, and the **representable matroid** $M(\mathbf{A})$ generated by (the columns of) $\mathbf{A}$ capturing combinatorial data of linear independence. This can be applied to subsets of simple roots in type A, B, C, and D, where the combinatorics can be nicely explained by (signed) graphs.

Matroids have a notion of duality, and if $M(\mathbf{A})$ is a matroid represented by a full rank matrix $\mathbf{A} \in \mathbb{R}^{d\times m}$, then its dual $M^{\triangle}(\mathbf{A})$ has a representation given by Gale duality: choose a basis $\mathbf{d}_1, \ldots, \mathbf{d}_{m-d} \in \mathbb{R}^m$ for the kernel of $\mathbf{A}$, write $\mathbf{d}_1, \ldots, \mathbf{d}_{m-d}$ as the columns in the matrix $\mathbf{D} \in \mathbb{R}^{n\times(m-d)}$. Then the matroid dual to $M(\mathbf{A})$ is represented by the columns in the transposed matrix $\mathbf{D}^T$, i.e., $M^{\triangle}(\mathbf{A}) = M(\mathbf{D}^T)$. The combinatorial structure of the zonotope $\mathsf{Z}(\mathbf{D}^T)$ was studied by McMullen [McM71] under the name *derived zonotopes*.

In Chapter 4, based on joint work with Eleon Bach and Matthias Beck [BBR24], we study the Ehrhart polynomial of $\mathsf{Z}(\mathbf{D}^T)$. In order to ensure that the Ehrhart polynomial of the dual zonotope is well defined, we specialize the above construction: choose a *lattice* basis $\mathbf{d}_1, \ldots, \mathbf{d}_{m-d} \in \mathbb{Z}^m$ for the $\mathrm{kern}(\mathbf{A}) \cap \mathbb{Z}^m$. We call $\mathbf{D}$ the **lattice Gale dual**. This is an instance of arithmetic matroids [DM13; DM12].

Theorem 4.6. *Let $\mathbf{A} \in \mathbb{Z}^{d\times m}$ be of rank d, with lattice Gale dual $\mathbf{D} \in \mathbb{Z}^{m\times(m-d)}$. Then we can compute the Ehrhart polynomial of the associated lattice Gale zonotope $\mathsf{Z}(\mathbf{D}^T)$ as*

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{D}^T); n) = \sum_S \frac{g(\mathbf{A}_S)}{g(\mathbf{A})}\, n^{m-|S|}$$

where the sum is over all spanning sets $S \subseteq [m]$ in the matroid represented by $\mathbf{A}$.

A prime example of the above dictionary between zonotopes, hyperplane arrangements and matroids is given by subsets of simple roots of types A, B, C, and D. Here, we have a combinatorial model given by signed graphs Σ adding yet another class of objects to the dictionary [Zas81]. Let $\mathbf{A}_\Sigma$ denote the incidence matrix of a signed graph Σ, i.e., its columns are simple roots of type A, B, C, and D. If we only consider roots of type A (and the combinatorial model is a classical graph G), Greene and Zaslavsky [GZ83] showed that the vertices of $\mathsf{Z}(\mathbf{A}_G)$ (equivalently, the regions of $\mathcal{H}(\mathbf{A}_G)$) are in one-to-one correspondence with the acyclic orientations of G, and they gave analogous interpretations for all faces of $\mathsf{Z}(\mathbf{A}_G)$. Zaslavsky [Zas91] showed the parallel result for signed graphs and thus coined the charming term **acyclotope** for $\mathsf{Z}(\mathbf{A}_\Sigma)$.

The general formula for Ehrhart coefficients of zonotopes and the fact that the incidence matrix $\mathbf{A}_G$ of a graph G is totally unimodular imply that the k^{th} coefficient of the Ehrhart polynomial of the acyclotope $\mathsf{Z}(\mathbf{A}_G)$ counts the number of forests with k edges in G [Sta91].

We derive a similar theorem for zonotopes generated by subsets of roots in type B, C, and D. Here, the incidence matrix $\mathbf{A}_\Sigma$ is not totally unimodular, but its minors are powers of two. Extending results by [ABM20] we give an analogous combinatorial interpretation for Ehrhart coefficients in terms of pseudo-forests of the signed graph encoding the subset of simple roots, see Theorem 4.4.

It follows from [GZ83; Zas91] that the regions in the hyperplane arrangement dual to the one induced by $\mathbf{A}_\Sigma$ correspond to totally cyclic orientations of the signed graph Σ. Hence the vertices of the dual zonotope $\mathsf{Z}(\mathbf{D}_\Sigma^T)$ are in bijection with totally cyclic orientations and we call this zonotope **tocyclotope**. We give a combinatorial interpretation for the Ehrhart coefficients in terms of the signed graph Σ: The coefficients are again powers of two weighted according to circle and loop structures in spanning sets of the signed graph.

Theorem 4.14. *Let Σ be a connected signed graph whose incidence matrix has full rank. Choose a connected basis $T \subseteq E$ that contains a halfedge if Σ contains an halfedge. Then the Ehrhart polynomial of the tocyclotope $\mathsf{Z}([\mathbf{I} \,|\, -(\mathbf{T}^{-1}\mathbf{R})^T]) \subseteq \mathbb{R}^{m-d}$ is*

$$\operatorname{ehr}_{\mathbb{Z}}(\mathsf{Z}([\mathbf{I} \,|\, -\mathbf{B}^T]); n) = \begin{cases} \sum_S 2^{\operatorname{pc}(S)+\operatorname{lc}(S)} n^{m-|S|} & \text{if } \Sigma \text{ contains a halfedge,} \\ \sum_S 2^{\operatorname{pc}(S)+\operatorname{lc}(S)-1} n^{m-|S|} & \text{otherwise,} \end{cases}$$

where the sums run over all sets $S \subseteq E$ that contain a basis of Σ, i.e., $\Sigma(S)$ contains a maximal pseudo-forest of Σ, and $\operatorname{pc}(S) + \operatorname{lc}(S)$ is the smallest number of pseudo-tree components plus loop-tree components that a maximal pseudo-forest contained in S can have.

There seem to be connections to applications in scheduling periodic time tables [LM07] and tropical geometry for the type A case. In [BLM24], the tocyclotope of a graph is the cycle offset zonotopes and integer points correspond to cycle offsets of an periodic event scheduling problem.

Chapter 5: Rational Ehrhart theory

In some contexts, it is beneficial to express the Ehrhart polynomial in a binomial basis. This leads to the definition of h^*-coefficients. Understanding the connection between properties of h^*-coefficients (or similar ones) and the polytopes is an overarching research question. The h^*-coefficients of a polytope form a palindromic sequence if and only if the polytope belongs to a special class called **Gorenstein**. In Chapter 5, based on joint work with Matthias Beck and Sophia Elia [BER23], we extend classical Ehrhart theory by considering rational dilation factors for the polytope. Among other things, we extended structural results from classical Ehrhart theory to **rational Ehrhart theory**.

The **rational Ehrhart counting function** is defined as

$$\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda) \;:=\; \left| \lambda \mathsf{P} \cap \mathbb{Z}^d \right| ,$$

where $\lambda \in \mathbb{Q}$. This counting function has more subtle properties than the classical Ehrhart counting function. For example, it is not invariant under lattice translations and might not be monotone in λ. To the best of our knowledge, Linke [Lin11] initiated the study of the rational (and real) counting Ehrhart function. She proved several fundamental results starting with the fact that $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ is a **quasipolynomial** in the rational (equivalently, real) variable λ, that is,

$$\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda) \;=\; c_d(\lambda)\, \lambda^d + c_{d-1}(\lambda)\, \lambda^{d-1} + \cdots + c_0(\lambda)$$

where $c_0, c_1, \ldots, c_d \colon \mathbb{Q} \to \mathbb{Q}$ are periodic functions. The **period** of the quasipolynomial $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ is defined as the least common period of $c_0(\lambda), \ldots, c_d(\lambda)$.

We add a generating-function point of view to rational Ehrhart functions as in [Bal+13; Lin11], one that is inspired by [Sta08; Sta17]. To set it up, we need to make a definition. Suppose the rational d-polytope $\mathsf{P} \subseteq \mathbb{R}^d$ is given by the irredundant half-space description $\mathsf{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}$, where $\mathbf{A} \in \mathbb{Z}^{f \times d}$ and $\mathbf{b} \in \mathbb{Z}^f$ such that the greatest common divisor of b_i and the entries in the ith row of $\mathbf{A}$ equals 1, for every $i \in \{1, \ldots, n\}$. We define the **codenominator** r of P to be the least common multiple of the nonzero entries of the right hand side $\mathbf{b}$, i.e., $r := \operatorname{lcm}(b_1, \ldots, b_f)$. As we assume that P is full dimensional, the codenominator is well-defined. We show that $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ is fully determined by evaluations at rational numbers with denominator $2r$; if $\mathbf{0} \in \mathsf{P}$ then we actually need to know only evaluations at rational numbers with denominator r. For this exposition, we restrict to the case when $\mathbf{0} \in \mathsf{P}$.

We associate a generating series to the rational Ehrhart counting function of a full-dimensional rational polytope P with codenominator r, the **rational Ehrhart series**:

$$\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P}; t) := 1 + \sum_{n \in \mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{n}{r}\right) t^{\frac{n}{r}} .$$

Theorem 5.7. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathsf{P}$ is a lattice polytope. Then*

$$\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P}; t) = \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; m)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}}$$

where $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; m)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with nonnegative integral coefficients. Consequently, the counting function $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ agrees with a quasipolynomial and the period of $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ divides $\frac{m}{r}$, i.e., this period is of the form $\frac{j}{r}$ with $j \mid m$.

From this theorem we recover Linke's result [Lin11, Corollary 1.4] that $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ is a quasipolynomial with period dividing q, where q is the smallest positive rational number such that $q\mathsf{P}$ is a lattice polytope. We give structural theorems about these generating functions: rationality and its consequences for the quasipolynomial $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ (Theorem 5.7 and Theorem 5.13), nonnegativity of the coefficients in $\mathrm{h}^*_{\mathbb{Q}}$ (Corollary 5.12), connections to the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial in classical Ehrhart theory (Corollary 5.15), and combinatorial reciprocity theorems (Corollary 5.17 and Corollary 5.18). One can find a precursor of sorts to our generating functions $\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P}; t)$ and $\operatorname{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}; t)$ in work by Stapledon [Sta08; Sta17], and in fact this work was our initial motivation to look for and study rational Ehrhart generating functions.

We explain the connection of [Sta17] to our work in Section 5.2. We deduce that in the case $\mathbf{0} \in \mathsf{P}^\circ$ the generating function $\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P}; t)$ exhibits additional symmetry, i.e., $\mathrm{h}^*_{\mathbb{Q}}$ is palindromic.

A $(d+1)$-dimensional, pointed, rational cone $\mathsf{C} \subseteq \mathbb{R}^{d+1}$ is called **Gorenstein** if there exists a point $(p_0, \mathbf{p}) \in \mathsf{C} \cap \mathbb{Z}^{d+1}$ such that $\mathsf{C}^\circ \cap \mathbb{Z}^{d+1} = (p_0, \mathbf{p}) + \mathsf{C} \cap \mathbb{Z}^{d+1}$ (see, e.g., [BB97; BR07; Sta78]). We define the **homogenization** $\hom(\mathsf{P}) \subseteq \mathbb{R}^{d+1}$ of a rational polytope as $\hom(\mathsf{P}) := \operatorname{cone}(\{1\} \times \mathsf{P})$. A lattice polytope $\mathsf{P} \subseteq \mathbb{R}^d$ is **Gorenstein** if the homogenization $\hom(\mathsf{P})$ of P is Gorenstein. Gorenstein polytopes play an important role in Ehrhart theory, as they have palindromic $\mathrm{h}^*_{\mathbb{Z}}$-polynomials. We gave an analogous definition and characterization of Gorenstein polytopes in the setting of rational Ehrhart theory: A rational d-polytope $\mathsf{P} \subseteq \mathbb{R}^d$ is **γ-rational Gorenstein** if $\mathrm{hom}(\frac{1}{\gamma}\mathsf{P})$ is a Gorenstein cone.

Theorem 5.29. *Let P be a rational d-polytope with codenominator r and $\mathbf{0} \in \mathsf{P}$. Then the following are equivalent for $g, m \in \mathbb{Z}_{\geq 1}$ and $\frac{m}{r}\mathsf{P}$ a lattice polytope:*

(i) P *is r-rational Gorenstein.*

(ii) *There exists a (necessarily unique) integer solution $(g, \mathbf{y})$ to*

$$\begin{aligned} -\langle \mathbf{a}_j, \mathbf{y} \rangle &= 1 \qquad \text{for } j = 1, \ldots, i \\ b_j\, g - r\, \langle \mathbf{a}_j, \mathbf{y} \rangle &= b_j \qquad \text{for } j = i+1, \ldots, n\,. \end{aligned}$$

(iii) $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; m)$ *is palindromic:*

$$t^{(d+1)\frac{m}{r} - \frac{g}{r}}\, \mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}; m\right) \;=\; \mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; m)\,.$$

(iv) $(-1)^{d+1} t^{\frac{g}{r}}\, \mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}; t) = \mathrm{Ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}\right)$.

(v) $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}; \frac{n}{r}) = \mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; \frac{n+g}{r})$ *for all $n \in \mathbb{Z}_{\geq 0}$.*

(vi) $\mathrm{hom}(\frac{1}{r}\mathsf{P})^\vee$ *is the cone over a lattice polytope, i.e., there exists a lattice point $(g, \mathbf{y}) \in \mathrm{hom}(\frac{1}{r}\mathsf{P})^\circ \cap \mathbb{Z}^{d+1}$ such that for every primitive ray generator $(v_0, \mathbf{v})$ of* $\mathrm{hom}(\frac{1}{r}\mathsf{P})^\vee$

$$\langle (g, \mathbf{y})\,, (v_0, \mathbf{v}) \rangle \;=\; 1\,.$$

The equivalence of (i) and (vi) is well known (see, e.g., [BN08, Definition 1.8] or [BG09, Exercises 2.13 and 2.14]). It turns out that there are “many more” *rational* Gorenstein polytopes among rational polytopes than Gorenstein polytopes among lattice polytopes; e.g., any rational polytope containing the origin in its interior is r-rational Gorenstein.

Chapter 6: Poset permutahedra

In joint work with Alexander Black and Raman Sanyal (unpublished, extended abstract to appear in [ASR]), we define a new class of polytopes associated with posets. Recall that an order polytope is given as follows: Let $\mathcal{P}$ be a poset on $[d]$ and consider the **order cone** $\mathsf{C}_{\mathcal{P}} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_a \leq \mathbf{x}_b \text{ for all } a \preceq b \in \mathcal{P}\}$. Then the **order polytope** $\mathsf{O}(\mathcal{P})$ is the order cone intersected with the 0/1-hypercube: $\mathsf{O}(\mathcal{P}) = \mathsf{C}_{\mathcal{P}} \cap [0,1]^d$. We alter that construction here to introduce the **poset permutahedron** $\Pi_{\mathcal{P}}$, which is given by $\mathsf{C}_{\mathcal{P}} \cap \Pi_d$, where Π_d is the standard permutahedron on d elements in $\mathbb{R}^d$, i.e., the convex hull of all the permutations of the point $(1, 2, \ldots, d) \in \mathbb{R}^d$.

Poset permutahedra arise as the **monotone path polytopes** of order polytopes, i.e., a particular case of fiber polytopes introduced by Billera and Sturmfels [BS92].

Theorem 6.16. *Let $\mathcal{P} = ([d], \preceq)$ be a poset. Then the monotone path polytope $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P}))$ of the order polytope $\mathsf{O}(\mathcal{P})$ with respect to the linear function $\mathbf{1}(\mathbf{x}) = \mathbf{x}_1 + \cdots + \mathbf{x}_d$ is a translation of the poset permutahedron, i.e., $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1} = \Pi_{\mathcal{P}}$.*

Theorem 6.16 is our main tool to understand the face structure of poset permutahedra. We give explicit descriptions for vertices (Theorem 6.19), facets (Corollary 6.17), vertex-facet incidences (Corollary 6.24), edge directions (Theorem 6.22), subdivisions (Theorem 6.26) and the volumes (Corollary 6.27) of poset permutahedra in terms of the poset.

A **score sequence** is an integer sequence $0 \leq s_1 \leq \cdots \leq s_d \leq d-1$ that is a possible result of an d-team round-robin tournament. For $d = 2$, the only score sequence is $0 \leq 1$ and for $d = 3$, we have two score sequences: $0 \leq 1 \leq 2$ and $1 \leq 1 \leq 1$. A **score vector** is a tuple of integers $(t_1, \ldots, t_d)$ where t_i records the number of points team i wins during the tournament. For $d = 2$, there are two score vectors: $(0, 1)$ and $(1, 0)$; for $d = 3$, we have 7 score vectors: $(1, 1, 1)$ and the six permutations of $(0, 1, 2)$.

It follows from Landau's theorem [Lan53] that score *vectors* are the integer points in the standard permutahedron. Score *sequences* are the integer points in the standard permutahedron intersected with the order cone of the chain poset. Integer points in the n^{th} dilate of the permutahedron (resp. the permutahedron intersected with the order cone of a chain) correspond to score vectors (resp. score sequences) where n points are distributed in each game[1] (see Proposition 6.31).

We define $\mathcal{P}$**-score vectors** as tuples of integers $(t_1, \ldots, t_d) \in \{0, 1, \ldots, d-1\}^d$ that are possible results of an d-team round-robin tournament under the condition that $t_i \leq t_j$ whenever $i \preceq_{\mathcal{P}} j$ is a relation in the poset $\mathcal{P}$.

Theorem 6.32. *For a finite poset $\mathcal{P}$ the integer points in the n-th dilate of the poset permutahedron $n \cdot (\Pi_{\mathcal{P}} - \mathbf{1}) \cap \mathbb{Z}^d$ correspond to $\mathcal{P}$-score vector of an d-team round-robin tournament, where in each game n points are distributed between the two teams.*

Hence, integer points in poset permutahedra interpolate between score vectors and score sequences. This offers a unifying perspective to study $\mathcal{P}$-score vectors (including classical score sequences and vectors) for arbitrary posets $\mathcal{P}$ and arbitrary numbers of points awarded in each game of the tournament.

This interpretation of lattice points offers an easy argument to show that $2(\Pi_{\mathcal{P}} - \mathbf{1})$ (or equivalently $2\Pi_{\mathcal{P}}$) has the **integer decomposition property (idp)**, that is, for every lattice point $\mathbf{p} \in 2n(\Pi_{\mathcal{P}} - \mathbf{1})$ there exist lattice points $\mathbf{q}_1, \ldots, \mathbf{q}_n \in 2(\Pi_{\mathcal{P}} - \mathbf{1})$ such that $\mathbf{p} = \mathbf{q}_1 + \cdots + \mathbf{q}_n$ (see Corollary 6.33).

Special cases of poset permutahedra show up in several other contexts: The Newton polytope of the discriminant is unimodularly equivalent to the poset permutahedron of a chain. The stellahedron [PRW08, Section 10.4] is combinatorially equivalent the poset permutahedron of a claw of size $d+1$ (i.e., an antichain of size d and a unique maximum). For a disjoint union of two chains of length m and n, the face lattice of the poset permutahedron corresponds to a poset of subdivisions of an $(m+n+2)$-gon with vertices split into two color classes. These triangulations

1 The On-Line Encyclopedia of Integer Sequences (OEIS) entries A000571, A007747, A047729-A047731, and A047733-A047737.

appear in [AMV24] and extend a combinatorial description of the Newton polytope of the resultant from [GKZ90; GKZ08].

For a disjoint union of k chains of lengths $m_1, m_2, \dots, m_k$, the resulting poset permutahedron is a type A partitioned permutahedra recently introduced and studied in [Hor+24b; Hor+24a] in the context of Hessenberg varieties and Representation Theory. They show that partitioned permutahedra are simple and that their h-vectors were determined by using the cohomology of regular Hessenberg varieties. We show that $\Pi_d(K)$ is the poset permutahedron of a disjoint union of chains, which implies simplicity by our Theorem 6.25. We also provide a bijective proof using combinatorial and polyhedral techniques for the following description of the h-vector.

Theorem 6.40 ([Hor+24b, Proposition 7.4])**.** *The h-polynomial of the partitioned permutahedron for $K \subseteq [d-1]$ is given by*

$$h_{\Pi_d(K)}(x) = \sum_{\sigma \in W(K)} x^{des(\sigma)},$$

where $W(K)$ is the set of permutations σ such that $\sigma^{-1}(i) - \sigma^{-1}(i+1) \leq 1$ for all $i \in K$.

2 Background

In this chapter we introduce the basic notation used in this work as well as background on polyhedral geometry, Ehrhart theory, generalized (or deformed) permutahedra, root systems, and some examples of combinatorial structures with interesting geometric perspectives.

2.0 Basic Notation and Vector Spaces with Unordered Basis

We will work in real vector spaces, mostly $\mathbb{R}^d$. In some combinatorial settings it is convenient to work with vector spaces with an unordered basis. We briefly introduce the notation. For a non-empty finite set U let $\mathbb{R}U$ be the real vector space with distinguished, unordered basis U. The elements $i \in U$ are denoted $\mathbf{e}_i$ when we want to distinguish the elements i in the set U from the corresponding basis vector $\mathbf{e}_i$ in the vector space $\mathbb{R}U$. Moreover, we identify an element $\sum_{i\in U} \mathbf{x}_i\mathbf{e}_i$ in the vector space $\mathbb{R}U$ with the tuple $(\mathbf{x}_i)_{i\in U}$ for $\mathbf{x}_i \in \mathbb{R}$. For the disjoint union $U = S \uplus T$ of two finite sets S, T the equality

$$\mathbb{R}S \times \mathbb{R}T = \mathbb{R}U = \mathbb{R}T \times \mathbb{R}S$$

holds, which is handy in combinatorial contexts. For a finite set U with $|U| =: d$ we can identify $\mathbb{R}^d \cong \mathbb{R}U$ by fixing a bijection $\sigma\colon U \to [d] := \{1, \ldots, d\}$. Via this bijection we may also assume $U = [d]$.

The dual vector space $(\mathbb{R}U)^*$ can be interpreted as

$$(\mathbb{R}U)^* = \mathbb{R}^U := \{\text{maps } \mathbf{y}\colon U \to \mathbb{R}\}\,.$$

We call the elements $\mathbf{y} \in \mathbb{R}^U$ **directions**. They act as linear functionals on elements $\mathbf{x} = \sum_{i\in U} \mathbf{x}_i\mathbf{e}_i \in \mathbb{R}U$ via

$$\mathbf{y}\Big(\sum_{i\in U} \mathbf{x}_i\mathbf{e}_i\Big) = \sum_{i\in U} \mathbf{x}_i\mathbf{y}(i)\,.$$

For the vector space $\mathbb{R}^d$ and its dual $\left(\mathbb{R}^d\right)^*$ we identify

$$\left(\mathbb{R}^d\right)^* = \mathbb{R}^{[d]} := \{\text{maps } \mathbf{y}\colon [d] \to \mathbb{R}\}\,.$$

Again, directions $\mathbf{y} \in \mathbb{R}^{[d]}$ act as linear functionals on elements $\mathbf{x} \in \mathbb{R}^d$:

$$\mathbf{y}(\mathbf{x}) = \sum_{i=1}^{d} \mathbf{x}_i\mathbf{y}(i).$$

For elements $\mathbf{y} \in (\mathbb{R}^d)^*$ we will also denote the inner product $\mathbf{y}(\mathbf{x})$ by $\langle \mathbf{y}, \mathbf{x} \rangle$. Similarly, we identify $\mathbb{R}^U \simeq \left(\mathbb{R}^d\right)^*$ by fixing a bijection $\sigma \colon U \to [d] := \{1, \ldots, d\}$. In the context of this work these two notations can and will be used interchangeably. We will also exploit that primal and dual vector spaces are isomorphic.

As mentioned above, for $i \in U$ ($i \in [d]$) we denote by $\mathbf{e}_i \in \mathbb{R}U$ ($\mathbf{e}_i \in \mathbb{R}^d$) the corresponding standard basis vector and for $T \subseteq U$ ($T \subseteq [d]$) we denote by

$$\begin{aligned} \mathbb{1}_T &:= \sum_{i \in T} \mathbf{e}_i \in \mathbb{R}U \quad (\in \mathbb{R}^d) \\ \mathbf{1} &:= \mathbb{1}_U \in \mathbb{R}U \quad (\mathbf{1} := \mathbb{1}_{[d]} \in \mathbb{R}^d) \\ \mathbf{0} &:= \mathbb{1}_\varnothing \in \mathbb{R}U \quad (\in \mathbb{R}^d) \end{aligned}$$

the **indicator vector** $\mathbb{1}_T$ of the subset T, the vector $\mathbf{1}$ with all entries equal to one, and the zero element $\mathbf{0}$ in the vector spaces.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $\mathbf{x} = \lambda_1 \mathbf{x}_1 + \cdots + \lambda_k \mathbf{x}_k$, then $\mathbf{x}$ is called

- a **linear combination** of $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$,
- an **affine combination** of $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$, if $\lambda_1 + \cdots + \lambda_k = 1$.

We define

- the **linear hull** or **linear span** $\operatorname{span}_{\mathbb{R}}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ as the set of all linear combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$, or equivalently, the intersection of all linear subspace containing $\mathbf{x}_1, \ldots, \mathbf{x}_k$,
- the **affine hull** $\operatorname{aff}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ as the set of all affine combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$, or equivalently, the intersection of all affine subspace containing $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

2.1 Polytopes and Polyhedra

We first recall some basic notions from polytopes and polyhedra; for more detailed information consult, e.g., [Zie98; Grü03]. Polyhedra can be described as the intersection of finitely many half-spaces, so their point set is the set of solutions to finitely many linear inequalities, therefore they appear in many areas in Mathematics as well as applications, such as optimization.

A set $P \subseteq \mathbb{R}^d$ is **convex** if for any two points $\mathbf{x}, \mathbf{y} \in P$ we have that $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in P$ for all $\lambda \in [0, 1]$. A **polyhedron** $\mathsf{P} \subseteq \mathbb{R}^d$ is the intersection of finitely many half-spaces:

$$\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{A}\mathbf{x} \leq \mathbf{b}\},$$

where $\mathbf{A} \in \mathbb{R}^{f \times d}$ and $\mathbf{b} \in \mathbb{R}^f$. Such a half-space description is **irredundant** if deleting any row $i \in \{1, \ldots, f\}$ in $\mathbf{A}$ together with the entry $\mathbf{b}_i$ would define a different polyhedron. If the intersection is bounded it is called a **polytope** and can equivalently be described as the convex hull of finitely many points in $\mathbb{R}^d$. A **(polyhedral) cone** C is a polyhedron such that for $\mathbf{x} \in \mathsf{C}$ the point $\lambda \mathbf{x}$ is again contained in C for every $\lambda \in \mathbb{R}_{\geq 0}$. It follows that polyhedra are convex sets.

A **supporting hyperplane** $\mathsf{H} = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\right\}$ of a polyhedron P is a hyperplane such that the polyhedron is contained in one of the closed half-spaces bounded by H. The intersection of a polyhedron P with a supporting hyperplane H

is a **face** $\mathsf{F} = \mathsf{H} \cap \mathsf{P}$ of P. We consider the polytope P itself a face, since it is the intersection with $\mathbb{R}^d = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{0} \cdot \mathbf{x} = 0\right\}$, the degenerate hyperplane. Faces that are not the polytope itself and non-empty are called **proper faces**. The **dimension** $\dim(\mathsf{P})$ (resp. $\dim(\mathsf{F})$) of a polyhedron P (resp. face F) is the dimension of the affine hull of the polytope P (resp. face F). A d-dimensional polyhedron is called a **d-polyhedron** for short, 0-dimensional faces are called **vertices**, 1-dimensional faces are called **edges** and $(\dim(\mathsf{P}) - 1)$-dimensional faces are called **facets**. The empty set is a face of dimension -1. We denote the **set of vertices** of a polytope P by $\operatorname{vert}(\mathsf{P})$. The 1-dimensional faces of a polyhedral cone are called **rays**, an element $\mathbf{x} \in \mathsf{R} \setminus \{\mathbf{0}\}$ is called a **ray generator** and for rational cones we define the **primitive ray generator** of the ray R as an element $\mathbf{r} \in \mathsf{R} \cap \mathbb{Z}^d$ such that there exists no $\lambda \in (0,1)$ with $\lambda\mathbf{r} \in \mathbb{Z}^d$. The **codimension** $\operatorname{codim}(\mathsf{F})$ of a polyhedron F is the difference between the dimension of the ambient space and the dimension of the polyhedron $\dim(\mathsf{F})$. The set of faces can be ordered by inclusion. The resulting partial order (poset) is a lattice[1] called the **face lattice** of a polytope. Polytopes with isomorphic face lattice are called **combinatorially equivalent** or **combinatorially isomorphic**. Define the **boundary** $\partial(\mathsf{P})$ of a polyhedron P as

$$\partial(\mathsf{P}) \coloneqq \bigcup_{\mathsf{F} \text{ a proper face of } \mathsf{P}} \mathsf{F} \subseteq \mathbb{R}^d .$$

The **(relative) interior** P° of a polyhedron P is the point set

$$\mathsf{P}^\circ \coloneqq \mathsf{P} \setminus \partial(\mathsf{P}) .$$

We will call such a point set a **(relatively) open polyhedron**.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $\mathbf{x} = \lambda_1\mathbf{x}_1 + \cdots + \lambda_k\mathbf{x}_k$, then $\mathbf{x}$ is called

- a **conical combination** of $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$, if $\lambda_i \geq 0$,
- a **convex combination** of $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$, if $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_k = 1$,

Recall that a set $A \subset \mathbb{R}^d$ is called **convex** if for $\mathbf{x}, \mathbf{y} \in A$ and $\lambda \in [0,1]$ we have $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in A$. Note that polyhedra are convex sets. We define the

- the **conical hull** $\operatorname{cone}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ as the set of all conical combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$, or equivalently, the intersection of all cones containing $\mathbf{x}_1, \ldots, \mathbf{x}_k$,
- the **convex hull** $\operatorname{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ as the set of all convex combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$, or equivalently, the intersection of all convex sets containing the points $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

It is probably *the* main theorem in Polyhedral Geometry that

- every polyhedron $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{A}\mathbf{x} \leq \mathbf{b}\} \subseteq \mathbb{R}^d$ can be written as

  $$\mathsf{P} = \operatorname{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k) + \operatorname{cone}(\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n)$$

  for some $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^d$,
- every polytope $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{A}\mathbf{x} \leq \mathbf{b}\} \subseteq \mathbb{R}^d$ can be written as

  $$\mathsf{P} = \operatorname{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

  for some $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$, in particular, $\mathsf{P} = \operatorname{conv}(\operatorname{vert}(\mathsf{P}))$,

1 See Section 2.5.5 for definitions of poset and lattice.

- every (polyhedral) cone $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{A}\mathbf{x} \le \mathbf{0}\} \subseteq \mathbb{R}^d$ can be written as

$$\mathsf{P} = \operatorname{cone}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

  for some $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$, in particular, $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$ can be chosen so that they generate the rays of P.

A d-polytope is called a **simplex** if it has $d+1$ vertices. If every facet of a polytope is a simplex it is called **simplicial**. Similarly, a **simplicial polyhedral cone** of dimension $d+1$ has precisely $d+1$ facets. A d-polytope is called **simple** if every vertex is contained in exactly d facets, or equivalently, every vertex is contained in exactly d edges.

For a subset $K \subseteq \mathbb{R}^d$ we define the **polar dual**

$$K^{\vee} := \left\{\mathbf{y} \in \left(\mathbb{R}^d\right)^* : \langle \mathbf{y}, \mathbf{x}\rangle \ge -1 \text{ for all } \mathbf{x} \in K\right\} \subseteq \left(\mathbb{R}^d\right)^*.$$

If P is a full-dimensional polytope with the origin in the interior ($\mathbf{0} \in \mathsf{P}^{\circ}$), then $\mathsf{P}^{\vee}$ is again a polytope and the face lattice of P is anti-isomorphic to the face lattice of $\mathsf{P}^{\vee}$. In particular, the polar dual of a full-dimensional simplicial polytope is a full-dimensional simple polytope and vice versa.

For a d-polytope P we define the **f-vector** $f(\mathsf{P}) = (f_{-1}(\mathsf{P}), f_0(\mathsf{P}), \ldots, f_d(\mathsf{P}))$, where $f_i(\mathsf{P})$ denotes the number of i-dimensional faces in P. For example, for the f-vector of a d-simplex S we have

$$f_i(\mathsf{S}) = \binom{d+1}{i+1} \text{ for } i = -1, \ldots, d\,.$$

If P is a full-dimensional polytope with the origin in the interior, then

$$f_i(\mathsf{P}^{\vee}) = f_{d-i-1}(\mathsf{P}) \text{ for } i = -1, \ldots, d\,.$$

Note that polytopes may have the same f-vector without being combinatorially equivalent. For dimensions two and three, f-vectors of polytopes are completely classified. In higher dimensions f-vectors for simple and simplicial polytopes are completely classified by the g-Theorem, see, e.g., [Zie00, Section 8.6] and references therein. For general polytopes in dimensions ≥ 4 we have inequalities given by the Upper-Bound-Theorem, but no complete characterization of f-vectors.

For a *simple* d-polytope P we define the **h-vector** $h(\mathsf{P}) = (h_0(\mathsf{P}), \ldots, h_d(\mathsf{P}))$, where

$$h_i(\mathsf{P}) := \sum_{k=i}^{d} (-1)^{k-i} \binom{k}{i} f_k(\mathsf{P}) \text{ for } i = 0, 1, \ldots, d\,.$$

We can compute the h-vector of a simple polytope the following way (see, e.g., [Bar02, Chapter VI.6]): First choose a linear functional $\boldsymbol{\omega} \colon \mathbb{R}^d \to \mathbb{R}$, which is **edge-generic**, i.e., for any two vertices $\mathsf{v}, \mathsf{u} \in \operatorname{vert}(\mathsf{P})$ connected by an edge we have $\boldsymbol{\omega}(\mathsf{v}) \neq \boldsymbol{\omega}(\mathsf{u})$. Now, orient the edges of P along $\boldsymbol{\omega}$, that is, if $\boldsymbol{\omega}(\mathsf{v}) < \boldsymbol{\omega}(\mathsf{u})$ then the edge is oriented from v to u. For every vertex v we count its out-degree $\operatorname{outdeg}(\mathsf{v})$ with respect to

$\boldsymbol{\omega}$, i.e., the number of vertices u connected by an edge to v and with $\boldsymbol{\omega}(\mathsf{v}) < \boldsymbol{\omega}(\mathsf{u})$. Then we have

$$h_i(\mathsf{P}) = |\{\mathsf{v} \in |(\mathsf{P}) \colon \operatorname{outdeg}(\mathsf{v}) = i\}| \,. \tag{2.1}$$

We will use this to compute h-vectors in Section 6.5.

A **polyhedral complex** $\mathcal{P}$ is a non-empty finite collection of polyhedra in $\mathbb{R}^d$ such that

(i) if F is a face of some $\mathsf{P} \in \mathcal{P}$ then also $\mathsf{F} \in \mathcal{P}$,
(ii) if $\mathsf{P}, \mathsf{Q} \in \mathcal{P}$ then $\mathsf{P} \cap \mathsf{Q}$ is a face of both P and Q.

We also call elements in $\mathcal{P}$ **cells** in the polyhedral complex. If every cell P in a polyhedral complex $\mathcal{P}$ is a polytope then $\mathcal{P}$ is a **polytopal complex**. If every cell C in a polyhedral complex $\mathcal{P}$ is a polyhedral cone then $\mathcal{P}$ is a **fan**. A **subdivision** $\mathcal{S}$ of a polyhedron P is a polyhedral complex such that

$$\mathsf{P} = \bigcup_{\mathsf{R} \in \mathcal{S}} \mathsf{R} \,.$$

If P is a polytope and $\mathcal{S}$ is a subdivision of P such that every cell $\mathsf{R} \in \mathcal{S}$ is a simplex then we say $\mathcal{S}$ is a **triangulation** of P. In Chapter 6 we will use special types of subdivisions to construct polytopes and study their face lattices.

For a direction $\mathbf{y} \in \left(\mathbb{R}^d\right)^*$ we define the **y-maximal face** $\mathsf{P}^{\mathbf{y}}$ of a polytope P by

$$\mathsf{P}^{\mathbf{y}} \coloneqq \{\mathbf{x} \in \mathsf{P} \,:\, \mathbf{y}(\mathbf{x}) \geq \mathbf{y}(\mathbf{x}') \quad \text{for all } \mathbf{x}' \in \mathsf{P}\}.$$

For a face F of a polytope P define the **open** and **closed normal cone** $\mathsf{N}^{\circ}_{\mathsf{P}}(\mathsf{F})$ and $\mathsf{N}_{\mathsf{P}}(\mathsf{F})$ to be the set of all direction that (strictly) maximize F in P, that is,

$$\mathsf{N}_{\mathsf{P}}(\mathsf{F})^{\circ} \coloneqq \left\{\mathbf{y} \in \left(\mathbb{R}^d\right)^* \,:\, \mathsf{P}^{\mathbf{y}} = \mathsf{F}\right\}$$
$$\mathsf{N}_{\mathsf{P}}(\mathsf{F}) \coloneqq \left\{\mathbf{y} \in \left(\mathbb{R}^d\right)^* \,:\, \mathsf{P}^{\mathbf{y}} \supseteq \mathsf{F}\right\}.$$

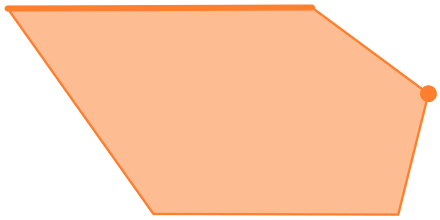
Collecting the normal cones $\mathsf{N}_{\mathsf{P}}(\mathsf{F})$ of all faces F of a polytope P defines the **normal fan**

$$\mathcal{N}(\mathsf{P}) \coloneqq \left\{\mathsf{N}_{\mathsf{P}}(\mathsf{F}) \,:\, \mathsf{F} \text{ a face of } \mathsf{P}\right\}.$$

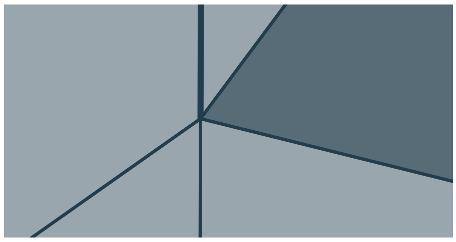
It can be checked that this is indeed a polyhedral fan as defined above. See Figure 2.1 for an example. Polytopes with the same normal fan are called **normally equivalent**. In particular, translating and scaling polytopes preserves the normal fan.

Lemma 2.1. *For a face* F *of a polytope* $\mathsf{P} \subseteq \mathbb{R}^d$ *with dimension* $\dim(\mathsf{F}) = k$ *the dimension of the normal cone is given by* $\dim(\mathcal{N}_{\mathsf{P}}(\mathsf{F})) = d - k = \operatorname{codim}(\mathsf{F})$*. For another face* G *of the polytope* P *we have* $\mathsf{F} \subseteq \mathsf{G}$ *if and only if* $\mathsf{N}_{\mathsf{P}}(\mathsf{F}) \supseteq \mathsf{N}_{\mathsf{P}}(\mathsf{G})$*.*

This implies that if we order the set of normal cones of a polytope by reverse inclusion, then the resulting poset is isomorphic to the face lattice of the polytope. Hence, normal equivalence of polytopes implies combinatorial equivalence.



(a)



(b)

Figure 2.1: Two-dimensional polytope (left) and its normal fan (right). Highlighted edge and vertex in the polytope correspond to highlighted ray and cone in the polyhedral fan, respectively.

Note that normal fans of polytopes are instances of **complete** fans in $\mathbb{R}^d$, that is, the union of the cones in the fan $\mathcal{N}$ covers the ambient space $\mathbb{R}^d$, i.e.,

$$\bigcup \mathcal{N} := \bigcup_{\mathsf{N} \in \mathcal{N}} \mathsf{N} = \mathbb{R}^d .$$

For an introduction to complete fans consult, e.g., [Zie98, Section 7.1]. A fan is called **rational** if all its cones $\mathsf{N} \in \mathcal{N}$ are rational.

For a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we define the **codimension-one fan**[2] $\mathcal{N}^{\mathrm{co}\,1}$ in $\mathbb{R}^d$ to contain the cones in $\mathcal{N}$ with codimension ≥ 1, that is, all but the full-dimensional cones in $\mathcal{N}$:

$$\mathcal{N}^{\mathrm{co}\,1} := \left\{ \mathsf{N} \in \mathcal{N} : \ \dim \mathsf{N} \leq d - 1 \right\} . \tag{2.2}$$

We think of the codimension-one fan as a **pruned** hyperplane arrangement, since cones of codimension one can be seen as parts of hyperplanes. This will be one of the central tools in Chapter 3.

Let $\mathsf{P} \subseteq \mathbb{R}^d$ and $\mathsf{Q} \subseteq \mathbb{R}^e$ be two polytopes, then we define their **Cartesian product** as $\mathsf{P} \times \mathsf{Q} := \{(\mathbf{p}, \mathbf{q}) \in \mathbb{R}^{d+e} : \mathbf{p} \in \mathsf{P}, \mathbf{q} \in \mathsf{Q}\}$. This is again a polytope. For example the unit square is the Cartesian product of two unit intervals, e.g., $[0,1]^2 = [0,1] \times [0,1]$.

The **Minkowski sum** of two sets $A, B \subseteq \mathbb{R}^d$ is defined as

$$A + B := \{\mathbf{x} + \mathbf{y} \in \mathbb{R}^d : \mathbf{x} \in A, \mathbf{y} \in B\} .$$

The Minkowski sum of two polytopes is again a polytope and the normal fan $\mathcal{N}(\mathsf{P} + \mathsf{Q})$ of the Minkowski $\mathsf{P} + \mathsf{Q} \subseteq \mathbb{R}^d$ sum of two polytopes $\mathsf{P}, \mathsf{Q} \subseteq \mathbb{R}^d$ is the (coarsest) common refinement of the normal fans $\mathcal{N}(\mathsf{P})$ and $\mathcal{N}(\mathsf{Q})$ of P and Q [Zie98, Proposition 7.12], i.e.,

$$\mathcal{N}(\mathsf{P} + \mathsf{Q}) = \mathcal{N}(\mathsf{P}) \wedge \mathcal{N}(\mathsf{Q}) := \{\mathsf{N} \cap \mathsf{M} : \mathsf{N} \in \mathcal{N}(\mathsf{P}), \mathsf{M} \in \mathcal{N}(\mathsf{Q})\} . \tag{2.3}$$

2 This is still a fan, but it is not complete anymore.

Minkowski sums are highly nontrivial operations, but they behave nicely in the following two ways. Linear functionals and Minkowski sums commute (see, e.g., [BS18, Lemma 7.5.1]), so for polytopes $\mathsf{P}, \mathsf{Q} \subset \mathbb{R}^d$ we have

$$(\mathsf{P}+\mathsf{Q})^{\mathbf{y}} = \mathsf{P}^{\mathbf{y}} + \mathsf{Q}^{\mathbf{y}} . \tag{2.4}$$

Furthermore, Minkowski sums and convex hulls commute, i.e., for finite sets $A, B \subset \mathbb{R}^d$ we have

$$\operatorname{conv}(A+B) = \operatorname{conv}(A) + \operatorname{conv}(B) .$$

A finite sum of line segments is called a **zonotope**. Up to a translation we may assume that all line segments defining a zonotope have the origin $\mathbf{0}$ as one of their end points. For $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^d$ we write

$$\mathsf{Z}(\mathbf{a}_1, \ldots, \mathbf{a}_m) \coloneqq \sum_{i=1}^{m} [\mathbf{0}, \mathbf{a}_i] . \tag{2.5}$$

Equivalently, a zonotope is a projection of a hypercube

$$\mathsf{Z}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \mathbf{A}([0,1]^m) , \tag{2.6}$$

where $\mathbf{A} \in \mathbb{R}^{d \times m}$ is the linear map defined by the matrix with $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^d$ as columns. Zonotopes will be one of the main objects to study in Chapter 4.

The normal fan of a line segment $[\mathbf{0}, \mathbf{a}]$ is defined by the hyperplane $\mathsf{H}(\mathbf{a}, 0) = \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{a} \cdot \mathbf{x} = 0\}$ and the two half-spaces defined by H. This implies that the normal fan of a zonotope $\mathsf{Z}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ is induced by a **hyperplane arrangement**, i.e., a set of hyperplanes, $\mathcal{H}(\mathbf{A}) \coloneqq \{\mathsf{H}(\mathbf{a}_1, 0), \ldots, \mathsf{H}(\mathbf{a}_m, 0)\}$ in the following way: consider the connected components of

$$\mathbb{R}^d \setminus \bigcup_{i=1}^{m} \mathsf{H}(\mathbf{a}_i, 0) ,$$

these form open polyhedral cones in $\mathbb{R}^d$; the normal fan $\mathcal{N}(\mathsf{Z}(\mathbf{a}_1, \ldots, \mathbf{a}_m))$ consists of the topological closures of those cones and all their faces.

So, every zonotope defines a unique hyperplane arrangement that captures the combinatorial data (face lattice) of the zonotope. However, several different (but normally equivalent) zonotopes may define the same hyperplane arrangement. For example, the unit cube

$$[\mathbf{0}, \mathbf{e}_1] + [\mathbf{0}, \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_3] \subset \mathbb{R}^3$$

and the axis-parallel box

$$[\mathbf{0}, 100 \cdot \mathbf{e}_1] + [\mathbf{0}, 13 \cdot \mathbf{e}_2] + [\mathbf{0}, -57 \cdot \mathbf{e}_3] \subset \mathbb{R}^3$$

have the same normal fan induced by the coordinate planes. We will return to the connections between zonotopes, hyperplane arrangements and vector configurations in Section 2.5.3 and Chapter 4.

2.2 Ehrhart Theory

Ehrhart theory is the study of lattice points in polytopes and their dilates. We will mostly consider the integer lattice $\mathbb{Z}^d$. Slightly simplified, we can say that Ehrhart theory is the study of integer points in polytopes and therefore the study of integer solutions to a finite number of linear inequalities. Ehrhart theory has connections to Number Theory, Computational Geometry, Commutative Algebra, Representation Theory, Combinatorics, and many other areas.

We start this section with a short introduction to general lattices before we define the Ehrhart counting function and state some of the fundamental results in the area.

Lattices[3] are points in a vector space with some periodic structure. As such they are important in Discrete Geometry, Cryptography, and Discrete Optimization, see, e.g., [CS99]. For an introduction to lattices, see, e.g., [Mat13, Section 2.2] or [Rot23]. For linearly independent vectors $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathbb{R}^d$ we define the **lattice** $\Lambda(\mathbf{b}_1, \dots, \mathbf{b}_k)$ as the $\mathbb{Z}$-span of $\mathbf{b}_1, \dots, \mathbf{b}_k$, that is,

$$\Lambda(\mathbf{b}_1, \dots, \mathbf{b}_k) \coloneqq \operatorname{span}_{\mathbb{Z}}(\mathbf{b}_1, \dots, \mathbf{b}_k) \coloneqq \left\{ \sum_{i=1}^{k} a_i \mathbf{b}_i \colon a_i \in \mathbb{Z} \text{ for } i = 1, \dots, k \right\}.$$

Note that different sets of linearly independent vectors may generate the same lattice. Let $\mathbf{B} \in \mathbb{R}^{d \times k}$ be the matrix with $\mathbf{b}_1, \dots, \mathbf{b}_k$ as columns. We will often also denote the set of vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ by $\mathbf{B}$ and write $\Lambda(\mathbf{B}) \coloneqq \Lambda(\mathbf{b}_1, \dots, \mathbf{b}_k)$. Algebraically, a lattice forms a free abelian group with vector addition and can be seen as a (discrete[4]) subgroup of $\mathbb{R}^d$. For example,

$$\Lambda(\mathbf{e}_1, \dots, \mathbf{e}_d) = \Lambda(\mathbb{1}_{\{1\}}, \mathbb{1}_{\{1,2\}}, \dots, \mathbb{1}_{\{1,\dots,d-1\}}, \mathbf{1}) = \mathbb{Z}^d \subset \mathbb{R}^d$$

is the **integer lattice**.

From now on we will restrict our discussion to lattices generated by integer vectors $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{Z}^d$, i.e., to sublattices of $\mathbb{Z}^d$. A square matrix $\mathbf{U} \in \mathbb{Z}^{d \times d}$ is called **unimodular** if it has determinant ± 1. A general integral matrix $\mathbf{U} \in \mathbb{Z}^{d \times k}$ is called **totally unimodular** if every square submatrix has determinant ± 1 or 0. Two lattices $\Lambda(\mathbf{A}) \subseteq \mathbb{Z}^d$ and $\Lambda(\mathbf{D}) \subseteq \mathbb{Z}^d$ are the same if and only if there exists a unimodular matrix $\mathbf{U} \in \mathbb{Z}^{d \times d}$ such that $\mathbf{A} = \mathbf{U}\mathbf{D}$. Then a matrix $\mathbf{U} \in \mathbb{Z}^{d \times d}$ defines a basis for $\mathbb{Z}^d$ if and only if it is unimodular.

The **half-open fundamental parallelepiped** of a lattice $\Lambda(\mathbf{A})$ is the set

$$\Pi(\mathbf{a}_1, \dots, \mathbf{a}_k) \coloneqq \left\{ \lambda_0 \mathbf{a}_1 + \dots + \lambda_k \mathbf{a}_k \in \mathbb{R}^d \ : \ 0 \le \lambda_i < 1 \text{ for } i = 0, \dots, k \right\}. \quad (2.7)$$

The half-open fundamental parallelepipeds tile the space $\mathbb{R}^d$, that means,

$$\mathbb{R}^d = \biguplus_{\mathbf{a} \in \Lambda(\mathbf{A})} \mathbf{a} + \Pi(\mathbf{A})$$

3 Note that these lattices are a fundamentally different concept from the special class of partially ordered sets, also called lattices, which we will introduce in Section 2.5.5.

4 Here, **discrete** means that all points in the lattice have distance at least ϵ from each other for some $\epsilon > 0$.

is a disjoint union. Said differently, every point $\mathbf{x} \in \mathbb{R}^d$ can be written uniquely as the sum of a lattice point $\mathbf{a} \in \Lambda(\mathbf{A})$ and a point $\mathbf{r} \in \Pi(\mathbf{A})$. We will see a similar argument for cones in Equation (2.15) and Figure 2.3.

For $k = d$, we can compute the volume $\mathrm{vol}(\Pi(\mathbf{A}))$ of the half-open fundamental parallelepiped by the absolute value of the determinant $\det(\mathbf{A})$. It also follows that this is precisely the number of integer points contained in the half-open fundamental parallelepiped (see, e.g., [BR15, Lemma 9.2]):

$$|\Pi(\mathbf{A}) \cap \mathbb{Z}^d| = |\det(\mathbf{A})| \,.$$

Let $\mathbf{A} \in \mathbb{Z}^{d\times d}$ and $\mathbf{D} \in \mathbb{Z}^{d\times d}$ be two bases for the same lattice, i.e., $\Lambda(\mathbf{A}) = \Lambda(\mathbf{D})$, then the two half-open fundamental parallelepiped $\Pi(\mathbf{A})$ and $\Pi(\mathbf{D})$ may be different, but they have the same volume $|\det(\mathbf{A})| = |\det(\mathbf{UD})| = |\det(\mathbf{D})|$. We define this as the **determinant of the lattice**: $\det(\Lambda(\mathbf{A})) := |\det(\mathbf{A})|$.

For $k \le d$, we can compute the *relative* volume of the half-open fundamental parallelepiped by using the Smith normal form (see, e.g., [KR22, Section 7.3]): For every matrix $\mathbf{A} \in \mathbb{Z}^{d\times k}$ there exist unimodular matrices $\mathbf{S} \in \mathbb{Z}^{d\times d}$ and $\mathbf{T} \in \mathbb{Z}^{k\times k}$ such that

$$\mathbf{S}\,\mathbf{A}\,\mathbf{T} \;=\; \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_k \\ & & & \\ & & & \end{pmatrix},$$

where every entry except on the diagonal equals zero. Note that here we assumed $\mathbf{A}$ to have full rank k. Then

$$d_1 \dots d_k = \gcd(\text{ all } k\times k \text{ minors of } \mathbf{A}\,) =: g(\mathbf{A})\,. \tag{2.8}$$

Since $\mathbf{S}$ and $\mathbf{T}$ preserve the integer lattice $\mathbb{Z}^d$ it also follows that

$$\mathrm{relvol}(\Pi(\mathbf{A})) = |\Pi(\mathbf{A}) \cap \mathbb{Z}^d| = d_1 \dots d_k = g(\mathbf{A}) \tag{2.9}$$

and this quantity defines the **determinant of the (lower-dimensional) lattice** $\Lambda(\mathbf{A})$, see, e.g., [BR15, Lemma 9.8]. Note that, by definition, $g(\mathbf{A}) = g(\mathbf{A}^T)$.

We want to give yet another interpretation for the quantity $g(\mathbf{A})$. For that, recall that (sub-)lattices have an abelian group structure. We call $\Lambda(\mathbf{B})$ a **sublattice** of $\Lambda(\mathbf{A})$ if it is a subset, i.e., $\Lambda(\mathbf{B}) \subseteq \Lambda(\mathbf{A})$. Since every lattice, by definition, comes with an abelian group structure, every sublattice is also a subgroup, and in fact a normal subgroup. Therefore, for two lattices $\Lambda(\mathbf{B}) \subseteq \Lambda(\mathbf{A})$ we can define its quotient, as the quotient of abelian groups, that is,

$$\Lambda(\mathbf{A})/\Lambda(\mathbf{B}) := \{\mathbf{a} + \Lambda(\mathbf{B}) \colon \mathbf{a} \in \Lambda(\mathbf{A})\}$$

is the set of all cosets of $\Lambda(\mathbf{B})$ in $\Lambda(\mathbf{A})$ and it has a well defined group structure. In particular, we want to consider the special case of the lattice $\mathbb{Z}^d \cap \mathrm{span}_{\mathbb{R}}(\mathbf{B})$ and its sublattice $\mathrm{span}_{\mathbb{Z}}(\mathbf{B}) = \Lambda(\mathbf{B})$. Then both lattices have the same **dimension**, i.e.,

$$\begin{aligned}\dim(\mathbb{Z}^d \cap \mathrm{span}_{\mathbb{R}}(\mathbf{B})) &:= \dim(\mathrm{span}_{\mathbb{R}}(\mathbb{Z}^d \cap \mathrm{span}_{\mathbb{R}}(\mathbf{B}))) \\ &= \dim(\mathrm{span}_{\mathbb{R}}(\mathbf{B})) =: \dim(\Lambda(\mathbf{B}))\,.\end{aligned}$$

The quotient group

$$(\mathbb{Z}^d \cap \mathrm{span}_{\mathbb{R}}(\mathbf{B}))/\Lambda(\mathbf{B}) = \left\{\mathbf{b} + \Lambda(\mathbf{B}) \colon \mathbf{b} \in \Pi(\mathbf{B}) \cap \mathbb{Z}^d\right\}$$

is hence finite. We can compute the size of the quotient group, i.e., its **index**, via

$$\left|(\mathrm{span}_{\mathbb{R}}(\mathbf{B}) \cap \mathbb{Z}^d)/\mathrm{span}_{\mathbb{Z}}(\mathbf{B})\right| = |\Pi(\mathbf{B}) \cap \mathbb{Z}^d| = g(\mathbf{B})\,. \tag{2.10}$$

So, $g(\mathbf{B})$ is also the number of cosets of the discrete subgroup generated by $\mathbf{B}$, considered as a sublattice of the integer points in the linear span of $\mathbf{B}$.

For general (sub-)lattices $\Lambda(\mathbf{B}) \subseteq \Lambda(\mathbf{A}) \subseteq \mathbb{Z}^d$ with the same dimension, i.e.,

$$\dim(\mathrm{span}_{\mathbb{R}}(\mathbf{A})) = \dim(\mathrm{span}_{\mathbb{R}}(\mathbf{B}))\,,$$

we have that $\Lambda(\mathbf{B}) \subseteq \Lambda(\mathbf{A}) \subseteq (\mathbb{Z}^d \cap \mathrm{span}_{\mathbb{R}}(\mathbf{A}))$ are normal subgroups with finite indices. From

$$\left|\mathbb{Z}^d/\mathrm{span}_{\mathbb{Z}}(\mathbf{B})\right| = \left|\mathbb{Z}^d/\mathrm{span}_{\mathbb{Z}}(\mathbf{A})\right| \cdot \left|\mathrm{span}_{\mathbb{Z}}(\mathbf{A})/\mathrm{span}_{\mathbb{Z}}(\mathbf{B})\right|\,,$$

see, e.g., [Rob22, (4.1.3)], it follows that

$$\left|\mathrm{span}_{\mathbb{Z}}(\mathbf{A})/\mathrm{span}_{\mathbb{Z}}(\mathbf{B})\right| = \frac{|\mathbb{Z}^d/\mathrm{span}_{\mathbb{Z}}(\mathbf{B})|}{|\mathbb{Z}^d/\mathrm{span}_{\mathbb{Z}}(\mathbf{A})|} = \frac{g(\mathbf{B})}{g(\mathbf{A})}\,. \tag{2.11}$$

Now we return to studying polytopes and connect them to the notion of lattices. A polytope $\mathsf{P} \subseteq \mathbb{R}^d$ is a **lattice polytope** if all its vertices lie in some lattice Λ. We usually assume this lattice to be the integer lattice $\mathbb{Z}^d$ and call such a polytope **integral polytope**. Most of the time we will use the terms lattice polytope and integral polytope interchangeably and assume the lattice to be $\mathbb{Z}^d$. A polyhedron P is a **rational polyhedron** if all its facet defining hyperplanes H can be described as $\mathsf{H} = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\right\}$ for some $\mathbf{a} \in \mathbb{Z}^d$ and $b \in \mathbb{Z}$. Equivalently, for a rational polytope all its vertices are contained in $\mathbb{Q}^d$.

For a polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ and a positive integer $n \in \mathbb{Z}_{>0}$ we define the $\mathbf{n^{th}}$ **dilate of** Q as

$$n\mathsf{Q} := \left\{\mathbf{x} \in \mathbb{R}^d : \tfrac{1}{n}\mathbf{x} \in \mathsf{Q}\right\} = \left\{n\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \in \mathsf{Q}\right\}.$$

The **Ehrhart counting function** $\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}; n)$ counts the number of integer point in the n^{th} dilate of the polytope Q:

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}; n) := \#\left(\tfrac{1}{n}\mathbb{Z}^d \cap \mathsf{Q}\right) = \#\left(\mathbb{Z}^d \cap n\mathsf{Q}\right)\,. \tag{2.12}$$

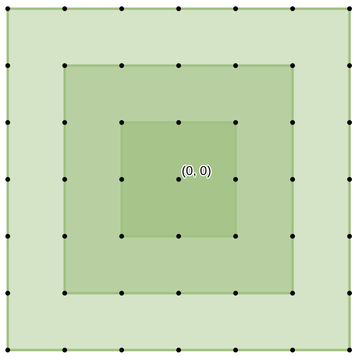


Figure 2.2: The cube $[-1,1]^2$ and its dilates $[-2,2]^2$ and $[-3,3]^2$ with Ehrhart function $\mathrm{ehr}_{\mathbb{Z}}([-1,1]^2;n) = (2n+1)^2$ and $\mathrm{ehr}_{\mathbb{Z}}((-1,1)^2;n) = (2n-1)^2$.

For the Cartesian product $\mathsf{P} \times \mathsf{Q} \subset \mathbb{R}^{d+e}$ of two polytopes $\mathsf{P} \subset \mathbb{R}^d$ and $\mathsf{Q} \subset \mathbb{R}^e$ we can check that for the Ehrhart counting function we have

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P} \times \mathsf{Q}; n) = \mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}; n) \cdot \mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}; n)\,.$$

For example, for the cube $[-1,1]^d$ we have $\mathrm{ehr}_{\mathbb{Z}}([-1,1]^d;n) = (2n+1)^d$. See Figure 2.2 for an example in dimension two. We say that two polytopes $\mathsf{P} \subset \mathbb{R}^d$ and $\mathsf{Q} \subset \mathbb{R}^d$ are **unimodular equivalent**, if there exists a **unimodular transformation** $f\colon \mathbb{R}^d \to \mathbb{R}^d$ given by matrix $\mathbf{U} \in \mathbb{Z}^{d\times d}$ such that

$$f(\mathsf{P}) := \{\mathbf{U}\mathbf{x} \colon \mathbf{x} \in \mathsf{P}\} = \mathsf{Q}\,.$$

We have seen that unimodular matrices leave the lattice $\mathbb{Z}^d$ invariant. This implies that for unimodular equivalent polytopes P and Q,

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}, n) = \mathrm{ehr}_{\mathbb{Z}}(f(\mathsf{P}), n) = \mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}, n)\,.$$

With a similar argument we have for polytopes $\mathsf{P} \subset \mathbb{R}^d$ that

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P} + \mathbf{t}, n) = \mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}, n)$$

for all lattice translations $\mathbf{t} \in \mathbb{Z}^d$.

For a rational polytope Q we define the **denominator of** Q as the smallest integer $k \in \mathbb{Z}_{>0}$ such that $k\mathsf{Q}$ is a lattice polytope. A **quasipolynomial** of degree d is a function $q\colon \mathbb{Z} \to \mathbb{R}$ of the form $q(t) = c_d(t)t^d + \cdots + c_1(t)t + c_0(t)$ where $c_0, c_1, \ldots, c_d \colon \mathbb{Z} \to \mathbb{R}$ are periodic functions. The least common period of $c_0(n), c_1(n), \ldots, c_d(n)$ is the **period** of $q(t)$.

Theorem 2.2 (Ehrhart's theorem [Ehr62])**.** *For a rational polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ and $n \in \mathbb{Z}_{>0}$ the Ehrhart counting function $\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}; n)$ agrees with a quasipolynomial of degree equal to the dimension of Q and period dividing the denominator of Q.*

For a lattice polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ Ehrhart's theorem implies that the Ehrhart counting function $\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}; n)$ is a polynomial. Therefore the Ehrhart counting function of a lattice polytope is called its **Ehrhart polynomial**. It is not hard to see that, if the polytope is rational and full-dimensional, the leading coefficient of the Ehrhart polynomial is constant and equals the volume of the polytope. To show that the constant term of the Ehrhart polynomial equals 1 requires some more work. The second highest coefficient can be computed via the relative volumes of the facets. One of the general research themes in Ehrhart theory is to understand the coefficients and the structure of the Ehrhart polynomial, as well as their connection to properties of the polytope.

It is an interesting research direction to find combinatorial formulas to describe the Ehrhart polynomial for special classes of polytopes. For lattice zonotopes this was done by Richard Stanley in the following theorem.

Theorem 2.3 ([Sta91, Theorem 2.2])**.** *Let* Z *be a zonotope generated by the integer vectors* $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{Z}^d$*. Then the Ehrhart polynomial of* Z *equals*

$$\operatorname{ehr}_{\mathbb{Z}}(\mathsf{Z}; n) = \sum_{\mathbf{F}} g(\mathbf{F})\, n^{\operatorname{rank}(\mathbf{F})}$$

where $g(\mathbf{F})$ *is the greatest common divisor of all maximal minors of* $\mathbf{F}$ *and* $\mathbf{F}$ *ranges over all matrices formed by linearly independent subsets of* $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ *as columns.*

This theorem has a beautiful proof via half-open zonotopal tilings, see, e.g., [BR15, Section 9.2]. The key observation is that every zonotope is the disjoint union of certain translates of half-open parallelepipeds

$$\mathsf{Z}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \biguplus_{\mathbf{F}} \left(\Pi(\mathbf{F}) + \mathbf{t}_{\mathbf{F}}\right) , \tag{2.13}$$

where $\mathbf{F}$ ranges over all linearly independent subsets of $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ and the half-open parallelepiped $\Pi(\mathbf{F})$ is define as in Equation (2.7).

In Chapter 4 we will apply this theorem to zonotopes associated with signed graphs and their "duals" to give combinatorial descriptions for the coefficients of the Ehrhart polynomial. See also Section 2.5.1 for a first instance.

Recall that Theorem 2.2 says that the Ehrhart counting function $\operatorname{ehr}_{\mathbb{Z}}$ *agrees with a polynomial* for positive integers. This means that there exists a polynomial that we can evaluate at positive integers and the result will give us the value of the Ehrhart counting function. Even though we can plug negative integers into that polynomial, there is a priori no reason to believe that this would give us meaningful results. This is where the beauty of **combinatorial reciprocity theorems** starts. There are numerous instances of combinatorial counting functions that agree with a polynomial for positive integers and where evaluating that polynomial at negative integers again agrees with a, different but related, counting function. See [BS18]. The following reciprocity theorem was conjectured and proved for various special cases by Eugéne Ehrhart and proved by Ian G. Macdonald.

Theorem 2.4 (Ehrhart–Macdonald reciprocity [Mac71])**.** *Let* $\mathsf{Q} \subseteq \mathbb{R}^d$ *be a rational polytope. Then*

$$(-1)^{\dim \mathsf{Q}} \operatorname{ehr}_{\mathbb{Z}}(\mathsf{Q}; -n) = \operatorname{ehr}_{\mathbb{Z}}(\mathsf{Q}^\circ; n) := \# \left(\mathbb{Z}^d \cap n\mathsf{Q}^\circ\right) \qquad \textit{for } n \in \mathbb{Z}_{>0} ,$$

where Q° *is the (relative) interior of the polytope* Q*.*

For example, if we count integer points in the interior of the cube $[-1,1]^d$

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}^\circ; n) = (2n-1)^d = (-1)^d(2(-n)+1)^d = (-1)^d\,\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Q}; -n)\,.$$

See Figure 2.2 for an example in dimension two.

We will expand Theorem 2.4 in Chapter 3 and apply it to derive new and old combinatorial reciprocity theorems.

It is a general technique in Combinatorics to study integer sequences via generating functions, see, e.g., [Wil94], [Sta12, Chapter 4]. We define the **Ehrhart series** of a polytope as the generating series of its Ehrhart (quasi-)polynomial:

$$\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P}; t) \;:=\; 1 + \sum_{n\in\mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}; n)\, t^n\,. \tag{2.14}$$

It is an easy exercise in generating functions to show that the generating series of a quasipolynomial with degree d and period k can be written as a rational function of the following form:

$$\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P}; t) \;=\; \frac{\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P}; t)}{(1-t^k)^{d+1}}\,,$$

where $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P}; t) \in \mathbb{Z}[t]$ is a polynomial with integral coefficients and of degree strictly smaller than $k\cdot(d+1)$. The step from $\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}; n)$ to $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P}; t)$ is essentially a change of basis; see, e.g., [BS18, Section 4.5].

We will take a little detour to introduce the integer point transform of a polyhedral cone and to discuss how the Ehrhart series is a specialization of the integer point transform. This is a useful technic both for actual computations as well as for theoretical results in Ehrhart theory.

The **integer point transform** of a polyhedral cone $\mathsf{C} \subseteq \mathbb{R}^{d+1}$ is defined as

$$\sigma(\mathsf{C}; \mathbf{z}) := \sum_{\mathbf{p}\in\mathsf{C}\cap\mathbb{Z}^{d+1}} \mathbf{z}^{\mathbf{p}}\,,$$

where $\mathbf{z}^{\mathbf{p}} = z_0^{\mathbf{p}_0}\cdot z_1^{\mathbf{p}_1}\cdot\dots\cdot z_d^{\mathbf{p}_d}$. For simplicial cones there is a nice recipe to compute a rational function expression for the interger point transform. Let $\mathsf{C} \subseteq \mathbb{R}^{d+1}$ be a rational simplicial $(d+1)$-cone generated by integer vectors $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(d)} \in \mathbb{Z}^{d+1}$, i.e., $\mathsf{C} = \mathrm{cone}\{\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(d)}\}$. Since C is simplicial, $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(d)} \in \mathbb{Z}^{d+1}$ are linearly independent and form a basis for $\mathbb{R}^{d+1}$. So every point in the cone C can be expressed uniquely as a positive linear combination using the cone generators $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(d)}$. Let $\mathbf{p} \in \mathsf{C}\cap\mathbb{Z}^{d+1}$, then there are unique coefficients $\mu_0, \dots, \mu_d \in \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned}
\mathbf{p} &= \mu_0\mathbf{v}^{(0)} + \mu_1\mathbf{v}^{(1)} + \dots + \mu_d\mathbf{v}^{(d)} \\
&= (\lfloor\mu_0\rfloor + \{\mu_0\})\mathbf{v}^{(0)} + \dots + (\lfloor\mu_d\rfloor + \{\mu_d\})\mathbf{v}^{(d)} \\
&= \underbrace{\lfloor\mu_0\rfloor\mathbf{v}^{(0)} + \dots + \lfloor\mu_d\rfloor\mathbf{v}^{(d)}}_{\in\mathbb{Z}^{d+1}} + \{\mu_0\}\mathbf{v}^{(0)} + \dots + \{\mu_d\}\mathbf{v}^{(d)} \in \mathbb{Z}^{d+1}\,,
\end{aligned} \tag{2.15}$$

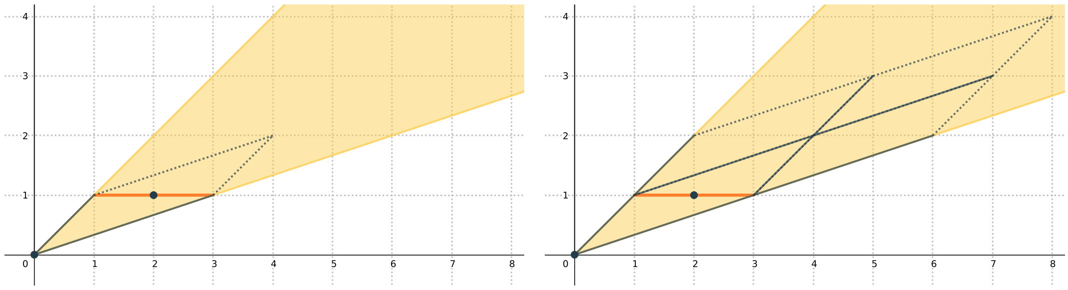


Figure 2.3: Cone C generated by $(1,1)$ and $(1,3)$ in yellow together with the half-open fundamental parallelepiped in blue (left) and the cone together with a few translates of the half-open fundamental parallelepiped (right).

where $\lfloor \mu \rfloor \in \mathbb{Z}$ denotes the largest integer smaller than or equal to $\mu \in \mathbb{R}$ and $\{\mu\}$ denotes the fractional part of μ, i.e., $\{\mu\} = \mu - \lfloor \mu \rfloor \in [0,1)$. It follows that

$$\{\mu_0\}\mathbf{v}^{(0)} + \cdots + \{\mu_d\}\mathbf{v}^{(d)} \in \mathbb{Z}^{d+1}$$

is a unique integer point in the **half-open fundamental parallelepiped**

$$\mathsf{\Pi}(\mathbf{v}^{(0)}, \ldots, \mathbf{v}^{(d)}) := \left\{\lambda_0\mathbf{v}^{(0)} + \cdots + \lambda_d\mathbf{v}^{(d)} \in \mathbb{R}^{d+1} \;:\; 0 \le \lambda_i < 1 \text{ for } i = 0, \ldots, d\right\} .$$

Hence, every integer point in the simplicial cone C has a unique expression as a nonnegative integer combination of the generators plus a lattice point in the half-open fundamental parallelepiped. See Figure 2.3 for an example.

So,

$$\begin{aligned}
\sigma(\mathsf{C};\mathbf{z}) &= \sum_{\mathbf{p}\in\mathsf{C}\cap\mathbb{Z}^{d+1}} \mathbf{z}^{\mathbf{p}} \\
&= \sum_{\mathbf{p}\in\mathsf{\Pi}\cap\mathbb{Z}^{d+1}} \mathbf{z}^{\mathbf{p}} \cdot \sum_{n_0,\ldots,n_d\in\mathbb{Z}_{\ge 0}} \mathbf{z}^{n_0\mathbf{v}^{(0)}+\cdots+n_d\mathbf{v}^{(d)}} \\
&= \frac{\sigma(\mathsf{\Pi};\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}^{(0)}}) \cdot \ldots \cdot (1-\mathbf{z}^{\mathbf{v}^{(d)}})} ,
\end{aligned}$$

where

$$\sigma(\mathsf{\Pi};\mathbf{z}) = \sum_{\mathbf{p}\in\mathsf{\Pi}\cap\mathbb{Z}^{d+1}} \mathbf{z}^{\mathbf{p}} .$$

By triangulation and inclusion-exclusion arguments it follows that the integer point transform of every polyhedral cone exhibits an expression as a rational function.

Example 2.5. We compute the integer point transform for the cone in Figure 2.3. The half-open fundamental parallelepiped $\mathsf{\Pi}((1,1),(1,3))$ contains two lattice points, $(0,0)$ and $(1,2)$. Hence the integer point transform

$$\sigma(\mathsf{C};\mathbf{z}) = \frac{1+\mathbf{z}^{(1,2)}}{(1-\mathbf{z}^{(1,1)})\cdot(1-\mathbf{z}^{(1,3)})} = \frac{1+\mathbf{z}_0\mathbf{z}_1^2}{(1-\mathbf{z}_0\mathbf{z}_1)\cdot(1-\mathbf{z}_0\mathbf{z}_1^3)} .$$

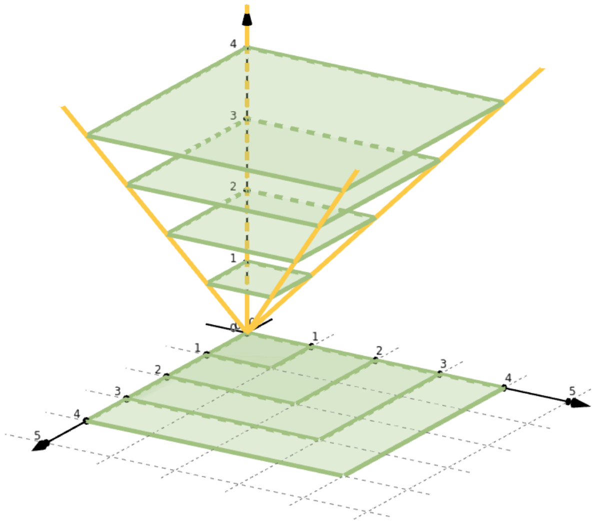


Figure 2.4: The cone $\hom([0,1]^2) = \mathrm{cone}(\{1\} \times [0,1]^2)$ (rays in yellow) with copies of $n\mathsf{P}$ at height n in the cone (in green).

We will now see that the Ehrhart series is a special case of an integer point transform. For that, we define the **cone over** $\mathsf{P} \subseteq \mathbb{R}^d$ (also the **homogenization of** P) of a rational polytope $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ as

$$\hom(\mathsf{P}) \;:=\; \mathrm{cone}(\{1\} \times \mathsf{P}) \;:=\; \Big\{(x_0, \mathbf{x}) \in \mathbb{R}^{d+1} :\, \mathbf{A}\mathbf{x} \le x_0 \mathbf{b}\,,\; x_0 \ge 0\Big\} \subseteq \mathbb{R}^{d+1}\,.$$

It follows that

$$\hom(\mathsf{P}) \cap \{(x_0, \mathbf{x}) \in \mathbb{R}^{d+1} \colon x_0 = n\} = \{n\} \times (n\mathsf{P})\,.$$

Hence, by intersecting the cone $\hom(\mathsf{P})$ with the hyperplane $\{(x_0, \mathbf{x}) \in \mathbb{R}^{d+1} \colon x_0 = n\}$ at height n we recover a copy of $n\mathsf{P}$ embedded in $\mathbb{R}^{d+1}$. We observe that the number of lattice points in $\hom(\mathsf{P})$ at height n is the same as the number of lattice point in $n\mathsf{P}$. See Figure 2.4 for an example. Therefore we can express the Ehrhart series of P as a specialization of the integer point transform of $\hom(\mathsf{P})$:

$$\begin{aligned}
\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P}; t) &= 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}; n)\, t^n \\
&= 1 + \sum_{n \in \mathbb{Z}_{>0}} \left| (\hom(\mathsf{P}) \cap \{(x_0, \mathbf{x}) \colon x_0 = n\}) \cap \mathbb{Z}^{d+1} \right| \, t^n \\
&= \sum_{(x_0, \mathbf{x}) \in \hom(\mathsf{P}) \cap \mathbb{Z}^{d+1}} (t, \mathbf{1})^{(x_0, \mathbf{x})} \\
&= \sigma(\hom(\mathsf{P}); (t, \mathbf{1}))\,.
\end{aligned}$$

Now, let k be the smallest integer so that all vertices of $k\mathsf{P}$ are integral, i.e., the lowest height in the cone $\hom(\mathsf{P})$ so that all the ray generators are integral and denote them by $\mathbf{v}^{(0)}, \ldots, \mathbf{v}^{(d)} \in \mathbb{Z}^{d+1}$. Then $\mathbf{v}^{(0)}_0 = \cdots = \mathbf{v}^{(d)}_0 = k$. Hence,

$$\begin{aligned}
\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P};t) &= \sigma(\mathrm{hom}(\mathsf{P});(t,\mathbf{1})) \\
&= \frac{\sigma(\Pi;(t,\mathbf{1}))}{(1-t^{\mathbf{v}_0^{(0)}})\cdot\ldots\cdot(1-t^{\mathbf{v}_0^{(d)}})} \\
&= \frac{\sigma(\Pi;(t,\mathbf{1}))}{(1-t^k)^{d+1}}\,.
\end{aligned}$$

Example 2.6. We continue Example 2.5 and compute the Ehrhart series for the cone in Figure 2.3. Note that the cone $\mathsf{C} = \mathrm{cone}\{(1,1),(1,3)\} = \mathrm{hom}([1,3])$ is the homogenization of the line segment $[1,3] \subset \mathbb{R}$. Then,

$$\mathrm{Ehr}_{\mathbb{Z}}([1,3];t) = \sigma(\mathrm{hom}([1,3]);(t,1)) = \frac{1+t\cdot 1^2}{(1-t\cdot 1)\cdot(1-t\cdot 1^3)} = \frac{1+t}{(1-t)^2}\,.$$

In the simplicial case, the polynomial $\sigma(\Pi;(t,\mathbf{1})) = \mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t)$ records the number of integer points at the different heights in the half-open fundamental parallelepiped. So the coefficients of the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial are nonnegative for simplices. For general polytopes one can use triangulation and inclusion-exclusion methods to compute the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial. However, those computations do not preserve nonnegativity. The following theorem shows that the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial for general polytopes has nonnegative coefficients nonetheless.

Theorem 2.7 ([Sta80, Theorem 2.1])**.** *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with denominator k. Then,*

$$\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P};t) = \frac{\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t)}{(1-t^k)^{d+1}}$$

and $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t) \in \mathbb{Z}_{\geq 0}[t]$ is a polynomial with nonnegative coefficients.

Richard Stanley proved this theorem in the setting of commutative algebra, today there are also more geometric proofs (see, e.g., [BR15]).

It follows from Theorem 2.4 that

$$\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P}^\circ;t) := \sum_{n\in\mathbb{Z}_{\geq 1}} \mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}^\circ;n)t^n = (-1)^{\dim(\mathsf{P})+1}\,\mathrm{Ehr}_{\mathbb{Z}}\left(\mathsf{P};\frac{1}{t}\right).$$

It is one of the main research directions within Ehrhart theory to investigate the structure of $\mathrm{h}^*_{\mathbb{Z}}$-polynomials, e.g., for special classes of polytopes. On example is the class of Gorenstein polytopes, which can be characterized by their $\mathrm{h}^*_{\mathbb{Z}}$-polynomials.

A lattice polytope P is called **reflexive** if there exists a lattice point $\mathbf{p}$ in the interior P° of P that has lattice distance one to every facet, i.e., for every facet F there exists a normal $\mathbf{n}_{\mathsf{F}} \in (\mathbb{Z}^d)^*$ such that

$$\langle \mathbf{n}_{\mathsf{F}}, \mathbf{p}\rangle = \langle \mathbf{n}_{\mathsf{F}}, \mathbf{v}\rangle + 1$$

for every vertex $\mathbf{v}$ in the facet F. A lattice polytope with the origin in its interior is reflexive if and only if its polar dual polytope is a lattice polytope. This is also equivalent to the polar dual polytope being reflexive. A lattice polytope P is called **Gorenstein of index** k if $k\mathsf{P}$ is a reflexive polytope.

Theorem 2.8. *Let $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \leq \mathbf{b}\} \subset \mathbb{R}^d$ be a lattice d-polytope and let s denote the degree of $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t)$. Then the following are equivalent:*

(i) P *is Gorenstein of index k.*

(ii) *There exists a (necessarily unique) integer solution $(k, \mathbf{x})$ to*

$$k - \langle \mathbf{a}_j, \mathbf{x} \rangle = 1 \quad \text{for } j = 1, \ldots, f\,.$$

(iii) *the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial is palindromic, i.e.,*

$$\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t) = t^s\,\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};\frac{1}{t})\,.$$

(iv) $(-1)^{d+1} t^k \operatorname{Ehr}_{\mathbb{Z}}(\mathsf{P};t) = \operatorname{Ehr}_{\mathbb{Z}}(\mathsf{P};\frac{1}{t})$.

(v) $\operatorname{ehr}_{\mathbb{Z}}(\mathsf{P},n) = (-1)^d \operatorname{ehr}_{\mathbb{Z}}(\mathsf{P};-n-k) = \operatorname{ehr}_{\mathbb{Z}}(\mathsf{P}^\circ;n+k)$ *for all $n \in \mathbb{Z}_{\geq 0}$.*

(vi) $\hom(\mathsf{P})^\vee$ *is the cone over a lattice polytope, i.e., there exists a lattice point $(x_0, \mathbf{x}) \in \hom(\mathsf{P})^\circ \cap \mathbb{Z}^{d+1}$ such that for every primitive ray generator $(w_0, \mathbf{w})$ of $\hom(\mathsf{P})^\vee$*

$$\langle (x_0, \mathbf{x})\,, (w_0, \mathbf{w}) \rangle = 1\,.$$

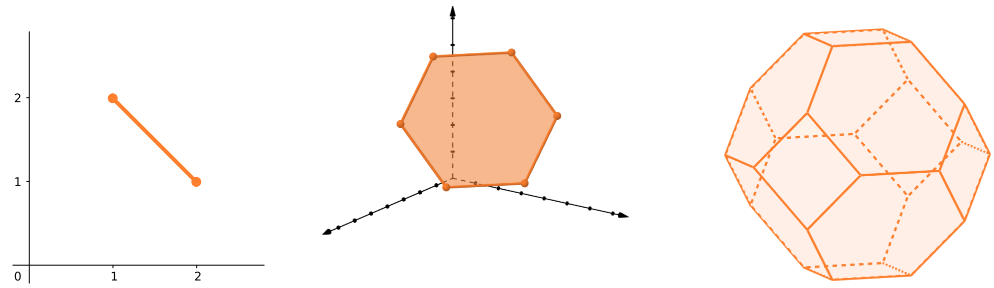
For proofs see, e.g., [HNP19, Section 7.3], [BR15, Exercise 4.8], [BN08, Definition 1.8], or [BG09, Exercises 2.13 and 2.14]. In Chapter 5 we generalize the definition of Gorenstein polytope to rational polytopes and give analogous characterization theorems.

2.3 Generalized Permutahedra

Generalized permutahedra or deformed permutahedra[5] are an interesting class of polytopes. Generalized permutahedra have several equivalent characterizations, as polymatroids [Edm70], as submodular functions [Fuj05], as deformations of standard permutahedra [CL20], by their edge directions, via their normal fans, and as (virtual) Minkowski sums of scaled standard simplices [ABD10], see also [Pos09], [AA23], and [PRW08]. Moreover, this class of polytopes contains many subclasses associated with combinatorial structures, as we will see in Section 2.5.

In this section we discuss in detail generalized permutahedra, which are also deformed Coxeter permutahedra in type A. We will introduce the latter in Section 2.4. Many of the properties discussed in this section translate to deformed Coxeter permutahedra.

5 In fact, *deformed permutahedra* is the more precise and descriptive term, as we will see later in this section.However, *generalized permutahedra* is more commonly used and known.



(a) $\Pi_2 \subseteq \mathbb{R}^2$ (b) $\Pi_3 \subseteq \mathbb{R}^3$ (c) $\Pi_4 \subseteq \mathbb{R}^4$

Figure 2.5: Three standard permutahedra.

We define the **standard permutahedron**[6] Π_d as the convex hull of the $d!$ permutations of the point $(1, 2, \ldots, d)$, that is, the standard permutahedron Π_d is defined by[7]

$$\Pi_d \coloneqq \operatorname{conv}\left\{(\mathbf{x}_i)_{i\in[d]} \in \mathbb{R}^d \colon \{\mathbf{x}_i\}_{i\in[d]} = [d]\right\} \subseteq \mathbb{R}^d.$$

Figure 2.5 shows examples in low dimensions. Note that the standard permutahedron is of dimension $d-1$ since all vertices are contained in a hyperplane with constant coordinate sum. In our definition, standard permutahedra are integer polytopes. The standard permutahedron can equivalently be described as the Minkowski sum of line segments:

$$\Pi_d = \sum_{i<j} \Delta_{\{i,j\}}, \tag{2.16}$$

where $\Delta_{\{i,j\}} \coloneqq [\mathbf{e}_i, \mathbf{e}_j] \coloneqq\coloneqq \operatorname{conv}\{\mathbf{e}_i, \mathbf{e}_j\}$ and $\mathbf{e}_i$ are standard basis vectors. This implies, in particular, that standard permutahedra are zonotopes. The facet description of the standard permutahedron is given by

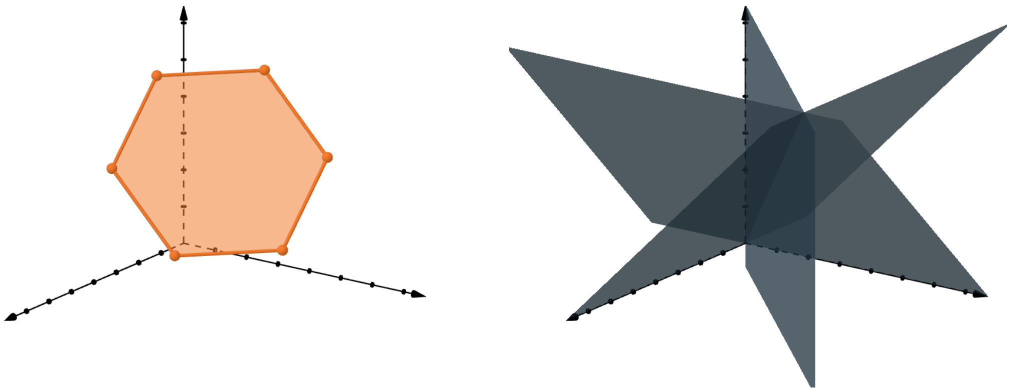
$$\begin{aligned} \sum_{i=1}^{d} \mathbf{x}_i &= d + (d-1) + \cdots + 1 = \frac{d(d+1)}{2} = \tbinom{d+1}{2} \\ \sum_{i\in T} \mathbf{x}_i &\leq d + (d-1) + \cdots + (d - |T| + 1) = \tbinom{d+1}{2} - \tbinom{|T|+1}{2} \quad \text{for all } T \subseteq [d]. \end{aligned} \tag{2.17}$$

Every face of the standard permutahedron can be described combinatorially by compositions, for details see, e.g., [AA23, Section 4.1.].

The normal fan of the standard permutahedron has a nice description via the **braid arrangement** $\mathcal{B}_d$, the hyperplane arrangement consisting of the finite set of hyperplanes $\mathsf{H}_{ij} \coloneqq \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_i = \mathbf{x}_j\}$ for $i, j \in [d]$, $i \neq j$. See Figure 2.6(b) for the example $\mathcal{B}_3$. The connected components of $\mathbb{R}^d \setminus \bigcup \mathcal{B}_d$ are the **(open) regions** of the arrangement. The **closed regions** of the braid arrangement are the topological closures of the open regions. They are polyhedral cones and their faces are the **faces** of the braid arrangement, also called **braid cones**. The braid cones can be described uniquely by compositions $[d] = T_1 \uplus \cdots \uplus T_k$ (Lemma 2.10). We therefore denote them by $\mathsf{B}_{T_1,\ldots,T_k}$. For more details about concepts on hyperplane arrangements see,

6 Also called permut*o*hedron.

7 The definition of standard permutahedron is not consistent within literature, e.g., Postnikov defines the standard permutahedron in a more general way: as the convex hull of all the points obtained by permuting the coordinates of an arbitrary point [Pos09, Definition 2.1].



(a) Standard permutahedron Π_3 (b) Braid arrangement $\mathcal{B}_3$

Figure 2.6: The standard permutahedron $\Pi_3 \subseteq \mathbb{R}^3$ (left) and the normal fan in $(\mathbb{R}^3)^*$ (right), where the intersection line is the normal cone $\mathsf{N}_{\Pi_3}(\Pi_3)$, the half hyperplanes are the normal cones of the edges and the full-dimensional cones are the normal cones of the vertices of Π_3.

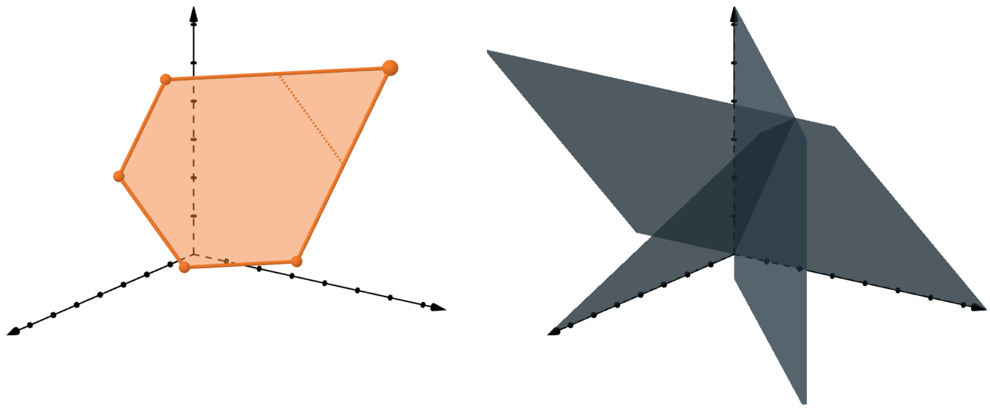


Figure 2.7: A generalization P of the standard permutahedron Π_3: here the up-right edge was moved outwards until it degenerated to a vertex. The normal cone of that "new" vertex is the union of the normal cones of the "old" degenerated edge and its adjacent vertices.

for example, [Sta07]. The faces of the braid arrangement $\mathcal{B}_d$ form the **braid fan** and the normal fan $\mathcal{N}(\Pi_d)$ of the standard permutahedron Π_d is precisely the braid fan (see, for example, [AA23, Section 4]).

We say a fan $\mathcal{N}$ is a **coarsening** of another fan $\mathcal{N}'$ if every cone in $\mathcal{N}$ is the union of some cones in $\mathcal{N}'$. A polytope $\mathsf{P} \subseteq \mathbb{R}^d$ is a **generalized permutahedron** or **deformed permutahedra** if its normal fan $\mathcal{N}(\mathsf{P})$ is a coarsening of the normal fan $\mathcal{N}(\Pi_d)$ of the standard permutahedron Π_d, that is, it is a coarsening of the fan induced by the braid arrangement $\mathcal{B}_d$.

This is equivalent to saying a polytope $\mathsf{P} \subseteq \mathbb{R}^d$ is a generalized permutahedron if and only if every edge is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in [d]$.

Since the normal fan $\mathcal{N}(\mathsf{P} + \mathsf{Q})$ of the Minkowski sum $\mathsf{P} + \mathsf{Q}$ of two polytopes P and Q is the common refinement of the two normal fans $\mathcal{N}(\mathsf{P})$ and $\mathcal{N}(\mathsf{Q})$, see Equation (2.3), generalized permutahedra are the **(weak) Minkowski summands** of standard permutahedra. That is, $\mathsf{P} \subseteq \mathbb{R}^d$ is a generalized permutahedron if and only if there exists a polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ and a real scalar $\lambda > 0$ such that $\mathsf{P} + \mathsf{Q} = \lambda\Pi_{[d]}$.

Every Minkowski sum of **standard simplices** $\Delta_I \coloneqq \operatorname{conv}(\mathbf{e}_i \colon i \in I) \subset \mathbb{R}^d$

$$\sum_{I \subseteq [d]} y(I)\Delta_I \qquad \text{for } y(I) \in \mathbb{R}_{\geq 0}$$

is a generalized permutahedron. We will see a proof of this fact for $y(I) \in \mathbb{Z}_{\geq 0}$ in Proposition 2.15 below.

Conversely, every generalized permutahedron $\mathsf{P} \subset \mathbb{R}^d$ can be uniquely written as a **signed Minkowski sum** of standard simplices

$$\mathsf{P} = \sum_{I \subseteq [d]} y(I)\Delta_I \qquad \text{for } y(I) \in \mathbb{R}\,,$$

where **Minkowski difference** is defined by $\mathsf{P} - \mathsf{Q} = \mathsf{R}$ if $\mathsf{P} = \mathsf{R} + \mathsf{Q}$. See [ABD10].

Finally, generalized permutahedra can be uniquely described as the base polytopes of submodular functions. Recall, a set function $z \colon 2^U \to \mathbb{R}$ is called **submodular** if for all $A, B \subseteq U$

$$z(A) + z(B) \geq z(A \cup B) + z(A \cap B)\,. \tag{2.18}$$

We define the **base polytope** $\mathsf{P}(z)$ of a submodular function $z \colon 2^U \to \mathbb{R}$ by

$$\mathsf{P}(z) \coloneqq \Big\{ \mathbf{x} \in \mathbb{R}U \;:\; \sum_{i \in U} \mathbf{x}_i = z(U) \;\text{ and }\; \sum_{i \in A} \mathbf{x}_i \leq z(A) \;\text{ for all } A \subseteq U \Big\}\,.$$

Theorem 2.9. *A polytope $\mathsf{P} \subset \mathbb{R}U$ is a generalized permutahedron if and only if it is the base polytope $\mathsf{P}(z)$ of a submodular function $z \colon 2^U \to \mathbb{R}$ with $z(\varnothing) = 0$.*

Note that we can interpret a submodular function $z \colon 2^{[d]} \to \mathbb{R}$ with $z(\varnothing) = 0$ as an element in $\mathbb{R}^{2^d - 1}$ and from Equations (2.18) it follows that the set of submodular functions forms a polyhedral cone, called the **deformation cone** of the standard permutahedron Π_d. More about deformation cones can be found, e.g., in [PRW08, Appendix].

Theorem 2.9 is well-known, see, e.g., [CL20, Theorem 3.11 and 3.17]. For the sake of completeness and the convenience of the reader we include a self-contained proof below. We do not claim the proof to be either new or original, but it seems to be hard to find in the literature.

As explained in Section 2.0 we will use the notations for $\mathbb{R}^d$ and $\mathbb{R}^U$ interchangeably. We further simplify notation by using

$$\mathbf{x}(A) \coloneqq \sum_{i \in A} \mathbf{x}_i \qquad \text{for } A \subseteq U\,.$$

With that at hand we can write the definition of base polytopes of submodular functions as

$$\mathsf{P}(z) = \Big\{ \mathbf{x} \in \mathbb{R}U \;:\; \mathbf{x}(U) = z(U) \;\text{ and }\; \mathbf{x}(A) \leq z(A) \;\text{ for all } A \subseteq U \Big\}\,.$$

As mentioned above the standard permutahedron Π_U is the base polytope of the submodular function

$$z(A) := |U| + (|U| - 1) + \cdots + (|U| - |A| + 1)\,.$$

We first give a description of the faces of the braid fan in terms of set compositions. A **composition** of a finite set U is an ordered sequence $(T_1, \ldots, T_k)$ of disjoint non-empty subsets $T_i \subseteq U$ such that $U = T_1 \uplus \cdots \uplus T_k$.

Lemma 2.10. *The faces of the braid arrangement $\mathcal{B}_U$, also called **braid cones**, can be described uniquely by compositions $U = T_1 \uplus \cdots \uplus T_k$:*

$$\begin{aligned}\mathsf{B}_{T_1,\ldots,T_k} :=& \Big\{\mathbf{y} \in \mathbb{R}^U : \mathbf{y}(i) = \mathbf{y}(j) \text{ for all } i, j \in T_a, \\ & \qquad \mathbf{y}(i) \geq \mathbf{y}(j) \text{ for all } i \in T_a, j \in T_b \text{ with } a < b\Big\} \\ =& \operatorname{cone}\{\mathbb{1}_{T_1}, \mathbb{1}_{T_1 \cup T_2}, \ldots, \mathbb{1}_{T_1 \cup \cdots \cup T_{k-1}}\} + \operatorname{span}_{\mathbb{R}}\{\mathbb{1}_U\}\end{aligned}$$

with

$$\dim \mathsf{B}_{T_1,\ldots,T_k} = k\,,$$

where $\mathbb{1}_T$ for some subset $T \subseteq U$ is the $0/1$-vector with entries equal to one for indices in the subset T and zero otherwise.

Proof of Theorem 2.9. For a submodular function $z\colon 2^U \to \mathbb{R}$ we show that $\mathsf{P}(z)$ is a generalized permutahedron by showing that every braid cone $\mathsf{B}_{T_1,\ldots,T_k} \subseteq \mathbb{R}^U$ is contained in a normal cone of $\mathsf{P}(z)$. Since $\mathsf{P}(z)$ is contained in the hyperplane $\{\mathbf{x} \in \mathbb{R}U : \mathbf{x}(U) = z(U)\}$ the normal cone $\mathsf{N}_{\mathsf{P}(z)}(\mathsf{P}(z))$ contains the line spanned by $\mathbb{1}_U \in \mathbb{R}U$, hence every normal cone of $\mathsf{P}(z)$ contains that line.

The following part of the proof relies on [FT83]. Fujishige and Tomizawa show under which conditions a greedy-like algorithm gives an optimal solution in the base polytope of a submodular functions on a general distributive lattice. We adapt the proof to our special case.

Let $\mathsf{B}_{T_1,\ldots,T_k} \subseteq \mathbb{R}^U$ a braid cone. Choose a maximal chain $\mathcal{C}\colon \varnothing = C_0 \subsetneq \cdots \subsetneq C_n = U$ in the Boolean lattice[8] 2^U such that $T_1, T_1 \sqcup T_2, \ldots, T_1 \sqcup \cdots \sqcup T_k$ are sets in the chain $\mathcal{C}$. Then

$$|C_j \setminus C_{j-1}| = 1$$

for $j = 1, \ldots, n := |U|$ and we define a linear ordering on U by $i_j := C_j \setminus C_{j-1} \in U$ for $j = 1, \ldots, n$. Now, consider the point $\tilde{\mathbf{x}} \in \mathbb{R}U$ defined by

$$\tilde{\mathbf{x}}_{i_j} := z(C_j) - z(C_{j-1}) \quad \text{für } j = 1, \ldots, n.$$

We will show

(i) that $\tilde{\mathbf{x}}(C_j) = z(C_j)$ for $j = 1, \ldots, n$, and that the point $\tilde{\mathbf{x}}$ lies in $\mathsf{P}(z)$,

(ii) that $\tilde{\mathbf{x}}$ is maximal for all directions in the braid cone $\mathsf{B}_{T_1,\ldots,T_k}$.

Then it follows that the braid cone $\mathsf{B}_{T_1,\ldots,T_k}$ is contained in the normal cone $\mathsf{N}_{\mathsf{P}(z)}(\mathsf{F})$, where F is a face containing $\tilde{\mathbf{x}}$.

8 See Section 2.5.5 for a definition of Boolean lattice.

For $j = 1, \dots, n$ we compute

$$\tilde{\mathbf{x}}(C_j) = \sum_{l=1}^{j} \tilde{\mathbf{x}}_{i_l} = \sum_{l=1}^{j} \left(z(C_l) - z(C_{l-1})\right) = z(C_j),$$

in particular, $\tilde{\mathbf{x}}(U) = z(U)$. We show by induction on the cardinality $|A|$ of a subset $A \subseteq U$ that $\tilde{\mathbf{x}}(A) \leq z(A)$. For the empty set we have $0 = \tilde{\mathbf{x}}(\varnothing) = z(\varnothing)$. For an arbitrary set $A \subseteq U$ let j^* be the minimal index such that $A \subseteq C_{j^*}$ and define the element $i^* := A \setminus C_{j^*-1} \in U$. We compute using the induction hypothesis, Lemma 2.3, and submodularity of z together with $A \setminus \{i^*\} = A \cap C_{j^*-1}$ and $C_{j^*} = A \cup C_{j^*-1}$:

$$\begin{aligned} \tilde{\mathbf{x}}(A) \ &= \ \tilde{\mathbf{x}}(\{i^*\}) + \tilde{\mathbf{x}}(A \setminus \{i^*\}) \ \leq \ \tilde{\mathbf{x}}(\{i^*\}) + z(A \setminus \{i^*\}) \\ &= \ z(C_{j^*}) - z(C_{j^*-1}) + z(A \setminus \{i^*\}) \ \leq \ z(A)\,. \end{aligned}$$

Hence, $\tilde{\mathbf{x}} \in \mathsf{P}(z)$.

Now, choose an arbitrary direction $\mathbf{y} \in \mathsf{B}_{T_1,\dots,T_k}$. By Lemma 2.10 $\mathbf{y}(i) = \mathbf{y}(i')$ for $i, i' \in T_l$ so we can set $\hat{y}_l := \mathbf{y}(i)$ for $i \in T_l$ and $l = 1, \dots, k$. Moreover, $\hat{y}_l \geq \hat{y}_{l+1}$. For a point $\mathbf{x} \in \mathsf{P}(z)$ compute:

$$\begin{aligned} \langle \mathbf{y}, \tilde{\mathbf{x}} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle &= \sum_{i \in U} \tilde{\mathbf{x}}_i \mathbf{y}(i) - \sum_{i \in U} \mathbf{x}_i \mathbf{y}(i) = \sum_{l=1}^{k} \hat{y}_l \Big(\tilde{\mathbf{x}}(T_l) - \mathbf{x}(T_l) \Big) \\ &= \sum_{l=1}^{k} \Big(\hat{y}_l \Big(\tilde{\mathbf{x}}(T_1 \sqcup \dots \sqcup T_l) - \mathbf{x}(T_1 \sqcup \dots \sqcup T_l) \Big) \\ &\qquad - \hat{y}_l \Big(\tilde{\mathbf{x}}(T_1 \sqcup \dots \sqcup T_{l-1}) - \mathbf{x}(T_1 \sqcup \dots \sqcup T_{l-1}) \Big) \Big) \\ &= \sum_{l=1}^{k-1} (\hat{y}_l - \hat{y}_{l+1}) \Big(\tilde{\mathbf{x}}(T_1 \sqcup \dots \sqcup T_l) - \mathbf{x}(T_1 \sqcup \dots \sqcup T_l) \Big) \\ &\qquad\qquad + \hat{y}_k \Big(\tilde{\mathbf{x}}(U) - \mathbf{x}(U) \Big) \\ &= \sum_{l=1}^{k-1} \underbrace{(\hat{y}_l - \hat{y}_{l+1})}_{\geq 0} \Big(\underbrace{z(T_1 \sqcup \dots \sqcup T_l) - \mathbf{x}(T_1 \sqcup \dots \sqcup T_l)}_{\geq 0} \Big) \geq 0, \end{aligned} \tag{2.19}$$

where we use in the last equality, that the sets $T_1, T_1 \sqcup T_2, \dots, T_1 \sqcup \dots \sqcup T_k$ are contained in the chain $\mathcal{C}\colon \varnothing = C_0 \subsetneq \dots \subsetneq C_n = U$ and that we already know $\tilde{\mathbf{x}}(C_j) = z(C_j)$ for $j = 1, \dots, n$. Since the computation in (2.19) is independent from the actual values of the direction $\mathbf{y} \in \mathsf{B}_{T_1,\dots,T_k}$, the inequality $\mathbf{y}(\tilde{\mathbf{x}}) \geq \mathbf{y}(\mathbf{x})$ holds for every direction $\mathbf{y} \in \mathsf{B}_{T_1,\dots,T_k}$. So the braid cone $\mathsf{B}_{T_1,\dots,T_k}$ is contained in the normal cone $\mathsf{N}_{\mathsf{P}(z)}(\mathsf{F})$, where F is a face containing $\tilde{\mathbf{x}}$. Hence, $\mathsf{P}(z)$ is a generalized permutahedron.

For the opposite implication let P be a generalized permutahedron. We will define a submodular function z_{P} and show that $\mathsf{P} = \mathsf{P}(z_{\mathsf{P}})$. Since the generalized permutahedron P is contained in the hyperplane with constant coordinate sum, the following set function is well defined:

$$z_{\mathsf{P}}(U) := \sum_{i \in U} \mathbf{x}_i \quad \text{for } \mathbf{x} \in \mathsf{P}$$
$$z_{\mathsf{P}}(A) := \max_{\mathbf{x} \in \mathsf{P}} \left(\sum_{i \in A} \mathbf{x}_i \right) \quad \text{for } A \subseteq U\,.$$

We can immediately deduce that $z(\varnothing) = 0$ and $\mathsf{P} \subseteq \mathsf{P}(z_{\mathsf{P}})$.

First we show that z_{P} is submodular. For arbitrary $A, B \subseteq U$ find a chain $\mathcal{C}\colon \varnothing = C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_k = U$ in the Boolean lattice 2^U that contains $A \cap B$ and $A \cup B$. We set $T_i := C_i \setminus C_{i-1}$ for $i = 1, \ldots, k$ and consider the braid cone $\mathsf{B}_{T_1,\ldots,T_k} = \operatorname{cone}\{\mathbb{1}_{T_1}, \ldots, \mathbb{1}_{T_1 \sqcup \cdots \sqcup T_{k-1}}\} + \operatorname{span}_{\mathbb{R}}\{\mathbb{1}_U\}$. Then there exists a face F of P such that the normal cone $\mathsf{N}_{\mathsf{P}}(\mathsf{F})$ contains the braid cone $\mathsf{B}_{T_1,\ldots,T_k}$ and in particular every point $\mathbf{x} \in \mathsf{F}$ is maximal in the direction $\mathbb{1}_{A \cap B}, \mathbb{1}_{A \cup B} \in \mathsf{B}_{T_1,\ldots,T_k}$. Then,

$$z_{\mathsf{P}}(A) + z_{\mathsf{P}}(B) \geq \mathbf{x}(A) + \mathbf{x}(B) = \mathbf{x}(A \cup B) + \mathbf{x}(A \cap B) = z_{\mathsf{P}}(A \cup B) + z_{\mathsf{P}}(A \cap B)$$

and z_{P} is submodular.

Now it is left to show that $\mathsf{P} \supseteq \mathsf{P}(z_{\mathsf{P}})$. The main idea for this part of the proof can be found in [DF10]. For the sake of a contradiction, let us assume there is a point $\mathbf{u} \in \mathsf{P}(z_{\mathsf{P}}) \setminus \mathsf{P}$. Then there exists a separating hyperplane

$$\mathsf{H}_{\mathbf{t},c} := \{\mathbf{x} \in \mathbb{R}U \,:\, \langle \mathbf{t}, \mathbf{x} \rangle = c\}$$

such that

$$\langle \mathbf{t}, \mathbf{u} \rangle = \sum_{i \in U} \mathbf{t}_i \mathbf{u}_i > c \quad \text{and} \quad \langle \mathbf{t}, \mathbf{p} \rangle = \sum_{i \in U} \mathbf{t}_i \mathbf{p}_i \leq c \quad \text{for all } \mathbf{p} \in \mathsf{P}$$

Now choose a braid cone $\mathsf{B}_{T_1,\ldots,T_k}$ such that $\mathbf{t} \in \mathsf{B}_{T_1,\ldots,T_k}$ and set again $\hat{t}_l := \mathbf{t}_i$ for $i \in T_l$, $l = 1, \ldots, k$. For points $\mathbf{q}$ in the $\mathbf{t}$-maximal face $\mathsf{F} := \mathsf{P}^{\mathbf{t}}$ we know by the definition of z that $\mathbf{q}(T_1 \cup \cdots \cup T_l) = z(T_1 \cup \cdots \cup T_l)$ for $l = 1, \ldots, k$. Using telescoping sums we compute

$$\begin{aligned}
\langle \mathbf{t}, \mathbf{u} \rangle = \sum_{i \in U} \mathbf{t}_i \mathbf{u}_i > c \geq \langle \mathbf{t}, \mathbf{p} \rangle &= \sum_{i \in U} \mathbf{t}_i \mathbf{q}_i = \sum_{l=1}^{k} \hat{t}_j \mathbf{q}(T_l) \\
&= \hat{t}_k \mathbf{q}(T_1 \cup \cdots \cup T_k) + \sum_{l=k-1}^{1} (\hat{t}_l - \hat{t}_{l+1}) \mathbf{q}(T_1 \cup \cdots \cup T_l) \\
&= \hat{t}_k z(T_1 \cup \cdots \cup T_k) + \sum_{l=k-1}^{1} (\hat{t}_l - \hat{t}_{l+1}) z(T_1 \cup \cdots \cup T_l) \\
&\geq \hat{t}_k \mathbf{u}(T_1 \cup \cdots \cup T_k) + \sum_{l=k-1}^{1} (\hat{t}_l - \hat{t}_{l+1}) \mathbf{u}(T_1 \cup \cdots \cup T_l) \\
&= \sum_{l=1}^{k} \hat{t}_l \mathbf{u}(T_l) = \sum_{i \in U} \mathbf{t}_i \mathbf{u}_i = \langle \mathbf{t}, \mathbf{u} \rangle.
\end{aligned}$$

This is a contradiction and completes the proof. □

2.4 Root Systems and Deformed Coxeter Permutahedra

We briefly introduce root systems, Coxeter permutahedra and deformed Coxeter permutahedra, generalizing the permutahedra and generalized permutahedra from the previous section. For introductions to root systems and Coxeter groups see, e.g., [Hum90] or [BB05] and for introductions to (deformed) Coxeter permutahedra, see, e.g.,[ABM20] or [Ard+20]. Similar to the generalized permutahedra in Section 2.3, deformed Coxeter permutahedra give a general framework for various subclasses of polytopes associated with (signed) combinatorial structures. We will see this in action in Section 2.5.4 and Chapter 4.

A **root system** $\Phi \subset \mathbb{V}$ in a real vector space $\mathbb{V}$ with inner product $\langle \cdot, \cdot \rangle$ is a finite set of vectors such that

(i) $\operatorname{span}_{\mathbb{R}}(\Phi) = \mathbb{V}$,
(ii) the only scalar multiples of a root $\mathbf{v} \in \Phi$ that belong to Φ are $\mathbf{v}$ and $-\mathbf{v}$,
(iii) for every $\mathbf{v} \in \Phi$ we have $s_{\mathbf{v}}(\Phi) = \Phi$, where

$$s_{\mathbf{v}}(\mathbf{x}) := \mathbf{x} - \frac{2\langle \mathbf{x}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v}\,.$$

For every root $\mathbf{v} \in \Phi$ define a hyperplane $\mathsf{H}_{\mathbf{v}} := \{\mathbf{x} \in \mathbb{V} \colon \langle \mathbf{x}, \mathbf{v}\rangle = 0\}$. For a root system Φ, the hyperplane arrangement $\mathcal{H}_\Phi := \{\mathsf{H}_{\mathbf{v}} \colon \mathbf{v} \in \Phi\}$ is called **Coxeter arrangement**. We call the polyhedral fan induced by a Coxeter arrangement $\mathcal{H}_\Phi$ a **Coxeter fan** $\mathcal{N}_\Phi$, i.e., the open full-dimensional cones in $\mathcal{N}_\Phi$ are the connected components of $\mathbb{V} \setminus \bigcup \mathcal{H}_\Phi$.

The **direct sum** of two root systems $\Phi_i \subset \mathbb{V}_1$ and $\Phi_2 \subset \mathbb{V}_2$ is the root system

$$\Phi_1 \oplus \Phi_2 := \{(\mathbf{v}_1, \mathbf{0}) \in \mathbb{V}_1 \oplus \mathbb{V}_2 \colon \mathbf{v}_1 \in \Phi_1\} \cup \{(\mathbf{0}, \mathbf{v}_2) \in \mathbb{V}_1 \oplus \mathbb{V}_2 \colon \mathbf{v}_2 \in \Phi_2\}\,.$$

A root system is called **irreducible** if it is not a nontrivial direct sum of root systems.

Example 2.11. Here are four families of irreducible root systems:

- $A_{d-1} = \{\pm(\mathbf{e}_i - \mathbf{e}_j)\}_{i \neq j},$,
- $B_d = \{\pm(\mathbf{e}_i - \mathbf{e}_j)\}_{i \neq j} \cup \{\pm(\mathbf{e}_i + \mathbf{e}_j)\}_{i \neq j} \cup \{\pm \mathbf{e}_i\}$,
- $C_d = \{\pm(\mathbf{e}_i - \mathbf{e}_j)\}_{i \neq j} \cup \{\pm(\mathbf{e}_i + \mathbf{e}_j)\}_{i \neq j} \cup \{\pm 2\mathbf{e}_i\}$,
- $D_d = \{\pm(\mathbf{e}_i - \mathbf{e}_j)\}_{i \neq j} \cup \{\pm(\mathbf{e}_i + \mathbf{e}_j)\}_{i \neq j}$,

where $\mathbf{e}_i$ for $i = 1, \ldots, d$ are the standard basis vectors in $\mathbb{R}^d$. Note that A_{d-1} spans the $(d-1)$-dimensional vector space

$$\left\{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}_1 + \cdots + \mathbf{x}_d = 0\right\} \subset \mathbb{R}^d\,,$$

hence the shift in the index. We will see combinatorial models for these root systems, their subsets, and their hyperplane arrangements in Section 2.5.1 and Section 2.5.4.

Theorem 2.12 ([Hum90, Section 2])**.** *Irreducible root systems can be completely classified up to isomorphism: there are 5 exceptional root systems and the four infinite families in Example 2.11.*

A subset $\Phi^+ \subseteq \Phi$ is called **positive roots** if for each root $\mathbf{v}$ exactly one of the roots $\mathbf{v}, -\mathbf{v}$ is contained in Φ^+ and for two distinct $\mathbf{v}_1, \mathbf{v}_2 \in \Phi^+$ with $\mathbf{v}_1 + \mathbf{v}_2 \in \Phi$ we also have $\mathbf{v}_1 + \mathbf{v}_2 \in \Phi^+$. The **standard Coxeter permutahedron of type Φ** or **Φ-permutahedron** is defined as

$$\Pi_\Phi := \sum_{\mathbf{v} \in \Phi^+} \left[-\tfrac{1}{2}\mathbf{v}, \tfrac{1}{2}\mathbf{v}\right] .$$

It follows directly from the definitions that the normal fan of the Φ-permutahedron Π_Φ is the Coxeter fan $\mathcal{N}_\Phi$. Note that the Coxeter arrangement $\mathcal{H}_{A_{d-1}}$ in type A_{d-1} is the braid arrangement $\mathcal{B}_d$ from Section 2.3 and the Coxeter permutahedron of type A_{d-1} is a translate of the standard permutahedron Π_d from Section 2.3, i.e., for the choice $A_{d-1}^+ = \{\mathbf{e}_i - \mathbf{e}_j \in \mathbb{R}^d \colon i < j\}$ we have with Equation (2.16)

$$\Pi_{A_{d-1}} + \tfrac{d-1}{2}\mathbf{1} = \sum_{i<j} \left(\left[-\tfrac{1}{2}(\mathbf{e}_i - \mathbf{e}_j), \tfrac{1}{2}(\mathbf{e}_i - \mathbf{e}_j)\right] + \tfrac{1}{2}(\mathbf{e}_i + \mathbf{e}_j)\right) = \sum_{i<j} [\mathbf{e}_i, \mathbf{e}_j] = \Pi_d .$$

The Ehrhart polynomial for standard Coxeter permutahedra of type A, B, C, and D, was computed in [ABM20]. We will extend these results (for certain translates) to integral subzonotopes, in the sense of subsets of the root systems, in Chapter 4.

We call a polytope **generalized Coxeter permutahedron** or **deformed Coxeter permutahedron** if its normal fan coarsens a Coxeter fan $\mathcal{N}_\Phi$, i.e., it is a deformation of a Φ-permutahedron. As in the last section, generalized Coxeter permutahedra can be described by certain Coxeter submodular functions, see [Ard+20]. For example subzonotopes of the standard Coxeter permutahedra are deformed Coxeter permutahedra. In the case of root systems of types A, B, C, and D, we will study those in Chapter 4. In Chapter 3 we will derive general a combinatorial reciprocity theorem for deformed Coxeter permutahedra in types A, B, C, and D.

2.5 Combinatorial Structures and their Polytopes

In the following sections we introduce five combinatorial structures—graphs, hypergraphs, matroids, signed graphs and posets—as well as their associated polytopes. Graphical zonotopes, hypergraphical polytopes, and matroid base polytopes are subclasses of generalized permutahedra. Representable matroids also come with (general) associated zonotopes. While signed graphical zonotopes are subclasses of deformed Coxeter permutahedra, order polytopes are neither generalized permutahedra nor deformed Coxeter permutahedra, but they are closely related to Coxeter arrangements in type A, i.e., braid arrangements.

2.5.1 Graphs

A **graph** $G = (V, E)$ is a tuple consisting of a finite set V called **nodes**[9] and a finite set E of two-subsets of V called **edges**. Throughout we will denote the number of nodes by $d := |V|$ and the number of edges by $m := |E|$. A **path** in a graph a sequence $(v_1, e_1, v_2, e_2, \ldots, e_\ell, v_{\ell+1})$ of nodes v_i and edges e_i, such that $e_i = \{v_i, v_{i+1}\}$

9 We use the less common term *nodes* for graphs to distinguish them from the *vertices* of a polytope.

and all the edges and nodes in the sequences are pairwise distinct. A graph G is **connected** if there exists a path between any two nodes. A **circle** a closed path, i.e., a sequence $(v_1, e_1, v_2, e_2, \ldots, e_\ell, v_1)$ where only the first and last node are equal. A **loop** is a circle (v, e, v). A **forest** is a graph without circles and a **tree** is a connected forest. A subset $S \subseteq E$ of edges is called **spanning** (in $G = (V, E)$), is the graph $G(S) \coloneqq (V, S)$ is connected. For a subset $T \subseteq V$ of the nodes of the graph $G = (V, E)$ we define the **(node) induced subgraph** $G|_T \coloneqq (T, E|_T)$, where

$$E|_T \coloneqq \{\{u, v\} \in E \colon u, v \in T\}\,. \tag{2.20}$$

For a subset $F \subseteq E$ of the edges of the graph $G = (V, E)$ we define the **(edge) subgraph** $G(F) \coloneqq (V, F)$. If $G(F)$ is a forest we also say that "$F \subseteq E$ is a forest in G" or "$F \subseteq E$ forms a forest in G".

An **orientation** of G assigns to every edge $e = \{u, v\} \in E$ an order (or direction). A **cycle** in an orientation of a graph is a sequence $(u_1, v_1), (u_2, v_2), \ldots, (u_\ell, v_\ell)$ of directed edges such that $u_i \neq u_j$ for $i \neq j$ and $u_i = v_{i-1}$ for $i = 2, \ldots, \ell$ and $u_1 = v_\ell$. An orientation of a graph is called **acyclic** if it does not contain any cycles and it is called **totally cyclic** (or strongly connected) if every edge in the graph is contained in a cycle.

For a graph $G = (V, E)$ with $|V| = d$ we define the **graphical hyperplane arrangement**

$$\mathcal{H}_G = \{\mathsf{H}_{\{u,v\}} \colon \{u, v\} \in E\}\,,$$

where $\mathsf{H}_{\{u,v\}} \coloneqq \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}_u = \mathbf{x}_v\}$. Note that graphical hyperplane arrangements are subsets of the Coxeter arrangement in type A introduced in Section 2.4. For example the graphical hyperplane arrangement of the complete graph on three nodes is the braid arrangement, or equivalently the Coxeter arrangement $\mathcal{H}_{A_2}$ (see Figure 2.6(b)). Some combinatorial data of subsets of the Coxeter arrangement in type A, and hence their (sub-)root systems, can be studied in terms of graphs. The combinatorics of hyperplane arrangements in general and graphical hyperplane arrangements in particular was studied in [GZ83]. Greene–Zaslavsky show among other things that regions in the graphical hyperplane arrangement $\mathcal{H}_G$ are in bijection with acyclic orientations of G.

We want to associate a polytope, more specifically a zonotope, to every graph. Let $G = (V, E)$ be a graph, we define the **graphical zonotope** or **acyclotope** as

$$\mathsf{Z}_G \coloneqq \sum_{\{u,v\} \in E} [\mathbf{e}_u, \mathbf{e}_v] \subseteq \mathbb{R}^{|V|}\,. \tag{2.21}$$

Zaslavsky coined the charming term *acyclotope* in [Zas82a, Section 4] within the more general setting of signed graphs (see Section 2.5.4), one that we would like to revitalize. Zaslavsky's original definition of the acyclotope is slightly different: it is a centrally symmetric version, which is homothetic to our definition. It is a translation of the second dilate, in fact it is the second dilate of subzonotopes of the standard Coxeter permutahedra defined in Section 2.4. We chose the above definition because it features more natural arithmetic properties, it is widely used in the case of graphs, and it has a nice generalization for hypergraphs (Section 2.5.2).

Vertices of Z_G are in bijection with the acyclic orientations of the graph G, and lattice points in the acyclotope Z_G correspond to indegree vectors of orientations. We will prove this result in more generality for hypergraphs later (see Proposition 3.13). Note that the acyclotope for graphs is a subzonotope of the standard permutahedron $\Pi_{|V|}$ in type A in the sense that it is generated by a subset of the roots in $A_{|V|-1}$, see Section 2.3.

In order to rewrite this zonotope as in Equations (2.5) or (2.6) we need to choose a suitable lattice translation. This can be done by choosing an arbitrary acyclic orientation on the graph (which by the comment above corresponds to a vertex of Z_G). For such an orientation we define the **incidence matrix**

$$\mathbf{A}_G = (\mathbf{e}_v - \mathbf{e}_u)_{(u,v)\in E} \in \mathbb{R}^{d\times m} = \mathbb{R}^{|V|\times|E|} . \tag{2.22}$$

The acyclotope as defined in Equation (2.21) is a translate of the zonotope defined by the incidence matrix,

$$\mathsf{Z}_G = \mathsf{Z}(\mathbf{A}_G) + (\operatorname{outdeg}(u))_{u\in V} ,$$

where $(\operatorname{outdeg}(u))_{u\in V}$ denotes the vector of out-degrees with respect to the chosen orientation. The properties of acyclotopes we will be interested in, such as face structure and Ehrhart polynomial, are invariant under lattice translations. Hence, the orientation chosen to define the incidence matrix is irrelevant and depending on the context, we will either consider Z_G or $\mathsf{Z}(\mathbf{A}_G)$. Note that for $F \subseteq E$ the incidence matrix $\mathbf{A}_{G(F)}$ of the subgraph $G(F)$ is the submatrix with columns corresponding to edges in F, we denote this by $\mathbf{F} := \mathbf{A}_{G(F)} \in \mathbb{R}^{|V|\times|F|}$.

We can apply Stanleys Theorem 2.3 to describe the coefficients of the Ehrhart polynomial of acyclotopes. For that we need to understand the linear independent subsets of the columns in the incidence matrix $\mathbf{A}_G$.

Lemma 2.13. *A subset $\mathbf{F} = \mathbf{A}_{G(F)}$ of columns in the incidence matrix $\mathbf{A}_G$ of a graph G is linearly independent if and only if the corresponding set of edges $F \subseteq E$ forms a forest in G.*

By an inductive arguments it can be shown that he incidence matrix $\mathbf{A}_G$ is **totally unimodular**, i.e., every subdeterminant is either 0, $+1$, or -1. This implies, together with Theorem 2.3 and Lemma 2.13, the following Corollary.

Corollary 2.14 ([Sta91])**.** *Let G be a graph and $\mathsf{Z}(\mathbf{A}_G)$ its acyclotope, then*

$$\operatorname{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{A}_G); n) = \sum_{i=0}^{m} c_i n^i ,$$

where c_i counts the number of (labeled) forests with i edges in the graph G.

For acyclotopes of graphs this gives a combinatorial meaning for the coefficients of the Ehrhart polynomial and hence answers the research question (ii) mentioned in the introduction. We will extend this result to signed graphs in Chapter 4, see also Section 2.5.4.

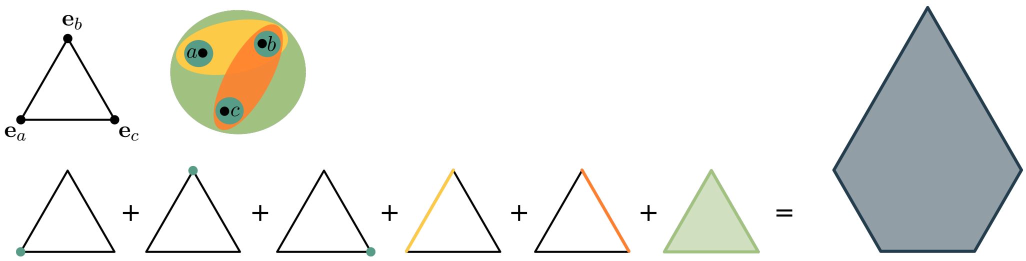


Figure 2.8: The hypergraph $h = (\{a,b,c\}, \{\{a,b,c\}, \{a,b\}, \{b,c\}, \{a\}, \{b\}, \{c\}\})$ and its hypergraphic polytope $\mathsf{P}(h)$.

2.5.2 Hypergraphs

A **hypergraph** $h = (U, E)$ is a pair of a finite set U of **nodes**[10] and a finite multiset E of non-empty subsets $e \subseteq U$ called **hyperedges**. Note that we allow multiple edges and edges consisting of only one node. As mentioned in Section 2.0, we can assume without loss of generality that the node set U equals $\{1, \ldots, d\} = [d]$ for $d = |U|$, since all the claims in this section are invariant under relabeling the set U. In a similar fashion recall that we can switch back and forth between the two vector space notations $\mathbb{R}U \simeq \mathbb{R}^d$ and $\mathbb{R}^U \simeq \left(\mathbb{R}^d\right)^*$.

For every hypergraph h we define the corresponding **hypergraphic polytope** $\mathsf{P}(h) \subseteq \mathbb{R}U$ as the following Minkowski sum of simplices:

$$\mathsf{P}(h) = \sum_{e \in E} \Delta_e \ \subseteq \mathbb{R}U$$

where

$$\Delta_e = \operatorname{conv}\{\mathbf{e}_i \ : \ i \in e\}, \quad \text{for a hyperedge } e \subseteq U$$

and $\mathbf{e}_i$ are the basis vectors for $i \in U$. An example is depicted in Figure 2.8. Hypergraphic polytopes have been studied (sometimes as Minkowski sum of simplices) in, e.g., [Agn17; BBM19]. Hypergraphs are in bijection with hypergraphic polytopes. A hypergraph with all hyperedges of size two is a (usual) graph and the hypergraphic polytope agrees with the acyclotope as defined in Section 2.5.1. We will study hypergraphs and their polytopes in more detail in Section 3.2.2.

Recall that the hypergraphic polytope $\mathsf{P}(h) \subseteq \mathbb{R}U$ of a hypergraph $h = (U, E)$ is defined as

$$\mathsf{P}(h) = \sum_{e \in E} \Delta_e \ \subseteq \mathbb{R}U$$

where

$$\Delta_e = \operatorname{conv}\{\mathbf{e}_i \ : \ i \in e\}, \quad \text{for a hyperedge } e \subseteq U$$

and $\mathbf{e}_i$ are the basis vectors for $i \in U$.

10 Again, we use the less common term *nodes* for hypergraphs to distinguish them from the *vertices* of a polytope.

Proposition 2.15 ([Pos09, Proposition 6.3.]). *For a hypergraph $h = (U, E)$ and its hypergraphic polytope $\mathsf{P}(h)$, the function $z\colon 2^U \to \mathbb{R}$ defined by*

$$z(T) := \sum_{\substack{e \in E \\ e \cap T \neq \varnothing}} 1 = \#(\textit{hyperedges in } h \textit{ that intersect } T) \quad \textit{for } T \subseteq U$$

is a submodular function with $z(\varnothing) = 0$ and

$$\mathsf{P}(h) = \left\{ \mathbf{x} \in \mathbb{R}U \,:\, \sum_{i \in U} \mathbf{x}_i = z(U) \quad \textit{and} \quad \sum_{i \in T} \mathbf{x}_i \leq z(T) \quad \textit{for } T \subseteq U \right\}.$$

Hence, hypergraphic polytopes are generalized permutahedra and in bijection with hypergraphs.

Remark 2.16. Postnikov uses a different convention for the facet description of a generalized permutahedron:

$$\mathsf{P}(z) := \left\{ \mathbf{x} \in \mathbb{R}^d \,:\, \sum_{i=1}^{d} \mathbf{x}_i = z([d])\,,\ \sum_{i \in J} \mathbf{x}_i \geq z(J)\,,\ \text{for } J \subseteq [d] \right\}.$$

This results in a differing formulation of Proposition 2.15 which is nevertheless equivalent.

For an interesting characterization when a submodular function gives rise to a hypergraphic polytope see [AA23, Proposition 19.4.].

2.5.3 Matroids

Matroids were introduced as an abstraction of the concept of (linear) independence by Whitney [Whi35] and independently by Nakasawa [Nak35; Nak36a; Nak36b; Nak38]. For more on the historical background see, e.g., [NK09]. There are numerous cryptomorphic, i.e., equivalent, descriptions for matroids. Here, we restrict ourselves to independent sets, bases, and rank functions. See, e.g., [Oxl03; GM12; Wel10; Whi86] for introductions to matroid theory.

We will start by describing the independent sets of a matroid and then consider two classes of examples that will serve as motivation and illustrate what is meant by "matroids are an abstraction of the concept of (linear) independence".

A **matroid** $M = (U, \mathcal{I})$ consists of a finite ground set U and a collection of **independent** subsets $\mathcal{I} \subseteq 2^U$ fulfilling the following **independence axioms**:

(I1) $\varnothing \in \mathcal{I}$,
(I2) if $F_1 \in \mathcal{I}$ and $F_2 \subseteq F_1$ then $F_2 \in \mathcal{I}$,
(I3) if $F_1, F_2 \in \mathcal{I}$ and $|F_2| < |F_1|$, then there exists an element $x \in F_1 \setminus F_2$ such that $F_2 \cup \{x\} \in \mathcal{I}$.

Example 2.17. Let $\mathbf{A} \in \mathbb{R}^{d \times m}$ be a matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^d$. We define the **linear matroid** $M(\mathbf{A}) = ([m], \mathcal{I}_{\mathbf{A}})$ with ground set $[m]$ and the collection of independent sets

$$\mathcal{I}_{\mathbf{A}} := \left\{ F \in 2^{[m]} \colon \{\mathbf{a}_i \colon i \in F\} \text{ are linearly independent} \right\}.$$

From Linear Algebra we know that $\mathcal{I}_{\mathbf{A}}$ fulfills the independence axioms. We call a matroid $M = (U, \mathcal{I})$ **representable over a field** $\mathbb{F}$ if there exists a matrix $\mathbf{A} \in \mathbb{F}^{d \times U}$ such that $M = M(\mathbf{A})$.

Example 2.18. Let $G = (V, E)$ be a graph. We define the **graphic matroid** $M(G) = (E, \mathcal{I}_G)$ with the ground set the edges of the graph E and the collection of independent sets

$$\mathcal{I}_G := \left\{ F \in 2^E : F \text{ forms a forest in } G \right\} .$$

It is easy to check that $\mathcal{I}_G$ fulfills the independence axioms.

Recall from Equation (2.22) the definition of incidence matrix $\mathbf{A}_G$ for a graph. From Lemma 2.13, it follows that every graphic matroid is representable over $\mathbb{R}$ and $M(G) \cong M(\mathbf{A}_G)$. Hence, the notion of independence captured by a matroid generalizes simultaneously the notion of linear independence in (real) vector spaces and the notion of subgraphs not containing cycles (i.e., a dependence).

We call an inclusion-maximal independent set a **basis** or **base** of a matroid. We can define a matroid also by its collection of bases, i.e., a **matroid** $M = (U, \mathcal{B})$ is tuple with a finite **ground set** U and a collection of subsets $\mathcal{B} \subseteq 2^U$ called **bases** such that the following **basis axioms** hold:

(B1) $\mathcal{B} \neq \varnothing$,

(B2) for two bases $B_1, B_2 \in \mathcal{B}$ and for every element $x \in B_1 \setminus B_2$ there exists an element $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

It can be shown that basis axioms and independence axioms are equivalent. It also follows from the axioms that every basis $B \in \mathcal{B}$ contains the same number of elements $r(B)$ and we call this the **rank** of the matroid M. A subset that is not contained in any base is called **dependent**. A **circuit** is an inclusion-minimal dependent set. We call a subset $S \subseteq U$ **spanning** if it contains a base, i.e., there exists a $B \in \mathcal{B}$ such that $S \supseteq B$. An element $x \in U$ is called **loop** if it is not contained in any base and it is called a **coloop** if it is contained in every base.

Note how some of the vocabulary in matroid theory stems from the two motivations discussed in the beginning, linear matroids and graphic matroids.

We give a third set of axioms for a matroid. The **rank function** $r\colon 2^U \to \mathbb{Z}_{\geq 0}$ is defined as

$$r(A) = \max_{B \in \mathcal{B}} |B \cap A| .$$

Then the rank function fulfills the following properties:

(R1) For every subset $A \subseteq U$, we have $r(A) \leq |A|$.

(R2) The rank function is **submodular**, i.e., for every two subsets $A_1, A_2 \subseteq U$, we have

$$r(A_1 \cup A_2) + r(A_1 \cap A_2) \leq r(A_1) + r(A_2) .$$

(R3) The rank function is **montone**, i.e., for every subset $A \subseteq U$ and every element $x \in U$, we have $r(A) \leq r(A \cup \{x\}) \leq r(A) + 1$.

Vice versa, every function $r\colon 2^U \to \mathbb{Z}_{\geq 0}$ that fulfills the axioms (R1), (R2), and (R3) defines a matroid.

Rank functions of matroids facilitate a polytopal perspective on matroids. Define the **matroid base polytope** P_M of a matroid $M = (U, \mathcal{B})$ as

$$\mathsf{P}_M := \operatorname{conv}(\mathbb{1}_B \in \mathbb{R}U \colon B \in \mathcal{B})\,.$$

Then matroid base polytopes are generalized permutahedra (Section 2.3). This can be seen from the fact that an inequality description of the matroid base polytope P_M is given by the rank function of the matroid M (compare Theorem 2.9), i.e.,

$$\mathsf{P}_M = \{\mathbf{x} \in \mathbb{R}U \colon \sum_{i \in U} \mathbf{x}_i = r(U)\,, \sum_{i \in A} \mathbf{x}_i = r(A) \text{ for every } A \subseteq U\}\,.$$

In fact, this is a defining property for matroids.

Theorem 2.19 ([Gel+87, Theorem 4.1])**.** *A collection $\mathcal{B} \subseteq 2^U$ of subsets of U is the set of bases of a matroid if and only if*

$$\operatorname{conv}(\mathbb{1}_B \in \mathbb{R}U \colon B \in \mathcal{B}) \subset \mathbb{R}U$$

is a generalized permutahedron.

In that sense, matroid base polytopes are a visualization of matroids and generalized permutahedra form a superclass.

There are numerous constructions on matroids. We will only discuss dual matroids and direct sums here. For every matroid $M = (U, \mathcal{B})$ there exists a **dual matroid** $M^{\triangle} = (U, \mathcal{B}^{\triangle})$ defined by

$$\mathcal{B}^{\triangle} := \{U \setminus B \colon B \in \mathcal{B}\}\,.$$

Equivalently, we can describe the independent sets of the dual matroid $M^{\triangle} = (U, \mathcal{I}^{\triangle})$ by

$$\mathcal{I}^{\triangle} = \{U \setminus S \colon S \text{ is a spanning set in } M\}\,. \tag{2.23}$$

Checking that the axioms (B1) and (B2), or (I1), (I2), and (I3), respectively, are fulfilled is an easy exercise.

The following theorem is true for any field $\mathbb{F}$, see, e.g., [Wel10, Section 9.3]. We only will use $\mathbb{F} = \mathbb{R}$ and therefore state and prove the theorem in that simplified version. To simplify notation we also assume the ground set of the matroid to be $[m]$.

Theorem 2.20. *Let $M = ([m], \mathcal{B})$ be a representable matroid over a field $\mathbb{R}$. Then the dual matroid $M^{\triangle} = ([m], \mathcal{B}^{\triangle})$ is also representable over $\mathbb{R}$.*

Proof. let $\mathbf{A} \in \mathbb{R}^{r \times m}$ be a representation for $M = ([m], \mathcal{B})$, i.e., $M = M(\mathbf{A})$. We can assume $\mathbf{A}$ to have full rank r. Then $\mathbf{A} \colon \mathbb{R}^m \to \mathbb{R}^r$ defines a linear map of rank r and with kernel $\operatorname{kern} \mathbf{A} \subseteq \mathbb{R}^m$ of dimension $\dim \operatorname{kern} \mathbf{A} = m - r$. Now choose a basis $\mathbf{b}_1, \ldots, \mathbf{b}_{m-r} \in \mathbb{R}^m$ for the kernel $\operatorname{kern} \mathbf{A}$ and write it as the columns of $\mathbf{B} \in \mathbb{R}^{m \times (m-r)}$.

Now, the transposed matrix $\mathbf{B}^T \in \mathbb{R}^{(m-r) \times m}$ is a representation of $M^{\triangle}$, i.e., $M^{\triangle} = M(\mathbf{B}^T)$. Indeed, after reordering columns it is enough to check that the first r columns of $\mathbf{A}$ are linearly dependent if and only if the last $m - r$ columns of $\mathbf{B}^T$

are linearly dependent. Assume the first r columns of $\mathbf{A}$ are linearly dependent, then there exists $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that

$$\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_r, 0, \dots, 0) \in \mathbb{R}^m \setminus \{\mathbf{0}\} \text{ and } \mathbf{A}\boldsymbol{\lambda}^T = \mathbf{0}\,.$$

That is, $\boldsymbol{\lambda} \in \operatorname{kern} \mathbf{A}$. Since $\mathbf{b}_1, \dots, \mathbf{b}_{m-r} \in \mathbb{R}^m$ generate the kernel $\operatorname{kern} \mathbf{A}$ there exists $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{m-r}) \in \mathbb{R}^{m-r} \setminus \{\mathbf{0}\}$ such that

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \boldsymbol{\lambda}^T = \mathbf{B}\boldsymbol{\omega}^T =: \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \boldsymbol{\omega}^T\,,$$

where $\mathbf{B}_1 \in \mathbb{R}^{r \times (m-r)}$ and $\mathbf{B}_2 \in \mathbb{R}^{(m-r)\times(m-r)}$, i.e., $\mathbf{B}_2^T$ consists of the last $m-r$ columns of $\mathbf{B}^T$. We have shown that $\mathbf{0} = \mathbf{B}_2\boldsymbol{\omega}^T$, hence $\mathbf{B}_2$ is singular and the last $m-r$ columns of $\mathbf{B}^T$ are linearly dependent. This argument can be reversed and finishes the proof. □

Let $M = ([m], \mathcal{B})$ be a representable matroid (over $\mathbb{R}$) of rank r. Assume that $\{1, \dots, r\} \in \mathcal{B}$ is a basis. Then there is a representation of M of the form

$$\left[\mathbf{R} \,|\, \mathbf{I}_r\right] \in \mathbb{R}^{r \times m}\,,$$

where $\mathbf{I}_r \in \mathbb{R}^{r\times r}$ is the identity matrix and $\mathbf{R} \in \mathbb{R}^{r\times(m-r)}$. We call this the **standard representation** of the matroid M.

Corollary 2.21. *If $M = ([m], \mathcal{B})$ is a matroid of rank r with a standard representation $[\mathbf{R} \,|\, \mathbf{I}_r] \in \mathbb{R}^{r\times m}$, then the dual matroid $M^{\triangle}$ has a representation of the form*

$$\left[\mathbf{I}_{m-r} \,|\, -\mathbf{R}^T\right] \in \mathbb{R}^{(m-r)\times m}\,.$$

Let $M_1 = (U_1, \mathcal{I}_1)$ and $M_2 = (U_2, \mathcal{I}_2)$ be two matroids with disjoint ground sets $U_1 \cap U_2 = \varnothing$. We define the **direct sum** $M_1 \oplus M_2 := (U_1 \uplus U_2, \mathcal{I}_1 \oplus \mathcal{I}_2)$, where the collection of independent sets $\mathcal{I}_1 \oplus \mathcal{I}_2$ is defined by

$$\mathcal{I}_1 \oplus \mathcal{I}_2 := \{I_1 \uplus I_2 \colon I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}\,.$$

It can be checked that this definition fulfills the independence axioms (I1), (I2), and (I3). See, e.g., [Whi86, Section 7.6]. Moreover, it follows that

$$(M_1 \oplus M_2)^{\triangle} = M_1^{\triangle} \oplus M_2^{\triangle}\,.$$

Similarly, for representable matroids $M_1 = (m_1, \mathcal{I}_1)$ and $M_2 = (m_2, \mathcal{I}_2)$ with representations if $\mathbf{A}_1 \in \mathbb{R}^{r_1 \times m_1}$ and $\mathbf{A}_2 \in \mathbb{R}^{r_2 \times m_2}$ we have

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{0}_{r_1 \times m_2} \\ \mathbf{0}_{r_2 \times m_1} & \mathbf{A}_2 \end{bmatrix} \in \mathbb{R}^{(r_1+r_2)\times(m_1+m_2)}$$

is a representation for $M_1 \oplus M_2$.

Remark 2.22. Matroids that are representable over $\mathbb{R}$ can be oriented, that is in addition to the matroid structure, i.e., a collection of subsets, we can construct a sign structure, i.e., a collection of signed subsets. For **oriented matroids** there are also various cryptomorphic axiomatizations, see [Bjö+99]. In the case of representable matroids over $\mathbb{R}$, we could work a bit harder and instead of capturing just the dependent sets, we could also record the signs of the coefficients of the linear dependencies. This would essentially define an oriented matroid. For oriented matroids there also exists a notion of duality and a theorem analogous to Theorem 2.20 defines **Gale transforms**. See, e.g., [Zie98, Chapter 6] for a geometric introduction to oriented matroids and Gale duality.

Recall that the normal fan of a zonotope $\mathsf{Z}(\mathbf{A}) \subset \mathbb{R}^d$ is induced by a hyperplane arrangement $\mathcal{H}(\mathbf{A})$ (Section 2.1). Then the linear matroid $M(\mathbf{A})$ captures the combinatorial information of the zonotope $\mathsf{Z}(\mathbf{A})$ and the hyperplane arrangement $\mathcal{H}(\mathbf{A})$. However, in each step, going from zonotope to hyperplane arrangement and from hyperplane arrangement to linear matroid, we loose some geometric information. Equivalently, in the reverse process we have to make a number of (non-unique choices): for every representable matroid M we can choose a (non-unique) representation $\mathbf{A} \in \mathbb{R}^{d\times m}$ so that $M = M(\mathbf{A})$. This uniquely defines a hyperplane arrangement $\mathcal{H}(\mathbf{A})$ and it also defines a zonotope $\mathsf{Z}(\mathbf{A})$ with the normal fan of $\mathsf{Z}(\mathbf{A})$ being induced by $\mathcal{H}(\mathbf{A})$. However, for any scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{R} \setminus \{0\}$ we have that $\mathsf{Z}(\lambda_1\mathbf{a}_1, \ldots, \lambda_m\mathbf{a}_m)$ has the same normal fan (as $\mathsf{Z}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$) induced by the hyperplane arrangement $\mathcal{H}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$.

Recall that the Ehrhart polynomial of a zonotope can be computed by

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{A}); n) = \sum_{\mathbf{F}} g(\mathbf{F})\, n^{\mathrm{rank}(\mathbf{F})},$$

where $\mathbf{F}$ ranges over submatrices of $\mathbf{A}$ with the linearly independent columns. So the Ehrhart polynomial encodes statistics of linearly independent sets, but also the arithmetic information $g(\mathbf{F})$. This arithmetic information gets lost when passing from a zonotope to its hyperplane arrangement of matroid. This is one motivation for defining arithmetic matroids [DM13].

That is, similarly as for oriented matroids, we can add more information or structure to a matroid. An **arithmetic matroids** is a pair (M, m), where M is a matroid on the ground set U and $m \colon 2^U \to \mathbb{Z}_{>0}$ is a **multiplicity function** with the following properties:

(A1) For $A \subseteq U$ and $x \in U$ with $r(A) = r(A \cup \{x\})$ we have $m(A \cup \{x\})$ divides $m(A)$.

(A2) For $A \subseteq U$ and $x \in U$ with $r(A) + 1 = r(A \cup \{x\})$ we have $m(A)$ divides $m(A \cup \{x\})$.

(A3) If $A \subseteq B \subseteq U$ with $B = A \sqcup F \sqcup T$ is a disjoint union such that for all $A \subseteq C \subseteq B$ we have $r(C) = r(A) + |C \cap F|$, then $m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T)$.

(A4) If $A \subseteq B \subseteq U$ with $r(A) = r(B)$, then

$$\sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(T) \geq 0\,.$$

(A5) If $A \subseteq B \subseteq U$ with $|A| + r(U \setminus A) = |B| + r(U \setminus B)$, then

$$\sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(U \setminus T) \geq 0\,.$$

See [DM13, Section 2.3] or [BM14, Section 2] for a more compact version.

For $\mathbf{A} \in \mathbb{Z}^{r \times m}$ the matroid $M(\mathbf{A})$ on the ground set $[m]$ together with the multiplicity function $m(B) := g(\mathbf{B})$, where $\mathbf{B}$ is the matrix with columns of $\mathbf{A}$ indexed by B and $g(\mathbf{B})$ is as defined in Equation (2.9), forms a **representable** arithmetic matroid, see [DM13, Section 2.4]. Arithmetic matroids have well-defined duals [DM13, Lemma 2.2]. We will see this construction in the special case of representable arithmetic matroids of the form $(M(\mathbf{A}), g)$ in Chapter 4.

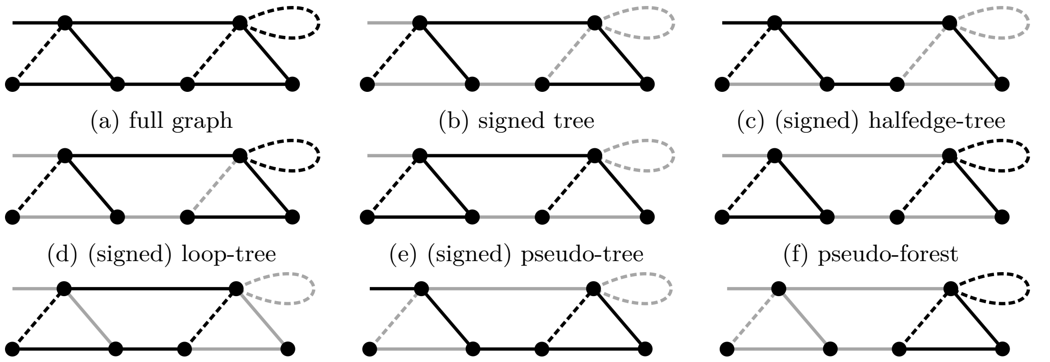
2.5.4 Signed Graphs

Signed graphs originated in the social sciences and have found applications also in biology, physics, computer science, and economics. Signed graphs have been extensively studied, generalized, applied and rediscovered since the first half of the 20$^{\text{th}}$ century. Thomas Zaslavsky's dynamic survey [Zas18] consists of over 500 pages of (commented) references regarding signed graphs and related concepts. Note that there is also a mathematically inconsistent use of the term "signed graph" within the literature, as explained in [Zas18].

Here, we will focus on signed graphs as a generalization of graphs, which allows us to give a combinatorial model for subsets of root systems of type A, B, C, and D, in analogy to (usual) graphs modelling subsets of root systems of type A. Excellent background references for this perspective are [Zas81; Zas10].

A **signed graph** $\Sigma = (\Gamma, \sigma)$ consists of a graph $\Gamma = (V, E)$ and a signature σ that assigns each link and loop of Γ either $+$ or $-$. The underlying graph Γ the edge set E may contain besides the usual, potentially multiple, links (two distinct endpoints) and loops (two endpoints that are the same), also halfedges (with only one endpoint) and loose edges (no endpoints), though the latter play no role in our work. An ordinary graph can be realized by a signed graph all of whose edges are labelled with $+$.

We recall some notions from the theory of signed graphs. Some of them are identical to notions from graph theory, but for the reader's convenience we repeat them here. For a subset $R \subseteq E$ of edges of a signed graph $\Sigma = (\Gamma, \sigma)$ with $\Gamma = (V, E)$ we define the **subgraph** $\Sigma(R)$ to be the signed graph with the underlying graph $\Gamma(R) = (V, R)$ and the same signature σ restricted to R.



(a) full graph

(b) signed tree

(c) (signed) halfedge-tree

(d) (signed) loop-tree

(e) (signed) pseudo-tree

(f) pseudo-forest

(g) balanced circle

(h) handcuff with a connecting path

(i) handcuff with degenerate connecting path

Figure 2.9: Various types of subgraphs.

- A **path** is a sequence $(v_1, e_1, v_2, e_2, \ldots, e_n, v_{n+1})$ of nodes v_i and edges e_i, such that $e_i = \{v_i, v_{i+1}\}$ and all the edges and vertices in the sequences are pairwise distinct.
- A signed graph Σ is **connected** if there exists a path between any two nodes.
- A **circle** a closed path, i.e., a sequence $(v_1, e_1, v_2, e_2, \ldots, e_n, v_1)$ where only the first and last node are equal. A loop is a circle but we will usually treat loops separately from circles.
- A **signed tree** is a connected signed graph with no circles, loops, or halfedges. See Figure 2.9(b).
- A **(signed) halfedge-tree** is a connected (signed) graph with no circles or loops, and a single halfedge. See Figure 2.9(c).
- A **(signed) loop-tree** is a connected (signed) graph with no circles or halfedges, and a single negative loop. See Figure 2.9(d).
- A **(signed) pseudo-tree** is a connected (signed) graph with no loops or halfedges that contains a single circle with an odd number of negative edges. See Figure 2.9(e).
- A **signed pseudo-forest** is a signed graph whose connected components are signed trees, signed halfedge-trees, signed loop-trees, or signed pseudo-trees. See Figure 2.9(f).
- A **circuit** is a subgraph with an inclusion minimal set of edges that is not a pseudo-forest. For signed graphs those can be a circle with an even number of negative edges, (positive loops, loose edges,) and a **handcuff**, i.e., a path (possibly consisting of only one node) that on each of its two (possibly identical) end-nodes is connected to a negative circle, halfedge, or negative loop. See Figure 2.9(g),2.9(h),2.9(i).

For a signed pseudo-forest Σ, let $\mathrm{tc}(\Sigma)$, $\mathrm{lc}(\Sigma)$ and $\mathrm{pc}(\Sigma)$ be the **number of tree components**, **loop-tree components**, and **pseudo-tree components**, respectively. For a subset $F \subseteq E$ of edges we use the shorthand

$$\mathrm{tc}(F) := \mathrm{tc}(\Sigma(F))$$

(a) positive link (bi)oriented from i to j (left) and from j to i (right), i.e., $\tau(\{i,j\},i) = -$ and $\tau(\{i,j\},j) = +$ (left) and with flipped signs on the right

(b) negative loop (bi)oriented extroverted (left) and (bi)oriented introverted (right), i.e., $\tau(\{i,i\},i) = +$ (left) and $\tau(\{i,i\},i) = -$ (right)

(c) negative link (bi)oriented introverted (left) and extroverted (right), i.e., $\tau(\{i,j\},i) = - = \tau(\{i,j\},j)$ (left) and with flipped signs on the right

(d) halfedge (no signature) pointing towards i (left) and pointing away from i (right), i.e., $\tau(\{i\},i) = +$ (left) and $\tau(\{i\},i) = -$ (right)

Figure 2.10: All different types of bioriented edges.

to count the number of signed tree components in the edge subgraph $\Sigma(F)$. We define $\mathrm{pc}(F)$ and $\mathrm{lc}(F)$ analogously. Note that the edge subgraph $\Sigma(F)$ still has the same number d of nodes as the signed graph Σ; in particular $\mathrm{tc}(\varnothing) = \mathrm{tc}(\Sigma(\varnothing)) = d$. For example, the subgraph in Figure 2.9(f) has one loop-tree component, one pseudo-tree component, and no tree component.

An **orientation** of a signed graph $\Sigma = (\Gamma, \sigma)$ is an assignment τ from the set of node-edge-incidences to $\{\pm\}$ such that $\sigma(e) = -\tau(e,v)\tau(e,u)$ for very edge $e = \{u,v\}$. Equivalently, choosing a **bidirection** τ for an unsigned graph $\Gamma = (V, E)$ first and setting $\sigma(e) = -\tau(e,v)\tau(e,u)$ for very edge $e = \{u,v\}$ defines an oriented signed graph $\Sigma = (\Gamma, \sigma)$ with orientation τ. Hence, oriented signed graphs and bidirected graphs are equivalent objects. We can interpret this as follows: if $\tau(e,v) = +$ the edge e enters node v, i.e., the head of the node-edge-incidence (e,v) points towards v, if $\tau(e,v) = -$ the edge e exits node v, i.e., the head of the node-edge-incidence (e,v) points away from v.[11] See Figure 2.10 for illustrations.

- A **source (sink)** is a node s with only outward (inward) pointing node-edge-incidences, i.e., $\tau(s,e) = -$ ($\tau(s,e) = +$) for all edges e incident to the source (sink) s.
- A **(bidirected) cycle** is an oriented circuit that has neither sinks nor sources.
- An oriented signed graph is called **acyclic** if it does not contain any cycles. It is called **totally cyclic** if every edge is contained in a cycle.

We want to emphasize that we carefully distinguish between the notions of circles, circuits, and cycles, which tend to have non-uniform meanings throughout the literature.

11 Thus, positive edges get oriented consistently with orienting an unsigned graph, whereas negative edges get oriented alongside the charming adjectives *introverted* or *extroverted*.

We now elaborate on the construction of an incidence matrix of a signed graph, hinted at in the beginning. For an oriented signed graph without positive loops or loose edges, we define the **incidence matrix** $\mathbf{A}_\Sigma \in \mathbb{R}^{d\times m}$ by

$$(\mathbf{A}_\Sigma)_{v,e} = \begin{cases} 0 & \text{if } v \text{ and } e \text{ are not incident,} \\ +1 & \text{if } e \text{ enters } v \text{, i.e., } \tau(v,e) = +\,, \\ -1 & \text{if } e \text{ exits } v \text{, i.e., } \tau(v,e) = -\,, \\ \pm 2 & \text{if } e \text{ is a negative loop at } v \text{ and } \tau(v,e) = \pm \text{, respectively.} \end{cases}$$

With our definition of subgraph (see above), the incidence matrix of $\mathbf{A}_{\Sigma(R)} =: \mathbf{R}$ is precisely the matrix formed by columns of $\mathbf{A}_\Sigma$ indexed by R. In order to highlight the already mentioned connection to root vectors of type B/C/D[12] we reformulate the definition of the incidence matrix of a given signed graph Σ with d nodes and m edges as the matrix $\mathbf{A}_\Sigma$ whose column corresponding to the edge e equals

- $\mathbf{e}_j - \mathbf{e}_k$ or $\mathbf{e}_k - \mathbf{e}_j$ if $e = (jk)$ is a positive link,
- $\mathbf{e}_j + \mathbf{e}_k$ or $-\mathbf{e}_k - \mathbf{e}_j$ if $e = (jk)$ is a negative link,
- $\mathbf{e}_j$ if e is a halfedge at j,
- $2\mathbf{e}_j$ or $-2\mathbf{e}_j$ if e is a negative loop at j.

The choices in the above list correspond to choosing a biorientation of Σ, in analogy to $\mathbf{A}_G$ depending on an orientation of G. In both cases, the combinatorial and arithmetic data we will compute are independent of the chosen (bi-)orientation.

Parallel to the graphic case, we define the **acyclotope** corresponding to Σ as the zonotope

$$\mathsf{Z}(\mathbf{A}_\Sigma) = \sum_{i=1}^{m} [\mathbf{0}, \mathbf{a}_i]\,.$$

As discussed in Section 2.5.1, Zaslavsky's original definition of the acyclotope in [Zas91, Section 4] is slightly different. The acyclotope for a signed graph is defined by a subset of a root system of type A, B, C, and D, and vice versa, any such subset defines a signed graph. Thus the acyclotope is a subzonotope (in the sense that we remove some of the generators) of a translate of the respective of the Coxeter permutahedra of type A, B, C, and D defined in Section 2.4. In Chapter 4 we will compute the Ehrhart polynomial for acyclotopes.

A signed graph $\Sigma = (\Gamma, \sigma)$ is called **balanced** if it does not contain any halfedges and every circle has an even number of negative links. An unsigned graph can be realized by a signed graph all of whose edges are labelled with $+$; it is automatically balanced. Vice versa, a balanced graph can be converted into a signed graph with only positive edges by switching operations, that is, for a fixed node v flipping the sign of $\tau(v,e)$ for all node-edge-incidences involving v. In terms of the incidence matrix, switching means flipping signs in one row. The incidence matrix of a connected signed graph is full rank if and only if the graph is not balanced. For more on switching equivalences and balance of signed graphs see, e.g., [Zas82b; Zas10].

12 This correspondence is one reason to leave out positive loops and loose edges when building the incidence matrix; neither do they play a role in our work.

To every signed graph Σ we can associate a **signed graphic matroid** $M(\Sigma) = (E, \mathcal{I})$, also called the **bias matroid** of Σ, which is the representable matroid with ground set E (indexing the columns of the incidence matrix A) and independent sets formed by selections of linearly independent columns in $\mathbf{A}_\Sigma$, i.e.,

$$M(\Sigma) := M(\mathbf{A}_\Sigma)\,.$$

Switching operations preserve all combinatorial data in the signed graphic matroid.

Proposition 2.23 ([Zas82b, Theorem 5.1],)**.** *We recall the signed graphic meaning of the relevant matroid notions:*

(i) *A subset $F \subseteq E$ is an independent set in $M(\Sigma)$ if and only if the subgraph $\Sigma(F)$ is a signed pseudo-forest.*

(ii) *A subset $T \subseteq E$ is a basis in $M(\Sigma)$ if and only if the subgraph $\Sigma(T)$ is an inclusion maximal signed pseudo-forest in Σ.*

(iii) *A subset $C \subseteq E$ is a circuit in $M(\Sigma)$ if and only if the subgraph $\Sigma(C)$ is a circuit together with isolated vertices (the set of isolated vertices might be empty).*

(iv) *A subset $C \subseteq E$ is dependent in $M(\Sigma)$ if and only if the subgraph $\Sigma(C)$ contains a circuit.*

Note that inclusion maximal edge sets that form a pseudo-forest as subgraphs are not necessarily connected, hence not necessarily a (signed) tree, loop-tree, halfedge-tree, or pseudo-tree, see, e.g., Figure 2.9(f). We will call an edge set $S \subseteq E$ **spanning** if it contains an inclusion maximal pseudo-forest. Again this does not imply that the subgraph is connected.

A **coloop** is an edge in a signed graph that corresponds to a coloop in the signed graphic matroid, i.e., an element that is contained in every basis. A coloop is not necessarily a bridge in the (underlying) graph, i.e., an edge that increases the number of connected components after removing, as it is in the case of unsigned graphs. Conversely, not every bridge in (the underlying graph of) a signed graph is necessarily a coloop. See [Bou83, Lemma 2.4 and Lemma 2.5].

Remark 2.24. Proposition 2.23 shows that every signed graph also has an associated matroid structure. This can be extended to oriented signed graphs having an oriented matroid structure. See [Zas91, Section 3].

2.5.5 Partially Ordered Sets

In this section we recall some background on posets, order polytopes, and order cones. These will be central objects in Chapter 6.

A finite partially ordered set, a **poset**, is a pair $\mathcal{P} = (U, \preceq_\mathcal{P})$ consisting of a finite ground set U and an order relation $\preceq_\mathcal{P} \subseteq U \times U$, such that $\preceq_\mathcal{P}$ is transitive, reflexive and antisymmetric. By a standard abuse of notation, we often identify the poset $\mathcal{P}$ with its ground set U. A poset $\mathcal{C} = (U, \preceq_\mathcal{C})$ is called a **total order** or **chain** if for any two elements $a, b \in \mathcal{C}$ either $a \preceq_{\mathcal{C}} b$ or $b \preceq_{\mathcal{C}} a$. An **antichain** $\mathcal{A} = (U, \preceq_{\mathcal{A}})$ is a poset without any relation, that is, $\preceq_{\mathcal{A}} = \varnothing$. An element $a \in \mathcal{P}$ is called a **minimal element** (resp. **maximal element**) if there is no element $c \in \mathcal{P}$ such that $c \preceq_{\mathcal{P}} a$ (resp. $a \preceq_{\mathcal{P}} c$). In an antichain every element is minimal and maximal, while a chain has precisely one minimal and one maximal element. For two elements $a, b \in \mathcal{P}$ we define the **interval** $[a, b] \subseteq \mathcal{P}$ as

$$[a, b] := \{c \in \mathcal{P} \colon a \preceq c \preceq b\} .$$

We say that b **covers** a (denoted by $a \prec b$) if $a \preceq b$, $a \neq b$, and there is no element c such that $a \preceq c \preceq b$. We call $a \prec b$ a **cover relation**. The **Hasse diagram** of a poset $\mathcal{P} = (U, \preceq)$ is a graphical representation of the poset in the following way: define an undirected graph $G(\mathcal{P}) = (U, \prec)$ with the ground set U as nodes and one undirected edge for each cover relation, i.e., $\{a, b\}$ is an edge in $G(\mathcal{P})$ if either $a \prec b$ or $b \prec a$. For example, the Hasse diagram of a chain is a path and the Hasse diagram of an antichain consists of isolated nodes. See Figure 2.11 for some Hasse digrams. We call a poset **connected**, if the Hasse diagram (considered as an undirected graph) is connected. For a poset $\mathcal{P} = (U, \preceq_{\mathcal{P}})$ and a subset $T \subseteq U$ we define the **poset restricted to** T as $\mathcal{P}[T] = (T, \preceq_T)$, where

$$\preceq_T := \preceq_{\mathcal{P}} |_{T \times T} = \{(a, b) \in T \times T \colon (a, b) \in \preceq_{\mathcal{P}} \subseteq U \times U\} .$$

The Hasse diagram of a restricted poset $\mathcal{P}[T]$ is the induced subgraph $G(\mathcal{P})|_T$ (2.20).

A **filter** $\mathcal{F}$ is a subset of the ground set U that is upwards closed, i.e., if $a \preceq b$ for some $a \in \mathcal{F}$ then $b \in \mathcal{F}$. We call a **chain of filters** $\mathfrak{F} \colon \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}$ **connected** if the poset restricted to $\mathcal{F}_{i+1} \setminus \mathcal{F}_i$ is connected for $i = 0, \ldots, k$.

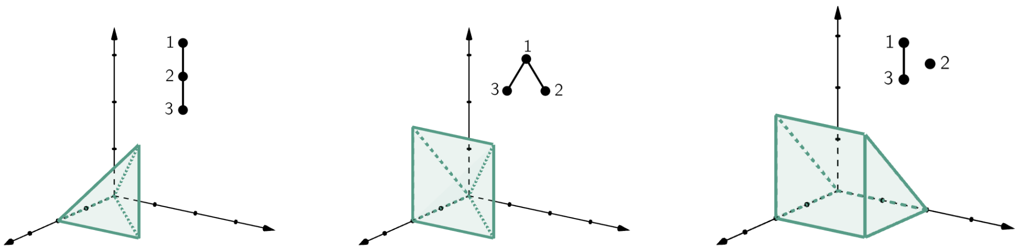
A poset $\mathcal{L}$ is called a **lattice**[13] if for every two elements $a, b \in \mathcal{L}$ there exists a lowest upper bound $a \vee b$, called **join**, and a greatest lower bound $a \wedge b$, called **meet**. A lattice $\mathcal{L}$ is called **distributive**, if for all $a, b, c \in \mathcal{L}$ we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) .$$

For any (finite) poset $\mathcal{P}$ the collection of filters together with the inclusion relation forms a finite distributive lattice called the **Birkhoff lattice of filters** $\mathcal{B}(\mathcal{P})$. For example, the Birkhoff lattice of an antichain $\mathcal{A} = ([d], \varnothing)$ is the **Boolean lattice** $\mathcal{B}(\mathcal{A}) = (2^{[d]}, \subseteq)$. The fundamental theorem for finite distributive lattices states that for every finite distributive lattice $\mathcal{L}$ there exists a (up to isomorphism unique) poset $\mathcal{P}$ such that $\mathcal{L} = \mathcal{B}(\mathcal{P})$, see, e.g., [Sta12, Section 3.4].

A **linear extension** $\mathcal{T} = (U, \preceq_{\mathcal{T}})$ of a poset $\mathcal{P} = (U, \preceq_{\mathcal{P}})$ is a total order (or chain) on the same ground set U that extends the partial order $\preceq_{\mathcal{P}}$, i.e., $\preceq_{\mathcal{P}} \subseteq \preceq_{\mathcal{T}}$. We denote the set of linear extensions of a poset $\mathcal{P}$ by $\mathfrak{L}(\mathcal{P})$. Note that linear extensions $\mathcal{T} \in \mathfrak{L}(\mathcal{P})$ are in bijection with full chains of filters $\mathfrak{F} \colon \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_{|U|} = \mathcal{P}$, i.e., $|\mathcal{F}_i \setminus \mathcal{F}_{i-1}| = 1$, or equivalently, inclusion-maximal chains in the Birkhoff lattice $\mathcal{B}(\mathcal{P})$. Indeed, for $\mathcal{T} = \{a_1 \prec a_2 \prec \cdots \prec a_{|U|}\}$ define $\mathcal{F}_i := \{a_{|U|-i+1}, \ldots, a_{|U|}\}$ for $i = 0, \ldots, k$. See, e.g., [BS18, Lemma 6.3.4.].

13 Note that a lattice $\mathcal{L}$ as a special partially ordered set essentially only has its name in common with the lattices (discrete subgroups) $\Lambda \subset \mathbb{R}^d$ from Section 2.2.



(a) Chain poset on three elements and its order simplex.

(b) A tree poset on three elements and its associated order polytope.

(c) Disjoint union of a chain on two elements and a singleton and its order polytope.

Figure 2.11: Three examples of order polytopes in three dimensions and the Hasse diagram of their corresponding posets on three elements.

We now want to recall order polytopes. The first reference systematically studying order polytopes was [Sta86]. Stanley writes in the introduction: "Much of what we say about the order polytope will be essentially a review of well-known results, albeit ones scattered throughout the literature, sometimes in a rather obscure form." For a textbook introduction to order polytopes and order cones, see, e.g., [BS18, Chapter 6].

Recall that $\mathbb{R}^{\mathcal{P}} = \{f : \mathcal{P} \to \mathbb{R}\}$ is a vector space of dimension $|\mathcal{P}|$. In this section we will mostly work in the vector space $\mathbb{R}^{\mathcal{P}}$. For some examples the notation will be simpler in $\mathbb{R}^d \cong \mathbb{R}^{\mathcal{P}}$ for $|\mathcal{P}| = d$ (see Section 2.0).

We define the **order polytope** $\mathsf{O}(\mathcal{P})$ as the subset of all functions $f : \mathcal{P} \to \mathbb{R}$ such that

$$\begin{aligned} 0 \leq f(a) &\leq 1\,, && \text{for all } a \in \mathcal{P}\,, \\ f(a) &\leq f(b) && \text{if } a \preceq b \text{ in } \mathcal{P}\,. \end{aligned} \tag{2.24}$$

This subset of $\mathbb{R}^{\mathcal{P}}$ is described by finitely many linear in equalities and from the conditions in (2.24) it follows that it is bounded, hence it is indeed a polytope. For example the order polytope of an antichain on d elements is the hypercube $[0,1]^d$. For more examples see Figure 2.11.

Facets are given by cover relations, minimal, and maximal elements, i.e., an irreducible description of the order polytope is given by

$$\begin{aligned} 0 \leq f(a)\,,&\ \text{for all minimal elements } a \in \mathcal{P}\,, \\ f(a) \leq f(b)\,,&\ \text{for all cover relations } a \prec\!\!\cdot\, b \text{ in } \mathcal{P}\,, \\ f(a) \leq 1\,,&\ \text{for all maximal elements } a \in \mathcal{P}\,. \end{aligned} \tag{2.25}$$

From that description it already follows that the vertices of order polytopes are $0/1$-vectors.

We will now give a complete combinatorial description of the faces of the order polytope in terms of partitions. Recall that a **(set) partition** $\pi = \{B_1, \ldots, B_k\}$ of a finite set U is a collection $\{B_1, \ldots, B_k\}$ of subsets of U called **blocks**, such that the blocks are pairwise disjoint and their union is U. Note that the blocks in a partition are not ordered, i.e., $\{\{1,2\},\{3\}\}$ and $\{\{3\},\{1,2\}\}$ are the same partition.

Here, it is convenient to embed the order polytope in a higher dimensional space by defining a new poset $\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}$, where the element $\widehat{0}$ is smaller than all elements in $\mathcal{P}$ and the element $\widehat{1}$ is larger than all other elements in $\mathcal{P}$. Now define $\widehat{\mathsf{O}}(\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}) \subset \mathbb{R}^{\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}}$ by the following (in-)equalities:

$$\begin{gathered} 0 = g(\widehat{0}), \quad g(\widehat{1}) = 1, \\ g(a) \leq g(b)\,, \text{ for all } a \prec b \text{ in } \mathcal{P} \cup \{\widehat{0}, \widehat{1}\}\,. \end{gathered}$$

The linear map given by forgetting the coordinates corresponding to $\widehat{0}$ and $\widehat{1}$ is clearly a linear bijection from $\widehat{\mathsf{O}}(\mathcal{P} \cup \{\widehat{0}, \widehat{1}\})$ to $\mathsf{O}(\mathcal{P})$.

Facets in $\widehat{\mathsf{O}}(\mathcal{P} \cup \{\widehat{0}, \widehat{1}\})$ are given by cover relations in $\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}$, i.e., facets are of the form

$$\mathsf{O}(\mathcal{P}) \cap \{f \in \mathbb{R}^{\mathcal{P}} \colon f(a) = f(b)\} \qquad \text{for some cover relation } a \prec b \text{ in } \mathcal{P}\,.$$

Every face is the intersection of some facets and hence every face $\mathsf{F} \subseteq \widehat{\mathsf{O}}(\mathcal{P} \cup \{\widehat{0}, \widehat{1}\})$ defines a partition of $\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}$ by putting elements $a, b \in \mathcal{P} \cup \{\widehat{0}, \widehat{1}\}$ into the same block if they are related by a cover relation $a \prec b$ corresponding to a facet containing the face F. We call such a partition a **face partition** and the coarsest partition defined by a particular face is called **closed**. A closed face partition has k blocks, if and only if the dimension of the corresponding face is $k - 2$.

However, not every partition of $\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}$ defines a face of the order polytope $\widehat{\mathsf{O}}(\mathcal{P} \cup \{\widehat{0}, \widehat{1}\})$. In order to characterize these partitions, we need more definitions: A partition $\pi = \{B_1, \ldots, B_k\}$ is called **connected**, if for every block B_i, $i = 1, \ldots, k$ the Hasse of the poset $\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}$ restricted to B_i is connected. Given a partition $\pi = \{B_1, \ldots, B_k\}$ of $\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}$ we define a binary relation on the blocks $\{B_1, \ldots, B_k\}$ by $B_i \prec_\pi B_j$ if $a \preceq_{\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}} b$ for some $a \in B_i$ and $b \in B_j$. We call the partition π **compatible** if the transitive closure of the relation $\preceq_\pi$ is anti-symmetric and hence defines a partial order. Note that connected compatible partitions of $\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}$ are exactly those defining poset quotients as described in [Wil24, Section 6.5].

Theorem 2.25 ([Sta86, Theorem 1.2])**.** *A partition $\pi = \{B_1, \ldots, B_k\}$ of $\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}$ with k blocks is a closed face partition for a face of dimension $k - 2$ if and only if it is connected and compatible. In particular, the partition into a single block $\pi = \{\mathcal{P} \cup \{\widehat{0}, \widehat{1}\}\}$ yields the empty face, which we regard as a face of dimension -1.*

This also implies that every face of an order polytope $\mathsf{O}(\mathcal{P})$ is again isomorphic to an order polytope (of a quotient poset of $\mathcal{P}$ as in [Wil24]).

Example 2.26. In figure Figure 2.11(b) we can see that the order polytope $\mathsf{O}(\mathcal{P})$ of the tree poset $\mathcal{P} = ([3], \preceq_{\mathcal{P}} = \{(3,1), (2,1)\})$ has one 2-face, that is a quadrilateral. This corresponds to the set partition $\{\{\widehat{0}\}, \{2\}, \{3\}, \{1, \widehat{1}\}\}$ of $[3] \cup \{\widehat{0}, \widehat{1}\}$. This face is isomorphic to the order polytope of an antichain with two elements.

The remaining 2-faces are triangles and correspond to the partitions

$$\begin{gathered} \{\{\widehat{0}, 2\}, \{1\}, \{3\}, \{\widehat{1}\}\}\,,\ \{\{\widehat{0}, 3\}, \{1\}, \{2\}, \{\widehat{1}\}\} \\ \{\{\widehat{0}\}, \{1, 2\}, \{3\}, \{\widehat{1}\}\}\,,\ \{\{\widehat{0}\}, \{2\}, \{1, 3\}, \{\widehat{1}\}\}\,. \end{gathered}$$

These faces are isomorphic to order polytopes of chains with two elements.

We collect some corollaries about the face structure of order polytopes, which we will use later. We start with a description of the vertices. Face partitions that give rise to a 0-dimensional face have two blocks $\{B_1, B_2\}$. Connectedness and compatibility imply that these two blocks are of the form $B_2 = \{\hat{1}\} \cup \mathcal{F}$ and $B_1 = \{\hat{0}\} \cup (\mathcal{P} \setminus \mathcal{F})$ for some filter $\mathcal{F} \subseteq \mathcal{P}$. For an alternative proof of the vertex description, see, e.g., [BS18, Corollary 6.3.1.].

Corollary 2.27. *Vertices of the order polytope* $\mathsf{O}(\mathcal{P})$ *are given by indicator vectors* $\mathbb{1}_\mathcal{F}$ *of filters* $\mathcal{F} \subseteq \mathcal{P}$*, i.e.,* $\mathrm{vert}(\mathsf{O}(\mathcal{P})) = \{\mathbb{1}_\mathcal{F} \colon \mathcal{F} \subseteq \mathcal{P} \text{ is a filter}\}$.

It follows that the order polytope is full-dimensional and for total orders $\mathcal{T}$ the order polytope $\mathsf{O}(\mathcal{T})$ is a simplex. E.g., for the total order $\mathcal{T} = ([d], \leq)$ the vertices $\mathbf{0}, \mathbb{1}_{\{n\}}, \mathbb{1}_{\{n-1,n\}}, \dots, \mathbb{1}_{\{2,\dots,n-1,n\}}, \mathbf{1}$ are affinely independent. See Figure 2.11(a) for a three-dimensional example.

Corollary 2.28. *Let* F *be a facet of* $\mathsf{O}(\mathcal{P})$ *corresponding to cover relation* $a \prec b$ *in* $\mathcal{P}$*, then* F *contains precisely all vertices* $\mathbb{1}_\mathcal{F}$ *for filters* $\mathcal{F}$ *with* $\{a, b\} \subseteq \mathcal{F}$.

Now we would like describe the edges of the order polytope using Theorem 2.25, hence we need face partitions $\{B_1, B_2, B_3\}$ with three blocks. It can be checked that these are connected and compatible if and only if they are of the form $B_3 = \{\hat{1}\} \cup \mathcal{F}'$, $B_2 = \mathcal{F}' \setminus \mathcal{F}$, and $B_1 = \{\hat{0}\} \cup (\mathcal{P} \setminus \mathcal{F})$ for two filters $\mathcal{F}, \mathcal{F}' \subseteq \mathcal{P}$ such that $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{F} \setminus \mathcal{F}'$ is a connected poset. A proof for the following Corollary can also be found in [CM10, Theorem 1].

Corollary 2.29. *Two filters* $\mathcal{F}, \mathcal{F}'$ *correspond to the vertices* $\mathbb{1}_\mathcal{F}, \mathbb{1}_{\mathcal{F}'}$ *of an edge of* $\mathsf{O}(\mathcal{P})$ *if and only if, say,* $\mathcal{F}' \subseteq \mathcal{F}$ *and* $\mathcal{F} \setminus \mathcal{F}'$ *is a connected poset.*

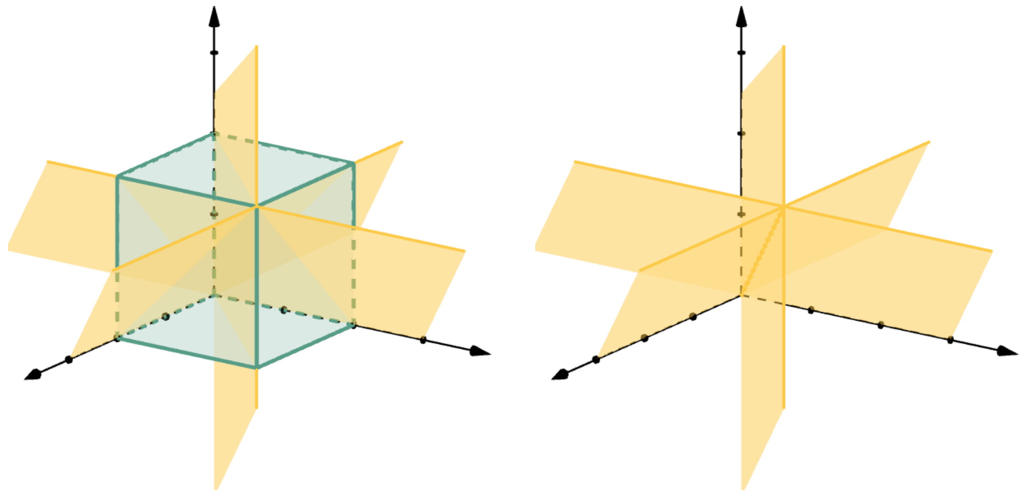
Every two dimensional faces of any 0/1-polytope is either a triangle or quadrilateral, see, e.g., [Zie00]. We can also see this from the interpretation that every face of an order polytope is (isomorphic to) an order polytope itself. Two-dimensional order polytopes $\mathsf{O}(\mathcal{P})$ correspond to posets $\mathcal{P}$ with two elements, either the two elements are comparable and $\mathsf{O}(\mathcal{P})$ is a triangle, or $\mathcal{P} = (\{a, b\}, \varnothing)$ and $\mathsf{O}(\mathcal{P})$ is a quadrilateral.

Corollary 2.30. *Let* F *be a two-dimensional face of* $\mathsf{O}(\mathcal{P})$*. Then* F *is either a triangle or a quadrilateral.*

We can also play that game the other way around: let us define a partition of the poset $\mathcal{P}$ that is connected and compatible and see what face in the order polytope $\mathsf{O}(\mathcal{P})$ it defines. Note that in the poset $\mathcal{P} \cup \{\hat{0}, \hat{1}\}$ every filter $\mathcal{F} \subseteq \mathcal{P}$ is connected. Similarly, every complement of a filter $\mathcal{P} \setminus \mathcal{F}$ is connected in $\mathcal{P} \cup \{\hat{0}, \hat{1}\}$. Finally, every singleton set $\{a\}$ for $a \in \mathcal{P}$ is trivially connected. Now it is easy to check that for any two filters $\mathcal{F}, \mathcal{F}' \subseteq \mathcal{P}$ with $\mathcal{F} \subseteq \mathcal{F}'$

$$\pi = \Big\{\{a\} \colon a \in \mathcal{F}' \setminus \mathcal{F}\Big\} \ \cup \ \Big\{\mathcal{F} \cup \{1\}\Big\} \ \cup \ \Big\{(\mathcal{P} \setminus \mathcal{F}') \cup \{0\}\Big\}$$

defines a connected and compatible partition, hence by Theorem 2.25 a face partition.



(a) The cube, also the order polytope of the antichain on three elements and its triangulation by the order cones of the six linear extensions.

(b) The three dimensional order cones of the six total orders on three elements inducing a complete polyhedral fan in $\mathbb{R}^3$.

Figure 2.12: Polyhedral fan induced by all total orders on three elements forming a complete fan in $\mathbb{R}^3$ (right) and subdividing the unit cube $[0,1]^3$ (left).

Corollary 2.31. *For any two filters $\mathcal{F} \subseteq \mathcal{F}'$ in a poset $\mathcal{P}$ there is a unique face $\mathsf{F}(\mathcal{F}, \mathcal{F}') \subseteq \mathsf{O}(\mathcal{P})$ whose vertices are $\mathbb{1}_{\mathcal{G}}$ for filters $\mathcal{G}$ in the interval $[\mathcal{F}, \mathcal{F}'] \subseteq \mathcal{B}(\mathcal{P})$ in the Birkhoff lattice of $\mathcal{P}$.*

For a poset $\mathcal{P}$ we define the **order cone**

$$\mathsf{C}(\mathcal{P}) := \left\{ f \in \mathbb{R}^{\mathcal{P}} \colon f(a) \leq f(b) \text{ if } a \preceq_{\mathcal{P}} b \right\} .$$

It follows from the transitivity of posets that an irredundant description of order cones is given by the cover relations in the poset $\mathcal{P}$:

$$\mathsf{C}(\mathcal{P}) = \left\{ f \in \mathbb{R}^{\mathcal{P}} \colon f(a) \leq f(b) \text{ if } a \prec\!\!\cdot_{\mathcal{P}} b \right\} .$$

We have seen these cones already in Section 2.3: Let us consider all total orders on the ground set $[d]$. Then

$$\{\mathsf{C}(\mathcal{T}) \subset \mathbb{R}^d \colon \mathcal{T} \text{ is a total order on } [d]\}$$

defines a complete polyhedral fan in $\mathbb{R}^d$, in fact, this is the braid fan $\mathcal{B}_d$, or equivalently, the Coxeter fan $\mathcal{N}_{A_{d-1}}$ in type A. See Figure 2.12(b) for an illustration in three dimensions. That is full-dimensional cones in a coarsening of the braid fan are order cones and the description of faces in the braid fan given in Lemma 2.10 is essentially equivalent to Theorem 2.25 for total orders and reformulated for cones.

From the inequality descriptions of the order cones we can also easily check that for total orders on U we have that $\mathsf{C}(\mathcal{T}) \subseteq \mathsf{C}(\mathcal{P})$ for a poset $\mathcal{P} = (U, \preceq)$ if and only if $\mathcal{T} \in \mathfrak{L}(\mathcal{P})$ (see, e.g., [BS18, Proposition 6.2.1]). Therefore

$$\{\mathsf{C}(\mathcal{T}) \colon \mathcal{T} \in \mathfrak{L}(\mathcal{P})\}$$

forms a polyhedral subdivision of the order cone $\mathsf{C}(\mathcal{P})$.

Now we return to order *polytopes*. For every poset $\mathcal{P}$

$$\mathsf{O}(\mathcal{P}) = [0,1]^{\mathcal{P}} \cap \mathsf{C}(\mathcal{P}) \,.$$

Recall that the order polytope of a total order is a simplex.

Proposition 2.32. *For every poset $\mathcal{P} = (U, \preceq)$ the collection of order simplices $\{\mathsf{O}(\mathcal{T}) \colon \mathcal{T} \in \mathfrak{L}(\mathcal{P})\}$ forms a triangulation of $\mathsf{O}(\mathcal{P})$.*

See Figure 2.12(a) for an illustration of the triangulation of the 3-cube, i.e., the order polytope of the antichain $\mathcal{A} = ([3], \varnothing)$.

One can consider various constructions on posets. We will only use disjoint unions and how they translate on the order polytope side. For a proof of the following proposition, see, e.g., [BS18, Lemma 6.1.3].

Proposition 2.33. *Let $\mathcal{P} = (U, \preceq_{\mathcal{P}})$ and $\mathcal{Q} = (V, \preceq_{\mathcal{Q}})$ be two posets. Then $\mathcal{P} \cup \mathcal{Q} :=$ $(U \cup V, \preceq_{\mathcal{P}} \cup \preceq_{\mathcal{Q}})$ is a poset and $\mathsf{O}(\mathcal{P} \cup \mathcal{Q}) = \mathsf{O}(\mathcal{P}) \times \mathsf{O}(\mathcal{Q})$.*

See Figure 2.11(c) for an example. Here, the poset is a disjoint union of a two-chain and an antichain with one element. The order polytope of a two-chain is a triangle, the order polytope of the antichain with one element is the line segment $[0,1] \subset \mathbb{R}$ and their product is the triangular prism in the figure.

3

Pruned Inside-out Polytopes

A *combinatorial reciprocity theorem* can be described as a result that relates two classes of combinatorial objects via their enumeration problems (see, e.g., [Sta74; BS18]). For example, the number of proper m-colorings of a graph $g = (U, E)$ agrees with a polynomial $\chi(g)(m)$ of degree $d = |U|$ for positive integers $m \in \mathbb{Z}_{>0}$, and $(-1)^d \chi(g)(-m)$ counts the number of pairs of compatible acyclic orientations and m-colorings of the graph g [Sta73]. Another example is Ehrhart–Macdonald-reciprocity (Theorem 2.4).

As we have seen in Section 2.3 and Section 2.5, generalized permutahedra are a class of polytopes containing numerous subclasses of polytopes defined via combinatorial structures, such as graphic zonotopes, hypergraphic polytopes (Minkowski sums of simplices), simplicial complex polytopes, matroid polytopes, associahedra, and nestohedra. Generalized permutahedra themselves are closely related to submodular functions, which have applications in optimization.

One of the main results in this chapter is a combinatorial reciprocity theorem for generalized permutahedra counting integral directions with k-dimensional maximal faces:

Theorem 3.10. *For a generalized permutahedron* $\mathsf{P} \subseteq \mathbb{R}^d$ *and* $k = 0, \ldots, d-1$,

$$\chi_{d,k}(\mathsf{P})(m) := \#\Big\{\mathbf{y} \in [m]^d \ : \ \mathbf{y}\text{-maximum face } \mathsf{P}^{\mathbf{y}} \text{ is a } k\text{-face}\Big\}$$

agrees with a polynomial of degree $d-k$, *and*

$$(-1)^{d-k}\chi_{d,k}(\mathsf{P})(-m) = \sum_{\mathbf{y}\in[m]^d} \#\left(k\text{-faces of } \mathsf{P}^{\mathbf{y}}\right).$$

We will use integer point counting in dissected and dilated cubes to prove this result and comment on further generalizations in Remark 3.11.

The special case of this theorem for $k = 0$, i.e., generic directions, was obtained by Aguiar and Ardila [AA23], and earlier by Billera, Jia, and Reiner [BJR09] in a slightly different language. The $k = 0$ case was also recently extended in [Kar22]. As shown for some examples in [AA23, Section 18] the application of such a result to the various subclasses of generalized permutahedra yields already known combinatorial reciprocity theorems for their related combinatorial structures such as matroid polynomials [BJR09], Bergmann polynomials of matroids and Stanley's famous reciprocity theorem for graph colorings [Sta73].

Aguiar and Ardila develop a Hopf monoid structure on the species of generalized permutahedra, work with polynomial invariants defined by characters, and apply their antipode formula to get the combinatorial interpretation of the reciprocity result for generalized permutahedra for $k = 0$ (Theorem 3.19, below) [AA23, Sections 16, 17]. This method is also used in [Kar22]. The approach in [BJR09] is similar to the one by Aguiar and Ardila. Billera, Jia, and Reiner use Hopf algebras of matroids and quasisymmetric functions, as well as a multivariate generating function as isomorphism invariants of matroids. The reciprocity providing ingredient is again the antipode of a Hopf algebra together with Stanley's reciprocity for P-partitions [BJR09, Sections 6 and 9].

We give a different, geometric perspective. In order to prove Theorem 3.10 we apply Ehrhart–Macdonald reciprocity to pruned inside-out polytopes. A pruned inside-out polytope $\mathsf{Q} \setminus \bigcup \mathcal{N}^{\mathrm{co}\,1}$ consist of the points that lie inside a polytope Q but not in the codimension one cones $\mathcal{N}^{\mathrm{co}\,1}$ of a complete polyhedral fan $\mathcal{N}$. This is a generalization of inside-out polytopes introduced by Beck and Zaslavsky [BZ06b]. We think of the codimension-one cones $\mathcal{N}^{\mathrm{co}\,1}$ defining a pruned inside-out polytope as *pruned* hyperplanes, hence the name. One of the many applications of inside-out polytopes [BZ06c; BZ06a; BZ10; BS18] is yet a different proof of Stanley's reciprocity result for graph colorings [Sta73].

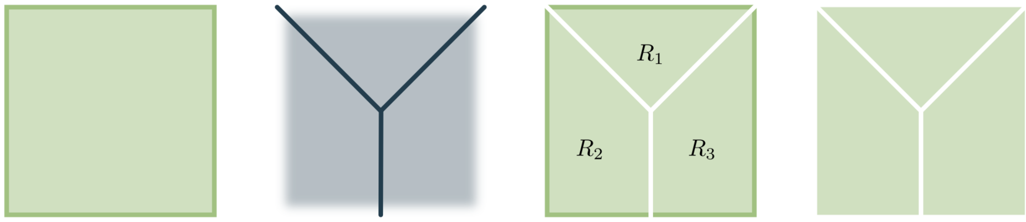
Aval, Karaboghossian, and Tanasa presented a reciprocity theorem for hypergraph colorings [AKT20], generalizing Stanley's result for graph colorings. A main tool in the paper is a Hopf monoid structure on hypergraphs defined in [AA23, Section 20.1.] and the associated basic polynomial invariant. However, they do *not* use the antipode as reciprocity inducing element, but rather technical computations involving Bernoulli numbers.

In Section 3.2.2 we show how the reciprocity theorem for hypergraph colorings in [AKT20] is a consequence of the reciprocity for generalized permutahedra. Our main tool is a vertex description of hypergraphic polytopes in terms of acyclic orientations of hypergraphs (Proposition 3.13). More recent work by Karaboghossian [Kar20; Kar22] presents a more general version of the combinatorial reciprocity result for hypergraphs and an alternative proof with similar techniques as we present in Section 3.2.2.

As spelled out in [AA23, Sections 21–25] and [AKT20, Section 4] hypergraphs and hypergraphic polytopes contain a number of interesting combinatorial subclasses such as simple hypergraphs, graphs, simplicial complexes, building sets, set partitions, and paths, together with their associated polytopes such as graphical zonotopes, simplicial complex polytopes, nestohedra, and graph associahedra.

This chapter is organized as follows: In Section 3.1 we introduce the notion of pruned inside-out polytopes, define two counting functions on pruned inside-out polytopes, and derive (quasi-)polynomiality and reciprocity results. Section 3.2 provides three applications of the results in Section 3.1; first, to generalized permutahedra, giving a new geometric perspective on reciprocity theorems in [BJR09; AA23; Kar22] and, moreover, presenting generalized versions for arbitrary face dimensions (Section 3.2.1). The relationship between our approach and the polynomial invariants for Hopf monoids is analyzed in Section 3.3. Secondly, we apply the reciprocity theorem for generalized permutahedra to the subclass of hypergraphic polytopes giving an elementary combinatorial and geometric proof of the reciprocity theorem for hypergraph colorings in [AKT20] (Section 3.2.2). Finally, in Section 3.2.3, we briefly discuss the application of our tools to deformed Coxeter permutahedra in types A, B, C, and D (Section 2.4).

Most of this chapter is published in [Reh22], except for Section 3.2.3 (unpublished).



(a) Polytope $[-1,1]^2$.

(b) The complete polyhedral fan $\mathcal{M}$ and its codimension-1 fan $\mathcal{M}^{\mathrm{co}\,1}$ (rays).

(c) The pruned inside-out polytope $[-1,1]^2 \setminus \bigcup \mathcal{M}^{\mathrm{co}\,1}$ with its regions R_1, R_2, and R_3.

(d) The open pruned inside-out polytope $(-1,1)^2 \setminus \bigcup \mathcal{M}^{\mathrm{co}\,1}$ with open regions.

Figure 3.1: Construction of pruned inside-out polytopes and their regions as in Example 3.1.

3.1 Pruned Inside-out Polytopes and Ehrhart Theory

In [BZ06b] Beck and Zaslavsky develop the notion of an inside-out polytope, that is, a polytope dissected by hyperplanes. Counting integer points in a polytope but off certain hyperplanes turns out to be a useful tool to derive (quasi-)polynomiality results and reciprocity laws for various applications such as graph colorings and signed graph colorings, composition of integers, nowhere-zero flows on graphs and signed graphs, antimagic labellings, as well as magic, semimagic, and magic latin squares [BZ06c; BZ06a; BZ10]. We introduce a generalization of inside-out polytopes, which we call pruned inside-out polytopes and develop Ehrhart-theoretic results (Section 3.1).

Recall that a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ is a family of polyhedral cones that is closed under taking faces, where intersections of cones form faces and that covers $\mathbb{R}^d$. Recall that the codimension one fan $\mathcal{N}^{\mathrm{co}\,1}$ contains all cones of the complete fan $\mathcal{N}$ with codimension at least one. This implies in particular that the codimension-one fan of the normal fan $\mathcal{N}^{\mathrm{co}\,1}(\mathsf{P})$ of a polytope P defined in (2.2) can be described as

$$\begin{aligned}\mathcal{N}^{\mathrm{co}\,1}(\mathsf{P}) &= \mathcal{N}(\mathsf{P}) \setminus \Big\{\mathsf{N}_{\mathsf{P}}(v) \ : \ v \text{ vertex of } \mathsf{P}\Big\}\\ &= \Big\{\mathsf{N}_{\mathsf{P}}(\mathsf{F}) \ : \ \mathsf{F} \text{ a face of } \mathsf{P} \text{ with } \dim(\mathsf{F}) \geq 1\Big\}.\end{aligned}$$

For a polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we call

$$\mathsf{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1}\right) = \biguplus_{\substack{\mathsf{N}\in\mathcal{N},\\ \mathsf{N} \text{ full-dimensional}}} (\mathsf{Q} \cap \mathsf{N}^{\circ})$$

a **pruned inside-out polytope** and we call the connected components in the pruned inside-out polytope $\mathsf{Q} \setminus (\bigcup \mathcal{N}^{\mathrm{co}\,1})$ **regions**. So, a pruned inside-out polytope $\mathsf{Q} \setminus (\bigcup \mathcal{N}^{\mathrm{co}\,1})$ is the disjoint union of its regions $\mathsf{Q} \cap \mathsf{N}^{\circ}$, where N° is an open full-dimensional cone in $\mathcal{N}$. We will mostly consider open pruned inside-out polytopes $\mathsf{Q}^{\circ} \setminus (\bigcup \mathcal{N}^{\mathrm{co}\,1})$, which decompose into disjoint open polytopes, the regions. A pruned inside-out polytope is **rational** if the topological closures of all its regions are rational polytopes.

Example 3.1. Let $[-1,1]^2 \subseteq \mathbb{R}^2$ be a square (see Figure 3.1(a)). We consider the complete fan $\mathcal{M} := \{\mathsf{N}_1, \mathsf{N}_2, \mathsf{N}_3, \dots\}$ consisting of all faces of the three full--dimensional cones

$$\begin{aligned}\mathsf{N}_1 &:= \{\mathbf{x} \in \mathbb{R}^2 \colon \mathbf{x}_2 + x_1 \geq 0,\ \mathbf{x}_2 - \mathbf{x}_1 \geq 0\}\,,\\ \mathsf{N}_2 &:= \{\mathbf{x} \in \mathbb{R}^2 \colon \mathbf{x}_2 \leq 0,\ \mathbf{x}_2 + \mathbf{x}_1 \leq 0\}\,,\\ \mathsf{N}_3 &:= \{\mathbf{x} \in \mathbb{R}^2 \colon \mathbf{x}_2 \geq 0,\ \mathbf{x}_2 - \mathbf{x}_1 \leq 0\}\,.\end{aligned}$$

Then the codimension-one fan $\mathcal{M}^{\mathrm{co}\,1} = \{n_1, n_2, n_3\}$ consists of the three rays

$$\begin{aligned}n_1 &:= \{(\lambda, \lambda) \in \mathbb{R}^2 \colon \lambda \geq 0\}\,,\\ n_2 &:= \{(-\lambda, \lambda) \in \mathbb{R}^2 \colon \lambda \geq 0\}\,,\\ n_3 &:= \{(0, -\lambda) \in \mathbb{R}^2 \colon \lambda \geq 0\}\,.\end{aligned}$$

See Figure 3.1(b). The pruned inside-out polytope

$$[-1,1]^2 \setminus \bigcup \mathcal{M}^{\mathrm{co}\,1} = [-1,1]^2 \setminus \bigcup_{i=1}^{3} n_i = \bigcup_{i=1}^{3} \left([-1,1]^2 \cap \mathsf{N}_i^\circ\right)$$

is composed of three half-open regions R_1, R_2, R_3, see Figure 3.1(c). Their topological closures can be described as

$$\overline{R}_1 = \mathrm{conv}\{(0,0),(1,1),(-1,1)\}\,, \quad \overline{R}_2 = \mathrm{conv}\{(0,0),(-1,1),(-1,-1),(0,1)\}\,,$$
$$\overline{R}_3 = \mathrm{conv}\{(0,0),(0,1),(1,-1),(1,1)\}\,.$$

The open pruned inside-out polytope

$$(-1,1)^2 \setminus \bigcup \mathcal{M}^{\mathrm{co}\,1} = (-1,1)^2 \setminus \bigcup_{i=1}^{3} n_i = \bigcup_{i=1}^{3} R_i^\circ$$

is depicted in Figure 3.1(d).

For a positive integer $n \in \mathbb{Z}_{>0}$ we define the **inner pruned Ehrhart function** as

$$\mathrm{in}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n) := \#\left(\frac{1}{n}\mathbb{Z}^d \cap \left(\mathsf{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1}\right)\right)\right) = \#\left(\mathbb{Z}^d \cap n \cdot \left(\mathsf{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1}\right)\right)\right),$$

where

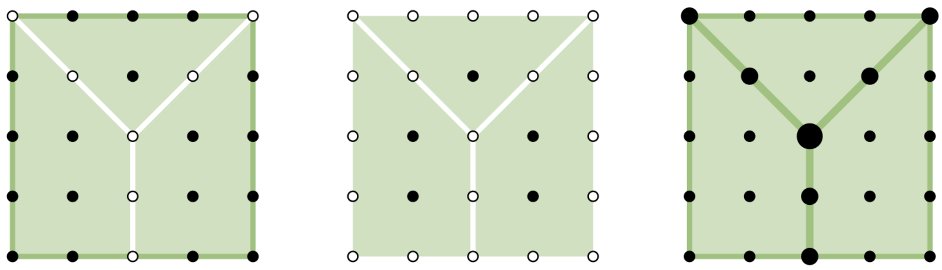
$$n \cdot \left(\mathsf{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1}\right)\right) := n \cdot \mathsf{Q} \setminus \left(\bigcup n \cdot \mathcal{N}^{\mathrm{co}\,1}\right).$$

See Figure 3.2(a) and Figure 3.2(b) for illustrations.

Lemma 3.2. *For a polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$,*

$$\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(n) = \sum_{i=1}^{k} \mathrm{ehr}_{\mathbb{Z}}(R_i^\circ; n)$$

where R_i° are the open regions of the open pruned inside-out polytope $\mathsf{Q}^\circ \setminus (\bigcup \mathcal{N}^{\mathrm{co}\,1})$.



(a) $\mathrm{in}_{[-1,1]^2,\mathcal{M}^{\mathrm{co}\,1}}(2) = 18$ (b) $\mathrm{in}_{(-1,1)^2,\mathcal{M}^{\mathrm{co}\,1}}(2) = 5$ (c) $\mathrm{cu}_{[-1,1]^2,\mathcal{M}^{\mathrm{co}\,1}}(2) = 33$

Figure 3.2: Inner and cumulative pruned Ehrhart functions of the pruned inside-out polytope $[-1,1]^2 \setminus \bigcup \mathcal{M}^{\mathrm{co}\,1}$ and the open pruned inside-out polytope $(-1,1)^2 \setminus \bigcup \mathcal{M}^{\mathrm{co}\,1}$ illustrated. White dots are not counted, black dots are counted according to their size with multiplicity one, two, or three. The corresponding computations can be found in Example 3.4.

Proof. We decompose the pruned inside-out polytope $\mathsf{Q} \setminus (\bigcup \mathcal{N}^{\mathrm{co}\,1})$ into its regions $R_1, \dots, R_k$. Then the open pruned inside-out polytope $\mathsf{Q}^\circ \setminus (\bigcup \mathcal{N}^{\mathrm{co}\,1}) = \biguplus_{i=1}^k R_i^\circ$ is the disjoint union of the open polytopes $R_1^\circ, \dots, R_k^\circ$. The result follows since counting lattice points is a valuation (see, e.g., [BS18, Section 3.4]). □

Furthermore, we define a second counting function for pruned inside-out polytopes, the **cumulative pruned Ehrhart function** $\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(\mathbb{Z}^d)$, for a positive integer $n \in \mathbb{Z}_{>0}$ as

$$\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n) := \sum_{\mathbf{y} \in \frac{1}{n}\mathbb{Z}^d} \mathrm{mult}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(\mathbf{y}) = \sum_{\mathbf{y} \in \mathbb{Z}^d} \mathrm{mult}_{(n \cdot \mathsf{Q}, n \cdot \mathcal{N}^{\mathrm{co}\,1})}(\mathbf{y}) \,,$$

where

$$\mathrm{mult}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(\mathbf{y}) := \begin{cases} \#\,(\text{closed full-dim. normal cones in } \mathcal{N} \text{ containing } \mathbf{y}) \,, & \text{if } \mathbf{y} \in \mathsf{Q} \,, \\ 0, & \text{otherwise.} \end{cases}$$

See Figure 3.2(c) for an illustration.

Lemma 3.3. *For a polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$,*

$$\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n) = \sum_{i=1}^{k} \mathrm{ehr}_{\mathbb{Z}}(\overline{R}_i; n) \,,$$

where $\overline{R}_i$ are the topological closures of the regions R_i of the pruned inside-out polytope $\mathsf{Q} \setminus (\bigcup \mathcal{N}^{\mathrm{co}\,1})$.

Proof. The right hand side of the equation counts lattice points in the interior of the regions tR_i precisely once and lattice points in the boundaries of the regions once for every closed region the lattice point is contained in. The closed regions are the intersections of the polytope Q with the closed full-dimensional cones in $\mathcal{N}$. Hence every lattice y point in $t\mathsf{Q}$ is counted with multiplicity $\mathrm{mult}_{(n \cdot \mathsf{Q}, n \cdot \mathcal{N}^{\mathrm{co}\,1})}(\mathbf{y})$. □

Example 3.4. We compute the counting functions for the pruned inside-out polytopes introduced in Example 3.1:

$$\begin{aligned}\mathrm{in}_{(-1,1)^2,\mathcal{M}^{\mathrm{co}\,1}}(n) &= (n^2-2n+1)+2(\tfrac{3}{2}n^2-\tfrac{5}{2}n+1) = 4n^2-7n+3\\ \mathrm{cu}_{[-1,1]^2,\mathcal{M}^{\mathrm{co}\,1}}(n) &= (n^2+2n+1)+2(\tfrac{3}{2}n^2+\tfrac{5}{2}n+1) = 4n^2+7n+3\,.\end{aligned}$$

See Figure 3.2 for illustrations.

Theorem 3.5. *Let* $\mathsf{Q}\setminus(\bigcup\mathcal{N}^{\mathrm{co}\,1})\subseteq\mathbb{R}^d$ *be a rational pruned inside-out polytope. Then the inner pruned Ehrhart function* $\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(n)$ *and the cumulative pruned Ehrhart function* $\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n)$ *agree with quasipolynomials in* n *of degree* d *for* $n\in\mathbb{Z}_{>0}$ *and are related by reciprocity:*

$$(-1)^d\,\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(-n) = \mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n).$$

Proof. We first use Lemma 3.2 to get

$$\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(n) = \sum_{i=1}^{k}\mathrm{ehr}_{\mathbb{Z}}(R_i^\circ;n)\,.$$

For every $i=1,\dots,k$ we can apply Ehrhart's Theorem 2.2 to $\mathrm{ehr}_{\mathbb{Z}}(R_i^\circ;n)$, hence the counting function $\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(n)$ is a sum of quasipolynomials, which is again a quasipolynomial.

For the second part of the claim we use Ehrhart–Macdonald reciprocity (Theorem 2.4) and compute

$$\begin{aligned}\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(n) &= \sum_{i=1}^{k}\mathrm{ehr}_{\mathbb{Z}}(R_i^\circ;n) = \sum_{i=1}^{k}(-1)^d\,\mathrm{ehr}_{\mathbb{Z}}(\overline{R}_i;-n)\\ &= (-1)^d\,\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(-n),\end{aligned}$$

where the last equality follows from Lemma 3.3. □

Remark 3.6. In the case that the polytope Q and the complete fan intersect such that all the closed regions $\overline{R}=\mathsf{Q}\cap\mathsf{N}$ of the pruned inside-out polytope $\mathsf{Q}\setminus(\bigcup\mathcal{N}^{\mathrm{co}\,1})$ are integer polytopes, the counting functions $\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(n)$ and $\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(n)$ agree with a *polynomial* of degree d, by Theorem 2.2 and Theorem 2.4. We will use this fact in the proof of Theorem 3.8.

Remark 3.7. One can certainly generalize this setting, e.g., to polyhedral complexes. The framework here is motivated by the applications below. However, it would be nice to find more applications for this framework.

3.2 Applications

We will now show how the tools from Section 3.1 can be applied to generalized permutahedra introduced in Section 2.3 to derive known and unknown reciprocity results for generalized permutahedra (Section 3.2.1). Reciprocity theorems for generalized permutahedra by Ardila and Aguiar ([AA23, Propositions 17.3 and 17.4], see Theorem 3.19) and extended by Karaboghossian ([Kar22, Theorem 2.5 and Theorem 2.8], see Theorem 3.20), were developed by introducing a Hopf monoid structure on the vector species of generalized permutahedra and using their antipode formula to derive polynomial invariants. We give a new interpretation from a discrete-geometric perspective as integer point counting functions. In Section 3.3 we give an explanation on the relation between the results in this chapter and prior results developed with Hopf-algebraic tools. We demonstrate why generalized permutahedra are such an interesting class of polytopes by translating the reciprocity result for hypergraphic polytopes to combinatorial statements about hypergraphs (Section 3.2.2). In Section 3.2.3 we briefly describe how to apply our perspective to deformed Coxeter permutahedra in types A, B, C, and D.

3.2.1 Generalized Permutahedra

We restate the combinatorial reciprocity result for generalized permutahedra by [AA23, Propositions 17.3 and 17.4] in a slightly different language (see Theorem 3.19 for the original statement) and prove it using Ehrhart theory.

Theorem 3.8. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a generalized permutahedron and $m \in \mathbb{Z}_{>0}$. Then*

$$\chi_d(\mathsf{P})(m) := \# \left(\mathsf{P}\text{-}\textit{generic directions } \mathbf{y} \in \left(\mathbb{R}^d\right)^* \textit{ with } \mathbf{y} \in [m]^d\right)$$

agrees with a polynomial in m of degree d. Moreover,

$$(-1)^d \chi_d(\mathsf{P})(-m) = \sum_{\mathbf{y} \in [m]^d} \# \left(\textit{vertices of } \mathsf{P}^{\mathbf{y}}\right) .$$

While we will extend Theorem 3.8 (and our proof) in Theorem 3.10 and Remark 3.11 below, we provide a self-contained proof here to present a flavor of our method. In contrast to [BJR09; AA23; Kar22] we will prove these results without using any Hopf-algebraic method. Our proof gives a geometric point of view by counting integer points in pruned inside-out cubes. That is, we will consider the cube

$$[1, m]^d := \{\mathbf{x} \in \mathbb{R}^d : 1 \le x_i \le m \text{ for } i = 1, \dots, d\} \subseteq \mathbb{R}^d$$

and intersect it with the integer lattice:

$$[1, m]^d \cap \mathbb{Z}^d = \left\{\mathbf{x} \in \mathbb{R}^d : x_i \in \{1, \dots, m\} \text{ for } i = 1, \dots, d\right\} = \{1, \dots, m\}^d = [m]^d.$$

The same holds in the dual space $\left(\mathbb{R}^d\right)^*$. Now, a direction $y \colon [d] \to [m] \in \left(\mathbb{R}^d\right)^*$ can be identified with an integer point $\mathbf{y}$ in the cube $\{1, \dots, m\}^d = [m]^d$ in the dual space. See Figure 3.3. Before we start the proof of Theorem 3.8 we need the following result.

Lemma 3.9. *The intersections of the unit cube $[0, 1]^d$ and the braid cones $\mathsf{B}_{T_1,\dots,T_k}$ for compositions $[d] = T_1 \uplus \cdots \uplus T_k$ are integer polytopes.*

Proof. It is enough to consider the full-dimensional braid cones, since lower dimensional braid cones are faces of full-dimensional braid cones and faces of an integer polytope are integer polytopes. Full-dimensional braid cones $\mathsf{B}_{T_1,\dots,T_d}$ correspond to

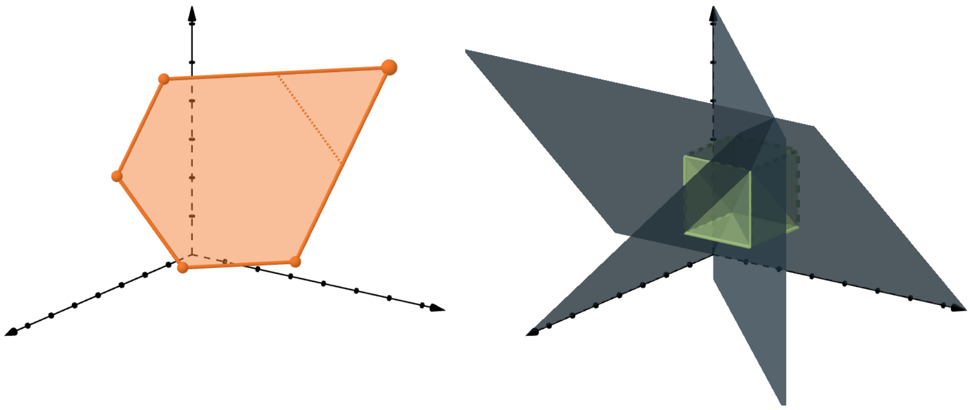


Figure 3.3: A generalized permutahedron in $\mathbb{R}^3$ (left) and its normal fan intersecting the cube $[1,2]^3$ (right).

permutations of the coordinates, i.e., total orders on $[d]$. Recall from Section 2.5.5 that we can think of the intersection $[0,1] \cap \mathsf{B}_{T_1,\ldots,T_d}$ as an order simplex of the total order on $[d]$ given by $T_1, \ldots, T_d$. Order polytopes are 0/1-polytopes and therefore integral polytopes. $\square$

Proof of Theorem 3.8. We will argue in the dual space $\left(\mathbb{R}^d\right)^*$ and its integer lattice; to simplify notation we will not always explicitly point that out. Let us recall that $\mathbf{y} \in \left(\mathbb{R}^d\right)^*$ being P-generic means that the $\mathbf{y}$-maximal face of P is a vertex, that is, $\mathbf{y}$ is contained in a full-dimensional cone of the normal fan $\mathcal{N}(\mathsf{P})$. So the direction $\mathbf{y}$ is *not* contained in any cone N in the codimension-one fan $\mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}$. Hence,

$$\begin{aligned}
\chi_d(\mathsf{P})(n) &= \#\left(\mathsf{P}\text{-generic directions } \mathbf{y} \in \left(\mathbb{R}^d\right)^* \text{ with } \mathbf{y} \in [n]^d\right) \\
&= \#\{\mathbf{y} \in [1,n]^d \cap \mathbb{Z}^d \,:\, \mathbf{y}\text{-maximum face } \mathsf{P}^{\mathbf{y}} \text{ is a vertex}\} \\
&= \#\{\mathbf{y} \in [1,n]^d \cap \mathbb{Z}^d \,:\, \mathbf{y} \in \mathsf{N} \in \mathcal{N}(\mathsf{P}) \text{ with } \mathsf{N} \text{ full-dimensional}\} \\
&= \#\{\mathbf{y} \in [1,n]^d \cap \mathbb{Z}^d \,:\, \mathbf{y} \notin \mathsf{N} \text{ for all } \mathsf{N} \in \mathcal{N}(\mathsf{P}) \text{ with codimension } \geq 1\} \\
&= \#\left(\left([1,n]^d \setminus \bigcup \mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}\right) \cap \mathbb{Z}^d\right) \\
&= \mathrm{in}_{(0,1)^d, \mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}}(n+1)\,,
\end{aligned}$$

where we use in the last line that $[1,n]^d \cap \mathbb{Z}^d = (0,n+1)^d \cap \mathbb{Z}^d = (n+1)\cdot(0,1)^d \cap \mathbb{Z}^d$. With Lemma 3.9 we know that the unit cube and the normal fan $\mathcal{N}(\mathsf{P})$ intersect producing integer regions. Therefore, using Theorem 3.5 and Remark 3.6, polynomiality of $\chi_d(\mathsf{P})(n)$ follows. With the above equality and Theorem 3.5 at hand, we compute

$$\begin{aligned}
(-1)^d \chi_d(\mathsf{P})(-n) &= (-1)^d \operatorname{in}_{(0,1)^d, \mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}}(-n+1) \\
&= (-1)^d \operatorname{in}_{(0,1)^d, \mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}}(-(n-1)) \\
&= \operatorname{cu}_{[0,1]^d, \mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}}(n-1) \\
&= \sum_{\mathbf{y} \in \frac{1}{n-1}\mathbb{Z}^d} \operatorname{mult}_{[0,1]^d, \mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}}(\mathbf{y}) \\
&= \sum_{\mathbf{y} \in \mathbb{Z}^d} \operatorname{mult}_{[0,n-1]^d, \mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}}(\mathbf{y}) \,.
\end{aligned}$$

Every cone in the braid fan contains the line $L = \lambda(1, \ldots, 1)$. Therefore, the fans $\mathcal{N}(\mathsf{P})$ and $\mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}$ are invariant under translations by vectors in the line L and scaling. So we can shift the cube $[0, n-1]^d$ to $[1, n]^d$ and this bijection not only preserves the number of integer points but also their multiplicities $\operatorname{mult}_{[1,n]^d, \mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}}$ with respect to the fan $\mathcal{N}(\mathsf{P})$. Hence,

$$\begin{aligned}
(-1)^d \chi_d(\mathsf{P})(-n) &= \sum_{\mathbf{y} \in \mathbb{Z}^d} \operatorname{mult}_{[1,n]^d, \mathcal{N}(\mathsf{P})^{\mathrm{co}\,1}}(\mathbf{y}) \\
&= \sum_{\mathbf{y} \in [1,n]^d \cap \mathbb{Z}^d} \#\,(\text{closed full-dim. normal cones that contain } \mathbf{y}) \\
&= \sum_{\mathbf{y} \in [n]^d} \#\,(\text{closed normal cones of vertices that contain } \mathbf{y}) \\
&= \sum_{\mathbf{y} \in [n]^d} \#\,(\text{vertices of } \mathsf{P}^{\mathbf{y}}) \,,
\end{aligned}$$

where we make use of Lemma 2.1. □

We can extend Theorem 3.8 above to faces of arbitrary dimension.

Theorem 3.10. *For a generalized permutahedron $\mathsf{P} \subseteq \mathbb{R}^d$ and $k = 0, \ldots, d-1$,*

$$\chi_{d,k}(\mathsf{P})(n) := \#\{\mathbf{y} \in [n]^d \,:\, \mathbf{y}\text{-maximum face } \mathsf{P}^{\mathbf{y}} \text{ is a } k\text{-face}\}$$

agrees with a polynomial of degree $d-k$, and

$$(-1)^{d-k} \chi_{d,k}(\mathsf{P})(-n) = \sum_{\mathbf{y} \in [n]^d} \#\,(k\text{-faces of } \mathsf{P}^{\mathbf{y}}) \,.$$

Before we prove the theorem we extend the notion of codimension-one fans to arbitrary dimensions by defining the **codimension-k fan** $\mathcal{N}^{\mathrm{co}\,k}$ as

$$\mathcal{N}^{\mathrm{co}\,k} := \{\mathsf{N} \in \mathcal{N}(\mathsf{P}) :\ \operatorname{codim}(\mathsf{N}) \geq k\},$$

that is, for a polytope P,

$$\mathcal{N}(\mathsf{P})^{\mathrm{co}\,k} = \{\mathsf{N}_{\mathsf{P}}(\mathsf{F}) \,:\ \mathsf{F} \text{ a face of } \mathsf{P} \text{ with } \dim(\mathsf{F}) \geq k\} \,.$$

For a polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ and $k \geq 0$ we define the k**-pruned inside-out polytope** as

$$\left(\mathsf{Q} \cap \bigcup \mathcal{N}^{\operatorname{co} k}\right) \setminus \left(\bigcup \mathcal{N}^{\operatorname{co} k+1}\right) = \mathsf{Q} \cap \biguplus \left\{\mathsf{N}^\circ : \mathsf{N} \in \mathcal{N}^{\operatorname{co} k}\right\} .$$

Note this is consistent with the notation in the beginning of this section. As before, for a polytope $\mathsf{Q} \subseteq \mathbb{R}^d$ the open k-pruned inside-out polytope $\left(\mathsf{Q}^\circ \cap \bigcup \mathcal{N}^{\operatorname{co} k}\right) \setminus \left(\bigcup \mathcal{N}^{\operatorname{co} k+1}\right)$ is the disjoint union of relatively open $(d-k)$-dimensional polytopes, namely, the intersection of Q° with the relatively open cones in $\mathcal{N}$ of codimension k.

Proof of Theorem 3.10. We compute

$$\begin{aligned}
\chi_{d,k}(\mathsf{P})(n) &= \#\left\{\mathbf{y} \in [1,n]^d \cap \mathbb{Z}^d \ : \ \mathbf{y}\text{-maximum face } \mathsf{P}^{\mathbf{y}} \text{ is a } k\text{-face}\right\} \\
&= \#\left(\left(\mathbb{Z}^d \cap (0,n+1)^d \cap \bigcup \mathcal{N}(\mathsf{P})^{\operatorname{co} k}\right) \setminus \left(\bigcup \mathcal{N}(\mathsf{P})^{\operatorname{co} k+1}\right)\right) \\
&= \#\left(\left(\biguplus_{\substack{\mathsf{N} \in \mathcal{N}(\mathsf{P}) \\ \dim \mathsf{N} = d-k}} \mathsf{N}^\circ \cap (0,n+1)^d\right) \cap \mathbb{Z}^d\right) .
\end{aligned}$$

The intersection $\mathsf{N}^\circ \cap (0,n+1)^d$ is the relative interior of a polytope. Moreover, since N° is an open cone containing the origin, $\mathsf{N}^\circ \cap (0,n+1)^d$ is the $(n+1)^{\text{st}}$ dilate of $\mathsf{N}^\circ \cap (0,1)^d$. Hence,

$$\begin{aligned}
\chi_{d,k}(\mathsf{P})(n) &= \#\left(\left(\biguplus_{\substack{\mathsf{N} \in \mathcal{N}(\mathsf{P}) \\ \dim \mathsf{N} = d-k}} \mathsf{N}^\circ \cap (0,n+1)^d\right) \cap \mathbb{Z}^d\right) \qquad (3.1) \\
&= \sum_{\substack{\mathsf{N} \in \mathcal{N}(\mathsf{P}) \\ \dim \mathsf{N} = d-k}} \operatorname{ehr}_{\mathbb{Z}}(\mathsf{N}^\circ \cap (0,1)^d; n+1) .
\end{aligned}$$

Using again Lemma 3.9 and Ehrhart's Theorem 2.2 we obtain polynomiality for $\chi_{d,k}(\mathsf{P})(n)$.

With Ehrhart–Macdonald reciprocity (Theorem 2.4) we compute

$$\begin{aligned}
(-1)^{d-k}\chi_{d,k}(\mathsf{P})(-n) &= (-1)^{d-k} \sum_{\substack{\mathsf{N} \in \mathcal{N}(\mathsf{P}) \\ \dim \mathsf{N} = d-k}} \operatorname{ehr}_{\mathbb{Z}}(\mathsf{N}^\circ \cap (0,1)^d; -n+1) \\
&= \sum_{\substack{\mathsf{N} \in \mathcal{N}(\mathsf{P}) \\ \dim \mathsf{N} = d-k}} (-1)^{d-k} \operatorname{ehr}_{\mathbb{Z}}(\mathsf{N}^\circ \cap (0,1)^d; -(n-1)) \\
&= \sum_{\substack{\mathsf{N} \in \mathcal{N}(\mathsf{P}) \\ \dim \mathsf{N} = d-k}} \operatorname{ehr}_{\mathbb{Z}}(\mathsf{N} \cap [0,1]^d; n-1) \qquad (3.2) \\
&= \sum_{\substack{\mathsf{N} \in \mathcal{N}(\mathsf{P}) \\ \dim \mathsf{N} = d-k}} \#\left(\mathsf{N} \cap [0,n-1]^d \cap \mathbb{Z}^d\right) .
\end{aligned}$$

Here, we use, as in the proof of Theorem 3.8, that the normal fan of a generalized permutahedron is a coarsened braid fan and therefore is invariant under scaling and shifts by $\lambda(1,\ldots,1)$ for $\lambda \in \mathbb{R}$. So,

$$\begin{aligned}
(-1)^{d-k}\chi_{d,k}(\mathsf{P})(-n) &= \sum_{\substack{\mathsf{N}\in\mathcal{N}(\mathsf{P})\\ \dim\mathsf{N}=d-k}} \#\left(\mathsf{N}\cap[1,n]^d\cap\mathbb{Z}^d\right)\\
&= \sum_{\mathbf{y}\in[1,n]^d\cap\mathbb{Z}^d} \#\left((d-k)\text{-dim. cones } \mathsf{N}\in\mathcal{N}(\mathsf{P}) \text{ that contain } \mathbf{y}\right)\\
&= \sum_{\mathbf{y}\in[n]^d} \#\left(k\text{-faces of } \mathsf{P}^{\mathbf{y}}\right)\,,
\end{aligned}$$

applying Lemma 2.1 in the last equality. □

Remark 3.11. At the heart of the proofs of Theorem 3.8 and Theorem 3.10 lie sums of Ehrhart polynomials and the reciprocity results are applications of Ehrhart-Macdonald reciprocity (Theorem 2.4): Recall (3.1) and (3.2) from the proof of Theorem 3.10. One can see that for a generalized permutahedron P any combination of Ehrhart polynomials as in (3.1) and (3.2) results in a polynomial counting function

$$\begin{aligned}
\chi_{d,\boldsymbol{\alpha}}(\mathsf{P})(n) &:= \sum_{\mathsf{N}\in\mathcal{N}(\mathsf{P})} \alpha_{\mathsf{N}}\,\mathrm{ehr}_{\mathbb{Z}}(\mathsf{N}^\circ\cap(0,1)^d;n+1)\\
&= \sum_{\mathsf{F}\text{ a face of }\mathsf{P}} \alpha_{\mathsf{N}_{\mathsf{P}}(\mathsf{F})}\,\mathrm{ehr}_{\mathbb{Z}}(\mathsf{N}_{\mathsf{P}}(\mathsf{F})^\circ\cap(0,1)^d;n+1)
\end{aligned} \tag{3.3}$$

for coefficients α_{N}. This provides a combinatorial reciprocity result

$$\begin{aligned}
\chi_{d,\boldsymbol{\alpha}}(\mathsf{P})(-n) &= \sum_{\mathsf{N}\in\mathcal{N}(\mathsf{P})} (-1)^{\dim\mathsf{N}}\alpha_{\mathsf{N}}\,\mathrm{ehr}_{\mathbb{Z}}(\mathsf{N}\cap[0,1]^d;n-1)\\
&= \sum_{\mathsf{F}\text{ a face of }\mathsf{P}} (-1)^{d-\dim\mathsf{F}}\alpha_{\mathsf{F}}\,\mathrm{ehr}_{\mathbb{Z}}(\mathsf{N}_{\mathsf{P}}(\mathsf{F})\cap[0,1]^d;n-1)\,.
\end{aligned}$$

Theorem 3.10 (and therefore also Theorem 3.8) is a reformulation of this general result with coefficients

$$\alpha_{\mathsf{N}} = \begin{cases} 1 & \text{if } \dim\mathsf{N} = d-k\\ 0 & \text{else}\end{cases} \qquad \text{for } k = 0,1,\ldots,d-1\,.$$

Remark 3.12. We observe that we used the following properties of generalized permutahedra in the proofs of Theorem 3.8 and Theorem 3.10

(i) the intersection of the unit cube and the normal fan of a generalized permutahedron form integer pruned inside-out polytopes,
(ii) every cone N in the normal fan $\mathcal{N}(\mathsf{P})$ of a generalized permutahedron P contains the line $L = \{\lambda(1,\ldots,1) : \lambda\in\mathbb{R}\}$.

The first property (i) can be weakened to rational intersections leading to a quasipolynomiality result. Considering normal fans without property (ii) produces similar but more complicated statements, since the shift of the cube $[0,n-1]^d$ to the cube $[1,n]^d$

can not be performed in general. Nevertheless, the framework of pruned inside-out polytopes can be applied to generate reciprocity results for generalized permutahedra in other types (see, e.g., [Ard+20]). See Section 3.2.3.

3.2.2 Hypergraphs

Generalized permutahedra are an especially interesting class of polytopes, due to their numerous combinatorial subclasses such as graphical zonotopes, matroid polytopes, hypergraphic polytopes, and many more. In this section we illustrate this fruitful connection between combinatorics and geometry proving a combinatorial reciprocity result for hypergraphs, which generalizes Stanley's famous theorem about the chromatic polynomial for graphs. Aval, Karaboghossian, and Tanasa use a Hopf-theoretic ansatz similar to that of Ardila and Aguiar to derive the reciprocity theorem for hypergraph colorings [AKT20]. They define a basic polynomial invariant on hypergraphs and give combinatorial interpretations. A general version of this can be found in [Kar22]. For convenience we demonstrate the technique for a special case of orientation, that we call heading. We give another perspective and proof by applying Theorem 3.19 (reciprocity for generalized permutahedra) and exploiting geometric and combinatorial properties of the hypergraph and its associated polytope. This approach is also described as alternative proof for the general case in [Kar22][1].

A **hypergraph** $h = (U, E)$ is a pair of a finite set U of **nodes**[2] and a finite multiset E of non-empty subsets $e \subseteq U$ called **hyperedges**. Note that we allow multiple edges and edges consisting of only one node. For simplicity we will often assume without loss of generality that the node set U equals $\{1, \ldots, d\} = [d]$ for $d = |U|$, since all the claims in this section are invariant under relabeling the set U. In a similar fashion we might switch back and forth between the two vector space notations $\mathbb{R}U \simeq \mathbb{R}^d$ and $\mathbb{R}^U \simeq \left(\mathbb{R}^d\right)^*$ (see Section 2.0).

For every hypergraph h we define the corresponding **hypergraphic polytope** $\mathsf{P}(h) \subseteq \mathbb{R}U$ as the following Minkowski sum of simplices:

$$\mathsf{P}(h) = \sum_{e \in E} \Delta_e \ \subseteq \mathbb{R}U$$

where

$$\Delta_e = \operatorname{conv}\{\mathbf{e}_i \ : \ i \in e\}, \quad \text{for a hyperedge } e \subseteq U\ \epsilon_i e \mathbf{e}_i e \boldsymbol{\epsilon}_i$$

and $\mathbf{e}_i$ are the basis vectors for $i \in U$. An example is depicted in Figure 2.8. Hypergraphic polytopes have been studied (sometimes as Minkowski sum of simplices) in, e.g., [Agn17; BBM19]. Hypergraphs are in bijection with hypergraphic polytopes and they form a subclass of generalized permutahedra (see Proposition 2.15 or, e.g., [Pos09, Proposition 6.3.]).

The vertices of graphic polytopes are described by the acyclic orientations of the corresponding graph [Zas91, Corollary 4.2]. We will give an analogous statement

1 There also seems to be a polytopal approach by Alexander Postnikov, mentioned in [AKT20, Acknowledgments] and on `http://math.mit.edu/~apost/courses/18.218_2016/` (Lecture 19. W 03/16/2016), but to the best of our knowledge, no reference is available.

2 We decided to use the less common term *nodes* for hypergraphs to distinguish them from the *vertices* of a polytope.

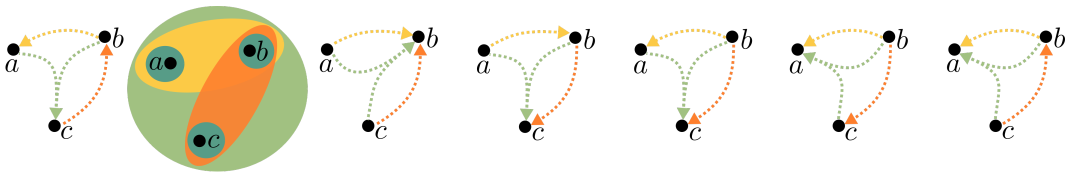


Figure 3.4: The hypergraph $h = (\{a, b, c\}, \{\{a, b, c\}, \{a, b\}, \{b, c\}, \{a\}, \{b\}, \{c\}\})$ with a cyclic heading (left) and all its acyclic headings (right). See Figure 2.8 for the corresponding hypergraphic polytope.

and proof for hypergraphic polytopes. In order to do so we need the subsequent definitions following[3] [AKT20]. A **heading**[4] σ of a hypergraph $h = (U, E)$ is a map $\sigma\colon E \to U$ such that for every hyperedge $e \in E$ we have $\sigma(e) \in e$. In other words the heading σ picks for every hyperedge e a node $i = \sigma(e) \in e$ within that hyperedge. We will call that node $\sigma(e)$ the **head of the hyperedge** e. An **oriented cycle** in a heading σ of a hypergraph h is a sequence $e_1, \ldots, e_\ell$ of hyperedges such that

$$\begin{aligned}
&\sigma(e_1) \in e_2 \setminus \sigma(e_2)\\
&\sigma(e_2) \in e_3 \setminus \sigma(e_3)\\
&\qquad\vdots\\
&\sigma(e_{\ell-1}) \in e_{\ell-1} \setminus \sigma(e_{\ell-1})\\
&\sigma(e_\ell) \in e_1 \setminus \sigma(e_1).
\end{aligned}$$

A heading σ of a hypergraph h is called **acyclic** if it does not contain any oriented cycle. See Figure 3.4 for some examples. Note that the notions of heading and acyclic here are special cases of the notions in [BBM19; Kar22; RR12; Rus13].

The following description of the vertices of the hypergraphic polytope in terms of acyclic orientations plays a central role in the remainder of this paper and is a particular instance of, e.g., [BBM19, Theorem 2.18.]. Proposition 3.13 was stated without proof in [CF18]. For convenience we give an elementary proof generalizing the proof idea for graphs presented in [CF18].

Proposition 3.13. *For a hypergraph $h = (U, E)$ the hypergraphic polytope $\mathsf{P}(h)$ can be described as*

$$\mathsf{P}(h) = \operatorname{conv}\{\ \delta(\sigma) \in \mathbb{R}U \ :\ \sigma \textit{ is an } \text{acyclic} \textit{ heading of } h\}$$

where

$$\delta(\sigma)_i = |\sigma^{-1}(i)| \quad \textit{for } i \in U,$$

i.e., $\delta(\sigma) \in \mathbb{R}U$ is the vector of in-degrees of the nodes $i \in U$ in the heading σ.

3 Some of the definitions are also mentioned by Postnikov (`http://math.mit.edu/~apost/courses/18.218_2016/` Problem set 2, Problem 6).

4 We have chosen to call this generalization of orientations *heading* to distinguish it from other definitions of orientations for hypergraphs. Thanks to Thomas Zaslavsky for suggesting this wording.

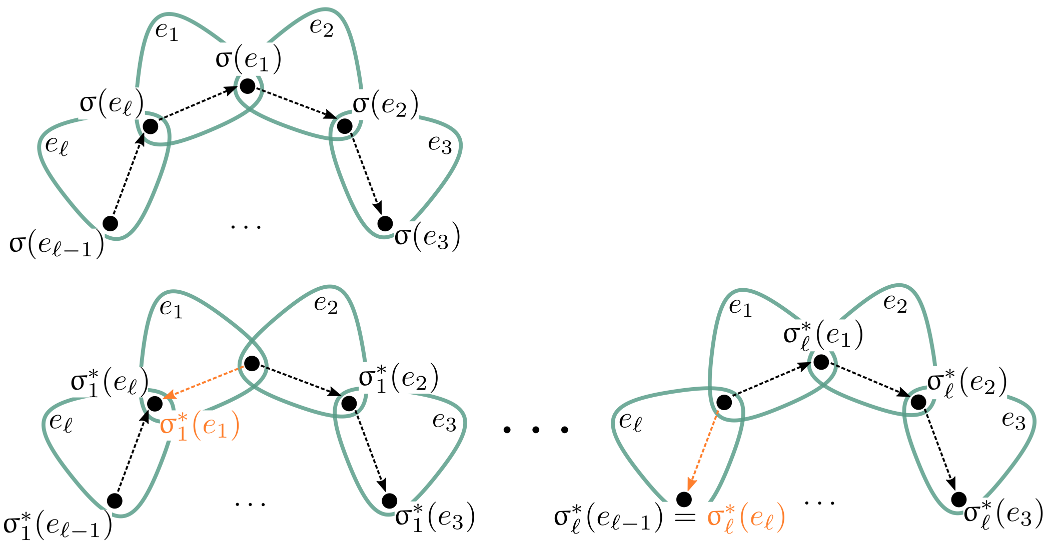


Figure 3.5: An oriented cycle $e_1, \dots, e_\ell$ with heading σ (top) and the new headings $\sigma_1^*, \dots, \sigma_\ell^*$ on the edges $e_1, \dots, e_\ell$ (below).

Proof. Since the Minkowski sum of convex hulls of point sets is the same as the convex hull of the Minkowski sum of the points sets, we have

$$\mathsf{P}(h) = \sum_{e \in E} \operatorname{conv}\Big(\{\mathbf{e}_i \,:\, i \in e\}\Big) = \operatorname{conv}\left(\sum_{e \in E} \{\mathbf{e}_i \,:\, i \in e\}\right).$$

Every point in the convex hull on the right-hand side is the vector of in-degrees of the nodes for some heading σ. Indeed, choosing some $\mathbf{e}_i$ in every summand corresponds to choosing $i \in e$ as the head for the hyperedge e, and vice versa. It is left to show that $\delta(\sigma)$ is a vertex of $\mathsf{P}(h)$ if and only if the heading σ is acyclic.

First, consider a heading containing an oriented cycle $e_1, \dots, e_\ell$. Then

$$\sigma(e_1) \in e_2 \setminus \sigma(e_2), \dots, \sigma(e_\ell) \in e_1 \setminus \sigma(e_1)$$

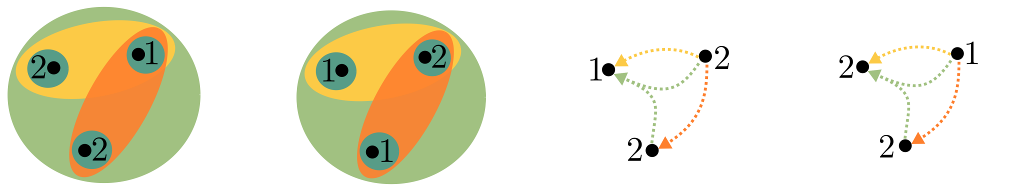
holds. We will construct new headings $\sigma_1^*, \dots, \sigma_\ell^*$ such that their vectors of in-degrees $\delta(\sigma_1^*), \dots, \delta(\sigma_\ell^*)$ convex combine the vector of in-degrees $\delta(\sigma)$ of the original heading σ. We define the new headings σ_j^* by changing the orientation of the hyperedge e_j in the cycle, as depicted in Figure 3.5:

$$\sigma_1^*(e) := \begin{cases} \sigma(e_\ell), & \text{if } e = e_1 \\ \sigma(e), & \text{otherwise} \end{cases}, \quad \sigma_j^*(e) := \begin{cases} \sigma(e_{j-1}), & \text{if } e = e_j \\ \sigma(e), & \text{otherwise.} \end{cases} \quad \text{for } j = 2, \dots, \ell\,.$$

Then

$$\delta(\sigma) = \sum_{j=1}^{\ell} \frac{1}{\ell} \delta(\sigma_j^*).$$

Therefore, the vector of in-degrees $\delta(\sigma)$ of a heading σ containing a cycle cannot be a vertex.



(a) Not a proper coloring $c_1\colon \{a,b,c\} \to \{1,2\}$. (b) A proper coloring $c_2\colon \{a,b,c\} \to \{1,2\}$. (c) Incompatible heading and coloring. (d) Compatible heading and coloring.

Figure 3.6: Hypergraph $h = (\{a,b,c\}, \{\{a,b,c\}, \{a,b\}, \{b,c\}, \{a\}, \{b\}, \{c\}\})$ with colorings $c_i\colon \{a,b,c\} \to \{1,2\}$.

Now, let σ be an acyclic heading and let us assume there are headings $\sigma_1^*, \dots, \sigma_l^*$ and scalars $0 \le \lambda_1, \dots, \lambda_l \in \mathbb{R}$ such that

$$\delta(\sigma) = \sum_{j=1}^{\ell} \lambda_j \delta(\sigma_j^*) \quad \text{and} \quad \sum_{j=1}^{\ell} \lambda_i = 1.$$

First note that hyperedges e with cardinality $|e| = 1$ have only one possible heading (the one choosing the only node in the hyperedge as head) and they do not appear in oriented cycles. Hence they are irrelevant when it comes to deciding whether an heading is acyclic or not. Therefore we delete all singleton hyperedges and adjust the values in $\delta(\sigma)$ as well as in $\delta(\sigma_1^*), \dots, \delta(\sigma_\ell^*)$.

Since the heading σ is acyclic and we deleted all singleton hyperedges, there exists at least one source $s \in U$ with $\delta(\sigma)_s = 0$. From Figure 3.2.2 it follows that $\delta(\sigma_j^*)_s = 0$ for all $j = 1, \dots, \ell$. So, for the node s the in-degree of all the headings is identical. We proceed by first deleting the source s in all hyperedges, then deleting all hyperedges e with cardinality $|e| = 1$, and adjusting the entries in $\delta(\sigma)$, $\delta(\sigma_1^*), \dots, \delta(\sigma_l^*)$. After finitely many iterations (the node set U is finite) we get $\delta(\sigma)_i = \delta(\sigma_j^*)_i$ for every node $i \in U$ and all $j = 1, \dots, \ell$ and the in-degree vector $\delta(\sigma)$ of the acyclic heading σ cannot be written as a convex combination, that is, $\delta(\sigma)$ is a vertex. $\square$

A **coloring of a hypergraph** $h = (U, E)$ with n colors is a map $c\colon U \to [n]$ that assigns a color $c(i) \in [n]$ to every node $i \in U$. A node $i \in e \in E$ is called a **maximal node** in the hyperedge e for the coloring c if the color $c(i)$ is maximal among the colors in the hyperedge e, that is $c(i) = \max_{j \in e} c(j)$. The color $\max_{j \in e} c(j)$ is called the **maximal color**. A coloring $c : U \to [n]$ of a hypergraph $h = (U, E)$ is called **proper** if every hyperedge $e \in E$ contains a unique maximal node $i \in e$. This definition of a proper coloring is the same as, e.g., in [AKT20], but different from the ones in [EH66; BTV15; BDK12; AH05]. A coloring $c\colon U \to [n]$ and a heading $\sigma\colon E \to U$ of a hypergraph $h = (U, E)$ are said to be **compatible** if $c(\sigma(e)) = \max_{j \in e} c(j)$, i.e., if the head $\sigma(e)$ of a hyperedge e has maximal color. See Figure 3.6 for some examples.

Remark 3.14. Considering usual graphs, the above definitions of (proper) colorings, (acyclic) headings and compatible pairs for hypergraphs specialize to those commonly used for graphs. In the same way the following Theorem 3.15 and Theorem 3.16 generalize Stanley's reciprocity theorem for chromatic polynomials of graphs [Sta73].

Theorem 3.15 ([AKT20, Theorem 18]). *For a hypergraph $h = (U, E)$ with $|U| =: d$ and a positive integer $n \in \mathbb{Z}_{>0}$,*

$$\chi_d(h)(n) := \#(\textit{proper colorings of } h \textit{ with } m \textit{ colors})$$

agrees with a polynomial in n of degree d.

Proof. Without loss of generality we assume $U = [d]$. For a hypergraph $h = (U, E)$ we consider its corresponding hypergraphic polytope $\mathsf{P}(h)$ and since $\mathsf{P}(h)$ is a generalized permutahedron we can apply Theorem 3.19. Hence we need to show

$$\#(\mathsf{P}(h)\text{-generic directions } \mathbf{y} \in [n]^d) = \#(\text{ proper colorings of } h \text{ with } n \text{ colors}).$$

We do so via a bijection. For $\mathbf{y} \in [n]^d$ we define the coloring $c_{\mathbf{y}}(i) := y_i$ for $i = 1, \ldots, d$ and vice versa, for a coloring $c\colon U \to [n]$ define $\mathbf{y}^c \in [n]^d$ by $y_i^c := c(i)$.

It is left to show that a direction $\mathbf{y} \in [n]^d$ is $\mathsf{P}(h)$-generic if and only if the coloring $c_{\mathbf{y}}$ is proper. Recall $\mathbf{y} \in \mathbb{R}^U$ is $\mathsf{P}(h)$-generic if the maximal face $(\mathsf{P}(h))^{\mathbf{y}}$ in direction $\mathbf{y}$ is a vertex. Linear functionals and Minkowski sums commute (see, e.g., [BS18, Lemma 7.5.1]), so

$$(\mathsf{P}(h))^{\mathbf{y}} = \left(\sum_{e \in E} \Delta_e\right)^{\mathbf{y}} = \sum_{e \in E} (\Delta_e)^{\mathbf{y}}. \tag{3.4}$$

Since the Minkowski sum is a point if and only if every summand is a point, the direction $\mathbf{y}$ is $\mathsf{P}(h)$-generic if and only if it is Δ_e-generic for every hyperedge $e \in E$. Finally, the direction $\mathbf{y}$ is Δ_e-generic if and only if $(\Delta_e)^{\mathbf{y}}$ is a vertex. Recall that $\Delta_e = \operatorname{conv}\{\mathbf{e}_i \,:\, i \in e\}$ is the convex hull of standard basis vectors $\mathbf{e}_i$, so

$$(\Delta_e)^{\mathbf{y}} = \operatorname{conv}\left\{\mathbf{e}_i \,:\, i \in e,\ \mathbf{y}(i) = \max_{j \in e} \mathbf{y}(j)\right\}. \tag{3.5}$$

Therefore $(\mathsf{P}(h))^{\mathbf{y}}$ is a vertex, if and only if for every hyperedge e the direction $\mathbf{y}$ has a unique maximal value among the entries $\mathbf{y}(i)$ with $i \in e$. The last statement is equivalent to the coloring $c_{\mathbf{y}}$ having a unique maximal node, i.e., being proper. In summary, for a positive integer $n \in \mathbb{Z}_{>0}$

$$\begin{aligned} \chi_d(h)(n) &= \#\,(\text{proper colorings of } h \text{ with } n \text{ colors}) \\ &= \#\left(\mathsf{P}(h)\text{-generic directions } \mathbf{y} \in [n]^d\right) = \chi_d(\mathsf{P}(h))(n) \end{aligned}$$

which is a polynomial in n of degree d. □

Theorem 3.16 ([AKT20, Theorem 24]). *Let $h = (U, E)$ be a hypergraph and $n \in \mathbb{Z}_{>0}$ a positive integer. Then*

$$\begin{aligned} (-1)^d \chi_d(h)(-n) = \#(&\textit{compatible pairs of acyclic headings of } h \\ &\textit{and colorings of } h \textit{ with } n \textit{ colors}). \end{aligned}$$

In particular, the number of acyclic headings of h equals $(-1)^d \chi_d(h)(-1)$.

Note that the colorings do not need to be proper here.

Proof. We follow the same idea as in the previous proof, that is, we use

$$(-1)^d \chi_d(h)(-n) = (-1)^d \chi_d(\mathsf{P}(h))(-n) = \sum_{\mathbf{y}\in[n]^d} \#\left(\text{vertices of } \mathsf{P}(h)^{\mathbf{y}}\right)$$

and need to show

$$\sum_{\mathbf{y}\in[n]^d} \#\left(\text{vertices of } \mathsf{P}(h)^{\mathbf{y}}\right) = \sum_{c\ n\text{-coloring}} \#\left(\text{acyclic headings of } h \text{ compatible to } c\right).$$

We use the same bijection between n-colorings $c_{\mathbf{y}}$ of h and directions $\mathbf{y}^c \in [n]^d$ as above. It is left show that for every direction $\mathbf{y} \in [m]^d$ the number of vertices of the maximal face $(\mathsf{P}(h))^{\mathbf{y}}$ in direction $\mathbf{y}$ equals the number of acyclic headings of h compatible to the coloring $c_{\mathbf{y}}$ defined by the direction $\mathbf{y}$. We compute the $\mathbf{y}$-maximum faces as in Equations (3.4) and (3.5):

$$\mathsf{P}(h)^{\mathbf{y}} = \left(\sum_{e\in E} \Delta_e\right)^{\mathbf{y}} = \sum_{e\in E} (\Delta_e)^{\mathbf{y}} = \sum_{e\in E} \operatorname{conv}\left\{\mathbf{e}_i \in \mathbb{R}U \ : \ i \in e,\ \mathbf{y}(i) = \max_{j\in e} \mathbf{y}(j)\right\}.$$

From Theorem 3.2.2 we can see that a vertex of $\mathsf{P}(h)^{\mathbf{y}}$ corresponds to choosing for every hyperedge $e \in E$ one of the nodes $i \in e$ with maximal entry $\mathbf{y}(i)$, i.e., maximal color $c_{\mathbf{y}}(i)$. This is, by definition, the same as constructing a compatible heading for the coloring $c_{\mathbf{y}}$. We know by Proposition 3.13 that vertices correspond to acyclic headings. Hence, vertices of $\mathsf{P}(h)^{\mathbf{y}}$ correspond to acyclic headings compatible to the coloring $c_{\mathbf{y}}$. Vice versa, for a coloring c the compatible acyclic headings are those with heads of hyperedges having a maximal coloring. That is, these acyclic headings correspond to those vertices, that are vertices of the maximum face $\mathsf{P}(h)^{\mathbf{y}^c}$ in direction $\mathbf{y}^c$. □

3.2.3 Deformed Coxeter Permutahedra

We extend our combinatorial reciprocity results to deformed Coxeter permutahedra in type A, B, C, and D (see Section 2.4). To do so, we first need to check that the resulting pruned inside-out polytopes are integral, then we state and proof the theorem.

Lemma 3.17. *Let $[-1,1]^d \subset \mathbb{R}^d$ be the d-dimensional hypercube and $\mathcal{N}$ a Coxeter fan in type A, B, C, or D. Then the pruned inside-out polytope $[-1,1]^d \setminus \mathcal{N}^{\mathrm{co}\,1}$ is integral.*

Proof. If we restrict our consideration to the non-negative orthant $\mathbb{R}^d_{\geq 0}$ then the statement is equivalent to Lemma 2.10. By symmetry the same follows for every other orthant and therefore $[-1,1]^d \setminus \mathcal{N}^{\mathrm{co}\,1}$ is integral. □

Theorem 3.18. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a deformed Coxeter permutahedron in type A, B, C, or D and $m \in \mathbb{Z}_{>0}$. Then*

$$\chi_d(\mathsf{P})(m) := \#\left(\mathsf{P}\text{-generic directions } \mathbf{y} \text{ with } \mathbf{y} \in \{-m,\dots,-1,0,1,\dots,m\}^d\right)$$

agrees with a polynomial in m of degree d. Moreover,

$$(-1)^d \chi_d(\mathsf{P})(-m) = \sum_{\mathbf{y} \in \{-m+1,\ldots,-1,0,1,\ldots,m-1\}^d} \#\left(\textit{vertices of } \mathsf{P}^{\mathbf{y}}\right) .$$

Proof. As before in the proof of Theorem 3.8, we compute:

$$\begin{aligned}
\chi_d(\mathsf{P})(m) &:= \#\left(\mathsf{P}\text{-generic directions } \mathbf{y} \text{ with } \mathbf{y} \in \{-m,\ldots,-1,0,1,\ldots,m\}^d\right) \\
&= \left|\left([-m,m]^d \setminus \mathcal{N}^{\mathrm{co}\,1}\right) \cap \mathbb{Z}^d\right| \\
&= \left|\left((-m-1,m+1)^d \setminus \mathcal{N}^{\mathrm{co}\,1}\right) \cap \mathbb{Z}^d\right| \\
&= \mathrm{in}_{(-1,1)^d,\mathcal{N}^{\mathrm{co}\,1}}(m+1) .
\end{aligned}$$

From Lemma 3.17 and Theorem 3.5 we get polynomiality and

$$\begin{aligned}
(-1)^d \chi_d(\mathsf{P})(-m) &= (-1)^d \,\mathrm{in}_{(-1,1)^d,\mathcal{N}^{\mathrm{co}\,1}}(-m+1) \\
&= \mathrm{cu}_{[-1,1]^d,\mathcal{N}^{\mathrm{co}\,1}}(m-1) \\
&= \sum_{\mathbf{y} \in \{-m+1,\ldots,-1,0,1,\ldots,m-1\}^d} \#\left(\text{vertices of } \mathsf{P}^{\mathbf{y}}\right) .
\end{aligned}$$

□

In a similar straightforward way we can extend Theorem 3.10 to deformed Coxeter permutahedra in type A, B, C, and D.

These results can easily be specialized to subclasses of deformed Coxeter permutahedra in type A, B, C, and D. This was done earlier for signed graphs, see [BZ06b; BZ06c].

3.3 Relation to Polynomial Invariants from Hopf Monoids

In this section we compare our results to the polynomial invariants from Hopf monoids developed in [AA23; Kar22]. This research was motivated by giving a geometric interpretation of the combinatorial reciprocity theorems in [AA23]. In this section it is convenient to work with unordered bases for vector spaces as introduced in Section 2.0.

An introduction to the theory of Hopf monoids can be found in, e.g., [AA23], [AM10] and is omitted here. For a Hopf monoid on the ground set U, a character ζ, and an element x in the Hopf monoid, there is a polynomial invariant

$$\chi_U^{\zeta}(x)(n) := \sum_{U = S_1 \sqcup \cdots \sqcup S_n} \left(\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}\right) \circ \Delta_{S_1,\ldots,S_n}(x) ,$$

where the sum is over all compositions and Δ denotes the coproduct of the Hopf monoid. Using the antipode s_U of the Hopf monoid one obtains the reciprocity relation

$$\chi_U^{\zeta}(x)(-n) = \chi_U\left(s_U(x)\right)(n)$$

which gives an interpretation for negative integers [AA23, Section 16]. In [AA23] Aguiar and Ardila define a Hopf monoid structure on the species of generalized permutahedra and then obtain combinatorial formulas for the polynomial invariant $\chi_U(x)(n)$ and $\chi_U(x)(-n)$ for $n \in \mathbb{Z}_{>0}$ using the basic character, which takes values in $\{0, 1\}$.

Theorem 3.19 ([AA23, Propositions 17.3 and 17.4])**.** *At a positive integer $n \in \mathbb{Z}_{>0}$ the basic polynomial invariant χ of a generalized permutahedron $\mathsf{P} \subseteq \mathbb{R}U$ is given by*

$$\chi_U(\mathsf{P})(n) = \#\left(\mathsf{P}\text{-}\textit{generic directions } y\colon U \to [n]\right)$$

and

$$(-1)^{|U|}\chi_U(\mathsf{P})(-n) = \sum_{y\colon U\to[n]} \#\left(\textit{vertices of } \mathsf{P}_y\right) .$$

This result was obtained earlier but stated differently by Billera, Jia, and Reiner using a similar Hopf-algebraic approach (using the antipode) on quasisymmetric functions and matroids [BJR09, Theorem 9.2. (v)]. We have seen in Section 3.2.1 how this result can be understood using pruned inside-out cubes. Recently, Theorem 3.19 was generalized in [Kar22].

Theorem 3.20 ([Kar22, Theorem 2.5 and Theorem 2.8])**.** *Let ζ be a character of the Hopf monoid of generalized permutahedra GP, U a finite set and $\mathsf{P} \in GP[U]$ a generalized permutahedron. Then,*

$$\chi^\zeta_U(\mathsf{P})(n) = \sum_{\mathsf{F} \textit{ a face of } \mathsf{P}} \zeta(\mathsf{F})|\mathsf{N}^\circ_\mathsf{P}(\mathsf{F})_n| ,$$

and

$$\chi^\zeta_U(\mathsf{P})(-n) = \sum_{\mathsf{F} \textit{ a face of } \mathsf{P}} (-1)^{|U|-\dim \mathsf{F}}\zeta(\mathsf{F})|\mathsf{N}_\mathsf{P}(\mathsf{F})_n| ,$$

Here, elements in the sets $\mathcal{N}^\circ_\mathsf{P}(\mathsf{F})_n = [n]^U \cap \mathcal{N}^\circ_\mathsf{P}(\mathsf{F})$ and $\mathcal{N}_\mathsf{P}(\mathsf{F})_n = [n]^U \cap \mathcal{N}_\mathsf{P}(\mathsf{F})$ are called the colorings $c\colon U \to [n]$ that are **strictly compatible**, respective **compatible** with F. These can be understood as the integer points in the open normal cone of the face F intersected with the $(n+1)^{\text{st}}$ dilate of the open unit cube $(0,1)^U$, so $|\mathsf{N}^\circ_\mathsf{P}(\mathsf{F})_n|$ agrees with the Ehrhart polynomial $\mathrm{ehr}_\mathbb{Z}(\mathsf{N}_\mathsf{P}(\mathsf{F})^\circ \cap (0,1)^d; n+1)$. Similarly, the set $\mathsf{N}_\mathsf{P}(\mathsf{F})_n$ can be recognized as the integer points in the closed normal cone of the face F intersected with the closed cube $[1,n]^U$. After shifting the cube as in the proof of Theorem 3.10, we can see that $|\mathsf{N}_\mathsf{P}(\mathsf{F})_n|$ agrees with the Ehrhart polynomial of $\mathsf{N}_\mathsf{P}(\mathsf{F}) \cap [0,1]^U$. Hence, the polynomial invariants can be interpreted as sums of Ehrhart polynomials that are weighted by the character $\zeta(\mathsf{F})$, compare Remark 3.11.

With a view towards applications our Ehrhart-theoretic approach has some advantages. One strength is that the weights in (3.3) can be chosen arbitrarily, while in the Hopf-theoretic setting the character needs to fulfill certain axioms. This does not allow to interpret the combinatorial reciprocity result in Theorem 3.10 as an instance of Theorem 3.20. A character taking value one on k-dimensional faces and zero elsewhere would not fulfill compatibility with multiplication in the Hopf monoid of generalized permutahedra. Another advantage is be the extension to generalized permutahedra in other types, as mentioned before (Remark 3.12, Section 3.2.3). This seems to be very hard from the Hopf monoid setting (see, e.g., [AA23, Theorem 6.1], [Ard+20, Section 9]).

4

Acyclotopes and Tocyclotopes

Let $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m] \in \mathbb{R}^{d\times m}$. There is a well-developed dictionary between the **zonotope** generated by $\mathbf{A}$,

$$\mathsf{Z}(\mathbf{A}) := \mathbf{A}\,[0,1]^m,$$

and the (central) **hyperplane arrangement** $\mathcal{H}(\mathbf{A})$ with normal vectors $\mathbf{a}_i$ for $i = 1, \dots, m$ (see Section 2.1), and in some instances, the zonotope and the arrangement encode certain combinatorial data captured by the linear matroid $M(\mathbf{A})$, see Section 2.5.3. A prime example is given by **graphic zonotopes/arrangements** as discussed in Section 2.5.1. Here, the generating matrix is $\mathbf{A}_G := [\mathbf{e}_j - \mathbf{e}_k : jk \in E]$, for a given graph $G = (V, E)$ and an (arbitrary but fixed) orientation on E; as usual we call $\mathbf{A}_G$ an **incidence matrix** of G. Greene and Zaslavsky [GZ83] showed that the vertices of $\mathsf{Z}(\mathbf{A}_G)$ (equivalently, the regions of $\mathcal{H}(\mathbf{A}_G)$) are in one-to-one correspondence with the acyclic orientations of G, and they gave analogous interpretations for all faces of $\mathsf{Z}(\mathbf{A}_G)$. Zaslavsky [Zas82a, Section 4] thus coined the charming term **acyclotope** for $\mathsf{Z}(\mathbf{A}_G)$.

Going back to general zonotopes, recall from Equation (2.13) that each $\mathsf{Z}(\mathbf{A})$ comes with a natural tiling into parallelepipeds [McM71; She74], whose (relative) volumes encode important arithmetic data of $\mathsf{Z}(\mathbf{A})$ (see Equation (2.9)). When $\mathbf{A} \in \mathbb{Z}^{d\times m}$, i.e., $\mathsf{Z}(\mathbf{A})$ is a **lattice zonotope**, this data can admit yet more combinatorial meaning. It is most easily packaged into the **Ehrhart polynomial** of $\mathsf{Z}(\mathbf{A})$ (2.12). In the above case that $\mathbf{A} = \mathbf{A}_G$ stems from a graph G, Stanley [Sta91] proved that the coefficient of t^j in the Ehrhart polynomial of $\mathsf{Z}(\mathbf{A}_G)$ equals the number of induced forests in G with j edges (see Corollary 2.14).

The goal in this chapter is twofold. First, we extend the interpretation of the Ehrhart coefficients for acyclotopes to signed graphs. The face structure of the acyclotope in this setting goes back to the same Greene–Zaslavsky paper [GZ83], but we could not find the analogue of Stanley's Ehrhart polynomial in the literature. Second, we define and study dual zonotopes in the sense of the underlying matroids (see Section 2.5.3), which we call **tocyclotopes**, as their vertices corresponds to the totally cyclic orientations of the given (signed) graph. The face structure of the tocyclotope in the case of an ordinary graph can once more be found in [GZ83]. Its Ehrhart polynomial must be known to experts, but we could not find it in the literature. Our results for tocyclotopes of signed graphs seem to be novel. Along the way, the construction and arithmetic of the tocyclotope suggest a general duality concept for zonotopes, one that was already employed by McMullen [McM71] and D'Adderio–Moci [DM12; DM13]: starting with $\mathbf{A} \in \mathbb{R}^{d\times m}$, construct a matrix that represents the matroid dual to that of $\mathbf{A}$; McMullen described the face structure of its associated zonotope entirely from the data of $\mathbf{A}$, and D'Adderio–Moci developed the general notion of an *arithmetic matroid*, whose duality nature applies here. In this context, we give a concrete computation that describes the Ehrhart polynomial of the zonotope of the dual arithmetic matroid in terms of $\mathbf{A}$.

Recall that we can phrase the above setting in the language of root systems, see Section 2.4. The generators $\mathbf{A}_G = [\mathbf{e}_j - \mathbf{e}_k : jk \in E]$ of the acyclotope for an ordinary graph $G = (V, E)$ form a subset of a root system of type A, and subsequently, the acyclotope is a subpolytope of the permutahedron, see Section 2.5.1. Recall that the **incidence matrix** of a given signed graph $\boldsymbol{\Sigma}$ with d nodes and m edges is an $d \times m$ matrix $\mathbf{A}_\Sigma$ whose column corresponding to the edge e equals

- $\mathbf{e}_j - \mathbf{e}_k$ or $\mathbf{e}_k - \mathbf{e}_j$ if $e = (jk)$ is a positive link,
- $\mathbf{e}_j + \mathbf{e}_k$ or $-\mathbf{e}_k - \mathbf{e}_j$ if $e = (jk)$ is a negative link,
- $\mathbf{e}_j$ if e is a halfedge at j,
- $2\mathbf{e}_j$ or $-2\mathbf{e}_j$ if e is a negative loop at j.

As already mentioned, these are root vectors of type A, B, C, D.[1] Similarly, the acyclotope for a signed graph is defined by a subset of a root system of type A, B, C, D, and vice versa, any such subset defines a signed graph, see Section 2.5.4. Thus the acyclotope is a subzonotope (in the sense that we remove some of the generators) of the respective root polytope (see, e.g., [ABM20]).[2]

Recall from Equation (2.23) that for a matroid $M = (E, \mathcal{I})$ its **dual matroid** $M^\triangle := (E, \mathcal{I}^\triangle)$ is defined via

$$\mathcal{I}^\triangle := \{J \subseteq E \mid E \setminus J \text{ is a spanning set of } M\},$$

where a subset $S \subseteq E$ is called **spanning** if it contains a basis. We are interested in the case that the matroid M is **representable** (over $\mathbb{R}$), i.e., a spanning set S consists of the columns of a given matrix $\mathbf{A} \in \mathbb{R}^{d \times m}$ and independence refers to linear independence, compare Example 2.17. In this case, and under the (reasonable) assumption that $\mathbf{A}$ has rank d, there is a well-known construction of $M^\triangle$, see Theorem 2.20. The standard representation (see Corollary 2.21) can be computed as follows: one uses elementary row operations on $\mathbf{A}$ resulting in a matrix of the form $[\mathbf{R} \,|\, \mathbf{I}] \in \mathbb{R}^{d \times m}$ where we denote by $\mathbf{I}$ the identity matrix of the appropriate dimension. The matrix $[\mathbf{R} \,|\, \mathbf{I}]$ also represents M. Now let $\mathbf{A}^\triangle := [\mathbf{I} \,|\, -\mathbf{R}^T] \in \mathbb{R}^{(m-d) \times m}$; by construction $\mathbf{A}^\triangle$ represents $M^\triangle$. We are particularly interested in the case when $\mathbf{A} = \mathbf{A}_\Sigma$ is the incidence matrix of a (signed) graph, i.e., consists of roots of type $A/B/C/D$.

In Section 4.1 we construct the tocyclotope $\mathsf{Z}(\mathbf{A}_G^\triangle)$ for a graph G and Theorem 4.1 gives its Ehrhart polynomial; and indeed, its coefficients enumerate (complements of) spanning sets in G.

In Section 4.2 we set up the necessary machinery from the theory of signed graphs and then study their acyclotopes, in particular we compute their Ehrhart polynomial in Theorem 4.4.

1 This correspondence is one reason to leave out positive loops and loose edges when building the incidence matrix; neither they do play a role for our work.

2 There are, unfortunately, conflicting definition of *root polytope* in the literature. Here we mean what might be more precisely called the *integral Coxeter permutahedron* $\Pi^{\mathbb{Z}}(\Phi) := \sum_{\alpha \in \Phi^+} [0, \alpha]$ of the finite root system Φ with a choice Φ^+ of positive roots.

In Section 4.3 we return to general lattice zonotopes. Given $\mathbf{A} \in \mathbb{Z}^{d\times m}$ of rank d, choose a lattice basis for the lattice $\ker(\mathbf{A}) \cap \mathbb{Z}^m$ and write them as the $m-d$ columns of $\mathbf{D} \in \mathbb{Z}^{m\times(m-d)}$. Theorem 4.6 describes the arithmetic of $\mathsf{Z}(\mathbf{D}^T)$ in terms of $\mathbf{A}$. We call $\mathsf{Z}(\mathbf{D}^T)$ the **lattice Gale zonotope** associated with $\mathbf{A}$. By construction, $\mathbf{D}^T$ represents the dual matroid of the matroid represented by $\mathbf{A}$, and our results give the afore-mentioned arithmetic extension of McMullen's study of the face structure of $\mathsf{Z}(\mathbf{D}^T)$ in the spirit of D'Adderio–Moci's arithmetic matroids.

Section 4.4 is devoted to the construction and study of tocyclotopes for a signed graph. We then apply Theorem 4.6 to compute their Ehrhart polynomials in Theorem 4.14.

This chapter is joint work with Eleon Bach and Matthias Beck [BBR24], Section 4.1 is based on Eleon Bachs master thesis.

4.1 Tocyclotopes and the Flow Space

This section is based on Eleon Bachs master thesis. We include it here since it serves as entry point and motivation for the remainder of this chapter.

We start with an (unsigned, simple) graph $G = (V, E)$ with incidence matrix $\mathbf{A}_G \in \mathbb{R}^{d\times m}$; it comes with a natural block form given by the connected components of G, and thus we may (and will) assume that G is connected. The matroid M defined by $\mathbf{A}_G$ can be given in terms of G (i.e., M is a **graphic matroid**): such a reduction corresponds to choosing a basis of the corresponding matroid, i.e., a spanning tree, whose edges then correspond each to the the columns of the identity matrix.

We first modify $\mathbf{A}_G$ (which has rank $d-1$) to a full-rank matrix that still represents M, using a well-known construction. Namely, one uses elementary row operations to create a row of zeros and then discards the latter. The result is a matrix of the form $[\mathbf{R} \,|\, \mathbf{I}] \in \mathbb{R}^{(d-1)\times m}$ known as a **network matrix** of G. It can be constructed, e.g., via a spanning tree of G, whose edges correspond to the identity matrix.

Matroid duality yields a representation matrix $\mathbf{A}_G^{\triangle} = [\mathbf{I} \,|\, -\mathbf{R}^T] \in \mathbb{R}^{(m-d+1)\times m}$ for $M^{\triangle}$, which is called the **cographic matroid**. It can be described purely in terms of G: its ground set is again E and the independent sets are precisely the complements of spanning sets of G. A **spanning set** of a connected graph is a subset of edges whose subgraph contains a spanning tree.

The **tocyclotope** of G is the zonotope $\mathsf{Z}(\mathbf{A}_G^{\triangle})$. We note that there were choices involved when constructing $\mathbf{A}_G^{\triangle}$; however, different choices correspond to resulting matrices that are unimodularly equivalent (since elementary row operations are unimodular). Thus $\mathsf{Z}(\mathbf{A}_G^{\triangle})$ is unique up to unimodular equivalence. Recall Stanley's result for Ehrhart polynomials of lattice zonotopes, Theorem 2.3, and the quantities $g(\mathbf{F})$ from Section 2.2 (they will also be rediscussed in Remark 4.10 below). These ingredients yield the companion result to Stanley's [Sta91] afore-mentioned Ehrhart polynomial structure of $\mathsf{Z}(\mathbf{A}_G)$, see Corollary 2.14.

Theorem 4.1. *Let $G = (V, E)$ be a simple and connected graph. Then the Ehrhart polynomial of the tocyclotope $\mathsf{Z}(\mathbf{A}_G^{\triangle})$ is*

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{A}_G^{\triangle}); n) = \sum_{k=0}^{|E|-|V|+1} d_k\, n^k$$

where the coefficient d_k equals the number of (complements of) spanning sets in G of size k, which equals the number of forests of size $m - k$ in G^{Δ}.

Proof. Both $\mathbf{A}_G$ and $\mathbf{A}_G^{\triangle}$ are totally unimodular. Further, the column vectors of $\mathbf{A}_G^{\triangle}$ are, by definition, linearly independent if and only if they induce complements of spanning sets on G as these induce the independent sets of the cographic matroid. Now apply Theorem 2.3. Counting the number of spanning sets or the number of their complements is the same. □

Example 4.2. The tocyclotope of the complete graph K_4 is the 3-permutahedron. We can thus calculate its Ehrhart polynomial as follows. The linear coefficient d_1 equals 6 since every edge of K_4 is a complement of a spanning set. Every choice of two edges of K_4 is a complement of a spanning set and so the second coefficient d_2 equals 15. Every choice of three edges besides the ones incident to a single node is a complement of a spanning set and thus $d_3 = 16$. In total we obtain

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{A}_{K_4}^{\triangle}); n) = 16\, n^3 + 15\, n^2 + 6\, n + 1\,.$$

We briefly comment on how $\mathsf{Z}(\mathbf{A}_G^{\triangle})$ connects to flows on G and its cographic arrangement; here we assume that G does not contain isthmi (in the graph case isthmi are bridges). The **flow space** (also called the *cycle space*) of G is defined as the kernel of $\mathbf{A}_G$, which is an $(m - d + 1)$-dimensional subspace of $\mathbb{R}^m$. The **cographic arrangement** $\mathcal{H}(\mathbf{A}_G^{\triangle})$ is the arrangement induced by the coordinate hyperplanes in $\mathbb{R}^m$ on $\ker(\mathbf{A}_G)$. Greene and Zaslavsky [GZ83] showed that the regions of the cographic arrangement are in one-to-one correspondence with the totally cyclic orientations of G.

Lemma 4.3. *Let $G = (V, E)$ be a simple, connected, and bridgeless graph. The linear surjection $\mathbf{A}_G^{\triangle} : \mathbb{R}^m \to \mathbb{R}^{m-d+1}$ maps the flow space $\ker(\mathbf{A}_G)$ bijectively to $\mathbb{R}^{m-d+1}$. Thus the columns of the matrix $\mathbf{A}_G^{\triangle}$ are normal vectors for an isomorphic copy of the cographic arrangement living in $\mathbb{R}^{m-d+1}$.*

Proof. We need to prove $\ker(\mathbf{A}_G) \cap \ker(\mathbf{A}_G^{\triangle}) = \{\mathbf{0}\}$. By construction, each row of $\mathbf{A}_G$ is perpendicular to each row of $\mathbf{A}_G^{\triangle}$. Thus, any $\mathbf{w} \in \ker(\mathbf{A}_G) \cap \ker(\mathbf{A}_G^{\triangle})$ is perpendicular to all row vectors in both $\mathbf{A}_G$ and in $\mathbf{A}_G^{\triangle}$, which implies $\mathbf{w} = \mathbf{0}$. □

Thus, we may think of $\mathbf{A}_G^{\triangle}$ as simultaneously generating the cographic arrangement $\mathcal{H}(\mathbf{A}_G^{\triangle})$ and the tocyclotope $\mathsf{Z}(\mathbf{A}_G^{\triangle})$, giving rise to geometric (e.g., the vertices of $\mathsf{Z}(\mathbf{A}_G^{\triangle})$ are given by totally cyclic orientations of G) and arithmetic (e.g., Theorem 4.1) structures.

4.2 Ehrhart Polynomial of Acyclotopes for Signed Graphs

Our definition of $\mathsf{Z}(\mathbf{A}_\Sigma)$ implicitly depends on a choice of orientation for each edge of Σ (except loops and halfedges). However, any property of the acyclotope discussed here is independent of these choices. For example, the face structure of the zonotope is determined by the combinatorial structure of the corresponding hyperplane arrangement, i.e., the poset of intersections. This in turn does not depend on the choice of orientation of the hyperplane normals.

The following result generalizes the Ehrhart polynomial of the type-B root polynomial [ABM20].

Theorem 4.4. *The Ehrhart polynomial of the acyclotope* $\mathsf{Z}(\mathbf{A}_\Sigma)$ *equals*

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{A}_\Sigma); n) = \sum_F 2^{\mathrm{pc}(F)+\mathrm{lc}(F)}\, n^{d-\mathrm{tc}(F)}\,,$$

where the sum is over all $F \subseteq E$ *such that* $\Sigma(F)$ *is a pseudo-forest.*

We recall that the number of nodes in every edge subgraph $\Sigma(F)$ is the same as the number d of nodes in Σ. Similarly, the number $\mathrm{tc}(F)$ of signed tree components counts, in particular, all the isolated vertices; e.g., for a graph with d nodes, $\mathrm{tc}(\varnothing) = d$.

The following lemma has appeared in various guises; see [ACH15, Lemmas 4.9. & 4.10.], [Kot02, Proposition 4.2.], and [Zas82b, Lemma 8A.2], see also Proposition 2.23.

Lemma 4.5. *Let* $\mathbf{F}$ *be a linearly independent subset of the columns of* $\mathbf{A}_\Sigma$*. The corresponding subset* F *of edges of* Σ *forms a pseudo-forest as subgraph. Then*

$$g(\mathbf{F}) = 2^{\mathrm{pc}(F)+\mathrm{lc}(F)} \quad \textit{and} \quad |F| = d - \mathrm{tc}(F)\,,$$

where $g(\mathbf{F})$ *is as in Equation 2.9.*

Proof of Theorem 4.4. We apply Stanley's Theorem 2.3. Linearly independent subsets of $\mathbf{A}_\Sigma$ correspond, by Lemma 4.5, to pseudo-forests of Σ. Lemma 4.5 also gives the dimension $(d - \mathrm{tc}(F))$ and volume $(2^{\mathrm{pc}(F)+\mathrm{lc}(F)})$ of the parallelepiped associated with a given linearly independent set. As we will discuss in Remark 4.10, the latter volume equals $g(\mathbf{F})$. $\square$

4.3 Lattice Gale Zonotopes

We now revisit the construction of the tocyclotope of a given graph G: starting with the incidence matrix of G, we constructed a matrix representing the cographic matroid, from which we generated a zonotope. This process is not confined to the incidence matrix of a graph, and so we now start with a general integral matrix $\mathbf{A} \in \mathbb{Z}^{d\times m}$ of rank d. Choose a lattice basis for the lattice $\ker(\mathbf{A}) \cap \mathbb{Z}^m$ and write them as the $m - d$ columns of $\mathbf{D} \in \mathbb{Z}^{m\times(m-d)}$, which we call a **lattice Gale dual** of $\mathbf{A}$.[3] By construction, $\mathbf{D}^T$ represents the dual of the matroid represented by $\mathbf{A}$.

3 Our terminology follows that of matrices/vector configurations over a field; see, e.g., [Zie98, Chapter 6].

This is reminiscent of the interplay of $\mathsf{Z}(\mathbf{A})$ and $\mathsf{Z}(\mathbf{A}^{\triangle})$ (which is combinatorially equivalent to $\mathsf{Z}(\mathbf{D}^T)$), which we alluded to in the introduction; McMullen [McM71] calls $\mathsf{Z}(\mathbf{A}^{\triangle})$ and $\mathsf{Z}(\mathbf{D}^T)$ *derived zonotopes* and the corresponding Gale diagrams *zonotopal diagrams*. He completely described the face structure of $\mathsf{Z}(\mathbf{A}^{\triangle})$. Our point is to extend this description to the arithmetic structure (in the sense of integer points) of the derived zonotope $\mathsf{Z}(\mathbf{D}^T)$; hence our construction of the lattice Gale dual.

Our goal is to describe the arithmetics of the **lattice Gale zonotope** $\mathsf{Z}(\mathbf{D}^T)$ in terms of $\mathbf{A}$. While this zonotope depends on the construction of the lattice basis that yields $\mathbf{D}$, Theorem 4.6 is the main result in this section and shows that the arithmetic of the lattice Gale zonotope depends only on $\mathbf{A}$.[4]

Theorem 4.6. *Let $\mathbf{A} \in \mathbb{Z}^{d\times m}$ be of rank d, with lattice Gale dual $\mathbf{D} \in \mathbb{Z}^{m\times(m-d)}$. Then we can compute the Ehrhart polynomial of the associated lattice Gale zonotope as*

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{D}^T); n) = \sum_{S} \frac{g(\mathbf{A}_S)}{g(\mathbf{A})}\, n^{m-|S|}$$

where the sum is over all spanning sets $S \subseteq [m]$ in the matroid represented by $\mathbf{A}$ and $g(\mathbf{A})$ is defined as in Equation 2.9.

Remark 4.7. As (the usual) Gale duality can be used to efficiently compute the face structure of a d-polytope with k vertices where $k-d$ is small (but d and k may be large), Theorem 4.6 can be used to efficiently compute the Ehrhart polynomial of a zonotope generated by $\mathbf{D}^T \in \mathbb{Z}^{(m-d)\times m}$ for large m but small d: here we have to understand only the arithmetic of the (much smaller) matrix $\mathbf{A} \in \mathbb{Z}^{d\times m}$. Note that every full rank integer matrix $\mathbf{A}$ can be seen as a lattice Gale dual. Moreover, the resulting matrix, after applying the lattice Gale dual construction twice, is unimodularly equivalent to the original matrix.

Theorem 4.1 is a special case of Theorem 4.6, because the incidence matrix of a graph is totally unimodular, and thus $g(\mathbf{A}_S) = 1$ for all $\mathbf{A}_S$. Indeed, the same reasoning implies the following specialization for any totally unimodular matrix $\mathbf{A}$, i.e., when the associated matroid is **regular**.

Corollary 4.8. *Let $\mathbf{A} \in \mathbb{Z}^{d\times m}$ be a totally unimodular matrix of rank d, with lattice Gale dual $\mathbf{D} \in \mathbb{Z}^{m\times(m-d)}$. Then the Ehrhart polynomial of the associated lattice Gale zonotope is given by*

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{D}^T); n) = \sum_{S} n^{m-|S|} = \sum_{k=d}^{m} d_k\, n^{m-k}$$

where the sum in the first sum is over all spanning sets S in the regular matroid represented by $\mathbf{A}$ and d_k is the number of spanning sets of size k in the matroid represented by $\mathbf{A}$.

4 In the language of D'Adderio–Moci, a lattice zonotope corresponds to representable, torsion-free arithmetic matroid with GCD property [DM13] and the Ehrhart polynomial is a specialization of the arithmetic Tutte polynomial [DM12]. Our lattice Gale zonotopes then correspond to the dual representable torsion-free arithmetic matroids.

There are two main ingredients we will need to prove Theorem 4.6. For the first we give an elementary proof here. Given a matrix $\mathbf{A}$, we denote by $\operatorname{span}_{\mathbb{R}}(\mathbf{A})$ the real vector space spanned by its columns and by $\operatorname{span}_{\mathbb{Z}}(\mathbf{A})$ the set of integer combination of its columns.

Lemma 4.9 (Gale duality for lattices)**.** *Let $\mathbf{A} \in \mathbb{Z}^{d\times m}$ be of rank d, with a lattice Gale dual $\mathbf{D} \in \mathbb{Z}^{m\times(m-d)}$. Every choice of $k \leq m$ linearly independent rows of $\mathbf{D}$ indexed by some $\overline{\rho} \subseteq [m]$ yields a submatrix $\mathbf{D}_{\overline{\rho}} \in \mathbb{Z}^{k\times(m-d)}$. Then the complement $\rho := [m] \setminus \overline{\rho}$ defines a matrix $\mathbf{A}_\rho \in \mathbb{Z}^{d\times(m-k)}$ consisting of the columns of $\mathbf{A}$ indexed by ρ, which are spanning and hence contain a basis of $\mathbb{R}^d$. Then there is a bijection*

$$\psi : \mathbb{Z}^k/\operatorname{span}_{\mathbb{Z}}(\mathbf{D}_{\overline{\rho}}) \;\to\; \operatorname{span}_{\mathbb{Z}}(\mathbf{A})/\operatorname{span}_{\mathbb{Z}}(\mathbf{A}_\rho)\,.$$

Proof. We may assume without loss of generality that ρ indexes the first $m-k$ columns of $\mathbf{A}$ and the first $m-k$ rows of $\mathbf{D}$. Let

$$\begin{aligned}\psi : \mathbb{Z}^k/\operatorname{span}_{\mathbb{Z}}(\mathbf{D}_{\overline{\rho}}) \;&\to\; \operatorname{span}_{\mathbb{Z}}(\mathbf{A})/\operatorname{span}_{\mathbb{Z}}(\mathbf{A}_\rho)\\ [\mathbf{v}] \;&\mapsto\; [\mathbf{A}_{\overline{\rho}}\mathbf{v}]\,.\end{aligned}$$

We first show that ψ is well defined and injective. Let $\mathbf{v} \in \mathbb{Z}^k$. Then

$$\psi[\mathbf{v}] = [\mathbf{A}_{\overline{\rho}}\mathbf{v}] = 0 \in \operatorname{span}_{\mathbb{Z}}(\mathbf{A})/\operatorname{span}_{\mathbb{Z}}(\mathbf{A}_\rho)$$

if and only if $\mathbf{A}_{\overline{\rho}}\mathbf{v} = \mathbf{A}_\rho\mathbf{w}$ for some $\mathbf{w} \in \mathbb{Z}^{m-k}$, that is,

$$\begin{pmatrix} -\mathbf{w} \\ \mathbf{v} \end{pmatrix} \in \ker(\mathbf{A}) \cap \mathbb{Z}^m = \operatorname{span}_{\mathbb{Z}}(\mathbf{D})\,.$$

This means $\binom{-\mathbf{w}}{\mathbf{v}} = \mathbf{D}\mathbf{u}$ for some $\mathbf{u} \in \mathbb{Z}^{m-d}$, i.e., $\mathbf{v} = \mathbf{D}_{\overline{\rho}}\mathbf{u}$, which in turn means

$$[\mathbf{v}] = 0 \in \mathbb{Z}^k/\operatorname{span}_{\mathbb{Z}}(\mathbf{D}_{\overline{\rho}})\,.$$

To show that ψ is surjective, let $\mathbf{y} \in \operatorname{span}_{\mathbb{Z}}(\mathbf{A})$, so $\mathbf{A}\mathbf{x} = \mathbf{y}$ for some $\mathbf{x} = \binom{\mathbf{x}_\rho}{\mathbf{x}_{\overline{\rho}}} \in \mathbb{Z}^m$. Let $\mathbf{v} = \mathbf{x}_{\overline{\rho}} \in \mathbb{Z}^k$ and $\mathbf{w} = -\mathbf{x}_\rho \in \mathbb{Z}^{m-k}$. Thus

$$\mathbf{y} = \mathbf{A}\mathbf{x} = -\mathbf{A}_\rho\mathbf{w} + \mathbf{A}_{\overline{\rho}}\mathbf{v}\,,$$

i.e., $\psi[\mathbf{v}] = [\mathbf{y}]$. $\square$

The second ingredient is the quantity $g(\mathbf{F})$ from Equation 2.9. Recall that, for a matrix $\mathbf{F}$, we defined $g(\mathbf{F})$ as the greatest common divisor of all maximal minors of $\mathbf{F}$.

Remark 4.10. We recall and slightly extend our discussion of the quantity $g(\mathbf{A})$ and the Smith Normal Form from the beginning of Section 2.2. The Smith Normal Form of a matrix $\mathbf{A} \in \mathbb{Z}^{d\times m}$ of rank r is

$$\mathbf{S\,A\,T} \;=\; \begin{pmatrix} d_1 & & & & & \\ & d_2 & & & & \\ & & \ddots & & & \\ & & & d_r & & \\ & & & & & \\ & & & & & \end{pmatrix}$$

where $\mathbf{S} \in \mathbb{Z}^{d\times d}$ and $\mathbf{T} \in \mathbb{Z}^{m\times m}$ are invertible matrices and $d_1\, d_2 \cdots d_r$ equals the gcd of all $r \times r$ minors of $\mathbf{A}$. Thus $\mathbf{S}$ and $\mathbf{T}$ are integer-lattice preserving, from which we deduce

$$\left| (\operatorname{span}_{\mathbb{R}}(\mathbf{A}) \cap \mathbb{Z}^n) \big/ \operatorname{span}_{\mathbb{Z}}(\mathbf{A}) \right| = d_1\, d_2 \cdots d_r\,.$$

Thus, if $\mathbf{A}$ has full (column or row) rank, as in Equation (2.10),

$$\left| (\operatorname{span}_{\mathbb{R}}(\mathbf{A}) \cap \mathbb{Z}^n) \big/ \operatorname{span}_{\mathbb{Z}}(\mathbf{A}) \right| = g(\mathbf{A})\,.$$

Note that, by definition, $g(\mathbf{A}) = g(\mathbf{A}^T)$. For some subset $\mathbf{F} \subseteq \mathbb{Z}^d$ of linearly independent vectors, i.e., the case of full column rank, we recall various interpretations for $g(\mathbf{F})$:

(i) $g(\mathbf{F})$ is (by definition) the greatest common divisor of all minors of size $|\mathbf{F}|$ of the matrix whose columns are the elements of $\mathbf{F}$, see Equation (2.8),

(ii) $g(\mathbf{F})$ is the $|\mathbf{F}|$-dimensional relative volume of the parallelepiped spanned by $\mathbf{F}$, see Equation (2.9),

(iii) $g(\mathbf{F})$ is the number of cosets of the discrete subgroup generated by $\mathbf{F}$, considered as a sublattice of the integer points in the linear span of $\mathbf{F}$, see Equation (2.10).

Let $\mathbf{A} \in \mathbb{Z}^{d\times m}$ be of rank d, with lattice Gale dual $\mathbf{D} \in \mathbb{Z}^{m\times(m-d)}$. Every choice of $k \le m$ linearly independent rows of $\mathbf{D}$ yields a matrix $\mathbf{D}_{\overline{\rho}} \in \mathbb{Z}^{k\times(m-d)}$ whose rows are indexed by $\overline{\rho} \subseteq [m]$ and induces a map $\mathbf{D}_{\overline{\rho}}\colon \mathbb{Z}^{m-d} \to \mathbb{Z}^k$. This on the other hand induces a matrix $\mathbf{A}_\rho \in \mathbb{Z}^{d\times(m-k)}$ consisting of the columns of $\mathbf{A}$ indexed by $\rho := [m] \setminus \overline{\rho}$; note that they contain a basis of $\mathbb{R}^d$.

Corollary 4.11. *With the same conditions and notations as in Lemma 4.9,*

$$g(\mathbf{D}_{\overline{\rho}}) = \frac{g(\mathbf{A}_\rho)}{g(\mathbf{A})}\,.$$

Proof. First note that

$$\operatorname{span}_{\mathbb{R}}(\mathbf{D}_{\overline{\rho}}) = \mathbb{R}^k\,,$$

since the rows of $\mathbf{D}_{\overline{\rho}}$ are linearly independent and $k \leq m - d$. By Remark 4.10, Lemma 4.9, and Equation (2.11)

$$g(\mathbf{D}_{\overline{\rho}}) = \left|\mathbb{Z}^k/\mathrm{span}_{\mathbb{Z}}(\mathbf{D}_{\overline{\rho}})\right| = \left|\mathrm{span}_{\mathbb{Z}}(\mathbf{A})/\mathrm{span}_{\mathbb{Z}}(\mathbf{A}_\rho)\right| = \frac{\left|\mathbb{Z}^n/\mathrm{span}_{\mathbb{Z}}(\mathbf{A}_\rho)\right|}{\left|\mathbb{Z}^n/\mathrm{span}_{\mathbb{Z}}(\mathbf{A})\right|} = \frac{g(\mathbf{A}_\rho)}{g(\mathbf{A})}\,.$$

□

Proof of Theorem 4.6. By Stanley's Theorem 2.3,

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{D}^T); n) = \sum_J g\left((\mathbf{D}^T)_J\right) n^{|J|}$$

where J indexes linearly independent subsets of columns of $\mathbf{D}^T$, i.e., the sum is over independent sets $J \subseteq [m]$ in the dual matroid. By matroid duality, these sets correspond to spanning sets $S = [m] \setminus J$ in the primal matroid. By Remark 4.10, $g((\mathbf{D}^T)_J) = g(\mathbf{D}_J)$, and from Corollary 4.11 we know $g(\mathbf{D}_J) = \frac{g(\mathbf{A}_S)}{g(\mathbf{A})}$, and so

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{D}^T); n) = \sum_S \frac{g(\mathbf{A}_S)}{g(\mathbf{A})}\, n^{m-|S|}$$

where the sum is now over spanning sets in the primal matroid. □

4.4 Tocyclotopes for Signed Graphs

The goal of this section is to construct the signed tocyclotope and then compute its Ehrhart polynomial in terms of signed graph-theoretic data. We could define the signed tocyclotope as a lattice Gale dual of the signed acyclotope as explained in the previous section. However, there is a more combinatorial and concrete route via bidirected network matrices, which can be seen in analogy to the graphic case. This construction is in fact a special case of our more general framework in Section 4.3, which will help us with the computation of the Ehrhart coefficients for the tocyclotopes.

As mentioned in Section 2.5.4, oriented signed graphs are equivalent to bidirected graphs. Bidirected graphs were first defined by Edmonds and Johnson [Jac67; JE70]. Appa and Kotnyek [AK06] studied a bidirectional analog of network matrices. One of their central results, which we will use, is conditions on when the duals of those matrices is integral. In general, those inverses are half-integral, a result that first appeared in [Bou83]. For more information see also the references in [BZ06d; Bou83; Kot02].

From now on we want to assume that our (signed) graph is connected. If it is not connected, we can apply the following results to each of the connected components and then take the appropriate product; we will give more details in Remark 4.17 below. Additionally, we will assume that the incidence matrix $\mathbf{A}_\Sigma \in \mathbb{Z}^{d\times m}$ has full rank, i.e., $\mathrm{rank}(\mathbf{A}_\Sigma) = d$. If this is not the case then the signed graph is balanced and can hence be considered as an unsigned graph.

4.4.1 Binet Matrices and the Tocyclotope

We will give the definition of a binet matrix[5] as it was introduced by Appa and Kotnyek [AK06] in order to generalize the dual of network matrices to bidirected graphs.[6]

Let $\mathbf{A}_\Sigma \in \mathbb{Z}^{d\times m}$ be the incidence matrix of the signed graph Σ and let $T \subseteq E$ be a subset of the edges of Σ that forms a basis, as discussed in Proposition 2.23. This implies that the submatrix $\mathbf{T} \in \mathbb{Z}^{d\times d}$ of $\mathbf{A}_\Sigma$ formed by choosing the columns indexed by T is invertible over $\mathbb{R}$. After reordering columns, we can write the incidence matrix $\mathbf{A}_\Sigma$ as $[\mathbf{R} \,|\, \mathbf{T}]$, where $\mathbf{R} \in \mathbb{Z}^{d\times(m-d)}$ is the matrix formed from columns indexed by $R := E \setminus T$. Then we multiply $\mathbf{A}_\Sigma = [\mathbf{R} \,|\, \mathbf{T}]$ with $\mathbf{T}^{-1}$ from the left to obtain

$$\mathbf{T}^{-1}\mathbf{A}_\Sigma = [\mathbf{T}^{-1}\mathbf{R} \,|\, \mathbf{I}] = [\mathbf{B} \,|\, \mathbf{I}]\,,$$

where $\mathbf{I} \in \mathbb{Z}^{d\times d}$ is the unit matrix and $\mathbf{B} := \mathbf{T}^{-1}\mathbf{R} \in \mathbb{R}^{d\times(m-d)}$. The matrix $\mathbf{B}$ is called the **binet matrix**. Appa and Kotnyek further present a graphical algorithm to compute binet matrices [AK06]. The algorithm gives an easier and more direct way of computing binet matrices, relies on the intuition of flows on bidirected graphs, and is a useful perspective to prove properties of binet matrices. The algorithm was reformulated in [BZ06d].

Lemma 4.12 ([AK06, Lemma 17])**.** *Let Σ be a signed graph and $T \subseteq E$ be a subset that forms a maximal pseudo-forest. The binet matrix $\mathbf{B} = \mathbf{T}^{-1}\mathbf{R}$ is integral if and only if one of the following conditions holds:*

1. *every connected component in the maximal pseudo-forest spanned by T is a (signed) halfedge-tree, or*
2. *Σ does not contain halfedges and T spans one connected component.*

Since we assumed the signed graph Σ to be connected, we can always choose a pseudo-tree $T \subseteq E$ that fulfills one of the conditions in Lemma 4.12: If the signed graph contains halfedges, choose a connected basis that contains one of the halfedges (case 1 in Lemma 4.12), otherwise choose any other connected basis (case 2 in Lemma 4.12). Then we know that the matrix $[\mathbf{B} \,|\, \mathbf{I}] \in \mathbb{Z}^{n\times m}$ has integral coefficients.

It is immediate from the construction that the rows of $\mathbf{D}^T := [\mathbf{I} \,|\, -\mathbf{B}^T] \in \mathbb{Z}^{(m-d)\times m}$, where here $\mathbf{I} \in \mathbb{Z}^{(m-d)\times(m-d)}$ and $-\mathbf{B}^T = -(\mathbf{T}^{-1}\mathbf{R})^T \in \mathbb{Z}^{(m-d)\times d}$, are contained in the kernel of $\mathbf{A}_\Sigma$. Since the matrix has full rank $m-d$, its rows span the kernel of $\mathbf{A}_\Sigma$. Note that $g([\mathbf{I} \,|\, -\mathbf{B}^T]) = 1$ because the maximal minor given by the identity matrix $\mathbf{I}$ equals one and hence the greatest common divisor of all minors as well. From Remark 4.10 it follows that the rows of $[\mathbf{I} \,|\, -\mathbf{B}^T]$ form a lattice basis for $\ker(\mathbf{A}_\Sigma) \cap \mathbb{Z}^m$. So $\mathbf{D}^T = [\mathbf{I} \,|\, -\mathbf{B}^T] \in \mathbb{Z}^{(m-d)\times m}$ is the transpose

5 "The term binet is used here as a short form for *bi*directed *net*work, but by coincidence it also matches the name of Jacques Binet (1786–1856) who worked on the foundations of matrix theory and gave the rule of matrix multiplication." [Kot02, page 46]

6 Note that the network matrix is the reduced incidence matrix of a graph, while the binet matrix is a part of the dual matrix in the signed graphic case.

of a lattice Gale dual of $\mathbf{A}_\Sigma$. Therefore this combinatorial construction fits into the general frame work from Section 4.3.

Hence, we define the **tocyclotope for signed graphs** as the integral zonotope $\mathsf{Z}([\mathbf{I}| - \mathbf{B}^T])$. As in the case of unsigned graphs, this zonotope depends on our choice of $\mathbf{T}$; however, not only is its face structure independent of this choice (by [McM71]) the same is true for its Ehrhart polynomial. This follows from Theorem 4.6) or from the observation that for every right choice of $\mathbf{T}$ the rows of the resulting matrix $[\mathbf{I}\,| - \mathbf{B}^T]$ form a lattice bases for $\ker(\mathbf{A}_\Sigma) \cap \mathbb{Z}^m$ and therefore are unimodular equivalent.

Parallel to the definitions for graphs, the **flow space** of the signed graph Σ is $\ker(\mathbf{A}_\Sigma)$, and the **signed cographic arrangement** is the hyperplane arrangement induced by the coordinate hyperplanes of $\mathbb{R}^m$ on $\ker(\mathbf{A}_\Sigma)$; see, e.g., [BZ06c; CW09; CWZ17].The proof of the following lemma is almost verbatim that of Lemma 4.3.

Lemma 4.13. *Let Σ be a signed graph without coloops and let $\mathbf{D}^T = [\mathbf{I}| - \mathbf{B}^T]$ be as described above. The linear surjection $\mathbf{D}^T : \mathbb{R}^m \to \mathbb{R}^{m-d}$ maps the flow space $\ker(\mathbf{A}_\Sigma)$ bijectively to $\mathbb{R}^{m-d}$. Thus the columns of the matrix $\mathbf{D}^T$ are normal vectors for an isomorphic copy of the signed cographic arrangement living in $\mathbb{R}^{m-d}$.*

Although this is not a main theme of this paper, we add a remark about the face structure of the tocyclotope, as it follows directly from (oriented matroid) duality. We recall that a cycle is a minimally dependent set of edges that is oriented in such a way that it has neither a sink nor a source. Recall that an orientation is totally cyclic if every (bioriented) edge is contained in a cycle. The regions of the signed cographic arrangement, and therefore the vertices of the signed tocyclotope, correspond bijectively to totally cyclic orientations of the signed graph Σ; see [BZ06c, proof of Theorem 4.5.(b)]. Higher dimensional faces of the signed tocyclotope can be understood via the flats of the signed corgraphic arrangement.

4.4.2 The Ehrhart Polynomial of the Tocyclotope

The goal of this section is to prove the combinatorial description of the coefficients in the Ehrhart polynomial of signed tocyclotopes.

Theorem 4.14. *Let Σ be a connected signed graph whose incidence matrix has full rank. Choose a connected basis $T \subseteq E$ that contains a halfedge if Σ contains an halfedge. Then the Ehrhart polynomial of the tocyclotope $\mathsf{Z}([\mathbf{I}\,| - (\mathbf{T}^{-1}\mathbf{R})^T])$ in $\mathbb{R}^{m-d}$ is*

$$\operatorname{ehr}_{\mathbb{Z}}(\mathsf{Z}([\mathbf{I}\,| - \mathbf{B}^T]); n) = \begin{cases} \sum_S 2^{\operatorname{mplc}(S)} n^{m-|S|} & \text{if } \Sigma \text{ contains a halfedge,} \\ \sum_S 2^{\operatorname{mplc}(S)-1} n^{m-|S|} & \text{otherwise,} \end{cases} \tag{4.1}$$

where the sums run over all sets $S \subseteq E$ that contain a basis of Σ, i.e., $\Sigma(S)$ contains a maximal pseudo-forest of Σ, and

$$\operatorname{mplc}(S) := \min_{\tilde{T} \subseteq S} \left(\operatorname{pc}(\tilde{T}) + \operatorname{lc}(\tilde{T})\right) ,$$

with the minimum taken over all maximal pseudo-forests $\tilde{T}$ in Σ contained in the spanning set S.

We will again apply Stanley's Theorem 2.3 for zonotopes. For that we need a combinatorial understanding of $g(\mathbf{J})$, where the columns in the submatrix $\mathbf{J}$ of $\mathbf{D}^T = [\mathbf{I} \,|\, -(\mathbf{T}^{-1}\mathbf{R})^T]$ are linearly independent. Recall that they correspond to independent sets J in the dual signed graphic matroid. Hence they correspond to subsets of edges $S = E \setminus J$ in the signed graph Σ that contain a basis, i.e., a maximal pseudo-forest.

Corollary 4.15. *A subset of columns $\mathbf{J}$ of $\mathbf{D}^T = [\mathbf{I} \,|\, -(\mathbf{T}^{-1}\mathbf{R})^T]$ (as constructed above) is linearly independent if and only if the subset $\mathbf{S}$ of columns in $\mathbf{A}_\Sigma$ indexed by $S = E \setminus J$ is a spanning set, i.e., S contains a maximal pseudo-tree in Σ. In this case,*

$$g(\mathbf{J}) = \frac{g(\mathbf{S})}{g(\mathbf{A}_\Sigma)} .$$

Proof. This follows from Corollary 4.11. □

Therefore, it remains to understand the parameter $g(\mathbf{S})$ for spanning sets in the signed graph.

Lemma 4.16. *Let $S \subseteq E$ be a spanning set. Then there exists a maximal forest $F \subseteq S$ such that $g(\mathbf{A}_{\Sigma(S)}) = g(\mathbf{A}_{\Sigma(F)})$. Moreover, this maximal forest F will be one with a minimal number of pseudo-tree components plus loop-tree components.*

Proof. From Remark 4.10 we know that $g(\mathbf{A}_{\Sigma(S)})$ is the greatest common divisor of all minors of size d in $\mathbf{S}$. Since all minors are powers of 2 (by Lemma 4.5), the greatest common divisor is the lowest power of 2 that appears. The selection of columns in $\mathbf{S}$ for which the minor attains its minimum corresponds to a forest $F \subseteq S$ of the kind that we are looking for. Then

$$g(\mathbf{A}_{\Sigma(S)}) = g(\mathbf{A}_{\Sigma(F)}) = 2^{\mathrm{pc}(F)+\mathrm{lc}(F)} = 2^{\mathrm{mplc}(S)} .$$

□

Proof of Theorem 4.14. By Theorem 4.6

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{T}(\Sigma); n) = \sum_S \frac{g(\mathbf{A}_S)}{g(\mathbf{A}_\Sigma)}\, n^{m-|S|}$$

where the sum is over all spanning sets S in the matroid represented by $\mathbf{A}_\Sigma$, i.e., over all subsets $S \subseteq E$ that contain a maximal pseudo-forest of Σ.

Note that for connected signed graphs Σ of full rank, $\mathrm{span}_{\mathbb{Z}}(\mathbf{A}_\Sigma) = \mathbb{Z}^d$ (and hence $g(\mathbf{A}_\Sigma) = 1$ by Remark 4.10) if and only if Σ contains a halfedge by Lemma 4.5. In the case of connected signed graphs without halfedges we can apply Corollary 4.11, and we will get a correction factor of 2 since then $g(\mathbf{A}_\Sigma) = 2$ again by Lemma 4.5. This explains the case distinction in (4.1) and the difference of a factor of 2 between the cases.

The last missing piece now is to understand $g(\mathbf{A}_S) = g(\mathbf{S})$. This is given in Lemma 4.16: we need to find the minimal possible number $\mathrm{mplc}(S)$ of loop-tree components plus pseudo-tree components in a maximal pseudo forest in the spanning set S. Then we arrive at

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{T}(\Sigma); n) = \begin{cases} \sum_S 2^{\mathrm{mplc}(S)} n^{m-|S|} & \text{if } \Sigma \text{ contains a halfedge,} \\ \sum_S 2^{\mathrm{mplc}(S)-1} n^{m-|S|} & \text{if } \Sigma \text{ does not contain any halfedges.} \end{cases}$$

□

We conclude this section with the extension of the results to signed graphs that are not connected.

Remark 4.17. Let Σ be an arbitrary signed graph with connected components $\Sigma_1, \ldots, \Sigma_c$. Then we can order nodes and edges so that the incidence matrix $\mathbf{A}_\Sigma$ has a block structure given by the connected components:

$$\mathbf{A}_\Sigma = \begin{bmatrix} \mathbf{A}_{\Sigma_1} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\Sigma_2} & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ldots & \ldots & \mathbf{0} & \mathbf{A}_{\Sigma_c} \end{bmatrix}.$$

This implies that the acyclotope of the signed graph Σ is simply the Cartesian product of the acyclotopes of the connected components:

$$\mathsf{Z}(\mathbf{A}_\Sigma) = \mathsf{Z}(\mathbf{A}_{\Sigma_1}) \times \cdots \times \mathsf{Z}(\mathbf{A}_{\Sigma_c}).$$

Hence the Ehrhart polynomial of $\mathsf{Z}(\mathbf{A}_\Sigma)$ is a product of Ehrhart polynomials

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{A}_\Sigma); n) = \mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{A}_{\Sigma_1}); n) \cdot \ldots \cdot \mathrm{ehr}_{\mathbb{Z}}(\mathsf{Z}(\mathbf{A}_{\Sigma_c}); n).$$

A similar decomposition property can be found on the level of matroids. Here the signed graphical matroid $M(\Sigma)$ is the direct sum

$$M(\Sigma) = M(\Sigma_1) \oplus \cdots \oplus M(\Sigma_c).$$

This structure is preserved under taking matroid duals, hence

$$M^\triangle(\Sigma) = M^\triangle(\Sigma_1) \oplus \cdots \oplus M^\triangle(\Sigma_c).$$

So we can also apply our duality construction block by block to achieve a dual representation

$$\mathbf{D}_\Sigma = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{D}_c \end{bmatrix}.$$

Then the signed tocyclotope is the Cartesian product $\mathsf{Z}(\mathbf{D}_\Sigma) = \mathsf{Z}(\mathbf{D}_1) \times \cdots \times \mathsf{Z}(\mathbf{D}_c)$, and hence its Ehrhart polynomial is again a product of Ehrhart polynomials

$$\mathrm{ehr}_\mathbb{Z}(\mathsf{Z}(\mathbf{D}_\Sigma); n) = \mathrm{ehr}_\mathbb{Z}(\mathsf{Z}(\mathbf{D}_1); n) \cdot \ldots \cdot \mathrm{ehr}_\mathbb{Z}(\mathsf{Z}(\mathbf{D}_c); n)\,.$$

We conclude with a concrete open questions. Recall that the lattice points in the acyclotope (for unsigned graphs) arise as indegree vectors from all orientations of the graph. While this correspondence is bijective for acyclic orientations, it is not for general orientations. For tocyclotopes we know that the vertices correspond to totally cyclic orientations. Is there a similar interpretation for all lattice points in the tocyclotope? One way to address this question might be via the algorithm in [AK06].

5

Rational Ehrhart Theory

The aim of this chapter is to study Ehrhart counting functions with a real dilation parameter. However, as P is a rational polytope, it suffices to compute this counting function at certain rational arguments to fully understand it; we will (quantify and) make this statement precise shortly (Corollary 5.6 below). We define the **rational Ehrhart counting function**

$$\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda) \ := \ \left|\lambda\mathsf{P}\cap\mathbb{Z}^d\right| ,$$

where $\lambda\in\mathbb{Q}$. To the best of our knowledge, Linke [Lin11] initiated the study of the rational (and real) counting function from the Ehrhart viewpoint. She proved several fundamental results starting with the fact that $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$ is a **quasipolynomial** in the rational (equivalently, real) variable λ, that is,

$$\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda) \ = \ c_d(\lambda)\,\lambda^d + c_{d-1}(\lambda)\,\lambda^{d-1} + \cdots + c_0(\lambda)$$

where $c_0, c_1, \dots, c_d \colon \mathbb{Q}\to\mathbb{Q}$ are periodic functions. The least common period of $c_0(\lambda),\dots,c_d(\lambda)$ is the **period** of $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$. For $x\in\mathbb{R}$, let $\lfloor x\rfloor$ (resp. $\lceil x\rceil$) denote the largest integer $\le x$ (resp. the smallest integer $\ge x$), and $\{x\} := x-\lfloor x\rfloor$. Here is a first example, which we will revisit below:

$$\begin{aligned}
\operatorname{ehr}_{\mathbb{Q}}([1,2];\lambda) \ &= \ \lfloor 2\lambda\rfloor - \lceil\lambda\rceil + 1 \\
&= \begin{cases} n+1 & \text{if } \lambda = n \quad\quad\quad\quad\quad\quad \text{for some } n\in\mathbb{Z}_{>0}\,, \\ n & \text{if } n<\lambda<n+\frac{1}{2} \quad\quad\quad \text{for some } n\in\mathbb{Z}_{>0}\,, \\ n+1 & \text{if } n+\frac{1}{2}\le\lambda<n+1 \quad \text{for some } n\in\mathbb{Z}_{>0}\,. \end{cases}
\end{aligned}$$

Rearranging gives the quasipolynomial in the format of the definition:

$$\operatorname{ehr}_{\mathbb{Q}}([1,2];\lambda) \ = \ \operatorname{vol}([1,2])\lambda + c_0(\lambda) \ = \ \lambda + (\{\lambda\} - \{2\lambda\})\,.$$

Linke views the coefficient functions of the rational Ehrhart quasipolynomial as piecewise-defined polynomials, which allows her, among many other things, to establish differential equations relating the coefficient functions. Essentially concurrently, Baldoni–Berline–Köppe–Vergne [Bal+13], inspired by [Bar06], developed an algorithmic theory of **intermediate sums** for polyhedra, which includes $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$ as a special case. We also mention more recent work of Royer [Roy17a; Roy17b], which, among many other things, also studies rational Gorenstein polytopes (see below).

Our goal is to add a generating-function viewpoint to [Bal+13; Lin11], one that is inspired by [Sta08; Sta17]. To set it up, we need to make a definition. Suppose the rational d-polytope $\mathsf{P} \subseteq \mathbb{R}^d$ is given by the irredundant half-space description

$$\mathsf{P} \;=\; \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}, \tag{5.1}$$

where $\mathbf{A} \in \mathbb{Z}^{f \times d}$ and $\mathbf{b} \in \mathbb{Z}^f$ such that the greatest common divisor of b_i and the entries in the ith row of $\mathbf{A}$ equals 1, for every $i \in \{1, \dots, n\}$.[1] We define the **codenominator** r of P to be the least common multiple of the nonzero entries of the right hand side $\mathbf{b}$:

$$r \;:=\; \operatorname{lcm}(\mathbf{b})\,.$$

As we assume that P is full dimensional, the codenominator is well-defined. Our nomenclature arises from determining r using polar duality, as follows. Recall that P° denotes the relative interior of P. Recall that for a rational polytope $\mathsf{P} \subseteq \mathbb{R}^d$ such that $\mathbf{0} \in \mathsf{P}^\circ$, the **polar dual polytope** is $\mathsf{P}^\vee := \{\mathbf{x} \in (\mathbb{R}^d)^\vee : \langle \mathbf{x}, \mathbf{y} \rangle \ge -1 \text{ for all } \mathbf{y} \in \mathsf{P}\}$, and the codenominator $r = \min\{q \in \mathbb{Z}_{>0} : q\,\mathsf{P}^\vee \text{ is a lattice polytope}\}$; see, e.g., [Bar02].

We will see in Section 5.1 that $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ is fully determined by evaluations at rational numbers with denominator $2r$ (see Corollary 5.6 below for details); if $\mathbf{0} \in \mathsf{P}$ then we actually need to know only evaluations at rational numbers with denominator r. Thus we associate two generating series to the rational Ehrhart counting function, the **rational Ehrhart series**, to a full-dimensional rational polytope P with codenominator r:

$$\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P}; t) \;:=\; 1 + \sum_{n \in \mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}\!\left(\mathsf{P}; \frac{n}{r}\right) t^{\frac{n}{r}}$$

and the **refined rational Ehrhart series**

$$\operatorname{Ehr}_{\mathbb{Q}}^{\mathsf{ref}}(\mathsf{P}; t) \;:=\; 1 + \sum_{n \in \mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}\!\left(\mathsf{P}; \frac{n}{2r}\right) t^{\frac{n}{2r}}\,.$$

Continuing our comment above, we typically study $\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P}; t)$ for polytopes such that $\mathbf{0} \in \mathsf{P}$, and $\operatorname{Ehr}_{\mathbb{Q}}^{\mathsf{ref}}(\mathsf{P}; t)$ for polytopes such that $\mathbf{0} \notin \mathsf{P}$. Our first main result is as follows.

Theorem 5.7. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathsf{P}$ is a lattice polytope. Then*

$$\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P}; t) \;=\; \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; m)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}}$$

where $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; m)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with nonnegative integral coefficients. Consequently, $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ is a quasipolynomial and the period of $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ divides $\frac{m}{r}$, i.e., this period is of the form $\frac{j}{r}$ with $j \mid m$.

1 If P is a *lattice* polytope then we do not need to include b_i in this gcd condition.

From this we recover Linke's result [Lin11, Corollary 1.4] that $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$ is a quasipolynomial with period dividing q, where q is the smallest positive rational number such that $q\mathsf{P}$ is a lattice polytope.

Section 5.1 contains structural theorems about these generating functions: rationality and its consequences for the quasipolynomial $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$ (Theorem 5.7 and Theorem 5.13), nonnegativity (Corollary 5.12), connections to the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial in classical Ehrhart theory (Corollary 5.15), and combinatorial reciprocity theorems (Corollary 5.17 and Corollary 5.18).

One can find a precursor of sorts to our generating functions $\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P};t)$ and $\mathrm{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t)$ in work by Stapledon [Sta08; Sta17], and in fact this work was our initial motivation to look for and study rational Ehrhart generating functions. We explain the connection of [Sta17] to our work in Section 5.2. In particular, we deduce that in the case $\mathbf{0} \in \mathsf{P}^\circ$ the generating function $\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P};t)$ exhibits additional symmetry (Corollary 5.28).

A $(d+1)$-dimensional, pointed, rational cone $\mathsf{C} \subseteq \mathbb{R}^{d+1}$ is called **Gorenstein** if there exists a point $(p_0,\mathbf{p}) \in \mathsf{C} \cap \mathbb{Z}^{d+1}$ such that $\mathsf{C}^\circ \cap \mathbb{Z}^{d+1} = (p_0,\mathbf{p}) + \mathsf{C} \cap \mathbb{Z}^{d+1}$ (see, e.g., [BB97; BR07; Sta78]). The point $(p_0,\mathbf{p})$ is called the **Gorenstein point** of the cone. Recall that the **homogenization** $\mathrm{hom}(\mathsf{P}) \subseteq \mathbb{R}^{d+1}$ of a rational polytope $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ is defined as

$$\mathrm{hom}(\mathsf{P}) \ := \ \mathrm{cone}(\{1\} \times \mathsf{P}) \ := \ \left\{(x_0,\mathbf{x}) \in \mathbb{R}^{d+1} : \ \mathbf{A}\mathbf{x} \le x_0\mathbf{b}\,, \ x_0 \ge 0\right\}$$

and that for a cone $C \subseteq \mathbb{R}^{d+1}$, the **polar dual cone** $C^\vee \subseteq (\mathbb{R}^{d+1})^\vee$ is

$$C^\vee \ := \ \left\{(y_0,\mathbf{y}) \in (\mathbb{R}^{d+1})^\vee : \ \langle (y_0,\mathbf{y}),(x_0,\mathbf{x})\rangle \ge 0 \text{ for all } (x_0,\mathbf{x}) \in C\right\}.$$

Another equivalent way of defining **Gorenstein** for a lattice polytope $\mathsf{P} \subseteq \mathbb{R}^d$ (compare Theorem 2.8) is that the homogenization $\mathrm{hom}(\mathsf{P})$ of P is a Gorenstein cone; in the special case where the Gorenstein point of that cone is $(1,\mathbf{q})$, for some $\mathbf{q} \in \mathbb{Z}^d$, we call P **reflexive** [Bat94; Hib92]. Recall that reflexive polytopes can alternatively be characterized as those lattice polytopes (containing the origin) whose polar duals are also lattice polytopes, i.e., they have codenominator 1. This definition has a natural extension to rational polytopes [FK08]. Gorenstein and reflexive polytopes (and their rational versions) play an important role in Ehrhart theory, as they have palindromic $\mathrm{h}^*_{\mathbb{Z}}$-polynomials. In Section 5.3 we give the analogous result in rational Ehrhart theory *without* reference to the polar dual:

Theorem 5.29. *Let $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ be a rational d-polytope with codenominator r and $\mathbf{0} \in \mathsf{P}$, as in Equation (5.1) and Equation (5.6) Then the following are equivalent for $g, m \in \mathbb{Z}_{\ge 1}$ and $\frac{m}{r}\mathsf{P}$ a lattice polytope:*

(i) P is r-rational Gorenstein with Gorenstein point $(g,\mathbf{y}) \in \mathrm{hom}(\frac{1}{r}\mathsf{P})$.

(ii) There exists a (necessarily unique) integer solution $(g,\mathbf{y})$ to

$$\begin{aligned} -\langle \mathbf{a}_j,\mathbf{y}\rangle &= 1 \qquad \text{for } j = 1,\dots,i \\ b_j\,g - r\,\langle \mathbf{a}_j,\mathbf{y}\rangle &= b_j \qquad \text{for } j = i+1,\dots,n\,. \end{aligned}$$

(iii) $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)$ *is palindromic:*

$$t^{(d+1)\frac{m}{r}-\frac{g}{r}}\,\mathrm{h}^*_{\mathbb{Q}}\Big(\mathsf{P};\frac{1}{t};m\Big) \;=\; \mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)\,.$$

(iv) $(-1)^{d+1}t^{\frac{g}{r}}\,\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P};t) = \mathrm{Ehr}_{\mathbb{Q}}\Big(\mathsf{P};\frac{1}{t}\Big)$.

(v) $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P};\frac{n}{r}) = \mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ;\frac{n+g}{r})$ *for all* $n\in\mathbb{Z}_{\geq 0}$.

(vi) $\mathrm{hom}(\frac{1}{r}\mathsf{P})^\vee$ *is the cone over a lattice polytope, i.e., there exists a lattice point* $(g,\mathbf{y})\in\mathrm{hom}(\frac{1}{r}\mathsf{P})^\circ\cap\mathbb{Z}^{d+1}$ *such that for every primitive ray generator* $(v_0,\mathbf{v})$ *of* $\mathrm{hom}(\frac{1}{r}\mathsf{P})^\vee$

$$\langle (g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle \;=\; 1\,.$$

The equivalence of (i) and (vi) is well known (see, e.g., [BN08, Definition 1.8] or [BG09, Exercises 2.13 and 2.14]). We will see that there are many more *rational* Gorenstein polytopes than among lattice polytopes; e.g., any rational polytope containing the origin in its interior is rational Gorenstein (Corollary 5.30).

We mention the recent notion of an l-reflexive polytope P ("reflexive of higher index") [KN12]. A lattice point $\mathbf{x}\in\mathbb{Z}^d$ is **primitive** if the gcd of its coordinates is equal to one. The **l-reflexive polytopes** are precisely the lattice polytopes of the form Equation (5.1) with $\mathbf{b}=(l,l,\dots,l)$ and primitive vertices; note that this means P has codenominator l and $\frac{1}{l}\mathsf{P}$ has denominator l.

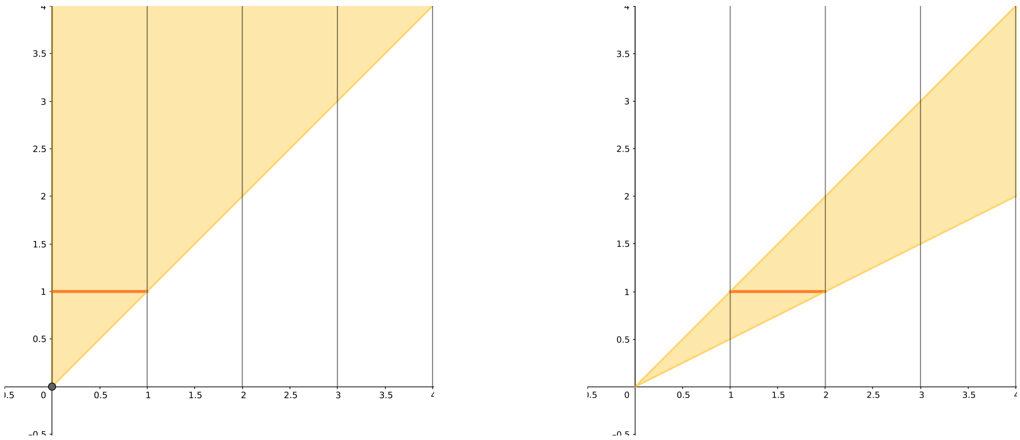
We conclude with two short sections further connecting our work to the existing literature. Section 5.4 exhibits how one can deduce a theorem of Betke–McMullen [BM85] (and also its rational analogue [BBV22]) from rational Ehrhart theory.

Ehrhart's theorem (Theorem 2.2) gives an upper bound for the period of the quasipolynomial $\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P};n)$, namely, the denominator of P. When the period of $\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P};n)$ is smaller than the denominator of P, we speak of **period collapse**. One can witness this phenomenon most easily in the Ehrhart series, as period collapse means that the rational function expression of the Ehrhart series (see Equation (2.14)) factors in such a way that one realizes there are no nontrivial roots of unity that are poles. It is an interesting question whether/how much period collapse happens in rational Ehrhart theory, and how it compares to the classical scenario. In Section 5.5, we offer some data points for period collapse for both rational and classical Ehrhart quasipolynomials.

This chapter is joint work with Matthias Beck and Sophia Elia [BER23].

5.1 Rational Ehrhart Dilations

We assume throughout this chapter that all polytopes are full-dimensional, and call a d-dimensional polytope in $\mathbb{R}^d$ a **d-polytope**. Recall that, consequently, the leading coefficient of $\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P};n)$ is constant (namely, the volume of P), and thus the rational generating function $\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P};t)$ has a unique pole of order $d+1$ at $t=1$. So we could write the rational generating function $\mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P};t)$ with denominator $(1-t)(1-t^k)^d$; in other words, $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t)$ always has a factor $(1+t+\cdots+t^{k-1})$. Recall, for $\lambda\in\mathbb{R}$,



(a) $\hom(\mathsf{P}_0) = \hom([0,1])$ (b) $\hom(\mathsf{P}_1) = \hom([1,2]) = \hom(\mathsf{P}_0 + 1)$

Figure 5.1: The rational Ehrhart counting function is not invariant under lattice translation: $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}_0;\lambda)$ is monotone, $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}_1;\lambda)$ is not monotone. See Example 5.1.

let $\lfloor\lambda\rfloor$ (resp. $\lceil\lambda\rceil$) denote the largest integer $\le \lambda$ (resp. the smallest integer $\ge \lambda$), and $\{\lambda\} = \lambda - \lfloor\lambda\rfloor$.

Example 5.1. We feature the following line segments as running examples. First, we compute the rational Ehrhart counting function.

(0) $\mathsf{P}_0 = [0,1]$, codenominator $r = 2$,

$$\begin{aligned}\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}_0;\lambda) &= \lfloor\lambda\rfloor + 1 \\ &= n+1 \qquad \text{if } n \le \lambda < n+1 \quad \text{for some } n \in \mathbb{Z}_{>0}\,.\end{aligned}$$

See Figure 5.1(a).

(i) $\mathsf{P}_1 := \left[-1, \frac{2}{3}\right]$, codenominator $r = 2$,

$$\begin{aligned}\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}_1;\lambda) &= \lceil\lambda\rceil + \left\lceil\tfrac{2}{3}\lambda\right\rceil + 1 \\ &= \begin{cases} \frac{5}{3}n+1 & \text{if } n \le \lambda < n+\frac{1}{2} & \text{for some } n \in 3\mathbb{Z}_{>0}\,, \\ \frac{5}{3}n+1 & \text{if } n+\frac{1}{2} \le \lambda < n+1 & \text{for some } n \in 3\mathbb{Z}_{>0}\,, \\ \frac{5}{3}n+2 & \text{if } n+1 \le \lambda < n+\frac{3}{2} & \text{for some } n \in 3\mathbb{Z}_{>0}\,, \\ \frac{5}{3}n+3 & \text{if } n+\frac{3}{2} \le \lambda < n+2 & \text{for some } n \in 3\mathbb{Z}_{>0}\,, \\ \frac{5}{3}n+4 & \text{if } n+2 \le \lambda < n+\frac{5}{2} & \text{for some } n \in 3\mathbb{Z}_{>0}\,, \\ \frac{5}{3}n+4 & \text{if } n+\frac{5}{2} \le \lambda < n+3 & \text{for some } n \in 3\mathbb{Z}_{>0}\,. \end{cases}\end{aligned}$$

(ii) $\mathsf{P}_2 := \left[0, \frac{2}{3}\right]$, codenominator $r = 2$,

$$\begin{aligned}\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}_2;\lambda) &= \left\lfloor\tfrac{2}{3}\lambda\right\rfloor + 1 \\ &= \tfrac{2}{3}n+1 \quad \text{if } n \le \lambda < n+\tfrac{3}{2} \quad \text{for some } n \in \tfrac{3}{2}\mathbb{Z}_{>0}\,.\end{aligned}$$

(iii) $\mathsf{P}_3 := [1,2]$, codenominator $r = 2$,

$$\begin{aligned}
\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}_3;\lambda) &= \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1 \\
&= \begin{cases} n+1 & \text{if } \lambda = n & \text{for some } n \in \mathbb{Z}_{>0}\,, \\ n & \text{if } n < \lambda < n + \frac{1}{2} & \text{for some } n \in \mathbb{Z}_{>0}\,, \\ n+1 & \text{if } n + \frac{1}{2} \le \lambda < n+1 & \text{for some } n \in \mathbb{Z}_{>0}\,. \end{cases}
\end{aligned}$$

See Figure 5.1(b).

(iv) $\mathsf{P}_4 := 2\mathsf{P}_3 = [2,4]$, codenominator $r = 4$,

$$\begin{aligned}
\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}_4;\lambda) &= \lfloor 4\lambda \rfloor - \lceil 2\lambda \rceil + 1 \;=\; \lfloor 4\lambda \rfloor + \lfloor -2\lambda \rfloor + 1 \\
&= 2\lambda + 1 - \{4\lambda\} + \{-2\lambda\} \\
&= \begin{cases} 2n+1 & \text{if } \lambda = n & \text{for some } n \in \frac{1}{2}\mathbb{Z}_{>0}\,, \\ 2n & \text{if } n < \lambda < n + \frac{1}{4} & \text{for some } n \in \frac{1}{2}\mathbb{Z}_{>0}\,, \\ 2n+1 & \text{if } n + \frac{1}{4} \le \lambda < n+\frac{1}{2} & \text{for some } n \in \frac{1}{2}\mathbb{Z}_{>0}\,. \end{cases}
\end{aligned}$$

Remark 5.2. If P is a lattice polytope, then the denominator of $\frac{1}{r}\mathsf{P}$ divides r. On the other hand, the denominator of $\frac{1}{r}\mathsf{P}$ need not equal r, as can be seen in the case of P_4 above.

Remark 5.3. If $\frac{1}{r}\mathsf{P}$ is a lattice polytope, its Ehrhart polynomial is invariant under lattice translations. Unfortunately, this does not clearly translate to invariance of $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$, as Linke already noted. Consider the line segment $[-1,1]$ and its translation $\mathsf{P}_4 = [2,4]$. For any $\lambda \in (0,\frac{1}{4})$, we have $\operatorname{ehr}_{\mathbb{Q}}([-1,1];\lambda) = 1$ and $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}_4;\lambda) = 0$. This observation raises the following two related questions. First, is there an example of a polytope and a translate with the same codenominator? We expect the answer is "no" in dimension one. Second, given a rational polytope P, for which r and $\tilde{\mathsf{P}}$ could $\mathsf{P} = \frac{1}{r}\tilde{\mathsf{P}}$? Royer shows in [Roy17a] that for every rational polytope P there is a integral translation vector $\mathbf{v}$ such that the functions $\operatorname{ehr}_{\mathbb{Q}}(k\mathbf{v}+\mathsf{P};\lambda)$ are all distinct for $k \in \mathbb{Z}_{\ge 0}$. Moreover, polytopes can be uniquely identified by knowing the rational Ehrhart counting function for each integral translate of the polytope.

Lemma 5.4. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope. If $\mathbf{0} \in \mathsf{P}$, then $\operatorname{ehr}_{\mathbb{Q}}(\lambda)$ is monotone for $\lambda \in \mathbb{Q}_{\ge 0}$.*

Proof. Let $\lambda < \omega$ be positive rationals. Suppose $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{x} \in \lambda\mathsf{P}$. Then $\mathbf{x}$ satisfies all n facet-defining inequalities of $\lambda\mathsf{P}$: $\langle \mathbf{a}_i, \mathbf{x} \rangle \le \lambda b_i$ for all $i \in [n]$. If $b_i = 0$, then $\langle \mathbf{a}_i, \mathbf{x} \rangle \le \lambda \cdot 0 = \omega \cdot 0$. Otherwise, $b_i > 0$, and $\langle \mathbf{a}_i, \mathbf{x} \rangle \le \lambda b_i < \omega b_i$. So $\mathbf{x} \in \omega\mathsf{P}$. □

Proposition 5.5. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r.*

(i) *The number of lattice points in $\lambda\mathsf{P}$ is constant for $\lambda \in (\frac{n}{r}, \frac{n+1}{r})$, $n \in \mathbb{Z}_{\ge 0}$.*

(ii) *If $\mathbf{0} \in \mathsf{P}$, then the number of lattice points in $\lambda\mathsf{P}$ is constant for $\lambda \in [\frac{n}{r}, \frac{n+1}{r})$, $n \in \mathbb{Z}_{\ge 0}$.*

Proof. (i). Suppose there exist two rationals λ and ω such that $\frac{n}{r} < \lambda < \omega < \frac{n+1}{r}$, and $\mathrm{ehr}_{\mathbb{Q}}(\lambda) \neq \mathrm{ehr}_{\mathbb{Q}}(\omega)$. Then there exists $\mathbf{x} \in \mathbb{Z}^d$ such that either ($\mathbf{x} \in \omega\mathsf{P}$ and $\mathbf{x} \notin \lambda\mathsf{P}$) or ($\mathbf{x} \in \lambda\mathsf{P}$ and $\mathbf{x} \notin \omega\mathsf{P}$). Suppose ($\mathbf{x} \in \omega\mathsf{P}$ and $\mathbf{x} \notin \lambda\mathsf{P}$). Then there exists a facet F with integral, reduced inequality $\langle \mathbf{a}, \mathbf{v} \rangle \leq b$ of P such that

$$\langle \mathbf{a}, \mathbf{x} \rangle \leq \omega b, \quad \langle \mathbf{a}, \mathbf{x} \rangle > \lambda b, \text{ and } \quad \langle \mathbf{a}, \mathbf{x} \rangle \in \mathbb{Z}\,.$$

As $\lambda < \omega$, this implies $b > 0$. We have

$$b\frac{n}{r} < \lambda b < \langle \mathbf{a}, \mathbf{x} \rangle \leq \omega b < \frac{n+1}{r} b.$$

As $r = bk$, with $k \in \mathbb{Z}_{>0}$, this is equivalent to

$$n < \lambda r < k\langle \mathbf{a}, \mathbf{x} \rangle \leq \omega r < n + 1. \tag{5.2}$$

This is a contradiction because $k\langle \mathbf{a}, \mathbf{x} \rangle$ is an integer. The second case is proved analogously: Assume ($\mathbf{x} \notin \omega\mathsf{P}$ and $\mathbf{x} \in \lambda\mathsf{P}$). Then there exists again a facet F with integral, reduced inequality $\langle \mathbf{a}, \mathbf{v} \rangle \leq b$ of P such that

$$\langle \mathbf{a}, \mathbf{x} \rangle > \omega b, \quad \langle \mathbf{a}, \mathbf{x} \rangle \leq \lambda b, \text{ and } \quad \langle \mathbf{a}, \mathbf{x} \rangle \in \mathbb{Z}.$$

As $\lambda < \omega$, this implies $b < 0$. We have

$$\frac{n+1}{r}|b| > \omega|b| > -\langle \mathbf{a}, \mathbf{x} \rangle \geq \lambda|b| > \frac{n}{r}|b|\,.$$

As $\frac{r}{|b|} \in \mathbb{Z}_{>0}$, this is equivalent to

$$n + 1 > \omega r > -\frac{r}{|b|}\langle \mathbf{a}, \mathbf{x} \rangle \geq \lambda r > n\,. \tag{5.3}$$

This leads to the same contradiction.

(ii) If $\mathbf{0} \in \mathsf{P}$ we know that $\mathbf{b} \geq \mathbf{0}$. So in the proof above only the first case applies. (This can also be seen as a consequence of Lemma 5.4.) Allowing $\frac{n}{r} \leq \lambda$ leads, with the same computations, to the following weakened version of Equation (5.2):

$$n \leq \lambda r < k\langle \mathbf{a}, \mathbf{x} \rangle \leq \omega r < n + 1\,,$$

which is still strong enough for the contradiction. Note that this is not the case in Equation (5.3). $\square$

We define the **real Ehrhart counting function**

$$\mathrm{ehr}_{\mathbb{R}}(\mathsf{P}; \lambda) \;:=\; \left|\lambda\mathsf{P} \cap \mathbb{Z}^d\right|,$$

for $\lambda \in \mathbb{R}$. It follows that we can compute the real Ehrhart function $\mathrm{ehr}_{\mathbb{R}}$ from the rational Ehrhart function $\mathrm{ehr}_{\mathbb{Q}}$:

Corollary 5.6. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r. Then*

$$\operatorname{ehr}_{\mathbb{R}}(\mathsf{P};\lambda) \;=\; \begin{cases} \operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda) & \text{if } \lambda \in \frac{1}{r}\mathbb{Z}_{\geq 0}\,, \\ \operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lfloor\lambda\rceil) & \text{if } \lambda \notin \frac{1}{r}\mathbb{Z}_{\geq 0}\,, \end{cases} \tag{5.4}$$

where

$$\lfloor\lambda\rceil := \frac{2j+1}{2r} \quad \text{for} \quad \left|\lambda - \frac{2j+1}{2r}\right| < \frac{1}{2r} \quad \text{and} \quad j \in \mathbb{Z}\,.$$

In words, $\lfloor\lambda\rceil$ is the element in $\frac{1}{2r}\mathbb{Z}$ with odd numerator that has the smallest Euclidean distance to λ on the real line. Furthermore, if $\mathbf{0} \in \mathsf{P}$ then

$$\operatorname{ehr}_{\mathbb{R}}(\mathsf{P};\lambda) \;=\; \operatorname{ehr}_{\mathbb{Q}}\left(\mathsf{P};\frac{\lfloor r\lambda\rfloor}{r}\right).$$

In light of this Corollary, any statement about the rational Ehrhart counting function $\operatorname{ehr}_{\mathbb{Q}}(\lambda)$ in this paper generalizes to the real Ehrhart counting function $\operatorname{ehr}_{\mathbb{R}}(\lambda)$ and we omit the latter versions for simplicity. We proceed to prove one of the main results.

Theorem 5.7. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathsf{P}$ is a lattice polytope. Then*

$$\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P};t) \;:=\; \sum_{n\in\mathbb{Z}_{\geq 0}} \operatorname{ehr}_{\mathbb{Q}}\left(\mathsf{P};\frac{n}{r}\right)t^{\frac{n}{r}} \;=\; \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}}$$

where $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with nonnegative integral coefficients. Consequently, the rational Ehrhart counting function $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$ is a quasipolynomial and the period of $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$ divides $\frac{m}{r}$, i.e., this period is of the form $\frac{j}{r}$ with $j \mid m$.

Proof. Our conditions imply that $\frac{1}{r}\mathsf{P}$ is a rational polytope with denominator dividing m. Thus by standard Ehrhart theory,

$$\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P};t) \;=\; \operatorname{Ehr}_{\mathbb{Z}}\left(\tfrac{1}{r}\mathsf{P};t^{\frac{1}{r}}\right) \;=\; \frac{\mathrm{h}^*_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P};t^{\frac{1}{r}}\right)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}}\,,$$

and $\mathrm{h}^*_{\mathbb{Z}}(\frac{1}{r}\mathsf{P};t)$ has nonnegative integral coefficients. □

Remark 5.8. Our implicit definition of $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t)$ depends on m. We will sometimes use the notation $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)$ to make this dependency explicit. Naturally, one often tries to choose m minimal, which gives a canonical definition of $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t)$, but sometimes it pays to be flexible.

Remark 5.9. Via Corollary 5.6, $\operatorname{ehr}_{\mathbb{R}}(\mathsf{P};\lambda)$ is a quasipolynomial and the period of $\operatorname{ehr}_{\mathbb{R}}(\mathsf{P};\lambda)$ divides $\frac{m}{r}$, i.e., this period is of the form $\frac{j}{r}$ with $j \mid m$.

Remark 5.10. By usual generatingfunctionology [Wil94], the degree of $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)$ is less than or equal to $m(d+1)-1$ as a polynomial in $t^{\frac{1}{r}}$.

We also recover the following result of Linke [Lin11].

Corollary 5.11. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathsf{P}$ is a lattice polytope. Then the period of the quasipolynomial $\operatorname{ehr}_{\mathbb{Z}}(\mathsf{P};\lambda)$ divides $\frac{m}{\gcd(m,r)}$.*

Proof. Viewed as a function of the integer parameter n, the function $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};\frac{n}{r})$ has period dividing m. Thus $\operatorname{ehr}_{\mathbb{Z}}(\mathsf{P};n) = \operatorname{ehr}_{\mathbb{Q}}(\mathsf{P};n)$ has period dividing $\frac{m}{\gcd(m,r)}$. □

Corollary 5.12. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r. Then*

$$\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P};t) \;=\; \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;r)}{(1-t)^{d+1}}$$

where $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;r)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with nonnegative coefficients.

For polytopes that do not contain the origin, the following variant of Theorem 5.7 is useful.

Theorem 5.13. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{2r}\mathsf{P}$ is a lattice polytope. Then*

$$\operatorname{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t) \;:=\; 1 + \sum_{n\in\mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}\left(\mathsf{P};\frac{n}{2r}\right) t^{\frac{n}{2r}} \;=\; \frac{\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t;m)}{\left(1-t^{\frac{m}{2r}}\right)^{d+1}}$$

where $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t;m)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{2r}}]$ with nonnegative coefficients.

The proof of Theorem 5.13 is virtually identical to that of Theorem 5.7. Similarly, many of the following assertions come in two versions, one for $\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P};t)$ and one for $\operatorname{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t)$. We typically write an explicit proof for only one version, as the other is analogous.

We recover another result of Linke [Lin11].

Corollary 5.14. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope. The rational Ehrhart function, $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P},\lambda)$, is given by a quasipolynomial of period 1.*

Corollary 5.15. *If $\frac{m}{r}$ (resp. $\frac{m}{2r}$) in Theorem 5.7 (resp. Theorem 5.13) is integral we can retrieve the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial from the $\mathrm{h}^*_{\mathbb{Q}}$-polynomial (resp. $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}$-polynomial) by applying the operator* Int *that extracts from a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ the terms with integer powers of t: $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t) = \operatorname{Int}(\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t))$ (resp. $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t) = \operatorname{Int}(\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t))$).*

Example 5.16 (continued)**.** Here are the (refined) rational Ehrhart series of the running examples. Recall that the rational Ehrhart series of P in the variable t can be computed as the Ehrhart series of $\frac{1}{r}\mathsf{P}$ in the variable $t^{\frac{1}{r}}$ (resp. the refined rational Ehrhart as the Ehrhart series of $\frac{1}{2r}\mathsf{P}$ in the variable $t^{\frac{1}{2r}}$).

(i) $\mathsf{P}_1 := [-1,\frac{2}{3}]$, $r=2$, $m=6$,

$$\begin{aligned}\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P}_1;t) &= \frac{1+t^{\frac{1}{2}}+t+t^{\frac{3}{2}}+t^2}{(1-t)\left(1-t^{\frac{3}{2}}\right)} \\ &= \frac{1+t^{\frac{1}{2}}+2t+3t^{\frac{3}{2}}+4t^2+4t^{\frac{5}{2}}+4t^3+4t^{\frac{7}{2}}+3t^4+2t^{\frac{9}{2}}+t^5+t^{\frac{11}{2}}}{(1-t^3)^2}.\end{aligned}$$

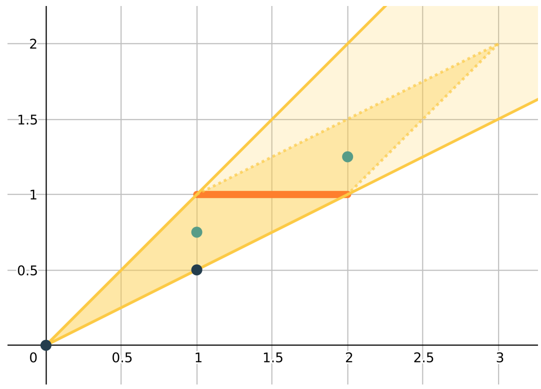


Figure 5.2: The cone $\hom(\mathsf{P}_3)$ over $\mathsf{P}_3 = [1,2]$. The lattice points in the fundamental parallelepiped with respect to the lattice $\frac{1}{4}\mathbb{Z} \times \mathbb{Z}$ are $(0,0)$, $(\frac{1}{2},1)$, $(\frac{3}{4},1)$, $(\frac{5}{4},2)$.

(ii) $\mathsf{P}_2 := [0, \frac{2}{3}]$, $r = 2$, $m = 3$,

$$\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}_2; t) \;=\; \frac{1}{\left(1-t^{\frac{1}{2}}\right)\left(1-t^{\frac{3}{2}}\right)} = \frac{1+t^{\frac{1}{2}}+t}{\left(1-t^{\frac{3}{2}}\right)^2}.$$

(iii) $\mathsf{P}_3 := [1,2]$, $r = 2$. $\frac{1}{4}\mathsf{P}_3 = [\frac{1}{4}, \frac{1}{2}]$ and $m = 4$, so $\frac{m}{2r} = 1$. See Figure 5.2.

$$\mathrm{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}(P_3; t) \;=\; \frac{1+t^{\frac{1}{2}}+t^{\frac{3}{4}}+t^{\frac{5}{4}}}{(1-t)^2} \;=\; \frac{\left(1+t^{\frac{3}{4}}\right)\left(1+t^{\frac{1}{2}}\right)}{(1-t)^2}.$$

(iv) $\mathsf{P}_4 := [2,4]$, $r = 4$. Then $\frac{1}{8}\mathsf{P}_4 = [\frac{1}{4}, \frac{1}{2}]$ and $m = 4$, so $\frac{m}{2r} = \frac{1}{2}$. See Figure 5.3.

$$\mathrm{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}_4; t) = \frac{1+t^{\frac{1}{4}}+t^{\frac{3}{8}}+2t^{\frac{1}{2}}+t^{\frac{5}{8}}+2t^{\frac{3}{4}}+2t^{\frac{7}{8}}+t+2t^{\frac{9}{8}}+t^{\frac{5}{4}}+t^{\frac{11}{8}}+t^{\frac{13}{8}}}{(1-t)^2}$$

$$= \frac{1+t^{\frac{1}{4}}+t^{\frac{3}{8}}+t^{\frac{5}{8}}}{(1-t^{\frac{1}{2}})^2}.$$

Choosing m to be minimal means $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}_4; t; 4) = (1+t^{\frac{3}{8}})(1+t^{\frac{1}{4}}) = 1+t^{\frac{1}{4}}+t^{\frac{3}{8}}+t^{\frac{5}{8}} = \mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}\left(\mathsf{P}_3; t^{\frac{1}{2}}; 4\right)$. The rational Ehrhart counting function agrees with a quasipolynomial for $\lambda \in \frac{1}{2r}\mathbb{Z}$.

From the (refined) rational Ehrhart series of these examples, we can recompute the quasipolynomials found earlier. For example, for P_3:

$$\begin{aligned}\mathrm{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}_3; t) &= \frac{1+t^{\frac{1}{2}}+t^{\frac{3}{4}}+t^{\frac{5}{4}}}{(1-t)^2} = \left(1+t^{\frac{1}{2}}+t^{\frac{3}{4}}+t^{\frac{5}{4}}\right)\sum_{j\ge 0}(j+1)\,t^j \\ &= \sum_{j\ge 0}(j+1)\,t^j + \sum_{j\ge 0}(j+1)\,t^{j+\frac{1}{2}} + \sum_{j\ge 0}(j+1)\,t^{j+\frac{3}{4}} + \sum_{j\ge 0}(j+1)\,t^{j+\frac{5}{4}}.\end{aligned}$$

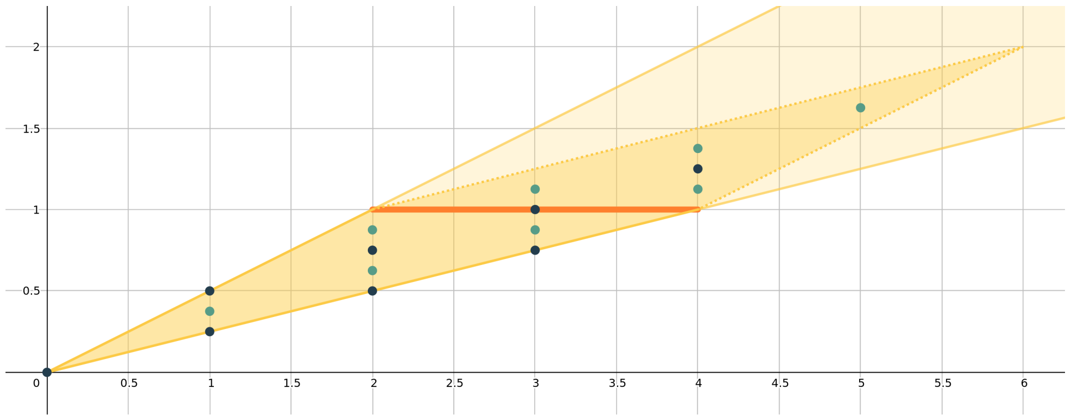


Figure 5.3: The cone $\hom(\mathsf{P}_4)$ over $\mathsf{P}_4 = [2,4]$. The lattice points in the fundamental parallelepiped with respect to the lattice $\frac{1}{8}\mathbb{Z} \times \mathbb{Z}$ are shown in the figure.

With a change of variables we compute for $\lambda \in \frac{1}{4}\mathbb{Z}$

$$\operatorname{ehr}_{\mathbb{Q}}(\lambda) \;=\; \begin{cases} \lambda + 1 & \text{if } \lambda \in \mathbb{Z}, \\ \lambda - \frac{1}{4} & \text{if } \lambda \equiv \frac{1}{4} \bmod 1, \\ \lambda + \frac{1}{2} & \text{if } \lambda \equiv \frac{1}{2} \bmod 1, \\ \lambda + \frac{1}{4} & \text{if } \lambda \equiv \frac{3}{4} \bmod 1. \end{cases}$$

Next we recover the reciprocity result for the rational Ehrhart function of rational polytopes proved by Linke [Lin11, Corollary 1.5].

Corollary 5.17. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope. Then $(-1)^d \operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; -\lambda)$ equals the number of interior lattice points in $\lambda\mathsf{P}$, for any $\lambda > 0$.*

Proof. Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r. The fact that $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda)$ is a quasipolynomial allows us to extend Equation (5.4) to the negative (and therefore all) rational numbers via

$$\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lambda) = \operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; \lfloor \lambda \rceil) \qquad \text{if } \lambda \notin \tfrac{1}{r}\mathbb{Z}\,.$$

By standard Ehrhart–Macdonald Reciprocity, $(-1)^d \operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}; -\frac{n}{2r}) = \operatorname{ehr}_{\mathbb{Z}}(\frac{1}{2r}\mathsf{P}; -n)$ equals the number of lattice points in the interior of $\frac{n}{2r}\mathsf{P}$. The result now follows from $\lfloor -\lambda \rceil = -\lfloor \lambda \rceil$. $\square$

Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope, let P° denote its interior and $\operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; \lambda) := |\lambda\mathsf{P}^\circ \cap \mathbb{Z}^d|$. We define the (refined) rational Ehrhart series of the interior of a polytope as follows:

$$\begin{aligned} \operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; t) \;&:=\; \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; \lambda)\, t^\lambda\,, \\ \operatorname{Ehr}_{\mathbb{Q}}^{\mathsf{ref}}(\mathsf{P}^\circ; t) \;&:=\; \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; \lambda)\, t^\lambda\,, \end{aligned}$$

where r as usual denotes the codenominator of P.

Corollary 5.18. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ be such that $\frac{m}{r}\mathsf{P}$ is a lattice polytope.*

(i) *The rational Ehrhart series of the open polytope P° has the rational expression*

$$\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; t) \;=\; \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}^\circ; t; m)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}}$$

where $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}^\circ; t; m)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$.

(ii) *The rational Ehrhart series fulfills the reciprocity relation*

$$\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; t) \;=\; (-1)^{d+1}\,\mathrm{Ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}\right).$$

(iii) *The $\mathrm{h}^*_{\mathbb{Q}}$-polynomial of the polytope P and its interior P° are related by*

$$\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}^\circ; t; m) \;=\; \left(t^{\frac{m}{r}}\right)^{d+1} \mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}; m\right).$$

Proof. Identity (i) follows from Ehrhart–Macdonald reciprocity (see, e.g., [BR15, Theorem 4.4]) and Remark 5.10:

$$\begin{aligned}
\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; t) \;&=\; \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; \lambda) t^\lambda \;=\; \sum_{n \in \mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}^\circ; n\right) t^{\frac{n}{r}} \;=\; \mathrm{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}^\circ; t^{\frac{1}{r}}\right) \\
&=\; (-1)^{d+1}\,\mathrm{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}; t^{-\frac{1}{r}}\right) \;=\; (-1)^{d+1}\,\frac{\mathrm{h}^*_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}; t^{-\frac{1}{r}}\right)}{\left(1 - t^{-\frac{m}{r}}\right)^{d+1}} \\
&=\; \frac{\left(t^{\frac{m}{r}}\right)^{d+1} \mathrm{h}^*_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}; t^{-\frac{1}{r}}\right)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}}.
\end{aligned}$$

For identities (ii) and (iii) we again apply Ehrhart–Macdonald reciprocity:

$$\begin{aligned}
\frac{\left(t^{\frac{m}{r}}\right)^{d+1} \mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}; m\right)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}} \;&=\; \frac{(-1)^{d+1}\,\mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}; m\right)}{\left(1 - \left(\frac{1}{t}\right)^{\frac{m}{r}}\right)^{d+1}} \;=\; (-1)^{d+1}\,\mathrm{Ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}\right) \\
&=\; (-1)^{d+1}\,\mathrm{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}; \frac{1}{t^{\frac{1}{r}}}\right) \;=\; \mathrm{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}^\circ; t^{\frac{1}{r}}\right) \\
&=\; \sum_{\lambda \in \mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}^\circ; \lambda\right) t^{\frac{\lambda}{r}} \;=\; \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P}^\circ; \frac{\lambda}{r}\right) t^{\frac{\lambda}{r}} \\
&=\; \mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; t) \;=\; \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}^\circ; t; m)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}}.
\end{aligned}$$

$\square$

As usual there is a refined version:

Corollary 5.19. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ be such that $\frac{m}{2r}\mathsf{P}$ is a lattice polytope.*

(i) *The refined rational Ehrhart series of the open polytope P° have the rational expressions*

$$\mathrm{Ehr}_{\mathbb{Q}}^{\mathsf{ref}}(\mathsf{P}^\circ; t) \;=\; \frac{\mathrm{h}_{\mathbb{Q}}^{*\mathsf{ref}}(\mathsf{P}^\circ; t; m)}{\left(1 - t^{\frac{m}{2r}}\right)^{d+1}},$$

where $\mathrm{h}_{\mathbb{Q}}^{*\mathsf{ref}}(\mathsf{P}^\circ; t; m)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{2r}}]$.

(ii) *The refined rational Ehrhart series fulfills the reciprocity relation*

$$\mathrm{Ehr}_{\mathbb{Q}}^{\mathsf{ref}}(\mathsf{P}^\circ; t) \;=\; (-1)^{d+1}\,\mathrm{Ehr}_{\mathbb{Q}}^{\mathsf{ref}}\left(\mathsf{P}; \frac{1}{t}\right).$$

(iii) *The $\mathrm{h}_{\mathbb{Q}}^{*\mathsf{ref}}$-polynomial of the polytope P and its interior P° are related by*

$$\mathrm{h}_{\mathbb{Q}}^{*\mathsf{ref}}(\mathsf{P}^\circ; t; m) \;=\; \left(t^{\frac{m}{2r}}\right)^{d+1}\,\mathrm{h}_{\mathbb{Q}}^{*\mathsf{ref}}\left(\mathsf{P}; \frac{1}{t}; m\right).$$

Remark 5.20. The **codegree** of a lattice polytope is defined as $\dim(\mathsf{P}) + 1 - \deg(h^*(t))$. Analogously, in the rational case, we define the **rational codegree of** $\mathrm{h}_{\mathbb{Q}}^*(\mathsf{P}; t; m)$ to be

$$\frac{m}{r}(\dim(\mathsf{P}) + 1) - \deg(\mathrm{h}_{\mathbb{Q}}^*(\mathsf{P}; t; m)),$$

where the degree of $\mathrm{h}_{\mathbb{Q}}^*(\mathsf{P}; t; m)$ is its (possibly fractional) degree as a polynomial in t. Likewise, the **rational codegree of** $\mathrm{h}_{\mathbb{Q}}^{*\mathsf{ref}}(\mathsf{P}; t; m)$ is defined as $\frac{m}{2r}(\dim(\mathsf{P}) + 1) - \deg(\mathrm{h}_{\mathbb{Q}}^{*\mathsf{ref}}(\mathsf{P}; t; m))$. As in the integral case, the rational codegree of $\mathrm{h}_{\mathbb{Q}}^*(\mathsf{P}; t; m)$ is the smallest integral dilate of $\frac{1}{r}\mathsf{P}$ containing interior lattice points. The proof requires no new insights and we omit it here.

5.2 Relation to Stapledon's work

We recall the setup from [Sta17]. Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r and $\mathbf{0} \in \mathsf{P}$. Let $\partial_{\neq 0}(\mathsf{P})$ denote the union of facets of P that do not contain the origin. In order to study all rational dilates of the boundary of P, Stapledon introduces the generating function

$$\mathrm{WEhr}(\mathsf{P}; t) \;:=\; 1 + \sum_{\lambda \in \mathbb{Q}_{>0}} \left|\partial_{\neq 0}(\lambda \mathsf{P}) \cap \mathbb{Z}^d\right| t^\lambda \;=\; \frac{\widetilde{\mathrm{h}}(\mathsf{P}; t)}{(1-t)^d}, \tag{5.5}$$

where $\widetilde{\mathrm{h}}(\mathsf{P}; t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with fractional exponents. The generating function WEhr is closely related to the (rational) Ehrhart series: the truncated sum $1 + \sum_{\lambda \in \mathbb{Q}>0}^{\omega} |\partial_{\neq 0}(\lambda \mathsf{P}) \cap \mathbb{Z}^d|$ equals the number of lattice points in $\omega\mathsf{P}$. Proposition 5.5 allows us to discretize this sum:

Corollary 5.21. *Let* $\mathsf{P} \subseteq \mathbb{R}^d$ *be a lattice d-polytope with codenominator* r *and* $\mathbf{0} \in \mathsf{P}$. *The number of lattice points in* $\lambda\mathsf{P}$ *equals* $1 + \sum_{\omega \in \frac{1}{r}\mathbb{Z}_{>0}, \omega < \lambda} |\partial_{\neq 0}(\omega\mathsf{P}) \cap \mathbb{Z}^d|$.

Proof. As $\mathbf{0} \in \mathsf{P}$, every nonzero lattice point in $\lambda\mathsf{P}$ occurs in $\partial_{\neq 0}(\omega\mathsf{P})$ for some unique $\omega \in \mathbb{Q}$ where $0 < \omega \leq \lambda$. Using Lemma 5.4,

$$\lambda\mathsf{P} \cap \mathbb{Z}^d \;=\; \mathbf{0} \cup \bigsqcup_{\omega \in \mathbb{Q}_{>0}}^{\lambda} \left(\partial_{\neq 0}(\omega\mathsf{P}) \cap \mathbb{Z}^d\right).$$

By Proposition 5.5, the union $\bigsqcup_{\omega \in \mathbb{Q}_{>0}}^{\lambda} (\partial_{\neq 0}(\omega\mathsf{P}) \cap \mathbb{Z}^d)$ is discrete and disjoint. □

Similarly, the polynomial $\widetilde{\mathrm{h}}(\mathsf{P};t)$ is related to $\mathrm{h}^*_{\mathbb{Z}}(\frac{1}{r}\mathsf{P}; t^{\frac{1}{r}})$ and to $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)$, as we show in Lemma 5.22 and Corollary 5.25. Recall that we use $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)$ to keep track of the denominator of $\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P};t) = \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)}{(1-t^{\frac{m}{r}})^{d+1}}$.

Lemma 5.22. *Let* $\mathsf{P} \subseteq \mathbb{R}^d$ *be a lattice d-polytope with codenominator* r *such that* $\mathbf{0} \in \mathsf{P}$. *Let* k *be the denominator of* $\frac{1}{r}\mathsf{P}$. *Then*

$$\mathrm{h}^*_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}; t^{\frac{1}{r}}\right) \;=\; \frac{\left(1 - t^{\frac{k}{r}}\right)^{d+1}}{\left(1 - t^{\frac{1}{r}}\right)(1-t)^d}\,\widetilde{\mathrm{h}}(\mathsf{P};t).$$

Proof. Applying classical Ehrhart theory, Proposition 5.5 and Corollary 5.21, we compute

$$\begin{aligned}
\frac{\mathrm{h}^*_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}; t^{\frac{1}{r}}\right)}{\left(1 - t^{\frac{k}{r}}\right)^{d+1}} \;=\; \mathrm{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}; t^{\frac{1}{r}}\right) &= 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}; n\right) t^{\frac{n}{r}} \\
&= 1 + \sum_{n \in \mathbb{Z}_{>0}} \left(1 + \sum_{j=1}^{n} \left|\partial_{\neq 0}\left(\frac{j}{r}\mathsf{P}\right) \cap \mathbb{Z}^d\right|\right) t^{\frac{n}{r}} \\
&= 1 + \sum_{n \in \mathbb{Z}_{>0}} t^{\frac{n}{r}} + \sum_{j>0} \sum_{n \geq j} \left|\partial_{\neq 0}\left(\frac{j}{r}\mathsf{P}\right) \cap \mathbb{Z}^d\right| t^{\frac{n}{r}} \\
&= 1 + \frac{t^{\frac{1}{r}}}{1 - t^{\frac{1}{r}}} + \sum_{j>0} \left|\partial_{\neq 0}\left(\frac{j}{r}\mathsf{P}\right) \cap \mathbb{Z}^d\right| \sum_{n \geq j} t^{\frac{n}{r}} \\
&= \frac{1 - t^{\frac{1}{r}} + t^{\frac{1}{r}} + \sum_{j>0} \left|\partial_{\neq 0}\left(\frac{j}{r}\mathsf{P}\right) \cap \mathbb{Z}^d\right| t^{\frac{j}{r}}}{1 - t^{\frac{1}{r}}} \\
&= \frac{\mathrm{WEhr}(\mathsf{P};t)}{1 - t^{\frac{1}{r}}} = \frac{\widetilde{\mathrm{h}}(\mathsf{P};t)}{\left(1 - t^{\frac{1}{r}}\right)(1-t)^d}.
\end{aligned}$$

□

Remark 5.23. The factor multiplying $\widetilde{\mathrm{h}}(\mathsf{P};t)$ in Lemma 5.22 can be rewritten in terms of finite geometric series. Let the codenominator $r = ks$ for some $s \in \mathbb{Z}_{\geq 1}$ (by Remark 5.2). Rewriting yields

$$\begin{aligned}\frac{\left(1-t^{\frac{k}{r}}\right)^{d+1}}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d} &= \frac{\left(1-t^{\frac{k}{r}}\right)}{\left(1-t^{\frac{1}{r}}\right)}\left(\frac{\left(1-t^{\frac{k}{r}}\right)}{(1-t)}\right)^d \\ &= \frac{\left(1-t^{\frac{1}{s}}\right)}{\left(1-t^{\frac{1}{ks}}\right)}\left(\frac{1}{1+t^{\frac{1}{s}}+\cdots+t^{\frac{s-1}{s}}}\right)^d = \frac{1+t^{\frac{1}{r}}+\cdots+t^{\frac{k-1}{r}}}{\left(1+t^{\frac{1}{s}}+\cdots+t^{\frac{s-1}{s}}\right)^d}\,.\end{aligned}$$

If $k = r$, this simplifies to $(1+t^{\frac{1}{r}}+\cdots+t^{\frac{r-1}{r}})$.

Remark 5.24. Lemma 5.22 corrects [Sta17, Remark 3], which was missing the factor between $\mathrm{h}^*_{\mathbb{Z}}(\frac{1}{r}\mathsf{P};t^{\frac{1}{r}})$ and $\widetilde{\mathrm{h}}(\mathsf{P};t)$.

Corollary 5.25. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r such that $\mathbf{0} \in \mathsf{P}$. Let k be the denominator of $\frac{1}{r}\mathsf{P}$. Then*

$$\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;k) \;=\; \mathrm{h}^*_{\mathbb{Z}}\left(\tfrac{1}{r}\mathsf{P};t^{\frac{1}{r}}\right) \;=\; \frac{\left(1-t^{\frac{k}{r}}\right)^{d+1}}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d}\,\widetilde{\mathrm{h}}(\mathsf{P},t)\,.$$

Remark 5.26. In [Sta08, Equation (14)] and [Sta17, Equation (6)], Stapledon shows that $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t) = \Psi(\widetilde{\mathrm{h}}(\mathsf{P};t))$, where $\Psi\colon \bigcup_{r\in\mathbb{Z}_{>0}} \mathbb{R}[t^{\frac{1}{r}}] \to \mathbb{R}[t]$ is defined by $\Psi(t^\lambda) = t^{\lceil\lambda\rceil}$. In the case of a lattice polytope with $\frac{m}{r} \in \mathbb{Z}$ we give a different construction to recover the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial from the $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}$- and $\mathrm{h}^*_{\mathbb{Q}}$-polynomial by applying the operator Int (see Corollary 5.15). Corollary 5.25 shows that, after a bit of computation, these two constructions are equivalent.

Remark 5.27. For a lattice d-polytope $\mathsf{P} \subseteq \mathbb{R}^d$ with codenominator r, $\mathbf{0} \in \mathsf{P}$, and denominator of $\frac{1}{2r}\mathsf{P}$ equal to k, we can relate $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t;k)$ and $\mathrm{h}^*_{\mathbb{Z}}(\frac{1}{2r}\mathsf{P};t^{\frac{1}{2r}})$ in a similar way. We again write $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t;k)$ to emphasize that it is the numerator of $\frac{\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t;k)}{(1-t^{\frac{k}{2r}})^{d+1}}$. Then

$$\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P};t;k) \;=\; \mathrm{h}^*_{\mathbb{Z}}\left(\frac{1}{2r}\mathsf{P};t^{\frac{1}{2r}}\right) \;=\; \frac{\left(1-t^{\frac{k}{2r}}\right)^{d+1}}{\left(1-t^{\frac{1}{2r}}\right)(1-t)^d}\,\widetilde{\mathrm{h}}(\mathsf{P};t)\,.$$

Corollary 5.28. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope with $\mathbf{0} \in \mathsf{P}^\circ$. Let r be the codenominator of P and k be the denominator of $\frac{1}{r}\mathsf{P}$. Then $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;k)$ is palindromic.*

Proof. From [Sta08, Corollary 2.12] we know that $\widetilde{\mathrm{h}}(\mathsf{P};t)$ is palindromic if $\mathbf{0} \in \mathsf{P}^\circ$. We compute, using Corollary 5.25,

$$\begin{aligned}
\mathrm{h}^*_{\mathbb{Q}}\Big(\mathsf{P}; t^{-1}; k\Big) &= \frac{\Big(1 - t^{\frac{-k}{r}}\Big)^{d+1}}{\Big(1 - t^{\frac{-1}{r}}\Big)\left(1 - t^{-1}\right)^d}\, \widetilde{\mathrm{h}}\Big(\mathsf{P}; t^{-1}\Big) \\
&= \frac{t^{\frac{-(d+1)k}{r}}}{t^{\frac{-1}{r}}}\, \frac{\Big(1 - t^{\frac{k}{r}}\Big)^{d+1}}{\Big(1 - t^{\frac{1}{r}}\Big)\left(1 - t\right)^d}\, \widetilde{\mathrm{h}}(\mathsf{P}; t) = \frac{1}{t^{\frac{k(d+1)-1}{r}}}\, \mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; k)\,.
\end{aligned}$$

Note that this implies, since the constant term of $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; k)$ is 1, that the degree of $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; k)$ (measured as a polynomial in $t^{\frac{1}{r}}$) equals $k(d+1)-1$. $\square$

This suggests that there is a 3-step hierarchy for rational dilations: $\mathbf{0} \in \mathsf{P}^\circ$ comes with extra symmetry, $\mathbf{0} \in \mathsf{P}$ comes with Proposition 5.5 (ii) and so we "only" have to compute $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; k) \in \mathbb{Z}[t^{\frac{1}{r}}]$, and $\mathbf{0} \notin \mathsf{P}$ means we have to compute $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}; t; k) \in \mathbb{Z}[t^{\frac{1}{2r}}]$. Corollary 5.28 is related to Gorenstein properties of rational polytopes, which we consider in the next section.

5.3 Gorenstein Musings

Our main goal in this section is to extend the notion of Gorenstein polytopes for lattice polytopes from Section 2.2 to the rational case. A rational d-polytope $\mathsf{P} \subseteq \mathbb{R}^d$ is **γ-rational Gorenstein** if $\hom(\frac{1}{\gamma}\mathsf{P})$ is a Gorenstein cone. See Figure 5.6 for an example. In this paper we explore this definition for parameters $\gamma = r$ and $\gamma = 2r$, other parameters are still to be investigated. The archetypal r-rational Gorenstein polytope is a rational polytope that contains the origin in its interior, see Corollary 5.30. The definition of γ-rational Gorenstein does not require that the origin is contained in the polytope, hence, it does not require the existence of a polar dual. A lattice polytope P is 1-rational Gorenstein if and only if it is a Gorenstein polytope in the classical sense.

Analogous to the lattice case (compare Theorem 2.8), the following theorem shows that a polytope containing the origin is r-rational Gorenstein if and only if it has a palindromic $\mathrm{h}^*_{\mathbb{Q}}$-polynomial. Let $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ be a rational d-polytope, as in Equation (5.1). We may assume that there is an index $0 \le i \le n$ such that $b_j = 0$ for $j = 1, \dots, i$ and $b_j \neq 0$ for $j = i+1, \dots, f$; thus we can write P as follows:

$$\mathsf{P} \;=\; \left\{ \mathbf{x} \in \mathbb{R}^d : \begin{array}{ll} \langle \mathbf{a}_j, \mathbf{x} \rangle \le 0 & \text{for } j = 1, \dots, i \\ \langle \mathbf{a}_j, \mathbf{x} \rangle \le b_j & \text{for } j = i+1, \dots, f \end{array} \right\}, \tag{5.6}$$

where $\mathbf{a}_j$ are the rows of $\mathbf{A}$.

Theorem 5.29. *Let $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ be a rational d-polytope with codenominator r and $\mathbf{0} \in \mathsf{P}$, as in Equation (5.1) and Equation (5.6). Then the following are equivalent for $g, m \in \mathbb{Z}_{\ge 1}$ and $\frac{m}{r}\mathsf{P}$ a lattice polytope:*

(i) P *is r-rational Gorenstein with Gorenstein point $(g, \mathbf{y}) \in \hom(\frac{1}{r}\mathsf{P})$.*

(ii) *There exists a (necessarily unique) integer solution* $(g, \mathbf{y})$ *to*

$$\begin{aligned} -\langle \mathbf{a}_j, \mathbf{y} \rangle &= 1 && \textit{for } j = 1, \dots, i \\ b_j\, g - r\, \langle \mathbf{a}_j, \mathbf{y} \rangle &= b_j && \textit{for } j = i+1, \dots, f\,. \end{aligned}$$

(iii) $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; m)$ *is palindromic:*

$$t^{(d+1)\frac{m}{r} - \frac{g}{r}}\, \mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}; m\right) \;=\; \mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; m)\,.$$

(iv) $(-1)^{d+1} t^{\frac{g}{r}}\, \mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}; t) = \mathrm{Ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}\right)$.

(v) $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}; \frac{n}{r}) = \mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; \frac{n+g}{r})$ *for all* $n \in \mathbb{Z}_{\geq 0}$.

(vi) $\hom(\frac{1}{r}\mathsf{P})^\vee$ *is the cone over a lattice polytope, i.e., there exists a lattice point* $(g, \mathbf{y}) \in \hom(\frac{1}{r}\mathsf{P})^\circ \cap \mathbb{Z}^{d+1}$ *such that for every primitive ray generator* $(v_0, \mathbf{v})$ *of* $\hom(\frac{1}{r}\mathsf{P})^\vee$

$$\langle (g, \mathbf{y})\,, (v_0, \mathbf{v}) \rangle = 1\,.$$

The equivalence of (i) and (vi) is well known (see, e.g., [BN08, Definition 1.8] or [BG09, Exercises 2.13, 2.14]); for the sake of completeness we include a proof below.

Corollary 5.30. *Let* $\mathsf{P} \subseteq \mathbb{R}^d$ *be a rational d-polytope with codenominator r. If* $\mathbf{0} \in \mathsf{P}^\circ$*, then* P *is r-rational Gorenstein with Gorenstein point* $(1, 0, \dots, 0)$ *and* $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}; t; m)$ *is palindromic.*

Before we prove Theorem 5.29, we compute some examples.

Example 5.31 (continued)**.** We check the Gorenstein criterion for the running examples such that $\mathbf{0} \in \mathsf{P}$.

(i) $\mathsf{P}_1 := \left[-1, \frac{2}{3}\right]$, $r = 2$, $m = 6$,

$$\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}_1; t; 6) = 1 + t^{\frac{1}{2}} + 2t + 3t^{\frac{3}{2}} + 4t^2 + 4t^{\frac{5}{2}} + 4t^3 + 4t^{\frac{7}{2}} + 3t^4 + 2t^{\frac{9}{2}} + t^5 + t^{\frac{11}{2}}\,.$$

The polynomial $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}_1; t; 6)$ is palindromic and therefore (by Theorem 5.29), P_1 is 2-rational Gorenstein. This is to be expected; as $\mathbf{0} \in \mathsf{P}^\circ$, Lemma 5.22 shows that $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}_1; t; 6)$ must be palindromic.

(ii) $\mathsf{P}_2 := \left[0, \frac{2}{3}\right]$, $r = 2$, $m = 3$,

$$\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}_2; t; 3) = 1 + t^{\frac{1}{2}} + t\,.$$

The polynomial $\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}_2; t; 3)$ is palindromic and P_2 is 2-rational Gorenstein with Gorenstein point $(g, \mathbf{y}) = (4, 1) \in \hom(\frac{1}{2}\mathsf{P}_2)$.

Example 5.32. The triangle $\nabla := \mathrm{conv}\{(0,0), (0,1), (3,1)\}$ has codenominator 1. It is not 1-rational Gorenstein as $|\nabla^\circ \cap \mathbb{Z}^2| = 0$ and $|(2\nabla)^\circ \cap \mathbb{Z}^2| = 2$.

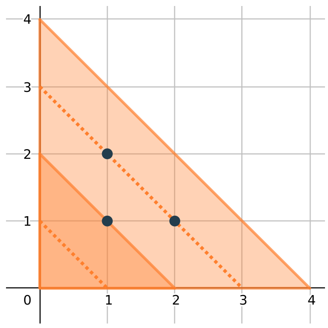


Figure 5.4: The triangle $\Delta := \operatorname{conv}\{(0,0),(2,0),(0,2)\}$ from Example 5.33 and some of its dilates demonstrating the property of being 2-rational Gorenstein but not Gorenstein in the classical setting.

Example 5.33. The *Haasenlieblingsdreieck* $\Delta := \operatorname{conv}\{(0,0),(2,0),(0,2)\}$ is not a Gorenstein polytope in the classic (integral) setting, but it is 2-rational Gorenstein (see Figure 5.4): we compute

$$\operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P};t) \;=\; \frac{1}{\left(1-t^{\frac{1}{2}}\right)^{3}} \;=\; \frac{1+3t^{\frac{1}{2}}+3t+t^{\frac{3}{2}}}{(1-t)^{3}}\,.$$

Example 5.34 (A polytope that is not γ-rational Gorenstein for any γ)**.** Let ∇ be the triangle as in Figure 5.5, i.e.,$\nabla = \operatorname{conv}\{(0,0),(0,2),(5,2)\}$. Then the inequality description is

$$\nabla \;=\; \left\{(x_1,x_2)\in\mathbb{R}^2\colon\; -x_1\le 0\,,\;\; x_2\le 2\,,\;\; 2x_1-5x_2\le 0\right\}.$$

We can read off the codenominator $r=2$ and compute its rational Ehrhart series with m chosen minimally as

$$\operatorname{Ehr}_{\mathbb{Q}}(\nabla;t) \;=\; \frac{1+4t^{\frac{1}{2}}+7t+6t^{\frac{3}{2}}+2t^2}{(1-t)^2}\,.$$

Hence, $\mathrm{h}^*_{\mathbb{Q}}(\nabla;t;2) = 1+4t^{\frac{1}{2}}+7t+6t^{\frac{3}{2}}+2t^2$ is not palindromic and ∇ is not rational Gorenstein.[2]

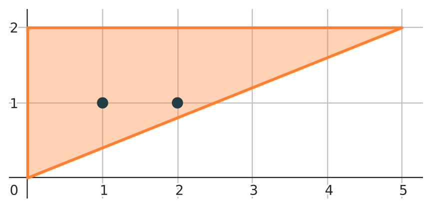


Figure 5.5: The triangle $\nabla = \operatorname{conv}\{(0,0),(0,2),(5,2)\}$, which is not rational Gorenstein. The cone $\operatorname{hom}(\frac{1}{\gamma}\nabla)$ contains two interior lattice points at lowest height, hence it does not posses a Gorenstein point.

2 We thank Esme Bajo for suggesting this example and helping with computing it. See [BB23] for symmetric decompositions and boundary h^*-polynomials.

Proof of Theorem 5.29.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) We compute using reciprocity (see Corollary 5.18):

$$1+\sum_{\lambda\in\frac{1}{r}\mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)t^{\lambda} \;=\; \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)}{\left(1-t^{\frac{m}{r}}\right)^{(d+1)}} \;=\; \frac{t^{(d+1)\frac{m}{r}-\frac{g}{r}}\,\mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P};\frac{1}{t};m\right)}{\left(1-t^{\frac{m}{r}}\right)^{(d+1)}}$$
$$= \; t^{-\frac{g}{r}}\frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P}^\circ;t;m)}{\left(1-t^{\frac{m}{r}}\right)^{(d+1)}} \;=\; t^{-\frac{g}{r}}\sum_{\lambda\in\frac{1}{r}\mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ;\lambda)t^{\lambda}\,.$$

That is equivalent to

$$t^{\frac{g}{r}}\,\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P};t) \;=\; t^{\frac{g}{r}}\left(1+\sum_{\lambda\in\frac{1}{r}\mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)t^{\lambda}\right) \;=\; \sum_{\lambda\in\frac{1}{r}\mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ;\lambda)t^{\lambda}$$
$$=\; \mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}^\circ;t) \;=\; (-1)^{d+1}\,\mathrm{Ehr}_{\mathbb{Q}}\left(\mathsf{P};\frac{1}{t}\right).$$

Comparing coefficients gives the third equivalence:

$$\mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P};\frac{n}{r}\right) \;=\; \mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P};\frac{n+g}{r}\right) \qquad \text{for } n\in\mathbb{Z}_{\geq 0}\,.$$

(v) $\Rightarrow$ (i) Since

$$\mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P};\frac{n}{r}\right) \;=\; \mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P};\frac{n+g}{r}\right) \qquad \text{for } n\in\mathbb{Z}_{\geq 0}$$

it suffices to show one inclusion:

$$\mathrm{hom}\left(\frac{1}{r}\mathsf{P}\right)^\circ\cap\mathbb{Z}^{d+1} \;\supseteq\; \left((g,\mathbf{y})+\mathrm{hom}\left(\frac{1}{r}\mathsf{P}\right)\right)\cap\mathbb{Z}^{d+1}\,,$$

where $\mathbf{y}$ is the unique interior lattice point in $\frac{g}{r}\mathsf{P}^\circ$. Indeed, for a point $(g,\mathbf{y})\in \mathrm{hom}(\frac{1}{r}\mathsf{P})^\circ\cap\mathbb{Z}^{d+1}$ it follows that the point $(g,\mathbf{y})+\mathbf{z}\in\mathrm{hom}(\frac{1}{r}\mathsf{P})^\circ\cap\mathbb{Z}^{d+1}$ for all $\mathbf{z}\in\mathrm{hom}(\frac{1}{r}\mathsf{P})\cap\mathbb{Z}^{d+1}$.

(i) $\Rightarrow$ (iii) By the definition of P being r-rational Gorenstein,

$$\mathrm{hom}\left(\frac{1}{r}\mathsf{P}\right)^\circ\cap\mathbb{Z}^{d+1} \;=\; (g,\mathbf{y})+\mathrm{hom}\left(\frac{1}{r}\mathsf{P}\right)\cap\mathbb{Z}^{d+1}.$$

Computing integer point transforms gives:

$$\sigma_{\mathrm{hom}\left(\frac{1}{r}\mathsf{P}\right)^\circ}(\mathbf{z}) \;=\; \mathbf{z}^{(g,\mathbf{y})}\sigma_{\mathrm{hom}\left(\frac{1}{r}\mathsf{P}\right)}(\mathbf{z})\,.$$

Applying reciprocity (see, e.g., [BR15, Theorem 4.3]) yields

$$\sigma_{\mathrm{hom}\left(\frac{1}{r}\mathsf{P}\right)^\circ}(\mathbf{z}) \;=\; (-1)^{d+1}\sigma_{\mathrm{hom}\left(\frac{1}{r}\mathsf{P}\right)}\left(\frac{1}{\mathbf{z}}\right) \;=\; \mathbf{z}^{(g,\mathbf{y})}\sigma_{\mathrm{hom}\left(\frac{1}{r}\mathsf{P}\right)}(\mathbf{z})\,. \tag{5.7}$$

By specializing $\mathbf{z} = (t^{\frac{1}{r}}, 1, \dots, 1)$ in Equation (5.7) we obtain the following relation between Ehrhart series for $\frac{1}{r}\mathsf{P}$ in the variable $t^{\frac{1}{r}}$ and $t^{-\frac{1}{r}}$:

$$(-1)^{d+1} \operatorname{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}, \frac{1}{t^{\frac{1}{r}}}\right) \;=\; t^{\frac{g}{r}} \operatorname{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}, t^{\frac{1}{r}}\right). \tag{5.8}$$

From (the proof of) Theorem 5.7 we know that

$$\operatorname{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathsf{P}, t^{\frac{1}{r}}\right) \;=\; \operatorname{Ehr}_{\mathbb{Q}}(\mathsf{P};t) \;=\; \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}},$$

where m is an integer such that $\frac{1}{r}\mathsf{P}$ is a lattice polytope. Substituting this into Equation (5.8) yields

$$\left(t^{\frac{m}{r}}\right)^{d+1} \frac{\mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P};\frac{1}{t};m\right)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}} \;=\; (-1)^{d+1} \frac{\mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P};\frac{1}{t};m\right)}{\left(1-\frac{1}{t^{\frac{m}{r}}}\right)^{d+1}} \;=\; t^{\frac{g}{r}} \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}}$$

and thus

$$t^{\frac{(d+1)m}{r}-\frac{g}{r}}\, \mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P};\frac{1}{t};m\right) \;=\; \mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;m)\,.$$

(ii) $\Leftrightarrow$ (vi) The primitive ray generators of $\hom(\frac{1}{r}\mathsf{P})^\vee$ are the primitive facet normals of $\hom(\frac{1}{r}\mathsf{P})$, that is,

$$(0,-\mathbf{a}_j) \text{ for } j=1,\dots,i \quad \text{ and } \quad \left(1,-\frac{r}{b_j}\mathbf{a}_j\right) \text{ for } j=i+1,\dots,f\,.$$

Note that, since $\mathbf{0}\in\mathsf{P}$, $b_j \ge 0$ for all $j=1,\dots,f$. The statement follows.

(vi) $\Rightarrow$ (i) Since $(g,\mathbf{y}) \in \hom(\frac{1}{r}\mathsf{P})^\circ \cap \mathbb{Z}^{d+1}$ is an interior point of the cone, it follows directly that $(g,\mathbf{y}) + \hom(\frac{1}{r}\mathsf{P}) \subseteq \hom(\frac{1}{r}\mathsf{P})^\circ$. Let $(x_0,\mathbf{x}) \in \hom(\frac{1}{r}\mathsf{P})^\circ$, then for any primitive ray generator $(v_0,\mathbf{v})$ of $\hom(\frac{1}{r}\mathsf{P})^\vee$ (being the primitive facet normals of $\hom(\frac{1}{r}\mathsf{P})$),

$$\langle (x_0,\mathbf{x})-(g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle \;=\; \underbrace{\langle (x_0,\mathbf{x})\,,(v_0,\mathbf{v})\rangle}_{>0} - \underbrace{\langle (g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle}_{=1} \;\ge\; 0\,.$$

Hence, $(x_0,\mathbf{x})-(g,\mathbf{y}) \in \hom(\frac{1}{r}\mathsf{P})$ and $(x_0,\mathbf{x}) \in (g,\mathbf{y}) + \hom(\frac{1}{r}\mathsf{P})$.

(i) $\Rightarrow$ (vi) From the definition of Gorenstein point we know that $(g,\mathbf{y}) \in \hom(\frac{1}{r}\mathsf{P})^\circ$ and hence

$$\langle (g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle > 0$$

for all primitive facet normals $(v_0,\mathbf{v})$ of $\hom(\frac{1}{r}\mathsf{P})$. Since the facet normals $(v_0,\mathbf{v})$ are primitive, i.e., $\gcd((v_0,\mathbf{v}))=1$, there exists an integer point in the shifted hyperplane H defined by

$$\mathsf{H} \;=\; \left\{(x_0,\mathbf{x}) \in \mathbb{R}^{d+1} \colon\; \langle (v_0,\mathbf{v}),(x_0,\mathbf{x})\rangle = 1\right\}$$

and hence H contains a d-dimensional sublattice. Since the intersection $\mathsf{H} \cap \hom(\frac{1}{r}\mathsf{P})^\circ$ contains a pointed cone (e.g., the shifted recession cone), it contains a lattice point $(z_0, \mathbf{z}) \in \hom(\frac{1}{r}\mathsf{P})^\circ$.

So, for any facet of $\hom\left(\frac{1}{r}\mathsf{P}\right)$ there exists a lattice point $(z_0, \mathbf{z})$ in the interior of $\hom(\frac{1}{r}\mathsf{P})$ at lattice distance one from the facet. Since $(g, \mathbf{y}) + \hom(\frac{1}{r}\mathsf{P}) = \hom(\frac{1}{r}\mathsf{P})^\circ$, there exists a point $(r_0, \mathbf{r}) \in \hom(\frac{1}{r}\mathsf{P})$ such that

$$(g, \mathbf{y}) + (r_0, \mathbf{r}) \;=\; (z_0, \mathbf{z})\,.$$

Then,

$$1 \;=\; \langle (z_0, \mathbf{z})\,, (v_0, \mathbf{v})\rangle \;=\; \underbrace{\langle (g, \mathbf{y})\,, (v_0, \mathbf{v})\rangle}_{>0} + \underbrace{\langle (r_0, \mathbf{r})\,, (v_0, \mathbf{v})\rangle}_{\geq 0}$$

and $\langle (g, \mathbf{y})\,, (v_0, \mathbf{v})\rangle = 1$. □

As usual we state a version of Theorem 5.29 for the refined rational Ehrhart series and the $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}$-polynomial. Here, the polytopes under consideration are not required to contain the origin. This means that in the description of the polytope as in Equation (5.6) the vector $\mathbf{b} \in \mathbb{Z}^n$ might have negative entries and we use absolute values when multiplying inequalities or facet normals with entries of $\mathbf{b}$. Except for this small difference, the proof is the same as that of Theorem 5.29 so we omit it.

Theorem 5.35. *Let $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \leq \mathbf{b}\}$ be a rational d-polytope with codenominator r, as in Equation (5.1) and Equation (5.6). Then the following are equivalent for $g, m \in \mathbb{Z}_{\geq 1}$ and $\frac{m}{2r}\mathsf{P}$ a lattice polytope:*

(i) *P is $2r$-rational Gorenstein with Gorenstein point $(g, \mathbf{y}) \in \hom(\frac{1}{2r}\mathsf{P})$.*

(ii) *There exists a (necessarily unique) integer solution $(g, \mathbf{y})$*

$$-\langle \mathbf{a}_j, \mathbf{y}\rangle = 1 \quad \text{for } j = 1, \ldots, i$$
$$b_j\, g - 2r\,\langle \mathbf{a}_j, \mathbf{y}\rangle = |b_j| \quad \text{for } j = i+1, \ldots, f\,.$$

(iii) *$\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}; t; m)$ is palindromic:*

$$t^{(d+1)\frac{m}{2r} - \frac{g}{2r}}\, \mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}; m\right) \;=\; \mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}; t; m)\,.$$

(iv) $(-1)^{d+1} t^{\frac{g}{2r}}\, \mathrm{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}; t) = \mathrm{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}\right)$.

(v) $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}; \frac{n}{2r}) = \mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}^\circ; \frac{n+g}{2r})$ *for all $n \in \mathbb{Z}_{\geq 0}$.*

(vi) $\hom(\frac{1}{2r}\mathsf{P})^\vee$ *is the cone over a lattice polytope, i.e., there exists a lattice point $(g, \mathbf{y}) \in \hom(\frac{1}{2r}\mathsf{P})^\circ \cap \mathbb{Z}^{d+1}$ such that for every primitive ray generator $(v_0, \mathbf{v})$ of* $\hom(\frac{1}{2r}\mathsf{P})^\vee$

$$\langle (g, \mathbf{y})\,, (v_0, \mathbf{v})\rangle = 1\,.$$

Theorem 5.35 could be generalized to ℓr-rational Gorenstein polytopes for $\ell \in \mathbb{Z}_{>0}$. However it is not clear that computationally this would provide any new insights to the (rational) Ehrhart theory of the polytopes.

Corollary 5.36.

(i) *If* $\mathbf{0} \in \mathsf{P}^\circ$*, then* P *is also* $2r$*-rational Gorenstein with the same Gorenstein point* $(1, 0 \ldots, 0)$ *(see Corollary 5.30).*

(ii) *If* $\mathbf{0} \in \mathsf{P}$ *and* P *is* r*-rational Gorenstein, then* P *is also* $2r$*-rational Gorenstein.*

(iii) *If* P *is* $2r$*-rational Gorenstein and the first coordinate* g *of the Gorenstein point* $(g, \mathbf{y})$ *is even, then* P *is also* r*-rational Gorenstein.*

Proof of (ii). Since $\mathbf{0} \in \mathsf{P}$ we know that $\mathrm{ehr}_{\mathbb{Q}}$ is constant on $[\frac{n}{r}, \frac{n+1}{r})$ and we compute

$$\begin{aligned}
\mathrm{Ehr}_{\mathbb{Q}}^{\mathsf{ref}}(\mathsf{P}; t) &= 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{n}{2r}\right) t^{\frac{n}{2r}} \\
&= 1 + \mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P}, \frac{1}{2r}\right) t^{\frac{1}{2r}} \\
&\quad + \sum_{n \in \mathbb{Z}_{>0}} \left(\mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{2n}{2r}\right) t^{\frac{2n}{2r}} + \underbrace{\mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{2n+1}{2r}\right)}_{=\mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{n}{r}\right)} t^{\frac{2n+1}{2r}} \right) \\
&= 1 + t^{\frac{1}{2r}} + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P}; \frac{n}{r}\right) t^{\frac{n}{r}} \left(1 + t^{\frac{1}{2r}}\right) \\
&= \left(1 + t^{\frac{1}{2r}}\right) \mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}; t),
\end{aligned}$$

where we also use that $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}, 0) = \mathrm{ehr}_{\mathbb{Q}}\left(\mathsf{P}, \frac{1}{2r}\right) = 1$. $\square$

Example 5.37. (continued) We check the Gorenstein criterion for the running examples such that $\mathbf{0} \notin \mathsf{P}$.

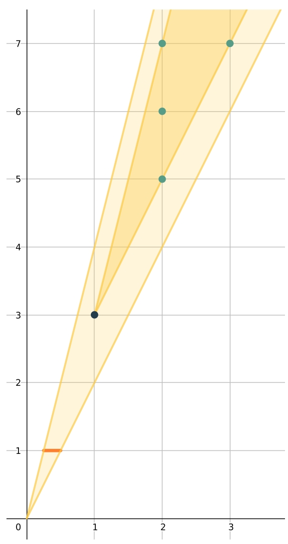


Figure 5.6: The cone $\hom(\frac{1}{4}\mathsf{P}_3) = \hom(\frac{1}{8}\mathsf{P}_4)$ with Gorenstein point $(3, 1)$ highlighted in dark blue. The other lattice points $\hom(\frac{1}{4}\mathsf{P}_3)^\circ \cap \mathbb{Z}^2$ are marked in blue. Observe that $(3, 1) + \hom(\frac{1}{4}\mathsf{P}_3) \cap \mathbb{Z}^2 = \hom(\frac{1}{4}\mathsf{P}_3)^\circ \cap \mathbb{Z}^2$.

(iii) $\mathsf{P}_3 := [1,2]$, $r=2$, $m=4$, $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}_3;t;4) = 1 + t^{\frac{2}{4}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}$.

(iv) $\mathsf{P}_4 := [2,4]$, $r=4$, $m=4$, $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}_4;t;4) = 1 + t^{\frac{1}{4}} + t^{\frac{3}{8}} + t^{\frac{5}{8}}$.

Both polynomials $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}_4;t;4)$ and $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}(\mathsf{P}_3;t;4)$ are palindromic and therefore P_3 is 4-rational Gorenstein and P_4 is 8-rational Gorenstein. In fact, $\frac{1}{4}\mathsf{P}_3 = \frac{1}{8}\mathsf{P}_4$ and so $\hom(\frac{1}{4}\mathsf{P}_3) = \hom(\frac{1}{8}\mathsf{P}_4)$. The Gorenstein point is $(g,\mathbf{y}) = (3,1)$.

Example 5.38 (A polytope that is not $2r$-rational Gorenstein)**.** Let $\mathsf{P}_5 = [1,4]$. Then $r = 4$ and $2r = 8$, so $\frac{1}{2r}\mathsf{P}_5 = [\frac{1}{8}, \frac{1}{2}]$. The first lattice point in the interior of $\hom(\frac{1}{8}\mathsf{P}_5)$ is $(g,\mathbf{y}) = (3,1)$. However, $(3,1)$ does not satisfy Condition (ii) from Theorem 5.29; it is at lattice distance 5 from one of the facets of $\hom(\frac{1}{8}\mathsf{P}_5)$.

Remark 5.39. Bajo and Beck [BB23, Section 5] essentially showed that the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial of a rational polytope P is palindromic if and only if $\hom(\mathsf{P})$ is a Gorenstein cone. Hence, polytopes with palindromic $\mathrm{h}^*_{\mathbb{Z}}$-polynomials, $\mathrm{h}^*_{\mathbb{Q}}$-polynomials, or $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}$-polynomials are fully classified. This implies, in particular, that polytopes with palindromic $\mathrm{h}^*_{\mathbb{Z}}$-polynomials also have palindromic $\mathrm{h}^*_{\mathbb{Q}}$ and $\mathrm{h}^{*\mathsf{ref}}_{\mathbb{Q}}$-polynomials.

5.4 Symmetric Decompositions

We now use the stipulations of the last section to give a new proof of the following theorem. As we will see, our proof will also yield a rational version (Theorem 5.42 below).

Theorem 5.40 (Betke–McMullen [BM85])**.** *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope that contains a lattice point in its interior. Then there exist polynomials $a(t)$ and $b(t)$ with nonnegative coefficients such that*

$$\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t) = a(t) + t\, b(t)\ , \qquad t^d\, a\left(\tfrac{1}{t}\right) = a(t)\ , \qquad t^{d-1}\, b\left(\tfrac{1}{t}\right) = b(t)\ .$$

Proof. Suppose P is a lattice d-polytope with codenominator r. If P contains a lattice point in its interior, we might as well assume it is the origin (the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial *is* invariant under lattice translations). Then Corollary 5.30 says

$$t^{d+1-\frac{1}{r}}\, \mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P}; \frac{1}{t}; r\right) \;=\; \mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;r)\,. \tag{5.9}$$

Note, since P is a lattice polytope we can choose $m = r$. On the other hand, as we noted in the beginning of Section 5.1, the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial of a rational d-polytope always has a factor, that carries over (by the proof of Theorem 5.7) to

$$\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;r) \;=\; \left(1 + t^{\frac{1}{r}} + \cdots + t^{\frac{r-1}{r}}\right) \widetilde{\mathrm{h}}(\mathsf{P};t)$$

for some $\widetilde{\mathrm{h}}(\mathsf{P};t) \in \mathbb{Z}[t^{1/r}]$ (which is, of course, very much related to Section 5.2). Moreover, by Equation (5.9) this polynomial satisfies $t^d\, \widetilde{\mathrm{h}}(\mathsf{P};\frac{1}{t}) = \widetilde{\mathrm{h}}(\mathsf{P};t)$. Note that

$$\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P};t) \;=\; \frac{\left(1 + t^{\frac{1}{r}} + \cdots + t^{\frac{r-1}{r}}\right) \widetilde{\mathrm{h}}(\mathsf{P};t)}{(1-t)^{d+1}} \;=\; \frac{\widetilde{\mathrm{h}}(\mathsf{P};t)}{\left(1 - t^{\frac{1}{r}}\right)(1-t)^d} \tag{5.10}$$

and the Gorenstein property of $\frac{1}{r}\mathsf{P}$ imply that $\widetilde{\mathrm{h}}(\mathsf{P};t)$ equals the h^*-polynomial (in the variable $t^{\frac{1}{r}}$) of the boundary of $\frac{1}{r}\mathsf{P}$. Indeed, the rational Ehrhart series of $\partial\mathsf{P}$ is

$$\begin{aligned}\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P};t) - \mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}^\circ;t) &= \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;r)}{(1-t)^{d+1}} - \frac{t^{d+1}\mathrm{h}^*_{\mathbb{Q}}\left(\mathsf{P};\frac{1}{t};r\right)}{(1-t)^{d+1}} \\ &= \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;r)}{(1-t)^{d+1}} - \frac{t^{\frac{1}{r}}\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;r)}{(1-t)^{d+1}} \\ &= \frac{(1-t^{\frac{1}{r}})\mathrm{h}^*_{\mathbb{Q}}(\mathsf{P};t;r)}{(1-t)^{d+1}} = \frac{\widetilde{\mathrm{h}}(\mathsf{P};t)}{(1-t)^{d}}\,.\end{aligned}$$

The (triangulated) boundary of a polytope is shellable [Zie98, Chapter 8], and this shelling gives a half-open decomposition of the boundary, which yields nonnegativity of the $\mathrm{h}^*_{\mathbb{Z}}$-vector. Hence, $\widetilde{\mathrm{h}}(\mathsf{P};t)$ has nonnegative coefficients.

Recall that Int is the operator that extracts from a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ the terms with integer powers of t. Thus

$$a(t) := \mathrm{Int}\left(\widetilde{\mathrm{h}}(\mathsf{P};t)\right)$$

is a polynomial in $\mathbb{Z}[t]$ with nonnegative coefficients satisfying $t^d\, a(\frac{1}{t}) = a(t)$. (Note that $a(t)$ can be interpreted as the h^*-polynomial of the boundary of P; see, e.g., [BB23].) Again, because we could choose $m = r$, we compute using Equation (5.10):

$$\begin{aligned}\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t) &= \mathrm{Int}\left(\left(1 + t^{\frac{1}{r}} + \cdots + t^{\frac{r-1}{r}}\right)\widetilde{\mathrm{h}}(\mathsf{P};t)\right) \\ &= a(t) + \mathrm{Int}\left(\left(t^{\frac{1}{r}} + t^{\frac{2}{r}} + \cdots + t^{\frac{r-1}{r}}\right)\widetilde{\mathrm{h}}(\mathsf{P};t)\right).\end{aligned}$$

Since $\beta(t) := \left(t^{\frac{1}{r}} + t^{\frac{2}{r}} + \cdots + t^{\frac{r-1}{r}}\right)\widetilde{\mathrm{h}}(\mathsf{P};t)$ satisfies $t^{d+1}\,\beta\left(\frac{1}{t}\right) = \beta(t)$, the polynomial

$$b(t) := \frac{1}{t}\,\mathrm{Int}\left(\left(t^{\frac{1}{r}} + t^{\frac{2}{r}} + \cdots + t^{\frac{r-1}{r}}\right)\widetilde{\mathrm{h}}(\mathsf{P};t)\right)$$

satisfies $t^{d-1}\,b\left(\frac{1}{t}\right) = b(t)$, and $\mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t) = a(t) + t\,b(t)$ by construction. □

Remark 5.41. We could have started the proof of Theorem 5.40 with Equation (5.5) and then used Stapledon's result [Sta17] that $\widetilde{\mathrm{h}}(\mathsf{P};t)$ is palindromic and nonnegative.

The rational version of this theorem is a special case of [BBV22, Theorem 4.7].

Theorem 5.42. *Let $\mathsf{Q} \subseteq \mathbb{R}^d$ be a rational d-polytope with denominator k that contains a lattice point in its interior. Then there exist polynomials $a(t)$ and $b(t)$ with nonnegative coefficients such that*

$$\mathrm{h}^*_{\mathbb{Z}}(\mathsf{Q};t) = a(t) + t\,b(t)\,, \qquad t^{k(d+1)-1}\,a\left(\frac{1}{t}\right) = a\,(t)\;, \qquad t^{k(d+1)-2}\,b\left(\frac{1}{t}\right) = b(t)\,.$$

Proof. We repeat our proof of Theorem 5.40 for $\mathsf{P} := k\,\mathsf{Q}$, except that instead of the operator Int, we use the operator Rat_k which extracts the terms with powers that are multiples of $\frac{1}{k}$. So now

$$\begin{aligned} a(t) &:= \mathrm{Rat}_k(\widetilde{\mathrm{h}}(\mathsf{P};t)), \\ b(t) &:= \frac{1}{t^{\frac{1}{k}}}\,\mathrm{Rat}_k\Big(\Big(t^{\frac{1}{r}} + t^{\frac{2}{r}} + \cdots + t^{\frac{r-1}{r}}\Big)\,\widetilde{\mathrm{h}}(\mathsf{P};t)\Big), \text{ and} \\ \mathrm{h}^*_{\mathbb{Z}}(\mathsf{P};t) &= a(t^k) + t\,b(t^k)\,. \end{aligned}$$

□

We conclude by remarking that there is a generalization of the Betke–McMullen theorem due to Stapledon [Sta09]; here the assumption of an interior lattice point is dropped, but the symmetric decomposition happens now with a modified $\mathrm{h}^*_{\mathbb{Z}}$-polynomial. A rational version is the afore-mentioned [BBV22, Theorem 4.7]; see also [BB23].

5.5 Period Collapse

One of the classic instances of period collapse in integral Ehrhart theory is the triangle

$$\Delta \;:=\; \mathrm{conv}\{(0,0),(1,\tfrac{p-1}{p}),(p,0)\} \tag{5.11}$$

where $p \ge 2$ is an integer [MW05]; see also [CLS19] for an irrational version. Here

$$\mathrm{Ehr}_{\mathbb{Z}}(\Delta;t) \;=\; \frac{1+(p-2)\,t}{(1-t)^3}$$

and so, while the denominator of Δ equals p, the period of $\mathrm{ehr}_{\mathbb{Z}}(\Delta;n)$ collapses to 1: the quasipolynomial $\mathrm{ehr}_{\mathbb{Z}}(\Delta;n) = \frac{p-1}{2}\,n^2 + \frac{p+1}{2}\,n + 1$ is a polynomial.

As mentioned in the Introduction, we offer data points towards the question of whether or how much period collapse happens in rational Ehrhart theory, and how it compares to the classical scenario.

Example 5.43. We consider the triangle Δ defined in Equation (5.11) with $p = 3$. Note that both denominator and codenominator of Δ equal 3. We compute

$$\mathrm{Ehr}_{\mathbb{Q}}(\Delta;t) \;=\; \frac{1+t^{\frac{5}{3}}}{\left(1-t^{\frac{1}{3}}\right)^2(1-t^3)}\,.$$

Note that the accompanying rational Ehrhart quasipolynomial $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$ thus has period 3. We can retrieve the integral Ehrhart series from the rational by rewriting

$$\mathrm{Ehr}_{\mathbb{Q}}(\Delta;t) \;=\; \frac{\left(1+t^{\frac{5}{3}}\right)\left(1+t^{\frac{1}{3}}+t^{\frac{2}{3}}\right)^2}{(1-t)^2\,(1-t^3)} \;=\; \frac{\left(1+t^{\frac{5}{3}}\right)\left(1+2t^{\frac{1}{3}}+3t^{\frac{2}{3}}+2t+t^{\frac{4}{3}}\right)}{(1-t)^2\,(1-t^3)}$$

and then disregarding the fractional powers in the numerator, which gives

$$\mathrm{Ehr}_{\mathbb{Z}}(\Delta;t) \;=\; \frac{1+2t+2t^2+t^3}{(1-t)^2\,(1-t^3)} \;=\; \frac{1+t}{(1-t)^3}\,.$$

Hence the classical Ehrhart quasipolynomial exhibits period collapse while the rational does not.

Example 5.44. The recent paper [Fer+21] studied certain families of polytopes arising from graphs, which exhibit period collapse. One example is the pyramid

$$\mathsf{P}_5 \;:=\; \operatorname{conv}\left\{(0,0,0)\,,\left(\tfrac{1}{2},0,0\right),\left(0,\tfrac{1}{2},0\right),\left(\tfrac{1}{2},\tfrac{1}{2},0\right),\left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2}\right)\right\}.$$

which has denominator 4 and codenominator 1. In particular, its rational Ehrhart series equals the standard Ehrhart series, and

$$\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}_5;t) \;=\; \mathrm{Ehr}_{\mathbb{Z}}(\mathsf{P}_5;t) \;=\; \frac{1+t^3}{(1-t)\,(1-t^2)^3}$$

shows that $\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}_5;n)$ and $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}_5;\lambda)$ both have period 2, i.e., they both exhibit period collapse.

Example 5.45. Recall the running examples $\mathsf{P}_1 = [-1,\frac{2}{3}]$ and $\mathsf{P}_2 = [0,\frac{2}{3}]$. Restricting the rational Ehrhart quasipolynomial from page 93 to positive integers we retrieve the Ehrhart quasipolynomials:

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}_1;n) \;=\; \begin{cases} \frac{5}{3}n+1 & \text{if } n\equiv 0 \bmod 3,\\ \frac{5}{3}n+\frac{1}{3} & \text{if } n\equiv 1 \bmod 3,\\ \frac{5}{3}n+\frac{2}{3} & \text{if } n\equiv 2 \bmod 3,\end{cases}$$

$$\mathrm{ehr}_{\mathbb{Z}}(\mathsf{P}_2;n) \;=\; \begin{cases} \frac{2}{3}n+1 & \text{if } n\equiv 0 \bmod 3,\\ \frac{2}{3}n+\frac{1}{3} & \text{if } n\equiv 1 \bmod 3,\\ \frac{2}{3}n+\frac{2}{3} & \text{if } n\equiv 2 \bmod 3.\end{cases}$$

We can observe the period 3 here for both functions. Recall the rational Ehrhart series from page 97:

$$\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}_1;t) \;=\; \frac{1+t^{\frac{1}{2}}+t+t^{\frac{3}{2}}+t^2}{(1-t)\left(1-t^{\frac{3}{2}}\right)}\,,$$

$$\mathrm{Ehr}_{\mathbb{Q}}(\mathsf{P}_2;t) \;=\; \frac{1}{\left(1-t^{\frac{1}{2}}\right)\left(1-t^{\frac{3}{2}}\right)} \;=\; \frac{1+t^{\frac{1}{2}}+t}{\left(1-t^{\frac{3}{2}}\right)^2}\,.$$

We can read off from the series that $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}_1;\lambda)$ has rational period 3, whereas $\frac{3}{2}$ is the rational period of $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P}_2;\lambda)$. Both P_1 and P_2 have codenominator $r=2$, but $m_{\mathsf{P}_1}=6$ and $m_{\mathsf{P}_2}=3$ (see computations on page 97). So the expected period is $\frac{6}{2}=3$ for P_1 and $\frac{3}{2}$ for P_2. Thus here neither the rational Ehrhart quasipolynomials nor the integral Ehrhart quasipolynomials exhibit period collapse.

We do not know any examples of polytopes with period collapse in their rational Ehrhart quasipolynomials but not in their integral Ehrhart quasipolynomials. The question about possible period collapse of an Ehrhart quasipolynomial is only one of many one can ask for a given rational polytope. For example, there are many interesting questions and conjectures on when the $\mathrm{h}^*_{\mathbb{Z}}$-polynomial is unimodal. One can, naturally, extend any such question to rational Ehrhart series. Finally, our results generalize to polynomial-weight counting functions of rational polytopes (see, e.g., [Bal+13]), where $\mathrm{ehr}_{\mathbb{Q}}(\mathsf{P};\lambda)$ gets replaced by $\sum_{\mathbf{x}\in\lambda\mathsf{P}\cap\mathbb{Z}^d} p(\mathbf{x})$ for a fixed polynomial $p(\mathbf{x}) \in \mathbb{C}[x_1,\dots,x_d]$.

6

Poset Permutahedra

Order polytopes [Sta86] provide a powerful link between polyhedral geometry and finite posets. Harnessing this connection resulted in many important results including the computation of order polynomials, the fundamental result that computing the volume of a polytope is ♯P-Hard [BW91], and that certain statistics on linear extensions are log-concave [Sta81]. Since the foundational work of Stanley, other poset polytopes have been introduced including marked poset polytopes [ABS11], double poset polytopes [CFS17], and poset associahedra [Gal24]. In this chapter we introduce another class of poset polytopes which provides a unified perspective of polytopes that have been studied recently. This is joint work with Alexander E. Black and Raman Sanyal (unpublished, extended abstract to appear in [ASR]).

Let $\mathcal{P} = ([d], \preceq)$ be a finite poset. The **order polytope** $\mathsf{O}(\mathcal{P})$ is the intersection of the 0/1-hypercube $[0,1]^d$ with the **order cone** $\mathsf{C}(\mathcal{P}) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_a \leq \mathbf{x}_b \text{ for all } a \preceq b\}$. This is a polytope with vertices in $\{0,1\}^d$ with remarkable combinatorics as we recalled in Section 2.5.5. For our construction, recall that the standard **permutahedron** $\Pi_d \subset \mathbb{R}^d$ is the convex hull of all $d!$ permutations of $(1, 2, \ldots, d)$. We define the **poset permutahedron** $\Pi_{\mathcal{P}}$ as the intersection of Π_d with the order cone $\mathsf{C}(\mathcal{P})$, i.e.,

$$\Pi_{\mathcal{P}} := \Pi_d \cap \mathsf{C}(\mathcal{P}) = \Pi_d \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}_a \leq \mathbf{x}_b \text{ for all } a \preceq b\} . \tag{6.1}$$

This is an $(d-1)$-dimensional polytope with half-integral vertices (see Corollary 6.21 below). See Figure 6.1 and Figure 6.2 for examples.

Poset permutahedra provide a unified construction principle for polytopes that have occurred in disparate areas. What follows is a non-exhaustive list:

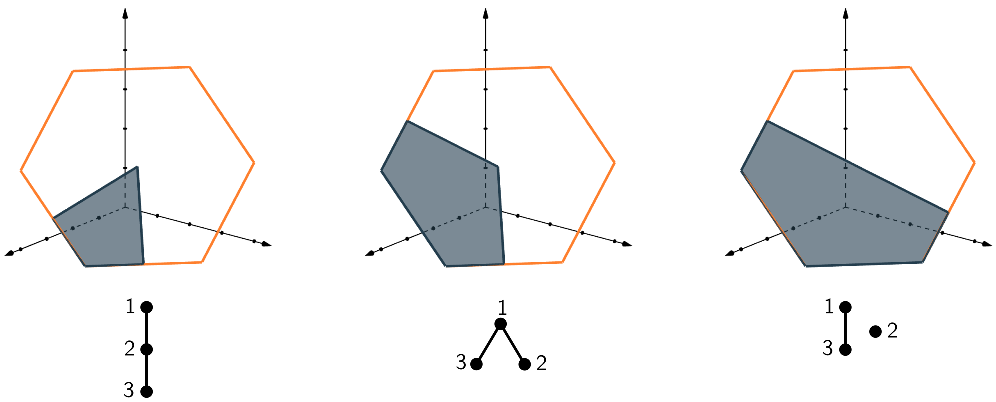


Figure 6.1: Poset permutahedra for posets on three elements are two-dimensional polytopes in $\mathbb{R}^3$. The boundary of the permutahedron Π_3 in orange, the poset permutahedron in dark blue and the Hasse diagram of the corresponding poset in black. Up to symmetry those are all of the two dimensional poset permutahedra other than the permutahedron itself.

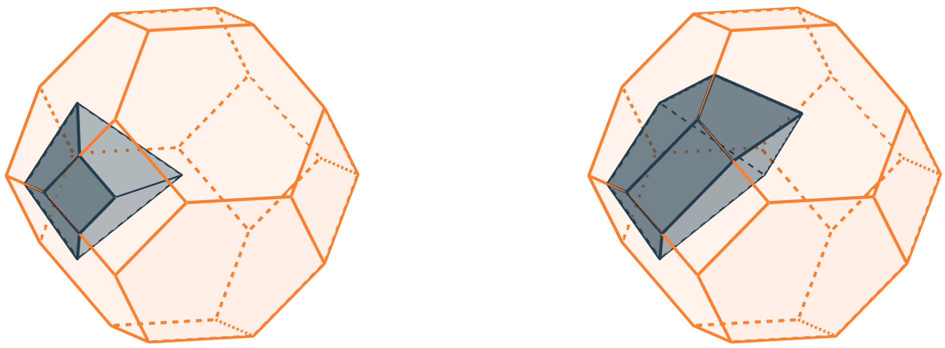


Figure 6.2: Two examples of three dimensional poset permutahedra, for a chain on four elements (left) and a poset on four elements with two linear extensions (right).

(i) If $\mathcal{P}$ is the antichain, then $\Pi_{\mathcal{P}} = \Pi_d$.

(ii) If $\mathcal{P}$ is a chain, then $2\Pi_{\mathcal{P}}$ is unimodularly equivalent to the Newton polytope of the discriminant [GKZ08, Section III.12.2].

(iii) If $\mathcal{P}$ is a chain, then the lattice points in $\Pi_{\mathcal{P}} - \mathbf{1}$ are the well-studied score sequences introduced by Landau [Lan53]; see below for more.

(iv) If $\mathcal{P}$ arises from the antichain by adjoining a maximial element, then $\Pi_{\mathcal{P}}$ is combinatorially equivalent to the stellahedron [PRW08, Section 10.4].

(v) If $\mathcal{P}$ is the disjoint union of two chains of length m and n, respectively, then the face lattice of $\Pi_{\mathcal{P}}$ is ismorphic to the poset of colorful subdivisions of an $(m+n+2)$-gon with bicolored vertices (cf. [AMV24]) and extends the combinatorial description of the Newton polytope of the classical resultant [GKZ08, Chapter 12].

(vi) If $\mathcal{P}$ is the disjoint union of k chains of lengths $m_1, m_2, \ldots, m_k$, then $\Pi_{\mathcal{P}}$ is the type-A partitioned permutahedron introduced and studied in [Hor+24b; Hor+24a] in the context of Hessenberg varieties and representation theory; see Section 6.5 for more.

Our key observation is that $\Pi_{\mathcal{P}}$ is a fiber polytope in the sense of Billera–Sturmfels [BS92]. More precisely, poset permutahedra are (translates of) monotone path polytopes of order polytopes, see Theorem 6.16. We will give detailed and formal definitions of monotone path polytopes in Section 6.1.2 below and a quick intuition here. For a (generic enough) linear functional φ a **φ-monotone path** in the polytope P is a vertex-edge path in P from the vertex minimizing φ to the vertex maximizing φ so that along each edge in the monotone path the value of φ increases. Then the vertices of the monotone path polytope $\Sigma_\varphi(\mathsf{P})$ correspond to the monotone paths in the polytope P. See Figure 6.3 and Figure 6.6 for examples. For a polytope P and a non-constant linear function φ on P, the notion of cellular strings (defined in detail below) generalizes that of maximal φ-monotone paths in the graph of P oriented by φ. The collection of cellular strings ordered by inclusion (i.e., one cellular string is contained within another if the union of cells of one is a subset of the union of cells of the other) is the **Baues poset** of (P, φ) from algebraic topology; see [BKS94].

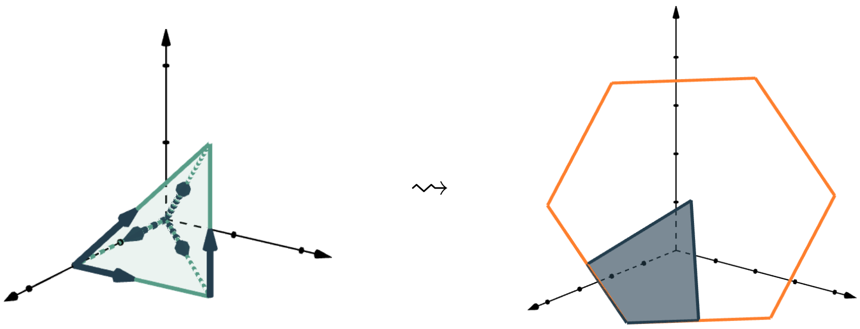


Figure 6.3: The four $\mathbf{1}$-monotone path from $\mathbf{0}$ to $\mathbf{1}$ in the order polytope $\mathsf{O}(\mathcal{C})$ of a three-chain $\mathcal{C}$ (left) correspond to the four vertices of the poset permutahedron $\Pi_{\mathcal{C}}$ (right). Monotone paths in $\mathsf{O}(\mathcal{C})$ corresponding to vertices connected by an edge in $\Pi_{\mathcal{C}}$ share a 2-face in the order polytope $\mathsf{O}(\mathcal{C})$.

While Baues posets can be rather wild, the subposet of coherent cellular strings is isomorphic to the face poset of the monotone path polytope $\Sigma_{\varphi}(\mathsf{P})$. The following theorem is our main tool to study poset permutahedra.

Theorem 6.16. *Let $\mathcal{P} = ([d], \preceq)$ be a poset. Then the poset permutahedron $\Pi_{\mathcal{P}}$ is a translate of the monotone path polytope of the order polytope $\mathsf{O}(\mathcal{P})$ with respect to the linear functional $\mathbf{1}(\mathbf{x}) = \mathbf{x}_1 + \cdots + \mathbf{x}_d$, i.e., $\Pi_{\mathcal{P}} = \Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1}$.*

It is typically nontrivial to determine if a cellular string is coherent. In our situation, however, it turns out that *all* cellular strings are coherent.

Corollary 6.10. *Let $\mathcal{P} = ([d], \preceq)$ be a poset. Then all $\mathbf{1}$-cellular strings of the order polytope $\mathsf{O}(\mathcal{P})$ are coherent. In particular the face lattice of $\Pi_{\mathcal{P}}$ is isomorphic to the Baues poset of $(\mathsf{O}(\mathcal{P}), \mathbf{1})$.*

Corollary 6.10 allows us to prove the following results about poset permutahedra:

(i) Vertices of $\Pi_{\mathcal{P}}$ are in bijection with chains of filters $\varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}$ such that the poset $\mathcal{F}_{i+1} \setminus \mathcal{F}_i$ is connected for all $i = 0, 1, \dots, k$ (Theorem 6.19).

(ii) Theorem 6.22 gives a combinatorial characterization of edges and the corresponding edge directions.

(iii) Facets of $\Pi_{\mathcal{P}}$ are in bijection with the set of proper filters of $\mathcal{P}$ and the cover relations of $\mathcal{P}$ (Corollary 6.17).

(iv) Corollary 6.24 yields the vertex-facet-incidences of $\Pi_{\mathcal{P}}$, and Theorem 6.25 shows that $\Pi_{\mathcal{P}}$ is simple if and only if the undirected Hasse diagram of $\mathcal{P}$ is a forest.

(v) Analogous to order polytopes, $\Pi_{\mathcal{P}}$ is subdivided by the set of all $\Pi_{\mathcal{T}}$, where $\mathcal{T}$ ranges over the linear extensions $\mathfrak{L}(\mathcal{P})$ (Theorem 6.26). This allows us to compute the volume of $\Pi_{\mathcal{P}}$ as $|\mathfrak{L}(\mathcal{P})|\frac{d^{d-2}}{d!}$ (Corollary 6.27).

(vi) For every poset $\mathcal{P}$, the scaled poset permutahedron $2 \cdot \Pi_{\mathcal{P}}$ is a lattice polytope that has the **integer decomposition property**, that is, if $p \in m \cdot 2\Pi_{\mathcal{P}}$ is a lattice point, then there are lattice points $p_1, \dots, p_m \in 2\Pi_{\mathcal{P}}$ with $p = p_1 + \dots + p_m$; see Section 6.4.

Consider a tournament with teams $1, 2, \dots, d$. Any two teams play against each other and during each match n points are distributed between the two teams. This gives rise to a score sequence $(s_1, s_2, \dots, s_d)$. We call $(s_1, s_2, \dots, s_d)$ a **$\mathcal{P}$-score sequence** if $s_i \le s_j$ whenever $i \preceq_{\mathcal{P}} j$ in the poset $\mathcal{P} = ([d], \preceq_{\mathcal{P}})$.

Theorem 6.32. *The $\mathcal{P}$-score sequences for n points are precisely the lattice points in $n \cdot (\Pi_{\mathcal{P}} - \mathbf{1})$.*

Corollary 6.17 gives a simple characterization of $(\mathcal{P}, n)$-score sequences. If $\mathcal{P}$ is a chain, then this characterization is classical and originally due to Landau [Lan53]. While the question of the number of score sequence for $n = 1$ and varying number of teams d has received considerable attention (cf. [Cla+23] and entries A000571, A007747, A047729-A047731, and A047733-A047737 in OEIS [OEI]), we are not aware of results pertaining to the number of score sequences with fixed number of teams d and varying the number of points n. Since $\Pi_{\mathcal{P}} - \mathbf{1}$ is a half-integral polytope it follows that this counting function is a quasipolynomial with period ≤ 2. Computations for small d show that there is no period collapse.

In [Hor+24b] a toric orbifold is associated with a Weyl group W and a choice of a parabolic subgroup W_K. In type A, they call the associated moment polytope a *partitioned permutahedron* $\Pi_d(K)$. It is shown that partitioned permutahedra are simple and their h-vectors were determined by using the cohomology of regular Hessenberg varieties.

Theorem 6.40 ([Hor+24b, Proposition 7.4])**.** *The h-polynomial of the partitioned permutahedron for $K \subseteq [d-1]$ is given by*

$$h_{\Pi_d(K)}(x) = \sum_{\sigma_{\mathsf{v}} \in W(K)} x^{\operatorname{des}(\sigma_{\mathsf{v}})},$$

where $W(K)$ is the set of permutations σ_{v} such that $\sigma_{\mathsf{v}}^{-1}(i) - \sigma_{\mathsf{v}}^{-1}(i+1) \le 1$ for all $i \in K$.

In Section 6.5, we show that $\Pi_d(K)$ is the poset permutahedron of a disjoint union of chains, which implies simplicity by Theorem 6.25. We provide a bijective proof of Theorem 6.40.

6.1 Preliminaries

We start by introducing and recalling properties of fiber polytopes and monotone path polytopes. Monotone path polytopes are a special case of fiber polytopes. We will briefly review the definition and some properties of fiber polytopes that will help us to compute monotone path polytopes.

6.1.1 Fiber Polytopes

Fiber polytopes were introduced by Billera and Sturmfels [BS92] and further discussed in [BS94] and [BKS94]. For more recent introductions to fiber polytopes see, e.g., [Zie98, chapter 9] or [Pou23] and [Bla24], as well as the references therein.

Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a polytope and $\pi\colon \mathsf{P} \to \mathsf{Q}$ be an affine projection. The fiber polytope is intuitively speaking the average of the fibers $\pi^{-1}(\mathbf{x}) \cap \mathsf{P}$ of the projection $\pi\colon \mathsf{P} \to \mathsf{Q}$. We define a **section** as a continuous map $\gamma\colon \mathsf{Q} \to \mathsf{P}$ such that $\pi \circ \gamma = \mathrm{id}_{\mathsf{Q}}$, i.e., $\pi(\gamma(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathsf{Q}$. The **fiber polytope** is defined as the Minkowski integral

$$\begin{aligned} \Sigma(\mathsf{P},\mathsf{Q}) &\coloneqq \frac{1}{\mathrm{vol}(\mathsf{Q})} \int_{\mathsf{Q}} \pi^{-1}(\mathbf{x}) \cap \mathsf{P}\, d\mathbf{x} \\ &\coloneqq \frac{1}{\mathrm{vol}(\mathsf{Q})} \left\{ \int_{\mathsf{Q}} \gamma(\mathbf{x}) d\mathbf{x} \colon \gamma \text{ is a section of } \pi \right\} . \end{aligned} \tag{6.2}$$

With this definition it is a priori not clear that the above set is a polytope. It turns out it is. In fact, it can be computed as a finite Minkowski sum in the following way.

Theorem 6.1 ([BS92, Theorem 1.5])**.** *Consider the subdivision that is the common refinement induced by the images $\pi(\mathsf{F})$ of faces F of P. Denote the maximal cells in that subdivision by $\mathsf{R}_1, \dots, \mathsf{R}_k$ and their respective centroids by $\mathbf{r}_1, \dots, \mathbf{r}_k$. Then*

$$\Sigma(\mathsf{P},\mathsf{Q}) = \frac{\mathrm{vol}(\mathsf{R}_1)}{\mathrm{vol}(\mathsf{Q})}\pi^{-1}(\mathbf{r}_1) + \frac{\mathrm{vol}(\mathsf{R}_2)}{\mathrm{vol}(\mathsf{Q})}\pi^{-1}(\mathbf{r}_2) + \dots + \frac{\mathrm{vol}(\mathsf{R}_k)}{\mathrm{vol}(\mathsf{Q})}\pi^{-1}(\mathbf{r}_k) ,$$

where $\pi^{-1}(\mathbf{r}_i) \subseteq \mathsf{P}$ denotes the fiber of the projection π over $\mathbf{r}_i$. Moreover, the dimension of $\Sigma(\mathsf{P},\mathsf{Q})$ is $\dim(\mathsf{P}) - \dim(\mathsf{Q})$.

In Section 6.1.2 we will define monotone path polytopes as a special case of fiber polytopes, but omit the scaling factor of $\mathrm{vol}(\mathsf{Q})^{-1}$. Characterizing the face lattice is a central tool we will use in this chapter. We first describe the face lattice of general fiber polytopes before specializing everything to monotone path polytopes. In order to do so, we dive into different types of subdivions of Q that arise from the projection $\pi\colon \mathsf{P} \to \mathsf{Q}$.

Recall the definition of subdivision on page 15. We now introduce classes of subdivisions constructed via projections of polytopes. Let $\pi : \mathsf{P} \to \mathsf{Q}$ be an affine projection of polytopes. A polyhedral subdivision of a polytope Q is called **induced** by π from P if there is a subset $\mathcal{F}$ of faces of P such that the subdivision is of the form $\mathcal{S}_\pi(\mathcal{F}) = \{\mathsf{R}_\mathsf{F} = \pi(\mathsf{F}) \colon \mathsf{F} \in \mathcal{F}\} = \pi(\mathcal{F})$. Note that $\dim(\pi(\mathsf{F})) \leq \dim(\mathsf{F})$. We call an induced subdivisions $\mathcal{S}_\pi$ **tight** if $\dim(\mathsf{R}_\mathsf{F}) = \dim(\mathsf{F})$ for every cell $\mathsf{R}_\mathsf{F} \in \mathcal{S}_\pi = \pi(\mathcal{F})$. We consider the cells in the π-induced subdivision $\mathcal{S}_\pi(\mathcal{F})$ as labeled by the faces $\mathsf{F} \in \mathcal{F}$, i.e., subdivisions that "look the same" in $\mathsf{P} = \pi(\mathsf{Q})$ but are induced by different subsets of faces are considered different subdivisions.

We define a partial order on induced subdivisions by **refinement**, that is, for $\mathcal{F}_1$ and $\mathcal{F}_2$ two subsets of faces in P inducing subdivisions in Q we define

$$\mathcal{F}_1 \preceq \mathcal{F}_2 \quad :\Leftrightarrow \quad \bigcup_{\mathsf{F} \in \mathcal{F}_1} \mathsf{F} \subseteq \bigcup_{\mathsf{F} \in \mathcal{F}_2} \mathsf{F} .$$

For a linear functional $\boldsymbol{\psi} \in (\mathbb{R}^d)^*$ we now define **$\boldsymbol{\psi}$-coherent** induced subdivisions $\mathcal{S}_\pi^{\boldsymbol{\psi}}$. For every $\mathbf{x} \in \mathsf{Q}$ consider first the fiber $\pi^{-1}(\mathbf{x}) \cap \mathsf{P}$ and then its subset maximized by the linear functional $\boldsymbol{\psi}$, i.e., $(\pi^{-1}(\mathbf{x}) \cap \mathsf{P})^{\boldsymbol{\psi}}$. This is a subset of the polytope P and hence, there is a unique minimal face $\mathsf{F}_{\mathbf{x}}$ of P containing $(\pi^{-1}(\mathbf{x}) \cap \mathsf{P})^{\boldsymbol{\psi}}$. Now $\mathcal{S}_\pi^{\boldsymbol{\psi}}$ is defined by the collection of those faces $\mathsf{F}_{\mathbf{x}}$, i.e.,

$$\mathcal{S}_\pi^{\boldsymbol{\psi}} := \left\{ \pi(\mathsf{F}_{\mathbf{x}}) \colon \mathbf{x} \in \mathsf{Q} \text{ and } \mathsf{F}_{\mathbf{x}} \text{ is the minimal face of } \mathsf{P} \text{ containing } (\pi^{-1}(\mathbf{x}) \cap \mathsf{P})^{\boldsymbol{\psi}} \right\} .$$

This is well defined, since it can be checked that if $\mathbf{y}$ lies in the relative interior $\pi(\mathsf{F}_{\mathbf{x}})^\circ$, then $\mathsf{F}_{\mathbf{y}} = \mathsf{F}_{\mathbf{x}}$. See also [Zie98, Def 9.2] for an equivalent definition.

Theorem 6.2 ([BS92, Theorem 2.1]). *The face lattice of the fiber polytope* $\Sigma(\mathsf{P}, \mathsf{Q})$ *is isomorphic to the poset of all coherent subdivisions of* Q*. Here, the vertices of* $\Sigma(\mathsf{P}, \mathsf{Q})$ *correspond to the tight coherent subdivisions of* Q*.*

Remark 6.3. The proof for Theorem 6.2 essentially shows that a face F of the fiber polytope $\Sigma(P, Q)$ is given by the linear functional $\boldsymbol{\psi}$, i.e., $\mathsf{F} = \Sigma(P, Q)^{\boldsymbol{\psi}}$, if and only if the corresponding π-induced subdivision is coherent with respect to $\boldsymbol{\psi}$. That is, fix a coherent π-induced subdivision $\mathcal{S}_\pi$ and define the polyhedral cone

$$\mathsf{C}(\mathcal{S}_\pi) := \{\boldsymbol{\psi} \in (\mathbb{R}^d)^* \colon \mathcal{S}_\pi \text{ is coherent w.r.t. } \boldsymbol{\psi}\} .$$

Then the normal fan of the fiber polytope $\Sigma(P, Q)$ is given by the collection of the cones $\mathsf{C}(\mathcal{S}_\pi)$ for all coherent π-induced subdivisions $\mathcal{S}_\pi$. Very broadly speaking this is a consequence of the fact that Minkowski sums behave well under maximizing with respect to a linear functional. (Compare [BS92, Corollary 1.4] and Equation (2.4).)

If the polytope $\mathsf{Q} = \{\mathbf{q}\}$ is just a point, then the fiber polytope $\Sigma(\mathsf{P}, \mathsf{Q})$ is the polytope P itself. Increasing the dimension of Q the next case is when Q is a line segment, i.e., $\Sigma(\mathsf{P}, \mathsf{Q})$ is a monotone path polytope, which we discuss in the next section.

6.1.2 Monotone Path Polytopes

In this section we define monotone path polytopes and discuss important properties that we will need later on. Monotone path polytopes have connections to linear programming since they encode shadow vertex rules for the simplex method, see, e.g., [Pou23, Chapter 4] or [Bla24, Appendix].

A **monotone path polytope** is a special fiber polytope where the projection $\pi \colon \mathsf{P} \to \mathsf{Q} \subseteq \mathbb{R}$ has a one dimensional image [BS92, Section 5]. So, the linear projection π is given by a linear functional $\boldsymbol{\varphi} \in (\mathbb{R}^d)^*$. We will assume $\boldsymbol{\varphi}$ to be **edge-generic** with respect to P, i.e., for any two vertices $\mathbf{u}$ and $\mathbf{v}$ connected by an edge in P we want $\boldsymbol{\varphi}(\mathbf{u}) \neq \boldsymbol{\varphi}(\mathbf{v})$. For the definition of the monotone path polytope, we will omit the scaling factor of $\mathrm{vol}(\mathsf{Q})^{-1} = \mathrm{vol}(\boldsymbol{\varphi}(\mathsf{P}))^{-1}$ from the original definition

(6.2), i.e., if we denote the monotone path polytope of P with respect to φ by $\Sigma_\varphi(\mathsf{P})$ and for $\varphi(\mathsf{P}) =: [t_1, t_2]$, then

$$\Sigma_\varphi(\mathsf{P}) \;:=\; \mathrm{vol}([t_1,t_2])\,\Sigma(\mathsf{P},[t_1,t_2]) \;=\; \int_{\mathbb{R}} \pi^{-1}(s) \cap \mathsf{P}\, ds\,. \tag{6.3}$$

It follows from Theorem 6.1 that the monotone path polytope of P with respect to edge generic φ has dimension $\dim(\mathsf{P}) - 1$.

A **cellular string** in P with respect to φ (we also say φ-cellular string) is a sequence $\mathcal{C} = (\mathsf{F}_1, \ldots, \mathsf{F}_k)$ of faces F_i in P such that $\mathsf{F}_1^{-\varphi} = \mathsf{P}^{-\varphi}$, $\mathsf{F}_k^{\varphi} = \mathsf{P}^{\varphi}$, and $\mathsf{F}_{i-1}^{\varphi} = \mathsf{F}_i^{-\varphi}$ for $i = 2, \ldots, k$. We may assume that faces in a cellular string have dimension at least one. That is, φ-cellular strings are precisely the φ-induced subdivisions of $\varphi(\mathsf{P})$ (as defined in Section 6.1.1). The tight φ-induced subdivisions are the **φ-monotone paths** in P, i.e., φ-cellular strings consisting only of edges.

In the same way that we defined a partial order on subdivisions, we define a **partial order on cellular strings** by

$$(\mathsf{F}_1, \ldots, \mathsf{F}_k) \preceq (\mathsf{G}_1, \ldots, \mathsf{G}_l) \quad :\Leftrightarrow \quad \bigcup_{i=1}^{k} \mathsf{F}_i \subseteq \bigcup_{j=1}^{l} \mathsf{G}_j\,.$$

We then also say the cellular string $(\mathsf{F}_1, \ldots, \mathsf{F}_k)$ is **contained** in $(\mathsf{G}_1, \ldots, \mathsf{G}_l)$. Similarly, we specialize the definition of coherence to cellular strings: a φ-cellular string $(\mathsf{F}_1, \ldots, \mathsf{F}_k)$ in P is **ψ-coherent** if for every $x \in \varphi(\mathsf{F}_i)$

$$(\varphi^{-1}(x) \cap \mathsf{P})^{\psi} \subseteq \mathsf{F}_i$$

and F_i is the minimal face of P with that property. We also call those **ψ-coherent φ-cellular strings** in P. Specializing Theorem 6.2 to monotone path polytopes, we can describe their face lattice.

Corollary 6.4. *The face lattice of a monotone path polytope $\Sigma_\varphi(\mathsf{P})$ is isomorphic to the poset of coherent φ-cellular strings in P.*

This means that vertices of $\Sigma_\varphi(\mathsf{P})$ correspond to smallest coherent φ-cellular strings, i.e., to the coherent φ-monotone paths, which explains the name monotone path polytope.

There are several equivalent and useful perspectives on coherent cellular strings, see, e.g., [Pou23, Chapter 4] or [Bla24, Appendix]. One is to specialize Remark 6.3 to monotone path polytope. Then the normal fan of monotone path polytopes is given by grouping linear functionals according to the coherent cellular strings they define. Recall that we assume φ to be generic with respect to the polytope P. Then $\mathsf{P}^{-\varphi}$ is a vertex v, equivalently, $-\varphi$ is contained in the interior of the full-dimensional normal cone $\mathsf{N}_\mathsf{P}(\mathsf{v})^\circ$ in the normal fan $\mathcal{N}(\mathsf{P})$. This means for any linear functional ψ there is a small enough $\epsilon > 0$ such that $\epsilon\psi - \varphi$ is still contained in $\mathsf{N}_\mathsf{P}(\mathsf{v})^\circ$, equivalently, for any linear functional ψ there is a big enough λ_- such that $\psi - \lambda\varphi$ is still contained in $\mathsf{N}_\mathsf{P}(\mathsf{v})^\circ$ for all $\lambda \geq \lambda_-$. Similarly, for $+\varphi$ there is a λ_+ such that $\psi + \lambda\varphi$ is still contained in the interior of the normal cone corresponding to P^{φ} for all $\lambda \geq \lambda_+$. Then the line segment

$$\{\boldsymbol{\psi} + \lambda \boldsymbol{\varphi} \in (\mathbb{R}^d)^* \colon -\lambda_- \leq \lambda \leq \lambda_+\}$$

starts in $\mathsf{N}_{\mathsf{P}}(\mathsf{P}^{-\varphi})^\circ$, it ends in $\mathsf{N}_{\mathsf{P}}(\mathsf{P}^{\varphi})^\circ$ and intersects with normal cones in $\mathcal{N}(\mathsf{P})$ corresponding to edges and vertices of P, due to the genericity assumption. In fact, this corresponds to $\boldsymbol{\varphi}$-monotone paths in P, which are $\boldsymbol{\psi}$-coherent.

Lemma 6.5. *Let $\mathsf{P} \subset \mathbb{R}^d$ be a polytope and let $\mathcal{C} = (\mathsf{F}_1, \ldots, \mathsf{F}_k)$ be a $\boldsymbol{\varphi}$-cellular string of P. Then $\mathcal{C} = (\mathsf{F}_1, \ldots, \mathsf{F}_k)$ is a $\boldsymbol{\psi}$-coherent if and only if*

$$\{\mathsf{P}^{\psi + \lambda\varphi} \colon \lambda \in \mathbb{R}\} = \{\mathsf{F}_i \colon i = 1, \ldots k\} \cup \{\mathsf{F}_i^{\pm\varphi} \colon i = 1, \ldots, k\}\,.$$

In general it is a hard and interesting question to determine which monotone path are coherent.

Example 6.6. Few examples of monotone path polytopes have been described:

(i) For $\mathsf{P} = [0,1]^d$ and the linear function $\mathbf{1}(\mathbf{x}) = \mathbf{x}_1 + \cdots + \mathbf{x}_d$, we get $\Sigma_{\mathbf{1}}([0,1]^d)$ is homothetic[1] to the standard permutahedron Π_d, as was observed in [BS92, Example 5.4]. We will see a detailed computation for this in Example 6.14.
(ii) The monotone path polytopes of simplices are combinatorial cubes and were described in the original paper on fiber polytopes by Billera and Sturmfels [BS92].
(iii) The monotone path polytope of cross-polytopes was studied in [BD23].
(iv) The monotone path polytope of hypersimplices was described in [Pou24].
(v) The monotone path polytope of (poly)matroid independence polytopes [BS24].

We will now turn to 0/1-polytopes and see that there we are in a very special situation.

6.1.3 Monotone Path Polytopes of $0/1$-polytopes

We now shift our focus to monotone path polytopes of 0/1-polytopes. The following results appeared in [Bla24]. We summarize the statements and give proofs, that we will apply in the next section to order polytopes.

Lemma 6.7 ([Bla24, Lemma 3.4.0.2, Corollary 3.4.2] reformulated)**.** *Let $\mathsf{Q} \subset \mathbb{R}^d$ be a 0/1-polytope. If every edge direction $\mathbf{u} - \mathbf{v}$ is of the form $\mathbb{1}_S$ for some subset $S \subseteq [d]$, then all $\mathbf{1}$-cellular strings of Q are coherent.*

In order to prove Lemma 6.7 we will first return to general polytopes. We call Q a **subpolytope** of P if $\operatorname{vert}(\mathsf{Q}) \subseteq \operatorname{vert}(\mathsf{P})$. For example, every 0/1-polytope, and in particular order polytopes, are subpolytopes of the hypercube $[0,1]^d$.

Lemma 6.8 ([Bla24, Lemma 3.4.0.1])**.** *Let $\mathsf{P} \subset \mathbb{R}^d$ be a polytope, and let Q be a subpolytope of P. Let $\boldsymbol{\varphi} \in (\mathbb{R}^d)^*$ be a linear functional and $(\mathsf{G}_1, \ldots, \mathsf{G}_k)$ be a $\boldsymbol{\psi}$-coherent $\boldsymbol{\varphi}$-cellular string on P for some $\boldsymbol{\psi} \in (\mathbb{R}^d)^*$. Suppose furthermore that there*

1 that is, up to scaling and translation.

exist $1 \leq i < j \leq k$ *such that* $\mathsf{Q}^{-\varphi} = \mathsf{G}_i^{-\varphi}$, $\mathsf{Q}^{\varphi} = \mathsf{G}_j^{\varphi}$ *and that* G_l^{φ} *and* $\mathsf{G}_l^{-\varphi}$ *are contained in* Q *for* $l = i, \dots, j$.

Then $(\mathsf{G}_i \cap \mathsf{Q}, \mathsf{G}_{i+1} \cap \mathsf{Q}, \dots, \mathsf{G}_j \cap \mathsf{Q})$ *is a* $\boldsymbol{\psi}$*-coherent* $\boldsymbol{\varphi}$*-cellular string on* Q.

Proof. Recall form Lemma 6.5 that

$$\{\mathsf{P}^{\boldsymbol{\psi}+\lambda\boldsymbol{\varphi}} \colon \lambda \in \mathbb{R}\} = \{\mathsf{G}_l \colon l = 1, \dots k\} \cup \{\mathsf{G}_l^{\pm\boldsymbol{\varphi}} \colon l = 1, \dots, k\}\,.$$

For $l = i, \dots, j$ we know that $\mathsf{G}_l \cap \mathsf{Q} \neq \varnothing$. For $\lambda_l \in \mathbb{R}$ such that $\mathsf{G}_l = \mathsf{P}^{\boldsymbol{\psi}+\lambda_l\boldsymbol{\varphi}}$ we have

$$\varnothing \neq \mathsf{G}_l \cap \mathsf{Q} = \mathsf{P}^{\boldsymbol{\psi}+\lambda_l\boldsymbol{\varphi}} \cap \mathsf{Q} = \mathsf{Q}^{\boldsymbol{\psi}+\lambda_l\boldsymbol{\varphi}}\,.$$

Then

$$\{\mathsf{Q}^{\boldsymbol{\psi}+\lambda\boldsymbol{\varphi}} \colon \lambda \in \mathbb{R}\} = \{\mathsf{G}_l \cap \mathsf{Q} \colon l = i, \dots j\} \cup \{\mathsf{G}_l^{\pm\boldsymbol{\varphi}} \colon l = i, \dots, j\}\,.$$

$\square$

Proof of Lemma 6.7. We assume $\mathbf{0} \in \mathsf{Q}$ and $\mathbf{1} \in \mathsf{Q}$. This will spare us some technical case distinctions in the following proof. Moreover, in this work, we want to apply Lemma 6.7 only to order polytopes, which always contain $\mathbf{0}$ and $\mathbf{1}$. At the end of the proof we will mention how it needs to be adapted for full generality.

Let $\mathcal{C} = (\mathsf{F}_1, \dots, \mathsf{F}_k)$ be a $\mathbf{1}$-cellular string. Since every edge direction is of the form $\mathbb{1}_S$, the faces $\mathsf{F}_i^{\mathbf{1}}$ and $\mathsf{F}_i^{-\mathbf{1}}$ are vertices for $i = 1, \dots, k$. Define $\mathbf{v}_{i-1} \coloneqq \mathbb{1}_{T_{i-1}} \coloneqq \mathsf{F}_i^{-\mathbf{1}} = \mathsf{F}_{i-1}^{\mathbf{1}}$ for $i = 1, \dots, k$ and $\mathbf{v}_k \coloneqq \mathbb{1}_{T_k} \coloneqq \mathsf{F}_k^{\mathbf{1}} = \mathbf{1}$. Then $\varnothing = T_0 \subsetneq \cdots \subsetneq T_k = [d]$ form a chain of subsets.

We now construct a $\mathbf{1}$-cellular string for the hypercube $[0,1]^d$ as follows:

$$\mathsf{G}_i \coloneqq [0,1]^d \cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}_a = 1 \text{ for all } a \in T_{i-1}\,,\ \mathbf{x}_b = 0 \text{ for all } b \in [d] \setminus T_i\} \tag{6.4}$$

for $i = 1, \dots, k$. Note that

$$\mathsf{G}_1^{-\mathbf{1}} = \mathbf{0}\,,$$
$$\mathsf{G}_k^{\mathbf{1}} = \mathbf{v}_k = \mathbb{1}_{T_k} = \mathbf{1}$$

and for $i = 1, \dots, k$ we have

$$\mathsf{G}_i^{-\mathbf{1}} = \mathsf{G}_{i-1}^{\mathbf{1}} = \mathbb{1}_{T_{i-1}} = \mathbf{v}_{i-1}\,.$$

Moreover, the faces G_i are the ones defined in Corollary 2.31, where the poset is the antichain $\mathcal{A} = ([d], \varnothing)$ and the filters are T_{i-1} and T_i. Hence, $(\mathsf{G}_1, \dots, \mathsf{G}_k)$ is a $\mathbf{1}$-cellular string for the hypercube $[0,1]^d$.

We claim that $(\mathsf{G}_1, \dots, \mathsf{G}_k)$ is coherent with respect to the linear functional $\boldsymbol{\psi} \in (\mathbb{R}^d)^*$ defined by

$$\boldsymbol{\psi} \coloneqq \sum_{i=1}^{k} (k-i)\mathbb{1}_{T_i \setminus T_{i-1}}\,. \tag{6.5}$$

Indeed, we can check that

$$([0,1]^d)^{\boldsymbol{\psi}+\lambda\mathbf{1}} = \begin{cases} \mathbf{0} & \text{for } \lambda < -k+1 \\ \mathsf{G}_i & \text{for } \lambda = -k+i \text{ and } i = 1,\dots,k \\ \mathbb{1}_{T_i} & \text{for } -k+i < \lambda < -k+i+1 \text{ and } i = 1,\dots,k-1 \\ \mathbf{1} & \text{for } 0 < \lambda \end{cases}$$

and use Lemma 6.5.

From Lemma 6.8 it follows that $(\mathsf{G}_1 \cap \mathsf{Q}, \dots, \mathsf{G}_k \cap \mathsf{Q})$ also is a $\boldsymbol{\psi}$-coherent $\mathbf{1}$-cellular string on Q. Recall that the face G_i contains all vertices $\mathbb{1}_R$ for $T_{i-1} \subseteq R \subseteq T_i$. Since $\mathsf{G}_i^{\pm\mathbf{1}} = \mathsf{F}_i^{\pm\mathbf{1}}$ and every edge direction in Q is of the form $\mathbb{1}_S$ for some $S \subseteq [d]$ it follows that $\mathsf{F}_i \subseteq \mathsf{G}_i \cap \mathsf{Q}$. It is left to show that $(\mathsf{F}_1, \dots, \mathsf{F}_k)$ is also coherent.

We will do so by inductively perturbing the linear functional $\boldsymbol{\psi}_0 := \boldsymbol{\psi}$, which certified the coherence for $(\mathsf{G}_1, \dots, \mathsf{G}_k)$ and $(\mathsf{G}_1 \cap \mathsf{Q}, \dots, \mathsf{G}_k \cap \mathsf{Q})$. Let $m \in \{1, \dots, k\}$ be the minimal index such that $\mathsf{F}_m \subsetneq \mathsf{G}_m \cap \mathsf{Q}$ and let $\boldsymbol{\psi}_{m-1}$ be the linear functional certifying coherence for the cellular string

$$(\mathsf{F}_1, \dots, \mathsf{F}_{m-1}, \mathsf{G}_m \cap \mathsf{Q}, \dots, \mathsf{G}_k \cap \mathsf{Q}) \,.$$

Since F_m is a face of $\mathsf{G}_m \cap \mathsf{Q}$ there exists a linear functional

$$\boldsymbol{\omega}_m \in \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}_a = 1 \text{ for all } a \in T_{m-1}\,,\ \mathbf{x}_b = 0 \text{ for all } b \in [d] \setminus T_m\}$$

such that $\mathsf{F}_m = (\mathsf{G}_m \cap \mathsf{Q})^{\boldsymbol{\omega}_m}$. Recall that

$$\mathsf{G}_m \cap \mathsf{Q} = ([0,1]^d)^{\boldsymbol{\psi}_{m-1}+\lambda_j\mathbf{1}} \cap \mathsf{Q} = \mathsf{Q}^{\boldsymbol{\psi}_{m-1}+\lambda_j\mathbf{1}} \,.$$

Then there exists an $\epsilon_m > 0$ small enough such that

$$\mathsf{F}_m = (\mathsf{G}_m \cap \mathsf{Q})^{\boldsymbol{\omega}_m} = (\mathsf{Q}^{\boldsymbol{\psi}_{m-1}+\lambda_j\mathbf{1}})^{\boldsymbol{\omega}_m} = \mathsf{Q}^{\epsilon_m\boldsymbol{\omega}_m+\boldsymbol{\psi}_{m-1}+\lambda_j\mathbf{1}}$$

(see, e.g., [Grü03, Lemma 3.1.5]). We choose ϵ_m small enough so that also the largest absolute value in $\epsilon_m\boldsymbol{\omega}_m$ is less than 1. Then we still have $\mathsf{G}_j = ([0,1]^d)^{\epsilon_m\boldsymbol{\omega}_m+\boldsymbol{\psi}+\lambda_j\mathbf{1}}$ for $j \neq m$, because adding $\epsilon_m\boldsymbol{\omega}_m$ adds a small constant ϵ to all entries in T_{m-1} and nothing to all entries in $[d] \setminus T_m$, only entries in $T_m \setminus T_{m-1}$ are changed substantially. Hence $(\mathsf{F}_1, \dots, \mathsf{F}_m, \mathsf{G}_{m+1} \cap \mathsf{Q}, \dots, \mathsf{G}_k \cap \mathsf{Q})$ is a $(\epsilon_m\boldsymbol{\omega}_m + \boldsymbol{\psi}_{m-1})$-coherent $\mathbf{1}$-cellular string.

If $\mathbf{0} \notin \mathsf{Q}$ or $\mathbf{1} \notin \mathsf{Q}$ we need to add faces G_0 or G_{k+1} in Equation (6.4) to construct a cellular string for the hypercube $[0,1]^d$ and then adapt every following step in the proof accordingly. □

Remark 6.9. By [Edm+21, Theorem 2.1] we can relax the conditions on the linear functional $\boldsymbol{\varphi}$ in Lemma 6.7: As long as the orientation on the edges of the 0/1-polytope $\mathsf{Q} \subset \mathbb{R}^d$ induced by a linear $\boldsymbol{\varphi}$ is the same as the orientation induced by $\mathbf{1}$, the poset of cellular strings will remain isomorphic to the poset of *coherent* cellular strings. Since we require every edge direction of Q to be parallel to $\mathbb{1}_S$ for some subset $S \subseteq [d]$, this is the case as long as φ is positive on the non-negative orthant $\mathbb{R}_{\geq 0}$.

Little is known about polytopes for which all cellular strings are coherent for some choice of linear functional φ. The above result adds to this list, which includes simplices and hypercubes [BS92], (poly)matroid independence polytopes [BS24], and certain zonotopes [Edm+21].

6.2 The Monotone Path Polytopes of Order Polytopes

The goal of this section is to show that poset permutahedra $\Pi_{\mathcal{P}}$ arise as monotone path polytopes of order polytopes $\mathsf{O}(\mathcal{P})$ with respect to the linear functional $\mathbf{1}$ (up to a translation), see Theorem 6.16. We will do so by deriving an irredundant inequality description of $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P}))$. We first find a combinatorial description of the facets of $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P}))$ (Theorem 6.12), then a description as Minkowski sum (Proposition 6.13), to finally derive the correct translation of the facet-defining inequalities (Proposition 6.15).

Corollary 6.10. *Let $\mathcal{P} = ([d], \preceq)$ be a poset. Then all $\mathbf{1}$-cellular strings of the order polytope $\mathsf{O}(\mathcal{P})$ are coherent.*

Remark 6.11. Following Remark 6.9 we could generalize Corollary 6.10 to linear functionals φ that are positive on $\mathbb{R}^d_{\geq 0}$ and the resulting monotone path polytope $\Sigma_{\varphi}(\mathsf{O}(\mathcal{P}))$ would have the same combinatorial structure. However, as we will see in the remainder of this chapter with the linear functional $\mathbf{1}$ we get a nice geometric realization, e.g., interesting interpretations for integer points (see Section 6.4).

To determine the facets of $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P}))$ it suffices by Corollary 6.10 to determine the coarsest, nontrivial cellular strings. Note that the coarsest (and trivial) cellular string is $(\mathsf{O}(\mathcal{P}))$.

Theorem 6.12. *The coarsest nontrivial cellular strings of $\mathsf{O}(\mathcal{P})$ with respect to $\mathbf{1}$ are of the following two forms:*

(i) (F), where F is a facet of the order polytope $\mathsf{O}(\mathcal{P})$ corresponding to a cover relation $a \prec\!\!\cdot\, b$.

(ii) $(\mathsf{F}_1, \mathsf{F}_2)$, where F_1, F_2 are faces of $\mathsf{O}(\mathcal{P})$ and there exists a filter $\mathcal{F}$ in $\mathcal{P}$ such that all vertices in F_1 correspond to filters contained in $\mathcal{F}$ and all vertices in F_2 correspond to filters containing $\mathcal{F}$.

Proof. Recall that facets of the monotone path polytope correspond to inclusion maximal nontrivial cellular strings, see Corollary 6.4. Consider an arbitrary cellular string $(\mathsf{F}'_1, \dots, \mathsf{F}'_k)$ in $\mathsf{O}(\mathcal{P})$.

If the cellular string consists of only one face F', then F' is a face containing the vertices $\mathbf{0}$ and $\mathbf{1}$. Every face is contained in a facet and the only facets of $\mathsf{O}(\mathcal{P})$ containing $\mathbf{0}$ and $\mathbf{1}$ are the ones defined by cover relations (see Equation (2.25)). So let F be a facet containing F' and defined by a cover relation in the poset $\mathcal{P}$. Then (F) forms a cellular string and since F is a facet the only face of $\mathsf{O}(\mathcal{P})$, it is contained in, is $\mathsf{O}(\mathcal{P})$ itself. So (F) is a coarsest nontrivial cellular string as described in (i).

Now we assume that the cellular string $(\mathsf{F}'_1, \dots, \mathsf{F}'_k)$ consists of at least two faces, i.e., $k \geq 2$. In the proof of Lemma 6.7, see Equation (6.4), we have shown that every such cellular string of a $0/1$-polytope is contained in a cellular string of the form $(\mathsf{G}_1 \cap \mathsf{O}(\mathcal{P}), \dots, \mathsf{G}_k \cap \mathsf{O}(\mathcal{P}))$, where

$$\mathsf{G}_i := [0,1]^d \cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}_a = 1 \text{ for all } a \in T_{i-1}\,,\ \mathbf{x}_b = 0 \text{ for all } b \in [d] \setminus T_i\}$$

for $\varnothing = T_0 \subsetneq \cdots \subsetneq T_k = [d]$.

For $i \in \{1, \dots, k-1\}$ we set $\mathcal{F} := T_i$, let F_1 be the face of $\mathsf{O}(\mathcal{P})$ containing all vertices corresponding to filters contained in $\mathcal{F}$ and let F_2 be the face of $\mathsf{O}(\mathcal{P})$ containing all vertices corresponding to filters containing in $\mathcal{F}$ (as described in (ii)). Then we have that for $1 \leq l \leq i-1$ the faces $\mathsf{G}_l \cap \mathsf{Q}$ are contained in F_1 and for $i \leq l \leq k$ the faces $\mathsf{G}_l \cap \mathsf{Q}$ are contained in F_2. Hence, the cellular string $(\mathsf{G}_1 \cap \mathsf{O}(\mathcal{P}), \dots, \mathsf{G}_k \cap \mathsf{O}(\mathcal{P}))$ in turn is coarsened by the cellular strings of the form $(\mathsf{F}_1, \mathsf{F}_2)$ as just defined.

It is left to show that these cellular strings cannot be coarsened. First we observe that F_1 is the inclusion-maximal face in $\mathsf{O}(\mathcal{P})$ containing the vertices $\mathbf{0}$ and $\mathbb{1}_{\mathcal{F}}$, similarly, F_2 is the inclusion-maximal face containing the vertices $\mathbb{1}_{\mathcal{F}}$ and $\mathbf{1}$. Hence the cellular string $(\mathsf{F}_1, \mathsf{F}_2)$ cannot be coarsened while retaining the vertex $\mathbb{1}_{\mathcal{F}}$ as the intersection point, i.e., there is no coarser cellular string with two cells.

Let us assume there is a cellular string (F) coarsening $(\mathsf{F}_1, \mathsf{F}_2)$. Then F has to contain F_1 and F_2. However, $\mathsf{F}_1 \cup \mathsf{F}_2$ contains a sequence of vertices of $\mathsf{O}(\mathcal{P})$ corresponding to a full chain of filters in $\mathcal{P}$ passing through $\mathcal{F}$. Such a full chain of filters defines a liner extension of $\mathcal{P}$, since the difference of two consecutive filters is just one element (see Section 2.5.5). Hence, these are the vertices of a full-dimensional simplex in the triangulation of $\mathsf{O}(\mathcal{P})$ and the only face that can contain $\mathsf{F}_1 \cup \mathsf{F}_2$ is $\mathsf{O}(\mathcal{P})$. So the only cellular string coarsening $(\mathsf{F}_1, \mathsf{F}_2)$ is the trivial string $(\mathsf{O}(\mathcal{P}))$.

We have shown that every cellular string is contained in (at least) one of the cellular strings described in (i) and (ii), and they are the coarsest nontrivial ones. ☐

Proposition 6.13. *Let $\mathcal{P} = ([d], \preceq)$ be a partially ordered set. We can write the monotone path polytope of the order polytope $\mathsf{O}(\mathcal{P}) \subset \mathbb{R}^d$ with respect to the linear functional $\mathbf{1}$ as the following Minkowski sum:*

$$\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) = \sum_{i=1}^{d-1} (\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i) + \frac{1}{2}\mathbf{1}\,,$$

where $\mathsf{H}_i := \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{1}(\mathbf{x}) = i\}$.

Proof. We apply Theorem 6.1. The image of $\mathbf{1} : \mathsf{O}(\mathcal{P}) \to \mathbb{R}$ is the line segment $[0, d] \subset \mathbb{R}$ and vertices of $\mathsf{O}(\mathcal{P})$ are mapped to $i \in \mathbb{Z}_{\geq 0}$ for $i \leq d$. So, the maximal cells in the subdivision induced by $\mathbf{1}(\mathsf{F})$ for faces F of $\mathsf{O}(\mathcal{P})$ are intervals $[i-1, i]$ and their centroids $\mathbf{r}_i$ are $\mathbf{r}_i = \frac{2i-1}{2}$ for $i \in \{1, \dots, d\}$ and the fibers $\mathbf{1}^{-1}(\mathbf{r}_i) = \mathsf{O}(\mathcal{P}) \cap \mathsf{H}_{\frac{2i-1}{2}}$.

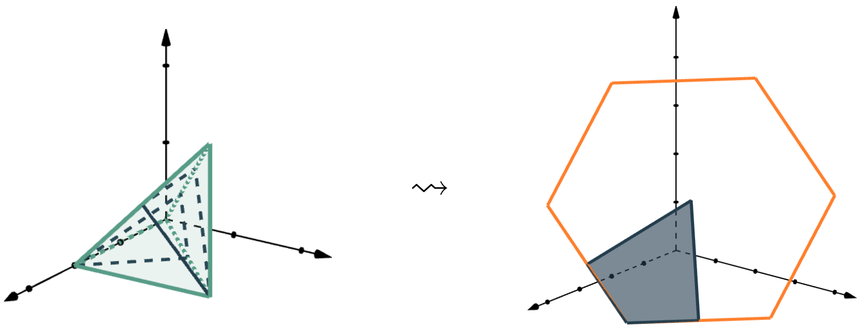


Figure 6.4: Order simplex $\mathsf{O}(\mathcal{C})$ (left) and the intersection with the hyperplanes H_1 and H_2 highlighted in dark blue. The Minkowski sum of those two slices is the poset permuathedron $\Pi_{\mathcal{C}}$ (right).

Recall from Equation (6.3) that we defined the monotone path polytope as a dilate of the original fiber polytope. Then

$$\begin{aligned}\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) &= \mathrm{vol}([0,d])\,\Sigma(\mathsf{O}(\mathcal{P}),[0,d])\\ &= \mathrm{vol}([0,d])\sum_{i=1}^{d}\frac{\mathrm{vol}([i-1,i])}{\mathrm{vol}([0,d])}\left(\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{\frac{2i-1}{2}}\right)\\ &= \sum_{i=1}^{d}\left(\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{\frac{2i-1}{2}}\right).\end{aligned}$$

We claim that:

$$\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{\frac{2i-1}{2}} = \mathrm{conv}\Big((\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{i-1})\cup(\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{i})\Big)\ \cap\ \mathsf{H}_{\frac{2i-1}{2}} \tag{6.6}$$

$$= \frac{1}{2}(\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{i-1}) + \frac{1}{2}(\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{i}). \tag{6.7}$$

For Equation (6.6), first note that

$$\mathsf{O}(\mathcal{P})\supseteq(\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{i-1})\cup(\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{i}).$$

This proves one inclusion. For the other inclusion, recall that the vertices of $\mathsf{O}(\mathcal{P})$ are $0/1$-vectors and therefore all the vertices are contained in the hyperplanes H_i for $i = 0,1,\ldots,d$. Then

$$\mathsf{O}(\mathcal{P})\cap\mathsf{H}_{\frac{2i-1}{2}}\subseteq\mathsf{H}^{\geq}_{i-1}\cap\mathsf{O}(\mathcal{P})\cap\mathsf{H}^{\leq}_{i},$$

with $\mathsf{H}^{\geq}_{i-1} := \{\mathbf{x}\in\mathbb{R}^d\colon \mathbf{1}(\mathbf{x})\geq i-1\}$ and $\mathsf{H}^{\leq}_{i} := \{\mathbf{x}\in\mathbb{R}^d\colon \mathbf{1}(\mathbf{x})\leq i\}$. Now, $\mathsf{H}^{\geq}_{i-1}\cap\mathsf{O}(\mathcal{P})\cap\mathsf{H}^{\leq}_{i}$ defines a polytope, again with all vertices contained in H_{i-1} and H_i. This proves the other inclusion and hence Equation (6.6).

For Equation (6.7), let $\mathbf{x} \in \operatorname{conv}\Big((\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_{i-1}) \cup (\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i)\Big) \cap \mathsf{H}_{\frac{2i-1}{2}}$ be arbitrary. Then there are points $\mathbf{p}_1, \dots, \mathbf{p}_k \in \mathsf{O}(\mathcal{P}) \cap \mathsf{H}_{i-1}$ and $\mathbf{q}_1, \dots, \mathbf{q}_l \in \mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i$, as well as $\lambda_1, \dots, \lambda_k \geq 0$, $\mu_1, \dots, \mu_l \geq 0$ such that

$$\mathbf{x} = \sum_{j=1}^{k} \lambda_j \mathbf{p}_j + \sum_{j=1}^{l} \mu_j \mathbf{q}_j \quad \text{and} \quad \sum_{j=1}^{k} \lambda_j + \sum_{j=1}^{l} \mu_j = 1\,.$$

We compute

$$\begin{aligned} \tfrac{2i-1}{2} = \mathbf{1}(\mathbf{x}) &= \sum_{j=1}^{k} \lambda_j \mathbf{1}(\mathbf{p}_j) + \sum_{j=1}^{l} \mu_j \mathbf{1}(\mathbf{q}_j) = (i-1)\sum_{j=1}^{k} \lambda_j + i \sum_{j=1}^{l} \mu_j \\ &= i \underbrace{\left(\sum_{j=1}^{k} \lambda_j + \sum_{j=1}^{l} \mu_j \right)}_{=1} - \sum_{j=1}^{k} \lambda_j = i - \sum_{j=1}^{k} \lambda_j\,. \end{aligned}$$

This implies

$$\sum_{j=1}^{k} \lambda_j = \frac{1}{2} = \sum_{j=1}^{l} \mu_j\,.$$

Then $\mathbf{x} = \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$, where

$$\mathbf{p} \coloneqq \sum_{j=1}^{k} 2\lambda_j \mathbf{p}_j \quad \in \mathsf{O}(\mathcal{P}) \cap \mathsf{H}_{i-1} \quad \text{and} \quad \mathbf{q} \coloneqq \sum_{j=1}^{l} 2\mu_j \mathbf{q}_j \quad \in \mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i\,.$$

Hence, $\mathbf{x} \in \frac{1}{2}(\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_{i-1}) + \frac{1}{2}(\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i)$. The other inclusion follows from similar computations.

With Equations (6.6) and (6.7) we have

$$\begin{aligned} \Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) = \sum_{i=1}^{d} \Big(\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_{\frac{2i-1}{2}} \Big) &= \sum_{i=1}^{d} \Big(\frac{1}{2}(\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_{i-1}) + \frac{1}{2}(\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i) \Big) \\ &= \sum_{i=1}^{d-1} (\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i) + \frac{1}{2}\mathbf{1} \end{aligned}$$

□

Example 6.14. The order polytope of the antichain on d elements is the unit cube $[0,1]^d$. With Proposition 6.13 we have:

$$\Sigma_{\mathbf{1}}([0,1]^d) + \frac{1}{2}\mathbf{1} = \sum_{i=1}^{d} \Big([0,1]^d \cap \mathsf{H}_i \Big) = \Pi_d\,.$$

The Minkowski sum of hypersimplices $[0,1]^d \cap \mathsf{H}_i$ is yet another representation of the standard permutahedron Π_d, see, e.g., [Pos09, Section 16].

Proposition 6.15. *An irredundant half-space description for the translated monotone path polytope of the order polytope $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1}$ is given by the following (in-)equalities:*

$$\mathbf{1}(\mathbf{x}) = \binom{d+1}{2}, \tag{6.8}$$

$$\mathbb{1}_{\mathcal{F}}(\mathbf{x}) \le f_d(\mathcal{F}), \quad \textit{for every proper non-empty filter } \mathcal{F} \textit{ in } \mathcal{P}, \tag{6.9}$$

$$\mathbf{x}_a \le \mathbf{x}_b, \quad \textit{for all cover relations } a \prec b \textit{ in } \mathcal{P}, \tag{6.10}$$

where $f_d \colon 2^{[d]} \to \mathbb{R}$ is defined by $f_d(S) := |S|d - \binom{|S|}{2}$.

Proof. We first check that the linear functionals $\mathbb{1}_{\mathcal{F}}$ and $\mathbb{1}_{\{a\}} - \mathbb{1}_{\{b\}}$ giving the inequalities, are the facet normals for the monotone path polytope of the order polytope $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1}$.

Recall from Theorem 6.12 the two types of cellular strings defining facets. Let us first consider a cellular strings of the form $(\mathsf{F}_1, \mathsf{F}_2)$ given by a filter $\mathcal{F}$ in $\mathcal{P}$ such that all vertices in F_1 correspond to filters contained in $\mathcal{F}$ and all vertices in F_2 correspond to filters containing $\mathcal{F}$. From the proof of Lemma 6.7, in particular Equation (6.5), we get that $(\mathsf{F}_1, \mathsf{F}_2)$ is coherent with respect to $\boldsymbol{\psi} = \mathbb{1}_{\mathcal{F}}$. (Here, the perturbation part is not needed, since $\mathsf{F}_i = \mathsf{G}_i \cap \mathsf{O}(\mathcal{P})$ for $i = 1, 2$.) Therefore, following Remark 6.3, the corresponding facet of the monotone path polytope is given by $(\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1})^{\mathbb{1}_{\mathcal{F}}}$.

Similarly, we consider a cellular string of the form (F), where F is a facet of $\mathsf{O}(\mathcal{P})$ given by a cover relation $a \prec b$ in $\mathcal{P}$. The proof of Lemma 6.7, in particular Equation (6.5) implies that $\boldsymbol{\psi} = \mathbf{0}$. However, in this case we need to perturb $\boldsymbol{\psi}$ and then (F) is coherent with respect to $\boldsymbol{\psi} + \epsilon(\mathbb{1}_{\{a\}} - \mathbb{1}_{\{b\}}) = \epsilon(\mathbb{1}_{\{a\}} - \mathbb{1}_{\{b\}})$.

Hence, the facet normals are precisely the ones used in the inequalities (6.9) and (6.10). It is left to check that every point in $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1}$ fulfills the inequalities and that for every inequality there are some points that achieve equality.

From Proposition 6.13 we have

$$\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \tfrac{1}{2}\mathbf{1} = \sum_{i=1}^{d-1} (\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i) + \mathbf{1},$$

where $\mathsf{H}_i := \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{1}(\mathbf{x}) = i\}$.

Let $\mathbf{x} \in \Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1}$ be arbitrary. Then there are points $\mathbf{p}_i \in \mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i$ for $i = 1, \dots, d-1$ such that

$$\mathbf{x} = \sum_{i=1}^{d-1} \mathbf{p}_i + \mathbf{1}.$$

Since every point $\mathbf{p}_i \in \mathsf{O}(\mathcal{P})$ fulfills the inequalities in (6.10), every finite sum of points in $\mathsf{O}(\mathcal{P})$ does so too (adding $\mathbf{1}$ does not alter that). For a fixed cover relation, the facet $\mathsf{O}(\mathcal{P}) \cap \{\mathbf{x}_a = \mathbf{x}_b\}$ intersects every hyperplane H_i for $i = 1, \dots, d-1$. So the inequalities in (6.10) are tight. We compute further

$$\mathbf{1}(\mathbf{x}) = \sum_{i=1}^{d-1} \mathbf{1}(\mathbf{p}_i) + d = \sum_{i=1}^{d-1} i + d = \binom{d+1}{2},$$

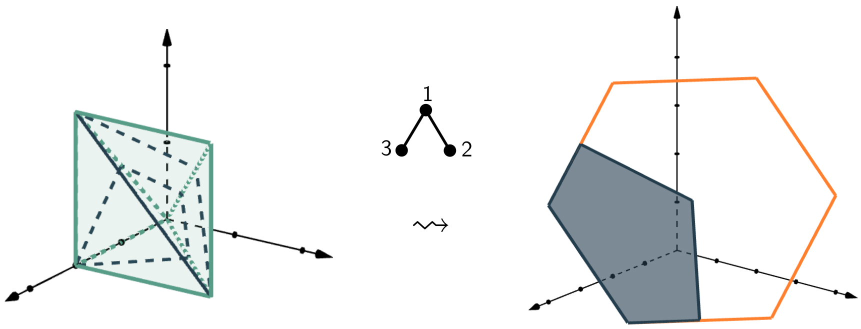


Figure 6.5: Order polytope $\mathsf{O}(\mathcal{P})$ of a tree poset on three elements (left) and the intersection with the hyperplanes H_1 and H_2 highlighted in dark blue. The Minkowski sum of those two slices is the poset permuathedron $\Pi_{\mathcal{P}}$ (right).

verifying (6.8), and for arbitrary proper, non-empty filters $\mathcal{F}$

$$\begin{aligned}
\mathbb{1}_{\mathcal{F}}(\mathbf{x}) &= \sum_{i=1}^{d-1} \mathbb{1}_{\mathcal{F}}(\mathbf{p}_i) + |\mathcal{F}| \\
&\leq \sum_{i=1}^{|\mathcal{F}|} i + \sum_{i=|\mathcal{F}|+1}^{d-1} |\mathcal{F}| + |\mathcal{F}| \;=\; \frac{|\mathcal{F}|(|\mathcal{F}|+1)}{2} + |\mathcal{F}|(d-|\mathcal{F}|) \\
&= \frac{|\mathcal{F}|(|\mathcal{F}|+1+2d-2|\mathcal{F}|)}{2} \;=\; \frac{|\mathcal{F}|(2d-|\mathcal{F}|+1)}{2} \;=\; |\mathcal{F}|d - \binom{|\mathcal{F}|}{2},
\end{aligned}$$

verifying the inequalities in (6.9). Again, for a fixed filter, let $\mathsf{F}_1, \mathsf{F}_2$ be the faces as defined Theorem 6.12(ii). Then $\mathsf{F}_1 \cup \mathsf{F}_2$ intersects every hyperplane H_i for $i = 1, \ldots, d-1$ nontrivially, proving the tightness of the inequalities given in (6.9). □

Now we are ready to prove the goal of this section: poset permutahedra are (translates of) monotone path polytopes of order polytopes.

Theorem 6.16. *Let $\mathcal{P} = ([d], \preceq)$ be a poset. Then the monotone path polytope of the order polytope with respect to the linear function $\mathbf{1}(\mathbf{x}) = \mathbf{x}_1 + \cdots + \mathbf{x}_d$ is a translation of the poset permutahedron, i.e., $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1} = \Pi_{\mathcal{P}}$.*

Proof. By Proposition 6.15, the translated monotone path polytope of the order polytope $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P}))$ has the following inequality description:

$$\begin{aligned}
\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \tfrac{1}{2}\mathbf{1} = \{&\mathbf{x} \in \mathbb{R}^d \colon \mathbf{1}(\mathbf{x}) = \tbinom{d+1}{2}\} \\
&\cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbb{1}_{\mathcal{F}_i}(\mathbf{x}) \leq |\mathcal{F}|d - \tbinom{|\mathcal{F}|}{2} \text{ for all proper non-empty filters } \mathcal{F}\} \\
&\cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}_a \leq \mathbf{x}_b \text{ for all cover relations } a \prec\!\!\cdot\, b \in \mathcal{P}\}.
\end{aligned}$$

By the definition of the poset permutahedron in Equation (6.1) and the inequality description of the permutahedron (see Equation (2.17)),

$$\begin{aligned}\Pi_{\mathcal{P}} = \Pi_d \cap \mathsf{C}_{\mathcal{P}} = \ &\{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{1}(\mathbf{x}) = \tbinom{d+1}{2}\} \\ &\cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbb{1}_T(\mathbf{x}) \leq \tbinom{d+1}{2} - \tbinom{|T|+1}{2} \text{ for all } T \subseteq [d]\} \\ &\cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}_a \leq \mathbf{x}_b \text{ for all cover realtions } a \prec b \in \mathcal{P}\}\end{aligned}$$

One can easily check that for any subset $T \subseteq [d]$ and its complement $T^C := [d] \setminus T$

$$\binom{d+1}{2} - \binom{|T|+1}{2} = \binom{d+1}{2} - \binom{d-|T^C|+1}{2} = |T^C|d - \binom{|T^C|}{2}.$$

Hence, $\Pi_{\mathcal{P}} \subseteq \Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1}$. At the same time, using Example 6.14, we have

$$\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \tfrac{1}{2}\mathbf{1} = \sum_{k=1}^{d} \mathsf{O}(\mathcal{P}) \cap \mathsf{H}_k \subseteq \sum_{k=1}^{d} [0,1]^d \cap \mathsf{H}_k = \Sigma_{\mathbf{1}}([0,1]^d) + \tfrac{1}{2}\mathbf{1} = \Pi_d\,.$$

It follows then that

$$\begin{aligned}\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) &= \Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) \cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}_a \leq \mathbf{x}_b \text{ for all cover realtions } a \prec b \in \mathcal{P}\} \\ &\subseteq \Pi_n \cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}_a \leq \mathbf{x}_b \text{ for all cover realtions } a \prec b \in \mathcal{P}\} = \Pi_{\mathcal{P}}\,.\end{aligned}$$

Therefore, both inclusions have been shown so that $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1} = \Pi_{\mathcal{P}}$ as desired. □

6.3 The Face Structure of Poset Permutahedra

In this section we will use the interpretation of poset permutahedra as monotone path polytopes to describe their face structures. We start by recording a direct consequence from Proposition 6.15 and Theorem 6.16.

Corollary 6.17. *Let $\mathcal{P} = ([d], \preceq)$ be a poset. A point $\mathbf{x} \in \mathbb{R}^d$ is contained in $\Pi_{\mathcal{P}}$ if and only if*

$$\begin{aligned}\mathbf{x}_a &\leq \mathbf{x}_b && \textit{for all cover relations } a \prec b && (6.11)\\ \mathbb{1}_{\mathcal{F}}(\mathbf{x}) &\leq f_d(\mathcal{F}) && \textit{for all proper, non-empty filters } \mathcal{F} \subsetneq \mathcal{P} && (6.12)\\ \mathbf{1}(\mathbf{x}) &= f_d([d]) = \tbinom{d+1}{2}\,,\end{aligned}$$

*where $f_d \colon 2^{[d]} \to \mathbb{R}$ is defined by $f_d(S) := |S|d - \binom{|S|}{2}$. The description is irredundant and the inequalities are facet defining. Facets defined by (6.11) are called **cover facet** and facets defined by (6.12) are called **filter facet**.*

Proposition 6.18. *The function $f_d(S) = |S|d - \binom{|S|}{2}$ defined in Proposition 6.15 is a submodular set function.*

Proof. We want to show that for $X \subseteq Y \subseteq [d]$ and $x \in [d] \setminus Y$

$$f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)\,.$$

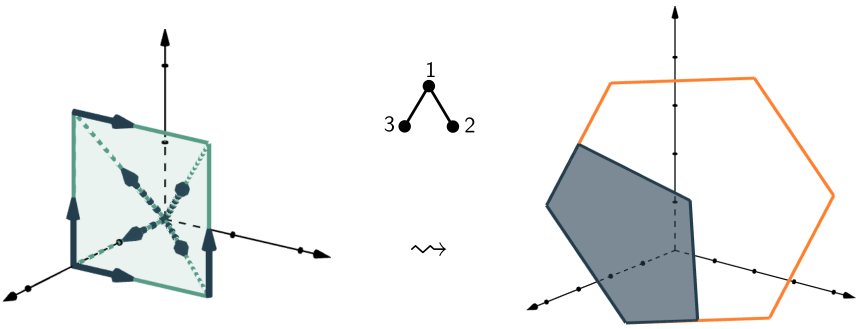


Figure 6.6: The five **1**-monotone path from **0** to **1** in the order polytope $\mathsf{O}(\mathcal{P})$ of a tree-poset $\mathcal{P}$ (left) correspond to the five vertices of the poset permutahedron $\Pi_{\mathcal{P}}$ (right). Monotone paths in $\mathsf{O}(\mathcal{P})$ corresponding to vertices connected by an edge in $\Pi_{\mathcal{P}}$ share a 2-face in the order polytope $\mathsf{O}(\mathcal{P})$.

Indeed,

$$\begin{aligned} f(X \cup \{x\}) - f(X) &= (|X|+1)d - \binom{|X|+1}{2} - |X|d + \binom{|X|}{2} \\ &= d + \frac{-(|X|+1)(|X|+2) + (|X|+1)(|X|)}{2} \\ &= d + |X| + 1 \\ &\geq d + |Y| + 1 = f(Y \cup \{x\}) - f(Y)\,. \end{aligned}$$

□

Recall, for $A \subseteq [d]$, we write $A^C = [d] \setminus A$ for the complement.

Theorem 6.19. *The vertices of $\Pi_{\mathcal{P}}$ are in bijection to connected chains of filters $\mathfrak{F}$ in $\mathcal{P}$. For $\mathfrak{F}\colon \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}$, the corresponding vertex is*

$$\mathsf{v}(\mathfrak{F}) \;=\; \frac{1}{2}\sum_{i=1}^{k}\Big(|\mathcal{F}_i^C| + |\mathcal{F}_{i-1}^C| + 1\Big)\,\mathbb{1}_{\mathcal{F}_i \setminus \mathcal{F}_{i-1}}\,. \tag{6.13}$$

Example 6.20. We consider the example from Figure 6.6. The vertex of $\Pi_{\mathcal{P}}$ located at the barycenter of the permutahedron Π_3 corresponds to the chain of filters $\varnothing \subsetneq [3]$. The two vertices of $\Pi_{\mathcal{P}}$ that coincide with vertices of Π_3 correspond to the full chains of filters $\varnothing \subsetneq \{1\} \subsetneq \{1,2\} \subsetneq [3]$ and $\varnothing \subsetneq \{1\} \subsetneq \{1,3\} \subsetneq [3]$, i.e., they correspond to the two linear extensions. The two vertices of $\Pi_{\mathcal{P}}$ located on the edges of Π_3 correspond to the connected two chains of filters $\varnothing \subsetneq \{1,2\} \subsetneq [3]$ and $\varnothing \subsetneq \{1,3\} \subsetneq [3]$.

Proof of Theorem 6.19. From Corollary 2.29 we have that vertices $\mathbb{1}_{\mathcal{F}_1}$ and $\mathbb{1}_{\mathcal{F}_2}$ of the order polytope are connected by an edge if and only if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and the induced subgraph of the Hasse diagram by $\mathcal{F}_2 \setminus \mathcal{F}_1$ is connected. Such an edge is **1**-improving from $\mathcal{F}_1$ to $\mathcal{F}_2$, so monotone paths correspond exactly to connected chains of filters. By Lemma 6.7, all monotone paths are coherent.

The vertex coordinates then follow from direct computation: From Proposition 6.13 and Theorem 6.16 we know that

$$\Pi_{\mathcal{P}} = \Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1} = \sum_{i=1}^{d-1} (\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i) + \mathbf{1}\,.$$

Fix a connected chain of filters $\mathfrak{F} : \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P} = [d]$. Then there is a linear functional $\boldsymbol{\psi} \in \mathbb{R}^d$ certifying coherence for the corresponding monotone path and the corresponding vertex $\mathsf{v}(\mathfrak{F})$ is

$$\mathsf{v}(\mathfrak{F}) = \Pi_{\mathcal{P}}^{\psi} = \sum_{i=1}^{d-1} (\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i)^{\psi} + \mathbf{1}\,,$$

where we use Equation (2.4). Then $(\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i)^{\psi}$ is a point for every $i = 1, \ldots, d-1$. If $i = |\mathcal{F}_l|$ for some $l = 1, \ldots, k-1$, then the corresponding vertex in the slice $\mathsf{O}(\mathcal{P}) \cap \mathsf{H}_i$ is precisely $\mathbb{1}_{\mathcal{F}_i}$. If $|\mathcal{F}_{l-1}| \leq i \leq |\mathcal{F}_l|$, then the vertex added in the Minkowski sum will be the convex combination

$$\frac{|\mathcal{F}_l| - i}{|\mathcal{F}_l \setminus \mathcal{F}_{l-1}|}\mathbb{1}_{\mathcal{F}_{l-1}} + \frac{i - |\mathcal{F}_{l-1}|}{|\mathcal{F}_l \setminus \mathcal{F}_{l-1}|}\mathbb{1}_{\mathcal{F}_l} = \mathbb{1}_{\mathcal{F}_{l-1}} + \frac{i - |\mathcal{F}_{l-1}|}{|\mathcal{F}_l \setminus \mathcal{F}_{l-1}|}\mathbb{1}_{\mathcal{F}_l \setminus \mathcal{F}_{l-1}}\,. \tag{6.14}$$

Note that the latter description holds in all the cases. In order to compute the vertex $\mathsf{v}(\mathfrak{F}) = \Pi_{\mathcal{P}}^{\psi}$ we add the vertices given in (6.14) for every $i = 1, \ldots, d-1$. We ignore the shift by $+\mathbf{1}$ in the following computation and add it in the end.

$$\begin{aligned}
&\sum_{l=1}^{k-1}\left(\sum_{i=|\mathcal{F}_{l-1}|+1}^{|\mathcal{F}_l|}\left(\mathbb{1}_{\mathcal{F}_{l-1}} + \frac{i - |\mathcal{F}_{l-1}|}{|\mathcal{F}_l| - |\mathcal{F}_{l-1}|}\mathbb{1}_{\mathcal{F}_l \setminus \mathcal{F}_{l-1}}\right)\right) \\
&\qquad\qquad + \sum_{i=|\mathcal{F}_{k-1}|+1}^{d-1}\left(\mathbb{1}_{\mathcal{F}_{k-1}} + \frac{i - |\mathcal{F}_{k-1}|}{d - |\mathcal{F}_{k-1}|}\mathbb{1}_{[d] \setminus \mathcal{F}_{k-1}}\right) \\
&= \sum_{l=1}^{k-1}\left((|\mathcal{F}_l| - |\mathcal{F}_{l-1}|)\,\mathbb{1}_{\mathcal{F}_{l-1}} + \sum_{i=1}^{|\mathcal{F}_l| - |\mathcal{F}_{l-1}|}\frac{i}{|\mathcal{F}_l| - |\mathcal{F}_{l-1}|}\mathbb{1}_{\mathcal{F}_l \setminus \mathcal{F}_{l-1}}\right) \\
&\qquad\qquad + (d - |\mathcal{F}_{k-1}| - 1)\mathbb{1}_{\mathcal{F}_{k-1}} + \sum_{i=1}^{d-1-|\mathcal{F}_{k-1}|}\left(\frac{i}{d - |\mathcal{F}_{k-1}|}\mathbb{1}_{[d] \setminus \mathcal{F}_{k-1}}\right) \\
&= \sum_{l=1}^{k-1}\left((|\mathcal{F}_l| - |\mathcal{F}_{l-1}|)\,\mathbb{1}_{\mathcal{F}_{l-1}} + \frac{|\mathcal{F}_l| - |\mathcal{F}_{l-1}| + 1}{2}\mathbb{1}_{\mathcal{F}_l \setminus \mathcal{F}_{l-1}}\right) \\
&\qquad\qquad + (d - |\mathcal{F}_{k-1}| - 1)\mathbb{1}_{\mathcal{F}_{k-1}} + \frac{d - 1 - |\mathcal{F}_{k-1}|}{2}\mathbb{1}_{[d] \setminus \mathcal{F}_{k-1}} \\
&\overset{(*)}{=} \sum_{l=1}^{k-1}\left((d - 1 - |\mathcal{F}_l|)\mathbb{1}_{\mathcal{F}_l \setminus \mathcal{F}_{l-1}}\right) + \sum_{l=1}^{k-1}\left(\frac{|\mathcal{F}_l| - |\mathcal{F}_{l-1}| + 1}{2}\mathbb{1}_{\mathcal{F}_l \setminus \mathcal{F}_{l-1}}\right) + \frac{d - 1 - |\mathcal{F}_{k-1}|}{2}\mathbb{1}_{[d] \setminus \mathcal{F}_{k-1}} \\
&= \sum_{l=1}^{k}\frac{(d - |\mathcal{F}_l|) + (d - |\mathcal{F}_{l-1}|) - 1}{2}\mathbb{1}_{\mathcal{F}_l \setminus \mathcal{F}_{l-1}}\,,
\end{aligned}$$

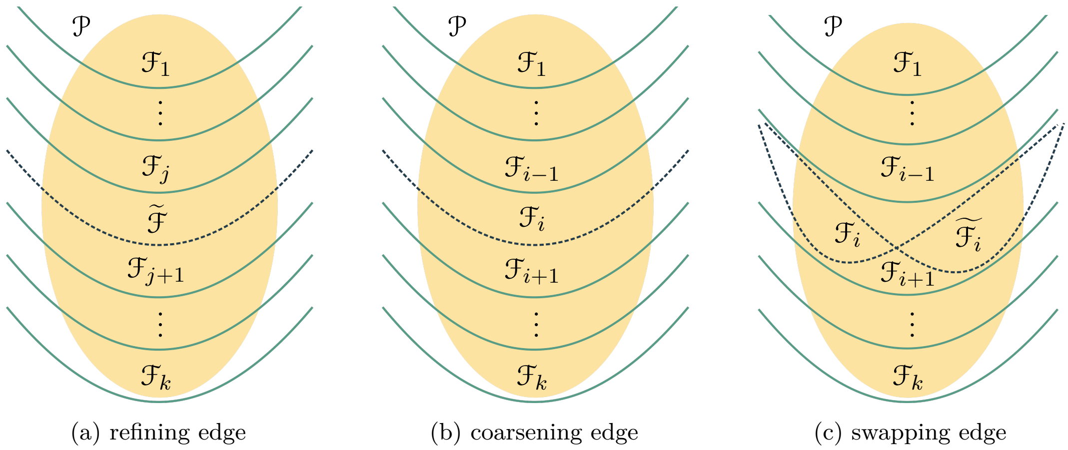


Figure 6.7: Schematic picture of connected chains of filters that correspond to neighboring vertices of $\mathsf{v}(\mathfrak{F})$.

where we used

$$\sum_{l=1}^{k-1}\Big((|\mathcal{F}_l|-|\mathcal{F}_{l-1}|)\mathbb{1}_{\mathcal{F}_{l-1}}\Big)+(d-1-|\mathcal{F}_{k-1}|)\mathbb{1}_{\mathcal{F}_{k-1}}=\sum_{l=1}^{k-1}(d-1-|\mathcal{F}_l|)\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}$$

for equality (*). Adding the shift of $+\mathbf{1}$ gives the desired result. $\square$

Corollary 6.21. *Poset permutahedra are half-integral polytopes. The poset permutahedron $\Pi_{\mathcal{P}}$ is a lattice polytope if and only if the poset $\mathcal{P}$ is an antichain.*

Proof. Recall that the poset permutahedron of an antichain is the standard permutahedron, hence a lattice polytope. Let $\mathcal{P}$ be a poset with at least one cover relation, i.e., not an antichain. Choose an element $b \in \mathcal{P}$ that is maximal and has another element $a \neq b$ below it, i.e., $a \prec b$ is a cover relation and b is a maximal element. Now consider the filter

$$\mathcal{F}_a := \{c \in \mathcal{P} \colon a \preceq c\}\,.$$

We make a case distinction:

If $|\mathcal{F}_a|$ is even, then let $\varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_a \subsetneq \dots$ be the start of a chain of filters. Note that $\mathcal{F}_a \setminus \varnothing = \mathcal{F}_a$ is connected. Choose the remaining filters so that $\varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_a \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}$ is a connected chain of filters. Using Equation (6.13) it follows that the corresponding vertex is half-integral.

If $|\mathcal{F}_a|$ is odd, then let $\varnothing = \mathcal{F}_0 \subsetneq \{b\} \subsetneq \mathcal{F}_a \subsetneq \dots$ be the start of a chain of filters. Note that $\mathcal{F}_a \setminus \{b\}$ is connected. Similarly, choose the remaining filters so that $\varnothing = \mathcal{F}_0 \subsetneq \{b\} \subsetneq \mathcal{F}_a \subsetneq \mathcal{F}_3 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}$ is a connected chain of filters and Equation (6.13) implies half-integrality of the corresponding vertex. $\square$

We move on to characterizing which vertices $\mathsf{v}, \mathsf{u} \in \operatorname{vert}(\Pi_{\mathcal{P}})$ are connected by an edge and computing the **edge vector** $\mathsf{v} - \mathsf{u} \in \mathbb{R}^d$.

Theorem 6.22. *Let $\Pi_{\mathcal{P}}$ be a poset permutahedron and $\mathfrak{F} : \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}$ a connected chain of filters. Then $\mathsf{v}(\mathfrak{F})$ is adjacent to a vertex $\mathsf{v}(\mathfrak{F}')$ if and only if one of the following three cases holds:*

(i) *there is an* $1 \leq j \leq k-1$ *such that* $|\mathcal{F}_{j+1} \setminus \mathcal{F}_j| \geq 2$ *and there is a filter* $\widetilde{\mathcal{F}}$ *with* $\mathcal{F}_j \subsetneq \widetilde{\mathcal{F}} \subsetneq \mathcal{F}_{j+1}$ *such that* $\mathcal{F}_{j+1} \setminus \widetilde{\mathcal{F}}$ *as well as* $\widetilde{\mathcal{F}} \setminus \mathcal{F}_j$ *are again connected and* $\mathfrak{F}'$ *is*

$$\mathfrak{F}' \colon \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_j \subsetneq \widetilde{\mathcal{F}} \subsetneq \mathcal{F}_{j+1} \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}\,.$$

We call these edges ***refining edges*** *and the corresponding edge vector is*

$$\mathsf{v}(\mathfrak{F}') - \mathsf{v}(\mathfrak{F}) = \frac{1}{2}\left(\left(|\mathcal{F}_{j+1}| - |\widetilde{\mathcal{F}}|\right)\mathbb{1}_{\widetilde{\mathcal{F}}\setminus\mathcal{F}_j} + \left(|\mathcal{F}_j| - |\widetilde{\mathcal{F}}|\right)\mathbb{1}_{\mathcal{F}_{j+1}\setminus\widetilde{\mathcal{F}}}\right) \tag{6.15}$$

(ii) *there is an* $1 \leq i \leq k-1$ *such that* $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ *is connected and* $\mathfrak{F}'$ *is*

$$\mathfrak{F}' \colon \varnothing = \mathcal{F}_0 \subsetneq \cdots \subsetneq \mathcal{F}_{i-1} \subsetneq \mathcal{F}_{i+1} \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}\,. \tag{6.16}$$

We call these edges ***coarsening edges*** *and the corresponding edge vector is*

$$\mathsf{v}(\mathfrak{F}') - \mathsf{v}(\mathfrak{F}) = \frac{1}{2}\left((|\mathcal{F}_i| - |\mathcal{F}_{i+1}|)\,\mathbb{1}_{\mathcal{F}_i\setminus\mathcal{F}_{i-1}} + (|\mathcal{F}_i| - |\mathcal{F}_{i-1}|)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\mathcal{F}_i}\right)\,.$$

(iii) *there is an* $1 \leq i \leq k-1$ *such that* $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ *consists of two connected components and* $\mathfrak{F}'$ *is*

$$\mathfrak{F}' \colon \varnothing = \mathcal{F}_0 \subsetneq \cdots \subsetneq \mathcal{F}_{i-1} \subsetneq \mathcal{F}_{i-1} \cup (\mathcal{F}_{i+1} \setminus \mathcal{F}_i) \subsetneq \mathcal{F}_{i+1} \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}\,. \tag{6.17}$$

We call these edges ***swapping edges*** *and the corresponding edge vector is*

$$\mathsf{v}(\mathfrak{F}') - \mathsf{v}(\mathfrak{F}) = (|\mathcal{F}_i| - |\mathcal{F}_{i+1}|)\,\mathbb{1}_{\mathcal{F}_i\setminus\mathcal{F}_{i-1}} + (|\mathcal{F}_i| - |\mathcal{F}_{i-1}|)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\mathcal{F}_i}\,. \tag{6.18}$$

Proof. Note that every refining edge from $\mathsf{v}(\mathfrak{F})$ to $\mathsf{v}(\mathfrak{F}')$ is a coarsening edge from $\mathsf{v}(\mathfrak{F}')$ to $\mathsf{v}(\mathfrak{F})$. Therefore, it is enough to show that coarsening and swapping as described above actually define edges in the poset permutahedron $\Pi_{\mathcal{P}}$ and vice versa that every edge in the poset permutahedron $\Pi_{\mathcal{P}}$ is either a coarsening edge or a swapping edge.

Let $\mathfrak{F} : \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}$ be a connected chain of filters. Note that $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1} = (\mathcal{F}_{i+1} \setminus \mathcal{F}_i) \cup (\mathcal{F}_i \setminus \mathcal{F}_{i-1})$ is a union of two subposets, each one being connected. So the number of connected components in $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ is either one or two. The vertex $\mathsf{v}(\mathfrak{F})$ is given by the cellular string, i.e., monotone path, $\mathcal{C} = (\mathsf{f}_1, \ldots, \mathsf{f}_k)$, where $\mathsf{f}_i = \operatorname{conv}(\mathbb{1}_{\mathcal{F}_{i-1}}, \mathbb{1}_{\mathcal{F}_i})$.

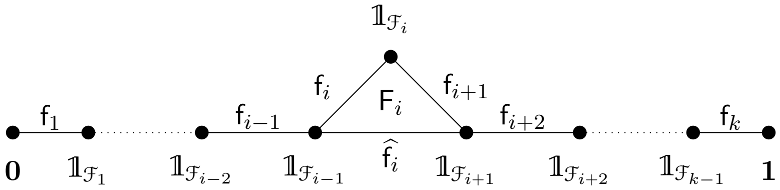


Figure 6.8: Schematic picture of cellular strings defining a coarsening edge: two cellular strings $\mathcal{C} = (\mathsf{f}_1, \ldots, \mathsf{f}_k)$ and $\mathcal{C}' = (\mathsf{f}_1, \ldots, \mathsf{f}_{i-1}, \widehat{\mathsf{f}_i}, \mathsf{f}_{i+2}, \ldots, \mathsf{f}_k)$ corresponding to vertices $\mathsf{v}(\mathfrak{F})$ and $\mathsf{v}(\mathfrak{F}')$ in $\Pi_{\mathcal{P}}$ that are connected by a coarsening edge, which corresponds to the cellular string $\mathcal{C}_i = (\mathsf{f}_1, \ldots, \mathsf{f}_{i-1}, \mathsf{F}_i, \mathsf{f}_{i+2}, \ldots, \mathsf{f}_k)$.

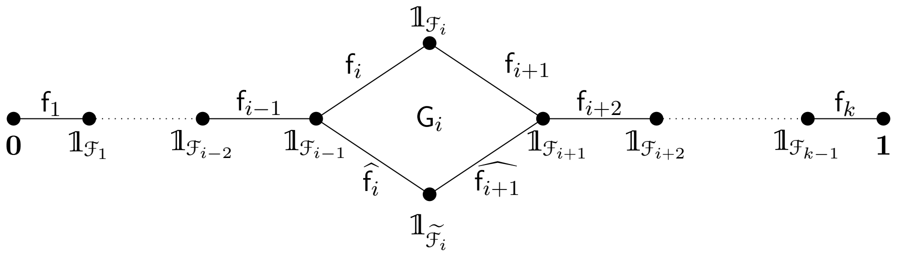


Figure 6.9: Schematic picture of cellular strings defining a swapping edge: two cellular strings $\mathcal{C} = (\mathsf{f}_1, \dots, \mathsf{f}_k)$ and $\mathcal{C}' = (\mathsf{f}_1, \dots, \mathsf{f}_{i-1}, \widehat{\mathsf{f}_i}, \widehat{\mathsf{f}_{i+1}}, \mathsf{f}_{i+2}, \dots, \mathsf{f}_k)$ corresponding to vertices $\mathsf{v}(\mathfrak{F})$ and $\mathsf{v}(\mathfrak{F}')$ in $\Pi_{\mathcal{P}}$ that are connected by a swapping edge, which corresponds to the cellular string $\mathcal{D}_i = (\mathsf{f}_1, \dots, \mathsf{f}_{i-1}, \mathsf{G}_i, \mathsf{f}_{i+2}, \dots, \mathsf{f}_k)$. Recall that $\widetilde{\mathcal{F}_i} \coloneqq \mathcal{F}_{i-1} \cup (\mathcal{F}_{i+1} \setminus \mathcal{F}_i)$.

First assume that $1 \le i \le k-1$ is such that $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ is connected. Then there is a 2-face $\mathsf{F}_i = \operatorname{conv}(\mathbb{1}_{\mathcal{F}_{i-1}}, \mathbb{1}_{\mathcal{F}_i}, \mathbb{1}_{\mathcal{F}_{i+1}})$. Indeed, this 2-face is given by the connected and compatible partition $\{\mathcal{F}_{i-1} \cup \{\widehat{1}\}, \mathcal{F}_i \setminus \mathcal{F}_{i-1}, \mathcal{F}_{i+1} \setminus \mathcal{F}_i, (\mathcal{P} \setminus \mathcal{F}_{i+1}) \cup \{\widehat{0}\}\}$ (compare Theorem 2.25). Now, we define a the cellular string $\mathcal{C}_i = (\mathsf{f}_1, \dots, \mathsf{f}_{i-1}, \mathsf{F}_i, \mathsf{f}_{i+2}, \dots, \mathsf{f}_k)$. See Figure 6.8 for a schematic illustration. Clearly, the cellular string $\mathcal{C} = (\mathsf{f}_1, \dots, \mathsf{f}_k)$ is contained in $\mathcal{C}_i$. The vertex $\mathsf{v}(\mathfrak{F}')$ defined by the coarser chain of filters $\mathfrak{F}'$ in Equation (6.16) is given by the cellular string $\mathcal{C}' = (\mathsf{f}_1, \dots, \mathsf{f}_{i-1}, \widehat{\mathsf{f}_i}, \mathsf{f}_{i+2}, \dots, \mathsf{f}_k)$, where $\widehat{\mathsf{f}_i} = \operatorname{conv}(\mathbb{1}_{\mathcal{F}_{i-1}}, \mathbb{1}_{\mathcal{F}_{i+1}})$ and the remaining edges are as in $\mathcal{C}$. Again, the cellular string $\mathcal{C}'$ is contained in $\mathcal{C}_i$. Moreover, $\mathcal{C}$ and $\mathcal{C}'$ are the only cellular strings refining $\mathcal{C}_i$. Then $\mathsf{v}(\mathfrak{F})$ and $\mathsf{v}(\mathfrak{F}')$ are the only vertices in the face given by the cellular string $\mathcal{C}_i$, so this face is an edge as claimed.

Second, assume that $1 \le i \le k-1$ is such that $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ consists of two connected components. We denote $\widetilde{\mathcal{F}_i} \coloneqq \mathcal{F}_{i-1} \cup (\mathcal{F}_{i+1} \setminus \mathcal{F}_i)$. Then there exists a 2-face $\mathsf{G}_i = \operatorname{conv}(\mathbb{1}_{\mathcal{F}_{i-1}}, \mathbb{1}_{\mathcal{F}_i}, \mathbb{1}_{\widetilde{\mathcal{F}_i}}, \mathbb{1}_{\mathcal{F}_{i+1}})$. Indeed, this 2-face is given by the connected and compatible partition $\{\mathcal{F}_{i-1} \cup \{\widehat{1}\}\,,\ \mathcal{F}_i \setminus \mathcal{F}_{i-1}\,,\ \mathcal{F}_{i+1} \setminus \mathcal{F}_i\,,\ (\mathcal{P} \setminus \mathcal{F}_{i+1}) \cup \{\widehat{0}\}\}$ (see Theorem 2.25). As above, we define a the cellular string $\mathcal{D}_i = (\mathsf{f}_1, \dots, \mathsf{f}_{i-1}, \mathsf{G}_i, \mathsf{f}_{i+2}, \dots, \mathsf{f}_k)$. See Figure 6.9 for a schematic illustration. Clearly, the cellular string $\mathcal{C} = (\mathsf{f}_1, \dots, \mathsf{f}_k)$ is contained in $\mathcal{D}_i$. The vertex $\mathsf{v}(\mathfrak{F}')$ defined by the chain of filters $\mathfrak{F}'$ as defined in Equation (6.17) is given by the cellular string $\mathcal{C}' = (\mathsf{f}_1, \dots, \mathsf{f}_{i-1}, \widehat{\mathsf{f}_i}, \widehat{\mathsf{f}_{i+1}}, \mathsf{f}_{i+2}, \dots, \mathsf{f}_k)$, where $\widehat{\mathsf{f}_i} = \operatorname{conv}(\mathbb{1}_{\mathcal{F}_{i-1}}, \mathbb{1}_{\widetilde{\mathcal{F}_i}})$ and $\widehat{\mathsf{f}_{i+1}} = \operatorname{conv}(\mathbb{1}_{\widetilde{\mathcal{F}_i}}, \mathbb{1}_{\mathcal{F}_{i+1}})$ and the remaining edges are as in $\mathcal{C}$. Again, the cellular string $\mathcal{C}'$ is contained in $\mathcal{D}_i$. Moreover, $\mathcal{C}$ and $\mathcal{C}'$ are the only cellular strings refining $\mathcal{D}_i$. Then $\mathsf{v}(\mathfrak{F})$ and $\mathsf{v}(\mathfrak{F}')$ are the only vertices in the face given by the cellular string $\mathcal{D}_i$, so this face is an edge as claimed.

Vice versa, every edge in the montone path polytope of an order polytope $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P}))$ arises that way. Indeed, let $\mathcal{C} = (\mathsf{F}_1, \dots, \mathsf{F}_k)$ be a cellular string in $\mathsf{O}(\mathcal{P})$ that corresponds to an edge in the monotone path polytope. Then $\mathcal{C}$ contains precisely two cellular strings $\mathcal{C}_{\mathsf{v}}, \mathcal{C}_{\mathsf{u}}$ refining it and they correspond to the two vertices v, u of the edge f in the monotone path polytope $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1}$, hence the two cellular strings $\mathcal{C}_{\mathsf{v}}, \mathcal{C}_{\mathsf{u}}$ are monotone paths. Therefore $\mathcal{C}_{\mathsf{f}}$ contains exactly one 2-face F_i for one $i \in \{1, \dots, k\}$ and the remaining faces F_j for $j \ne i$ are edges. This 2-face F_i is either a triangle or a quadrilateral (Corollary 2.30), giving rise to the two cases as described.

Finally we compute the edge vectors using Equation (6.13). For refining edges (i) we compute $\mathsf{v}(\mathfrak{F}') - \mathsf{v}(\mathfrak{F})$:

$$\begin{aligned}
&\frac{1}{2}\Bigg(\sum_{l=1}^{j}(n-|\mathcal{F}_l|+n-|\mathcal{F}_{l-1}|+1)\,\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}+(n-|\widetilde{\mathcal{F}}|+n-|\mathcal{F}_j|+1)\mathbb{1}_{\widetilde{\mathcal{F}}\setminus\mathcal{F}_j}\\
&\quad+(n-|\mathcal{F}_{j+1}|+n-|\widetilde{\mathcal{F}}|+1)\mathbb{1}_{\mathcal{F}_{j+1}\setminus\widetilde{\mathcal{F}}}+\sum_{l=j+2}^{k}(n-|\mathcal{F}_l|+n-|\mathcal{F}_{l-1}|+1)\,\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}\Bigg)\\
&\quad-\frac{1}{2}\sum_{l=1}^{k}(n-|\mathcal{F}_l|+n-|\mathcal{F}_{l-1}|+1)\,\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}\\
&=\frac{1}{2}\Bigg(\Big(n-|\widetilde{\mathcal{F}}|+n-|\mathcal{F}_j|+1\Big)\,\mathbb{1}_{\widetilde{\mathcal{F}}\setminus\mathcal{F}_j}+\Big(n-|\mathcal{F}_{j+1}|+n-|\widetilde{\mathcal{F}}|+1\Big)\,\mathbb{1}_{\mathcal{F}_{j+1}\setminus\widetilde{\mathcal{F}}}\\
&\qquad-(n-|\mathcal{F}_{j+1}|+n-|\mathcal{F}_j|+1)\,\mathbb{1}_{\mathcal{F}_{j+1}\setminus\mathcal{F}_j}\Bigg)\\
&=\frac{1}{2}\left(\Big(|\mathcal{F}_{j+1}|-|\widetilde{\mathcal{F}}|\Big)\,\mathbb{1}_{\widetilde{\mathcal{F}}\setminus\mathcal{F}_j}+\Big(|\mathcal{F}_j|-|\widetilde{\mathcal{F}}|\Big)\,\mathbb{1}_{\mathcal{F}_{j+1}\setminus\widetilde{\mathcal{F}}}\right)\,,
\end{aligned}$$

where we use that $\mathcal{F}_{j+1}\setminus\mathcal{F}_j=(\mathcal{F}_{j+1}\setminus\widetilde{\mathcal{F}})\sqcup(\widetilde{\mathcal{F}}\setminus\mathcal{F}_j)$ is a disjoint union.

For coarsening edges (ii) we compute $\mathsf{v}(\mathfrak{F}') - \mathsf{v}(\mathfrak{F})$:

$$\begin{aligned}
&\frac{1}{2}\Bigg(\sum_{l=1}^{i-1}(n-|\mathcal{F}_l|+n-|\mathcal{F}_{l-1}|+1)\,\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}+\sum_{l=i+2}^{k}(n-|\mathcal{F}_l|+n-|\mathcal{F}_{l-1}|+1)\,\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}\\
&\quad+(n-|\mathcal{F}_{i+1}|+n-|\mathcal{F}_{i-1}|+1)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\mathcal{F}_{i-1}}\Bigg)-\frac{1}{2}\sum_{l=1}^{k}(n-|\mathcal{F}_l|+n-|\mathcal{F}_{l-1}|+1)\,\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}\\
&=\frac{1}{2}\Bigg((n-|\mathcal{F}_{i+1}|+n-|\mathcal{F}_{i-1}|+1)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\mathcal{F}_{i-1}}\\
&\qquad-(n-|\mathcal{F}_i|+n-|\mathcal{F}_{i-1}|+1)\,\mathbb{1}_{\mathcal{F}_i\setminus\mathcal{F}_{i-1}}-(n-|\mathcal{F}_{i+1}|+n-|\mathcal{F}_i|+1)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\mathcal{F}_i}\Bigg)\\
&=\frac{1}{2}\left((|\mathcal{F}_i|-|\mathcal{F}_{i+1}|)\,\mathbb{1}_{\mathcal{F}_i\setminus\mathcal{F}_{i-1}}+(|\mathcal{F}_i|-|\mathcal{F}_{i-1}|)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\mathcal{F}_i}\right)\,.
\end{aligned}$$

where we use that $\mathcal{F}_{i+1}\setminus\mathcal{F}_{i-1}=(\mathcal{F}_{i+1}\setminus\mathcal{F}_i)\sqcup(\mathcal{F}_i\setminus\mathcal{F}_{i-1})$ is a disjoint union.

To compute edge vectors for swapping edges (iii) we denote $\widetilde{\mathcal{F}_i}\coloneqq\mathcal{F}_{i-1}\cup(\mathcal{F}_{i+1}\setminus\mathcal{F}_i)$ and note that

$$\widetilde{\mathcal{F}_i}\setminus\mathcal{F}_{i-1}=\mathcal{F}_{i+1}\setminus\mathcal{F}_i\quad\text{and}\quad\mathcal{F}_{i+1}\setminus\widetilde{\mathcal{F}_i}=\mathcal{F}_i\setminus\mathcal{F}_{i-1}\,.$$

Now we compute $\mathsf{v}(\mathfrak{F}') - \mathsf{v}(\mathfrak{F})$

$$\begin{aligned}
&\frac{1}{2}\Bigg(\sum_{l=1}^{i-1}(n-|\mathcal{F}_l|+n-|\mathcal{F}_{l-1}|+1)\,\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}\\
&+\Big(n-|\widetilde{\mathcal{F}_i}|+n-|\mathcal{F}_{i-1}|+1\Big)\,\mathbb{1}_{\widetilde{\mathcal{F}_i}\setminus\mathcal{F}_{i-1}}+\Big(n-|\mathcal{F}_{i+1}|+n-|\widetilde{\mathcal{F}_i}|+1\Big)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\widetilde{\mathcal{F}_i}}\\
&+\sum_{l=i+2}^{k}(n-|\mathcal{F}_l|+n-|\mathcal{F}_{l-1}|+1)\,\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}\Bigg)-\frac{1}{2}\sum_{l=1}^{k}(n-|\mathcal{F}_l|+n-|\mathcal{F}_{l-1}|+1)\,\mathbb{1}_{\mathcal{F}_l\setminus\mathcal{F}_{l-1}}
\end{aligned}$$

$$\begin{aligned}
=&\ \frac{1}{2}\Bigg(\Big(n-|\widetilde{\mathcal{F}_i}|+n-|\mathcal{F}_{i-1}|+1\Big)\,\mathbb{1}_{\widetilde{\mathcal{F}_i}\setminus\mathcal{F}_{i-1}}+\Big(n-|\mathcal{F}_{i+1}|+n-|\widetilde{\mathcal{F}_i}|+1\Big)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\widetilde{\mathcal{F}_i}}\\
&-(n-|\mathcal{F}_i|+n-|\mathcal{F}_{i-1}|+1)\,\mathbb{1}_{\mathcal{F}_i\setminus\mathcal{F}_{i-1}}-(n-|\mathcal{F}_{i+1}|+n-|\mathcal{F}_i|+1)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\mathcal{F}_i}\Bigg)\\
=&\ \frac{1}{2}\Bigg(\Big(|\mathcal{F}_i|+|\mathcal{F}_{i-1}|-|\mathcal{F}_{i+1}|-|\widetilde{\mathcal{F}_i}|\Big)\,\mathbb{1}_{\mathcal{F}_i\setminus\mathcal{F}_{i-1}}+\Big(|\mathcal{F}_{i+1}|+|\mathcal{F}_i|-|\widetilde{\mathcal{F}_i}|-|\mathcal{F}_{i-1}|\Big)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\mathcal{F}_i}\Bigg)\\
=&\ (|\mathcal{F}_i|-|\mathcal{F}_{i+1}|)\,\mathbb{1}_{\mathcal{F}_i\setminus\mathcal{F}_{i-1}}+(|\mathcal{F}_i|-|\mathcal{F}_{i-1}|)\,\mathbb{1}_{\mathcal{F}_{i+1}\setminus\mathcal{F}_i}
\end{aligned}$$

□

Example 6.23. We revisit the example from Figure 6.6, see also Example 6.20. The vertices $\mathsf{v}(\varnothing \subsetneq \{1\} \subsetneq \{1,2\} \subsetneq [3])$ and $\mathsf{v}(\varnothing \subsetneq \{1\} \subsetneq \{1,3\} \subsetneq [3])$ are connected by a swapping edge. Note, in higher dimensions swapping edges do not necessarily coincide with edges of the standard permutahedron Π_d. A coarsening resp. refining edge can be found, e.g., between the vertices $\mathsf{v}(\varnothing \subsetneq [3])$ and $\mathsf{v}(\varnothing \subsetneq \{1,2\} \subsetneq [3])$.

From Theorem 6.12 and Corollary 6.17 we can also derive explicit vertex-facet incidences for the poset permutahedron $\Pi_{\mathcal{P}}$.

Corollary 6.24. *A vertex given by the connected chain of filters $\varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}$ is contained in the facets corresponding to*

1. *the proper non-empty filters $\mathcal{F}_1, \ldots, \mathcal{F}_{k-1}$ and*
2. *cover relations $a \prec b$ such that $a, b \in \mathcal{F}_{i+1} \setminus \mathcal{F}_i$ for some $i \in \{0, 1, \ldots, k-1\}$.*

Applying this characterization together with a counting argument enables us to characterize the simple poset permutahedra.

Theorem 6.25. *A poset permutahedron $\Pi_{\mathcal{P}}$ is a simple polytope if and only if the undirected Hasse diagram of $\mathcal{P}$ is a forest.*

Proof. Suppose first that $\Pi_{\mathcal{P}}$ is simple, i.e., every vertex is contained in exactly $|\mathcal{P}| - 1 = d - 1$ facets. Let $\mathcal{P}_1, \ldots, \mathcal{P}_c$ denote the connected components of $\mathcal{P}$ and define $\mathcal{F}_i = \bigcup_{l=1}^{i} \mathcal{P}_i$ for $i = 0, \ldots, c$. Then $\mathfrak{F} : \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_c$ is a connected chain of filters and the poset permutahedron has a corresponding vertex $\mathsf{v}(\mathfrak{F})$. The Hasse diagram of $\mathcal{P}_i$ has at least $|\mathcal{P}_i| - 1$ edges (since it is connected), so $\mathcal{P}_i$ contains at least $|\mathcal{P}_i| - 1$ cover relations. The vertex $\mathsf{v}(\mathfrak{F})$ is contained in facets corresponding to the cover relations in every connected component $\mathcal{P}_i$, i.e., at least

$$\sum_{l=1}^{c}(|\mathcal{P}_i| - 1) = -c + \sum_{l=1}^{c}|\mathcal{P}_i| = |\mathcal{P}| - c$$

facets corresponding to cover relations. Moreover, the vertex $\mathsf{v}(\mathfrak{F})$ is contained in precisely $c-1$ facets corresponding to the proper non-empty filters in $\mathfrak{F}$. In total these are at least $|\mathcal{P}| - 1$ facets, since $\Pi_{\mathcal{P}}$ is simple, the vertex $\mathsf{v}(\mathfrak{F})$ is contained in exactly $|\mathcal{P}| - 1 = d - 1$ facets. Hence, every Hasse diagram of a connected component $\mathcal{P}_i$ has the minimum number of edges and therefore is a tree.

For the other direction, suppose that $\mathcal{P}$ is a forest. Let $\varnothing = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_k = \mathcal{P}$ be an arbitrary connected chain of filters corresponding to vertex v of $\Pi_{\mathcal{P}}$. We count cover facets and filter facets separately (see Corollary 6.17). First observe that the number of filter facets containing v is $k-1$. Then the set of cover facets containing v corresponds precisely to the set of cover relations in $\mathcal{F}_i \setminus \mathcal{F}_{i-1}$ for each $i = 1, \dots, k$. Since the Hasse diagram of $\mathcal{F}_i \setminus \mathcal{F}_{i-1}$ is connected, it is a tree and has precisely $|\mathcal{F}_i| - |\mathcal{F}_{i-1}| - 1$ edges. Therefore, the number of cover facets containing v is exactly $\sum_{i=1}^{k} |\mathcal{F}_i| - |\mathcal{F}_{i-1}| - 1 = |\mathcal{F}_k| - |\mathcal{F}_0| - k = d - k$. This means that v is contained in total in $d - k + k - 1 = d - 1$ facets, as desired.

$\square$

6.4 Subdivision, Volumes and Integer Points

In this section we will see a beautiful subdivision of poset permutahedra into combinatorial cubes, use this to compute volumes and give combinatorial interpretations for the integer points in poset permutahedra.

Recall that a linear extensions $\mathcal{T}$ of $\mathcal{P}$ is a refinement of $\preceq$ to a total order $[d]$ and that we denote the set of linear extensions of $\mathcal{P}$ by $\mathfrak{L}(\mathcal{P})$. Recall from Section 2.5.5 that the order polytope of a linear extension $\mathcal{T}$ is a simplex. Recall furthermore that $\{\mathsf{C}(\mathcal{T}) \colon \mathcal{T} \in \mathfrak{L}(\mathcal{P})\}$ forms a subdivision of the order cone $\mathsf{C}(\mathcal{P})$ and that this induces a triangulation $\{\mathsf{O}(\mathcal{T}) \colon \mathcal{T} \in \mathfrak{L}(\mathcal{P})\}$ of $\mathsf{O}(\mathcal{P})$. This argument carries over to poset permutahedra.

Theorem 6.26. *For any poset $\mathcal{P}$, the set $\{\Pi_{\mathcal{T}} : \mathcal{T} \in \mathfrak{L}(\mathcal{P})\}$ is a subdivision of $\Pi_{\mathcal{P}}$.*

Note that from the monotone path polytope perspective, this is a non-obvious statement. See Figure 6.10 and Figure 6.11 for illustrations in low dimensions. As mentioned in Example 6.6(ii), monotone path polytopes of simplices are combinatorial cubes, so the subdivision in Theorem 6.26 consists of combinatorial cubes, hence it is a **cubical subdivision**. However, the cells are far from being zonotopes.

If $\mathcal{P}$ is the antichain on d elements, then the symmetric group S_d acts simply transitively[2] on $\mathfrak{L}(\mathcal{P})$ and shows that any two $\Pi_{\mathcal{T}}$ are isometric. In particular $\operatorname{vol}(\Pi_{\mathcal{T}}) = \frac{1}{d!} \operatorname{vol} \Pi_d$. The volume of Π_d is famously known to be the number of spanning trees of the complete graph on d nodes, i.e., d^{d-2}.

2 I.e., for any two $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{L}(\mathcal{P})$ there exists a unique $\sigma \in S_d$ such that $\mathcal{T}_1 = \sigma \curvearrowright \mathcal{T}_2$.

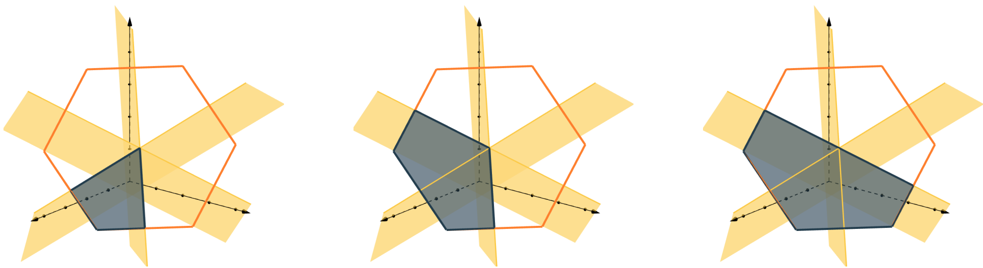


Figure 6.10: Up to symmetry all the poset permutahedra for posets on three elements (except the permutahedron itself) with their subdivisions induced by linear extensions of the poset.

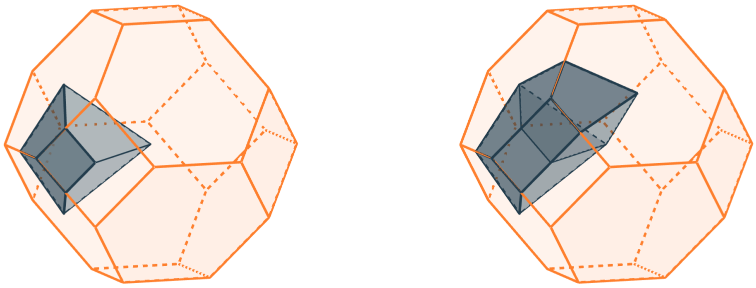


Figure 6.11: A chain permutahedron (left) and a poset permutahedron for a poset on four elements with two linear extensions and the induced subdivision into two chain permutahedra.

Corollary 6.27. *For a poset $\mathcal{P}$ on $[d]$, the volume of $\Pi_{\mathcal{P}}$ is $\frac{d^{d-2}}{d!}|\mathfrak{L}(\mathcal{P})|$. In particular, the probability that a random point of Π_d is in $\Pi_{\mathcal{P}}$ is precisely the probability that a random permutation is a linear extension of $\mathcal{P}$.*

In this section we want to analyse the number and structure of integer points in poset permutahedra and their dilates. We start with the chain permutahedron and show that in that case the integer points are related to score sequences.

A **score sequence** is an integer sequence $0 \leq s_1 \leq \cdots \leq s_d \leq n-1$ that is a possible result of an d-team round-robin tournament, that is, every team plays against each other team and gets one point if it wins. For $d = 2$ the only score sequence is $0 \leq 1$ and for $d = 3$ we have two score sequences: $0 \leq 1 \leq 2$ and $1 \leq 1 \leq 1$. Equivalently, a score sequence is a non-decreasingly ordered indegree sequence of a directed complete graph on d-nodes, such a graph is called a **tournament** T_d. Note that there are non-isomorphic tournaments that result in the same score sequence. For a more detailed introduction to score sequences see, e.g., [Moo13].

The following theorem is a classical result by Landau (1953) to characterize score sequences.

Theorem 6.28 ([Lan53])**.** *A sequence $s_1 \leq \cdots \leq s_d$ of integers is a score sequence of a tournament if and only if*

$$\sum_{i=1}^{d} s_i = \binom{d}{2} \quad \text{and} \quad \sum_{i=k+1}^{d} s_i \leq \binom{d}{2} - \binom{k}{2}. \tag{6.19}$$

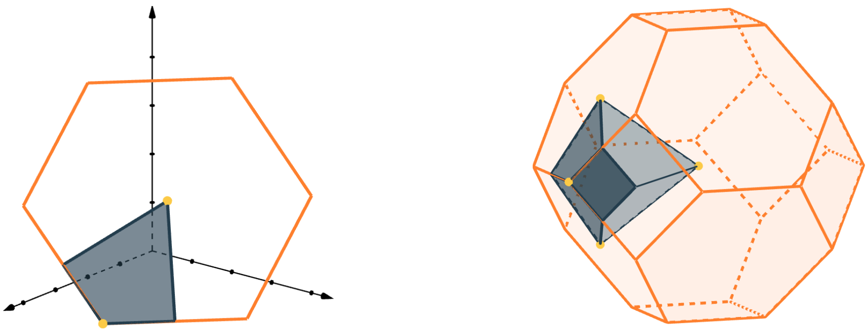


Figure 6.12: Chain permutahedra for chains on three elements (left) and four elements (right), together with their two, resp. four, integer points corresponding to the score sequences of tournaments with three, resp. four, teams in yellow.

Corollary 6.29. *The integer points in the translated chain permutahedron $\Pi_{\mathcal{C}_d} - \mathbf{1}$ are precisely the score sequences of length d.*

Proof. With Corollary 6.17 we can easily compute that a point is in $\Pi_{\mathcal{C}_d} - \mathbf{1}$ if and only if it fulfills the (in-)equalities in (6.19):

$$f_d([d]) - d = \binom{d+1}{2} - d = \binom{d}{2}$$

and since filters in the d-chain are precisely sets of the form $\{k+1, \ldots, d\}$ for $k = 0, \ldots, d-1$ we compute

$$f(\{k+1, \ldots, d\}) - \mathbb{1}_{\{k+1,\ldots,d\}}(\mathbf{1}) = (d-k)d - \binom{d-k}{2} - (d-k) = \binom{d}{2} - \binom{k}{2}.$$

□

A **score vector** is a tuple of integers $\mathbf{t} = (t_1, \ldots, t_d)$, where t_i records the number of points that team i wins during the tournament. For $d = 2$ there are two score vectors: $(0,1)$ and $(1,0)$; for $d = 3$ we have 7 score vectors: $(1,1,1)$ and the six permutations of $(0,1,2)$. Note that the score vectors of length d are precisely the integer points in the translated permutahedron $\Pi_d - \mathbf{1}$.

For a poset $\mathcal{P}$ on $[d]$ we define a **$\mathcal{P}$-score vector** as a tuple of integers $(s_1, \ldots, s_d) \in \{0, 1, \ldots, d-1\}^d$ that is a possible result of a d-team round-robin tournament under the condition that $s_i \leq s_j$ whenever $i \preceq_{\mathcal{P}} j$ is a relation in the poset $\mathcal{P}$. Hence for a chain $\mathcal{C}$ the $\mathcal{C}$-score vectors are precisely the score sequences and for an antichain $\mathcal{A}$ the $\mathcal{A}$-score vectors are all possible permutations of score sequences, i.e., the score vectors.

Corollary 6.30. *The integer points in a poset permutahedron $\Pi_{\mathcal{P}} - \mathbf{1}$ correspond to $\mathcal{P}$-score vectors.*

Proof. This follows from subdividing the poset poermutahedron $\Pi_{\mathcal{P}}$ according to its linear extensions (Theorem 6.26). □

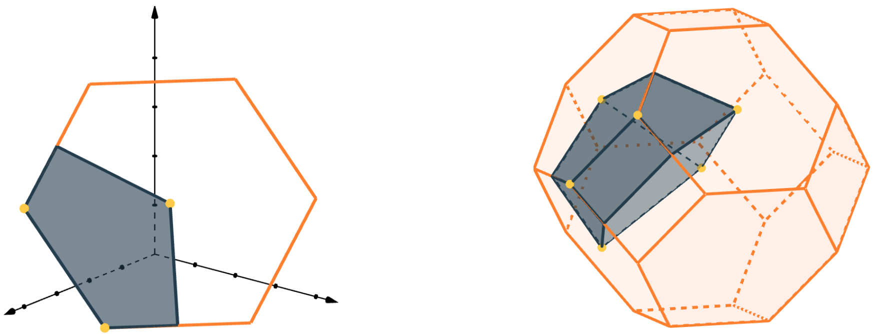


Figure 6.13: Poset permutahedra for posets on three elements (left) and four elements (right) with two linear extensions, together with their three, resp. six, integer points corresponding to the $\mathcal{P}$-score vectors highlighted in yellow.

Proposition 6.31. *The integer points in the n-th dilate of the chain permutahedron $n \cdot (\Pi_{\mathcal{C}_d} - \mathbf{1}) \cap \mathbb{Z}^d$ correspond to score sequences of an d-team round-robin tournament, where in each game n points are distributed between the two teams.*

Proof. First note that integer points in $n \cdot \Pi_{\mathcal{C}_d}$ are in bijection with points in $\Pi_{\mathcal{C}_d} \cap \frac{1}{n} \cdot \mathbb{Z}^d$. We can now apply a result by John Moon [Moo63]: Consider a generalized tournament, that is, in the game between team i and team j for $i < j$ we award a score of α_{ij} to team i and a score of $1 - \alpha_{ij}$ to team j for some real number $\alpha_{ij} \in [0, 1]$. Now a sequence of real numbers $(s_1 \leq s_2 \leq \cdots \leq s_d)$ is the resulting score sequence of a generalized tournament if and only if the conditions in (6.19) hold true. So any real point in $\Pi_{\mathcal{C}}$ is a score sequence of a generalized tournament as described above. Hence the claim follows from restricting values for α_{ij} to integral multiples of $\frac{1}{n}$. □

Combinatorially we can interpret integer points in $n \cdot \Pi_{\mathcal{C}_d}$ as indegree sequences of graphs with n arcs between any pair of the d nodes in the graph. For $n \in \{1, 2, \ldots, 10\}$ the number of integer points in $n \cdot \Pi_{\mathcal{C}_d}$ for increasing d are recorded on the On-Line Encyclopedia of Integer Sequences [OEI] entries A000571, A007747, A047729-A047731, and A047733-A047737.

Theorem 6.32. *For a finite poset $\mathcal{P}$ the integer points in the n-th dilate of the poset permutahedron $n \cdot (\Pi_{\mathcal{P}} - \mathbf{1}) \cap \mathbb{Z}^d$ correspond to $\mathcal{P}$-score vector of a d-team round-robin tournament, where in each game n points are distributed between the two teams.*

A polytope P is said to have the **integer decomposition property (idp)** if for every integer point $\mathbf{p} \in n\mathsf{P}$ in the n-th dilate of the polytope, $n \in \mathbb{Z}_{>0}$, there are n points $\mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathsf{P} \cap \mathbb{Z}^d$ such that $\mathbf{p} = \mathbf{q}_1 + \cdots + \mathbf{q}_n$.

From Theorem 6.32 one might think that $\Pi_{\mathcal{P}}$ has the integer decomposition property. However, this is not (quite) true, see Corollary 6.33 below. We will give two explanations for that, one geometric and one combinatorial:

Poset permutahedra (except the permutahedron itself) are rational polytopes (Corollary 6.21) and rational polytopes can never have the idp. Indeed, for a rational polytope $\mathsf{P} \subset \mathbb{R}^d$, let $\mathsf{v} \in \mathrm{vert}(\mathsf{P})$ be a rational vertex of P. Then there exists $n \in \mathbb{Z}_{>0}$ such that $n\mathsf{v} \in nP \cap \mathbb{Z}^d$. If P had the idp, then there would be

$\mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathsf{P} \cap \mathbb{Z}^d$ such that $n\mathsf{v} = \mathbf{q}_1 + \cdots + \mathbf{q}_n$. Then $\mathsf{v} = \frac{1}{n}(\mathbf{q}_1 + \cdots + \mathbf{q}_n)$ would be a nontrivial convex combination, contradicting v being a vertex of P.

Combinatorially, we can consider score sequences for a game with two points awarded. Here, a tie is possible, but this cannot be expressed as the sum score sequences of two games with only one point awarded if we do not allow both players to win, as is the case for score sequences, i.e., integer points in the chain permutahedron.

Corollary 6.33. *Let $\mathcal{P}$ be a poset. Then $2(\Pi_{\mathcal{P}} - \mathbf{1})$ (equivalently $2\Pi_{\mathcal{P}}$) has the integer decomposition property (idp).*

Proof. By Theorem 6.26 it is enough to show the claim for chain permutahedra $\Pi_{\mathcal{T}}$. Recall that we can interpret integer points in $n\Pi_{\mathcal{T}}$ as score sequences of multi-tournaments, i.e., indegree sequences of graphs with n directed edges between any two nodes. Therefore we can decompose every such multi-tournament for $2n$ into n tournaments with two directed edges between any pair of nodes. □

We end this section by briefly discussing strong tournaments and their potential connections. A tournament T_d on d nodes is called **strongly connected** or **strong** if between any two distinct nodes there exists a directed path. Let $s_1 \leq \cdots \leq s_d$ be the score sequence of a tournament T_d, then the tournament T_d is **reducible** if and only if $\sum_{i=1}^{k} s_i = \binom{k}{2}$ holds for some $k \in [d-1]$. See also [SB12].

Theorem 6.34 ([Moo13, Theorem 2])**.** *A tournament is strong if and only if it is irreducible.*

Corollary 6.35. *A score sequence $s_1 \leq \cdots \leq s_d$ corresponds to a strong tournament if and only if the integer point $(s_1, \ldots, s_d)$ is not contained in any filter facet of the chain permutahedron $\Pi_{\mathcal{C}_d}$.*

For increasing dimension, i.e., increasing number of teams d in the tournament the number of strong score sequences is recorded in [OEI, A351822]. See also [Cla+23].

Conjecture 6.36. *The number of irreducible score sequences for a tournament with d teams equals the leading $\mathrm{h}^*_{\mathbb{Z}}$-coefficient of $\Pi_{\mathcal{C}_d}$.*

6.5 Partitioned Permutahedra

In a recent paper, Horiguchi, Masuda, Shareshian, and Song [Hor+24b] defined partitioned permutahedra. They study these polytopes in the context of cohomology rings of the associated toric variety and regular Hessenberg varieties. In [Hor+24b] the setting is slightly more general, including parabolic subgroups of Weyl groups associated with Coxeter root systems of type A_n, B_n, C_n, and D_n.

In this section we show how partitioned permutahedra (in type A) are a special case of our poset permutahedra and recover some of the results in [Hor+24b] using only combinatorial and polyhedral techniques.

To define partitioned permutahedra, Horiguchi, Masuda, Shareshian, and Song first define a linear half-space for each $a \in [d-1]$ given by

$$\mathsf{H}(a) \ := \ \{\mathbf{x} \in \mathbb{R}^d : x_a \leq x_{a+1}\} \, .$$

For each subset $K \subseteq [d-1]$, they define the **partitioned permutahedron** $\Pi_d(K)$ via

$$\Pi_d(K) := \Pi_d \cap \bigcap_{a \in K} \mathsf{H}(a).$$

However, in [Hor+24b] the permutahedron Π_{S_d} (in type A) is defined more general as the convex hull of the S_d-orbit of a point $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_d)$ with $\mathbf{x}_i \neq \mathbf{x}_j$ for every $i \neq j$. We choose a particularly nice geometric realization with $\mathbf{x} = (1, \dots, n)$ here. All of the geometric realizations are normally, and hence also combinatorially, equivalent. Except for Corollary 6.43 every statement in this section is about the combinatorial structure of partitioned permutahedra.

For every such subset $K \subseteq [d-1]$, we associate the poset $\mathcal{P}(K) = ([d], \preceq)$ with cover relations given by $a \prec a+1$ if $a \in K$. The following proposition is a direct consequence of our definition of poset permutahedra and $\mathcal{P}(K)$.

Proposition 6.37. *For every $K \subseteq [d-1]$, we have $\Pi_d(K) = \Pi_{\mathcal{P}(K)}$.*

Now results on facets, vertices and incidences in [Hor+24b] are recovered by Theorem 6.19, Corollary 6.17, and Corollary 6.24. From the observation that $\mathcal{P}(K)$ is a disjoint union of chains and Theorem 6.25 we directly obtain the following Corollary 6.38. This was also noted in [Hor+24b, page 84].

Corollary 6.38. *Partitioned permutahedra are simple polytopes.*

In [Hor+24b, Proposition 7.4] the authors give a description of the h-vector of $\Pi_d(K)$ in terms of descent statistics restricted to the permutations σ of $[d]$ with $\sigma^{-1}(a) - \sigma^{-1}(a+1) \leq 1$ for all $a \in K$. This is shown by applying results on the cohomology of Hessenberg varieties. We give a combinatorial proof using the geometry of poset permutahedra (proof of Theorem 6.40). We start by establishing a bijection between the vertices of the partitioned permutahedron $\Pi_{\mathcal{P}(K)}$ and the permutations in $W(K)$.

Proposition 6.39. *For $K \subseteq [d-1]$, there exists a bijection between the following sets:*

1. *The set of vertices of the partitioned permutahedron $\Pi_d(K) = \Pi_{\mathcal{P}(K)}$.*
2. *The set*

$$W(K) := \{\sigma \in S_d \colon \sigma^{-1}(a) - \sigma^{-1}(a+1) \leq 1 \text{ for all } a \in K\}. \tag{6.20}$$

Proof. We first construct a map from the set of connected chains in $\mathcal{P}(K)$ to the subset $W(K)$ of permutation on $[d]$ as defined in Equation (6.20). Let $\mathfrak{F} \colon \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}(K)$ be a connected chain of filters, i.e.,

$$\mathcal{F}_i \setminus \mathcal{F}_{i-1} = \{a_i, a_i + 1, \dots, a_i + f_i - 1\} \text{ with } |\mathcal{F}_i| =: f_i \tag{6.21}$$

is a connected part of a chain. We construct a permutation σ in the following way:

$$\begin{aligned}
\sigma(d) &:= a_1, \\
\sigma(d-1) &:= a_1 + 1, \\
&\vdots \\
\sigma(d - f_1 + 1) &:= a_1 + f_1 - 1 \\
\sigma(d - f_1) &:= a_2, \\
\sigma(d - f_1 - 1) &:= a_2 + 1, \\
&\vdots \\
\sigma(d - f_1 - f_2 + 1) &:= a_2 + f_2 - 1 \\
\sigma(d - f_1 - f_2) &:= a_3 \\
&\vdots \\
\sigma(d - f_1 - \cdots - f_{k-1} + 1) &:= a_{k-1} + f_{k-1} + 1, \\
\sigma(d - f_1 - \cdots - f_{k-1}) &:= a_k, \\
\sigma(d - f_1 - \cdots - f_{k-1} - 1) &:= a_k + 1, \\
&\vdots \\
\sigma(1) = \sigma(d - f_1 - \cdots - f_k + 1) &:= a_k + f_k - 1\,.
\end{aligned} \tag{6.22}$$

For every $a \in K$ we need to check that

$$\sigma^{-1}(a) - \sigma^{-1}(a+1) \leq 1\,.$$

If $\{a, a+1\} \subseteq \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ for some $i = 1, \ldots, k$ the we have $\sigma^{-1}(a) - \sigma^{-1}(a+1) = 1$ by the construction in Equation (6.22). If $\{a, a+1\} \nsubseteq \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ for every $i = 1, \ldots, k$, then, since $a \prec a+1$ is a cover relation in $\mathcal{P}(K)$, we have that $a \in \mathcal{F}_i$ and $a+1 \in \mathcal{F}_j$ for some $i > j$. Then we have $\sigma^{-1}(a) < \sigma^{-1}(a+1)$ again by the construction in Equation (6.22). So, we have a well-defined map from the set of connected chains in the poset $\mathcal{P}(K)$ into the permutations in $W(K)$. It is easy to see that this map is injective.

We now construct an inverse map, i.e., for a permutation $\sigma \in S_d$, we build a chain of filters in $\mathcal{P}(K)$ inductively. See Figure 6.14 for an example. We set $\mathcal{F}_0 = \varnothing$. For $i \geq 0$, as long as $\mathcal{F}_i \neq [d] = \mathcal{P}(K)$, let b be maximal such that $\sigma(b) \notin \mathcal{F}_i$ and let $\mathcal{F}_{i+1}$ be the unique (inclusion-)minimal filter containing $\sigma(b) \cup \mathcal{F}_i$. Note that $\mathcal{F}_i \subsetneq \mathcal{F}_{i+1}$ and since $[d] = \mathcal{P}$ is finite, this induction terminates after (say) k steps. We need to check, that this defines a connected chain of filters, i.e., that for $i = 1, \ldots, k$ we have that $\mathcal{F}_i \setminus \mathcal{F}_{i-1}$ is connected. Indeed, for $i = 1$, we consider $\sigma(d) =: a_1$. Then the unique smallest filter containing $\sigma(d) = a_1$ is

$$\mathcal{F}_1 = \{a \in \mathcal{P}(K) \colon a_1 \preceq a\}$$

and $\mathcal{F}_1 \setminus \mathcal{F}_0 = \mathcal{F}_1$ is trivially connected.

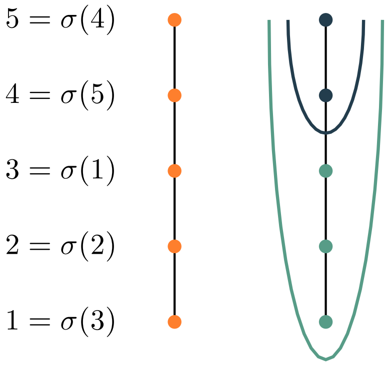


Figure 6.14: Pictured is an illustration of the bijection g used in the proofs of Proposition 6.39 and Theorem 6.40 for the permutation 32154. First consider $\sigma(5) = 4$. Then the smallest filter containing it is $\{4, 5\}$. The largest element whose image is not covered is 3. Since $\sigma(3) = 1$, and the smallest remaining filter containing 1 is the whole chain. This yields the resulting chain fo filters $\varnothing \subsetneq \{4, 5\} \subsetneq \{1, 2, 3, 4, 5\}$.

For $i \geq 1$, let b be maximal such that $a_{i+1} \coloneqq \sigma(b) \notin \mathcal{F}_i$. Then the unique minimal filter containing $\sigma(b) \cup \mathcal{F}_i$ is

$$\mathcal{F}_{i+1} = \{a \in \mathcal{P}(K) \setminus \mathcal{F}_i \colon a_{i+1} \preceq a\} \cup \mathcal{F}_i \,.$$

We have $\mathcal{F}_{i+1} \setminus \mathcal{F}_i = \{a \in \mathcal{P}(K) \setminus \mathcal{F}_i \colon a_{i+1} \preceq a\}$ and this is connected. Note that, this map is well-defined on every permutation $\sigma \in S_d$, however it is not injective on S_d and we will only use it on $W(K) = \{\sigma \in S_d \colon \sigma^{-1}(a) - \sigma^{-1}(a+1) \leq 1 \text{ for all } a \in K\}$.

It is left to check that these two constructions are indeed inverses of each other. Start with a connected chain of filters as defined in Equation (6.21) and construct the permutation $\sigma \in W(K)$ as given in Equation (6.22). Now apply the construction given to define a chain of filters, then we have

$$\mathcal{F}_1 = \{a \in \mathcal{P}(K) \colon a_1 \preceq a\} = \{a_1, a_1 + 1, \ldots, a_1 + f_1 - 1\}$$

and for $i \geq 1$

$$\mathcal{F}_{i+1} = \{a \in \mathcal{P}(K) \setminus \mathcal{F}_i \colon a_{i+1} \preceq a\} \cup \mathcal{F}_i = \{a_i, a_i + 1, \ldots, a_i + f_i - 1\} \cup \mathcal{F}_i$$

as desired. Similar arguments hold for the reverse concatenation of constructions. Hence, this defines a bijection. □

Recall that a **descent** in a permutation $\sigma \in S_d$ is a position $b \in [d]$ such that $\sigma(b) > \sigma(b+1)$. Similarly, an **ascent** in a permutation $\sigma \in S_d$ is a position $b \in [d]$ such that $\sigma(b) < \sigma(b+1)$. We denote the number of descents in a permutation σ by $\operatorname{des}(\sigma)$, i.e., $\operatorname{des}(\sigma) \coloneqq |\{b \in [d-1] \colon \sigma(b) > \sigma(b+1)\}|$.

Theorem 6.40 ([Hor+24b, Proposition 7.4])**.** *The h-polynomial of the partitioned permutahedron for $K \subseteq [d-1]$ is given by*

$$h_{\Pi_d(K)}(t) = \sum_{\sigma \in W(K)} t^{\operatorname{des}(\sigma)} \,,$$

where $W(K)$ is the set of permutations σ such that $\sigma^{-1}(a) - \sigma^{-1}(a+1) \leq 1$ for all $a \in K$ and $\operatorname{des}(\sigma)$ denotes the number of descents in σ.

Proof. Recall from Equation (2.1) in Section 2.1 that we can compute the h-vector of a simple polytope by choosing an edge-generic linear functional $\boldsymbol{\omega}\colon \mathbb{R}^d \to \mathbb{R}$, which induces an acyclic orientation on the graph of the polytope P and counting the number of vertices with i adjacent vertices that are $\boldsymbol{\omega}$-improving.

After applying Proposition 6.39, it is left to show that there exists a linear functional $\boldsymbol{\omega}$ such that the out-degree for every vertex $\mathsf{v} \in \Pi_{\mathcal{P}(K)}$ equals the number of descents in the corresponding permutation $\sigma_{\mathsf{v}} \in W(K)$. We claim that $\boldsymbol{\omega}(\mathbf{x}) := \sum_{l=1}^{d} 2^l \mathbf{x}_l$ for is such a linear functional.

Let us fix an arbitrary vertex v, the corresponding chain of connected filters $\mathfrak{F}_{\mathsf{v}}\colon \varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}(K)$ with filters defined as in Equation (6.21), and the corresponding permutation $\sigma_{\mathsf{v}} \in W(K)$ as defined in Equation (6.22).

Recall that $\Pi_{\mathcal{P}(K)}$ is simple and $(d-1)$-dimensional and hence very vertex is incident to $d-1$ edges. Recall also from Theorem 6.22 the three different types of edges: refining, coarsening, and swapping. With a similar counting argument as in the proof of Theorem 6.25 we can argue that these $d-1$ edges are given by

(i) $d-k$ refining edges, one for each cover relation $\bar{a} \prec\!\!\cdot\; \bar{a}+1$ contained in some $\mathcal{F}_{j+1} \setminus \mathcal{F}_j$ for $j = 0, \ldots, k-1$,

(ii)&(iii) $k-1$ coarsening or swapping edges, one for every $i \in \{1, \ldots, k-1\}$, where the type depends on the number of connected components in $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$.

We can map every edge bijectively to a unique position in $[d-1]$ in the permutation σ_{v} in the following way:

(i) for every cover relation $\bar{a} \prec\!\!\cdot\; \bar{a}+1$ contained in a filter, we map the refining edge to position $\sigma_{\mathsf{v}}^{-1}(\bar{a}+1)$,

(ii)&(iii) for every $i \in \{1, \ldots, k-1\}$ we map the coarsening or swapping edge to position $\sigma_{\mathsf{v}}^{-1}(a_{i+1})$ (recall Equation (6.21)).

Note that we do not use position $d = \sigma^{-1}(a_1)$ but every other position exactly once, so this is a well defined bijection as claimed.

We will now characterize those edges mapped to descent positions and those mapped to ascent positions. Finally, we will compute for each edge whether it is $\boldsymbol{\omega}$-improving and see that the $\boldsymbol{\omega}$-improving ones are precisely those mapped to descent positions.

(i) Let $\bar{a} \prec\!\!\cdot\; \bar{a}+1$ be a cover relation contained in a difference of filters $\mathcal{F}_{j+1} \setminus \mathcal{F}_j$, i.e., $\{\bar{a}, \bar{a}+1\} \subseteq \mathcal{F}_{j+1} \setminus \mathcal{F}_j = \{a_{j+1}, \ldots, a_{j+1} + f_{j+1} - 1\}$. Then the corresponding refining edge is mapped to the descent in the permutation σ_{v} at position $\sigma_{\mathsf{v}}^{-1}(\bar{a}+1)$, Equation (6.22).

(ii) If $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ is connected for some $i = 1, \ldots, k-1$, then deleting the filter $\mathcal{F}_i$ from the chain of filters defines a coarsening edge (see Theorem 6.22(ii)). Then this defines an ascent in σ_{v} at position $\sigma_{\mathsf{v}}^{-1}(a_{i+1})$.

Indeed, recall from Equation (6.22) that $\sigma_{\mathsf{v}}(d - f_1 - \cdots - f_i) = a_{i+1}$ and $\sigma_{\mathsf{v}}(d - f_1 - \cdots - f_i + 1) = a_i + f_i - 1$. Since $\mathcal{F}_i \subsetneq \mathcal{F}_{i+1}$ and $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ is a connected chain, it follows that

$$a_{i+1} \leq \max(\mathcal{F}_{i+1 \setminus \mathcal{F}_i}) = a_{i+1} + f_{i+1} - 1 < \min \mathcal{F}_i = a_i \leq a_i + f_i - 1\,.$$

Hence $\sigma_{\mathsf{v}}(d - f_1 - \cdots - f_i) < \sigma_{\mathsf{v}}(d - f_1 - \cdots - f_i + 1)$ and the permutation σ_{v} has an ascent at position $d - f_1 - \cdots - f_i = \sigma_{\mathsf{v}}^{-1}(a_{i+1})$.

(iii) If $\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ has two connected components for some $i = 1, \ldots, k-1$, there is a swapping edge as defined in Theorem 6.22(iii). Recall from Equation (6.22) that $\sigma_{\mathsf{v}}^{-1}(a_{i+1}) = d - f_1 - \cdots - f_i$ and that $\sigma_{\mathsf{v}}(d - f_1 - \cdots - f_i + 1) = a_i + f_i - 1$ again we need to compare a_{i+1} and $a_i + f_i - 1$:

a) If $a_{i+1} < a_i$, then also $a_{i+1} < a_i + f_i - 1$ and the permutation σ_{v} has an ascent at position $\sigma_{\mathsf{v}}^{-1}(a_{i+1}) = d - f_1 - \cdots - f_i$.

b) If $a_{i+1} > a_i$, then is also follows that $a_{i+1} > a_i + f_i - 1$ since $\{a_i, \ldots, a_i + f_i - 1\}$ is an interval of consecutive elements in $[d-1]$. So, the permutation σ_{v} has a descent at position $\sigma_{\mathsf{v}}^{-1}(a_{i+1}) = d - f_1 - \cdots - f_i$.

We now check that exactly the edges mapped to descent positions are $\boldsymbol{\omega}$-improving:

(i) At this cover relation the connected chain of filters $\mathfrak{F}$ can be refined to a chain of filter $\mathfrak{F}'$ as follows:

$$\varnothing = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_j \subsetneq \widetilde{\mathcal{F}} \subsetneq \mathcal{F}_{j+1} \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{P}(K)\,,$$

where $\widetilde{\mathcal{F}} := \{a \in \mathcal{P}(K) \colon \bar{a} + 1 \leq a\} \cup \mathcal{F}_j$. Then,

$$\begin{aligned} \widetilde{\mathcal{F}} \setminus \mathcal{F}_j &= \{\bar{a} + 1, \ldots, a_{j+1} + f_{j+1} - 1\} \\ \text{and} \quad \mathcal{F}_{j+1} \setminus \widetilde{\mathcal{F}} &= \{a_{j+1}, \ldots, \bar{a}\}\,. \end{aligned}$$

Note, this is still a connected chain of filters and the edge vector is (see Equation (6.15))

$$\frac{1}{2}\left(\left(|\mathcal{F}_{j+1}| - |\widetilde{\mathcal{F}}|\right) \mathbb{1}_{\widetilde{\mathcal{F}} \setminus \mathcal{F}_j} + \left(|\mathcal{F}_j| - |\widetilde{\mathcal{F}}|\right) \mathbb{1}_{\mathcal{F}_{j+1} \setminus \widetilde{\mathcal{F}}}\right)\,.$$

Now we can easily check that this edge direction is $\boldsymbol{\omega}$-improving:

$$\begin{aligned} &\frac{1}{2}\left(\left(|\mathcal{F}_{j+1}| - |\widetilde{\mathcal{F}}|\right) \boldsymbol{\omega}\left(\mathbb{1}_{\widetilde{\mathcal{F}} \setminus \mathcal{F}_j}\right) + \left(|\mathcal{F}_j| - |\widetilde{\mathcal{F}}|\right) \boldsymbol{\omega}\left(\mathbb{1}_{\mathcal{F}_{j+1} \setminus \widetilde{\mathcal{F}}}\right)\right) \\ =&\frac{1}{2}\left(\left(\underbrace{|\mathcal{F}_{j+1}| - |\widetilde{\mathcal{F}}|}_{\geq 1}\right) \sum_{l=\bar{a}+1}^{a_{j+1}+f_{j+1}-1} 2^l + \left(\underbrace{|\mathcal{F}_j| - |\widetilde{\mathcal{F}}|}_{\leq -1}\right) \sum_{l=a_{j+1}}^{\bar{a}} 2^l\right) \;\geq\; 0\,, \end{aligned}$$

since

$$\sum_{l=\bar{a}+1}^{a_{j+1}+f_{j+1}-1} 2^l \gg \sum_{l=a_{j+1}}^{\bar{a}} 2^l\,.$$

(ii) This also implies that every coarsening edge incident to v is not $\boldsymbol{\omega}$-improving.

(iii) For swapping edges, the edge direction (Equation (6.18)) is

$$(|\mathcal{F}_{j+1}| - |\mathcal{F}_j|)\,\mathbb{1}_{\mathcal{F}_j \setminus \mathcal{F}_{j-1}} + (|\mathcal{F}_{j-1}| - |\mathcal{F}_j|)\,\mathbb{1}_{\mathcal{F}_{j+1} \setminus \mathcal{F}_j}\,.$$

We compute again using Equation (6.21):

$$\begin{aligned}&(|\mathcal{F}_j| - |\mathcal{F}_{j+1}|)\,\boldsymbol{\omega}\left(\mathbb{1}_{\mathcal{F}_j \setminus \mathcal{F}_{j-1}}\right) + (|\mathcal{F}_j| - |\mathcal{F}_{j-1}|)\,\boldsymbol{\omega}\left(\mathbb{1}_{\mathcal{F}_{j+1} \setminus \mathcal{F}_j}\right)\\ =&\Big(\underbrace{|\mathcal{F}_j| - |\mathcal{F}_{j+1}|}_{\leq -1}\Big)\boldsymbol{\omega}\left(\sum_{l=a_j}^{a_j+f_j-1} 2^l\right) + \Big(\underbrace{|\mathcal{F}_j| - |\mathcal{F}_{j-1}|}_{\geq 1}\Big)\boldsymbol{\omega}\left(\sum_{l=a_{j+1}}^{a_{j+1}+f_{j+1}-1} 2^l\right).\end{aligned}$$

a) If $\sigma_{\mathsf{v}}^{-1}(a_{j+1})$ is an ascent position, we have $a_{j+1} < a_j + f_j - 1$, which implies

$$\sum_{l=a_j}^{a_j+f_j-1} 2^l \gg \sum_{l=a_{j+1}}^{a_{j+1}+f_{j+1}-1} 2^l\,,$$

hence these swapping edges are not $\boldsymbol{\omega}$-improving.

b) If $\sigma_{\mathsf{v}}^{-1}(a_{j+1})$ is an descent position, we have $a_{j+1} > a_j + f_j - 1$, which implies

$$\sum_{l=a_j}^{a_j+f_j-1} 2^l \ll \sum_{l=a_{j+1}}^{a_{j+1}+f_{j+1}-1} 2^l\,,$$

hence these swapping edges are $\boldsymbol{\omega}$-improving. □

We also offer an alternative characterization of the vertices of partitioned permutahedra $\Pi_{\mathcal{P}(K)}$. To do this, we require the notion of a high dimensional rook walk as found in [KZ11]. In high dimensions, one can consider any sequence from $(0, 0, \ldots, 0)$ to $(n_1, n_2, \ldots, n_c)$ such that at each step one can only increase (potentially by more than one) in a single coordinate at a time. See Equation (6.23) below, for an example.

Recall that $\mathcal{P}(K)$ is a disjoint union of chains. Let c be the number of connected components, i.e., the number of chains, and $n_1, \ldots, n_c$ the number of elements in those chains ($n_1 + \cdots + n_c = d$). We can read those off as follows: $K^C = [d] \setminus K = \{o_1, \ldots, o_c\}$ then $n_i = o_i - o_{i-1}$ for $i = 1, \ldots, c$ (set $o_0 := 0$, not that $o_c = d$). Further denote the minimal elements in the chains by $m_1, \ldots, m_c$ ($m_1 = 1$).

Proposition 6.41. *For $K \subseteq [d-1]$, the following sets have the same cardinality:*

1. *The set of vertices of the partitioned permutahedron $\Pi_d(K)$.*
2. *The set of high dimensional rook walks from $(0, 0, \ldots, 0)$ to $(n_1, \ldots, n_c)$,*

The bijection is best understood from an example. Writing down a formal proof is not to hard but heavy on notation. We give a sketch below, after an example.

Example 6.42. Let $\mathcal{P}(\{1, 2, 4, 5, 6\}) = ([3], \leq) \sqcup (\{4, \ldots, 7\}, \leq)$ the disjoint union of a 3-chain and a 4-chain, i.e., we have two connected components with $n_1 = 3$ and $n_2 = 4$. Let

$$\varnothing \subsetneq \{6, 7\} \subsetneq \{3, 6, 7\} \subsetneq \{1, 2, 3, 6, 7\} \subsetneq \{1, 2, 3, 4, 5, 6, 7\}$$

be a connected chain of filters. Then the corresponding rook walk is

$$(0,0) \to (0,2) \to (1,2) \to (3,2) \to (3,4)\,. \tag{6.23}$$

Proof sketch of Proposition 6.41. We will build a bijection between connected chains of filters of the poset $\mathcal{P}(K)$, which in turn correspond to the vertices of $\Pi_{\mathcal{P}(K)}$, and the high-dimensional rook walks from $(0,\dots,0)$ to $(n_1,\dots,n_c)$.

For $i = 1,\dots,k$ we define $\mathcal{F}_i \setminus \mathcal{F}_{i-1}$ from the ith step in the rook walk as follows: If in the rook walk the ith step is from $(x_1,\dots,x_l,\dots,x_c)$ to $(x_1,\dots,x_l+f_i,\dots,x_c)$ then set $\mathcal{F}_i \setminus \mathcal{F}_{i-1} \coloneqq \{d-n_c-\cdots-n_l+1+x_l+1,\dots,d-n_c-\cdots-n_l+1+x_l+f_i\}$. This is a connected chain of filters. This process can easily be shown to be a bijection. □

Thus vertices of $\Pi_{\mathcal{P}(K)}$ can be counted via high-dimensional rook walks. For this there is an easy recurrence: Let $v_{n_1,n_2,\dots,n_c}$ denote the number of high-dimensional rook walks from $(0,\dots,0)$ to $(n_1,\dots,n_c)$. Then

$$v_{n_1,n_2,\dots,n_c} = \sum_{i=1}^{c}\sum_{l=0}^{n_i-1} v_{n_1,n_2,\dots,n_{i-1},l,n_{i+1},\dots,n_c}\,.$$

Indeed, this is the sum over all possible last steps: Assume the last step is in coordinate i for $i \in \{1,\dots,c\}$ and of length $n_i - l$ for $l \in \{0,\dots,n_i-1\}$. For each possible last step we need to sum up the number of possible rook walks before that last step, i.e., $v_{n_1,n_2,\dots,n_{i-1},l,n_{i+1},\dots,n_c}$.

In [KZ11], Kauers and Zeilberger studied asymptotics and recurrences for the sequence of high dimensional rook walks from $(0,0,\dots,0)$ to $(d,d,\dots,d)$, and our results give a new perspective for arbitrary endpoints.

Finally, we note as a corollary of Corollary 6.27, we can compute the volumes of the partitioned permutahedra in our particular geometric realization.

Corollary 6.43. *The volume of the partitioned permutahedron $\Pi_d(K)$ for $K \subseteq [d-1]$ is given by*

$$\frac{d^{d-2}}{d!}\binom{d}{n_1,\dots,n_c}$$

with the same notation as above, i.e., $n_i = o_i - o_{i-1}$ and $[d] \setminus K = \{o_1,\dots,o_c\}$.

Proof. We need to compute the number of linear extensions of $\mathcal{P}(K)$. This is

$$\frac{d!}{n_1!\cdot\cdots\cdot n_c!} = \binom{d}{n_1,\dots,n_c}.$$

□

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