ON THE POLYHEDRAL GEOMETRY OF t–DESIGNS

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *ON THE POLYHEDRAL GEOMETRY OF t–DESIGNS* by Steven Collazos and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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ON THE POLYHEDRAL GEOMETRY OF t–DESIGNS

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Lisonek (2007) proved that the number of isomorphism types of $t-(v,k,\lambda)$ designs, for fixed t, v, and k, is quasi–polynomial in λ. We attempt to describe a region in connection with this result. Specifically, we attempt to find a region $\mathcal{F}$ of $\mathbb{R}^d$ with the following property: For every $x \in \mathbb{R}^d$, we have that $|\mathcal{F} \cap G_x| = 1$, where G_x denotes the G–orbit of x under the action of G. As an application, we argue that our construction could help lead to a new combinatorial reciprocity theorem for the quasi–polynomial counting isomorphism types of $t-(v,k,\lambda)$ designs.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Chapter 1

Introduction

In 1850, Thomas P. Kirkman, a priest who was an active mathematician, formulated the following question [4, p. 88]:

"Fifteen young ladies in a school walk out three abreast for seven days in succession: It is required to arrange them daily, so that no two walk twice abreast."

The question is whether it is possible to organize 15 students into five groups of three students each, in a way such that any pair of students are in the same triplet exactly once in the week. Refer to Figure 1.1 for one possible way of carrying out such a task, where we have identified each student with a number from 01 to 15 [4].

Kirkman's Schoolgirl Problem is one of the earliest examples of an object called a $\boldsymbol{t-(v,k,\lambda)}$ **design**, or t–design. We will define what a t–design is in Section 2.4.

A natural question to ask is whether the solution given in Figure 1.1 to Kirkman's Schoolgirl Problem is the only one. In general, it can be shown that if the parameters

Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
01,02,03	01,08,09	01,10,11	01,04,05	01,06,07	01,12,13	01,14,15
04,10,14	02,05,07	02,13,15	02,12,14	02,08,10	02,09,11	02,04,06
05,08,13	03,13,14	03,05,06	03,09,10	03,12,15	03,04,07	03,08,11
06,09,15	04,11,15	04,08,12	06,11,13	04,09,13	05,10,15	05,09,12
07,11,12	06,10,12	07,09,14	07,08,15	05,11,14	06,08,14	07,10,13

Figure 1.1: A solution to Kirkman's Schoolgirl Problem.

t, v, and k are fixed, then the function counting the number of different designs, or *isomorphism types* of designs, is a quasi–polynomial in the parameter λ, where λ is a natural number [13]

As one of the highlights of the proof for this result, we note the use of Ehrhart's theory of lattice–point enumeration in polyhedra, and a geometric model that is set up such that the integer points in a certain polytope $\mathcal{P}$ correspond to t–designs. However, lattice points in $\mathcal{P}$ are equivalent up to permutations of coordinates — where the permutations are the ones preserving the design's structure. Therefore, in order to avoid overcounting, it is necessary to count integer points in some $\mathcal{R} \subseteq \mathcal{P}$ such that every design is represented, and such that no two points in $\mathcal{R}$ are equivalent.

We study the combinatorial structure of $\mathcal{R}$, but instead of working with $\mathcal{R}$, we will focus on a *fundamental domain* $\mathcal{F} \subseteq \mathbb{R}^d$ because if the the problem of constructing $\mathcal{F}$ is solved, a method for constructing $\mathcal{R}$ would be immediate. We construct $\mathcal{F}$ to be the set of lexicographically largest elements in their respective orbits.

We have two main results. In Theorem 4.4, we prove that the set of lexicographically largest elements is *strongly pure*, that is, if σ is an i–face of our fundamental domain $\mathcal{F}$, then σ is a face of another face in $\mathcal{F}$ of dimension $i+1$. This result holds for any $G \subseteq S_d$ acting by permuting the coordinates of points in $\mathbb{R}^d$. The second outcome of our investigation, Theorem 4.9, is a system of linear inequalities for the interior of $\mathcal{F}$ in the case when G is $\mathrm{Aut}(\mathcal{D})$, where $\mathrm{Aut}(\mathcal{D})$ is the set of structure–preserving permutations for a t–design $\mathcal{D}$. We assume in our construction that $\mathcal{D}$ is a t–design with parameter $k=2$.

We assume that the reader is comfortable with groups and linear algebra. We will provide an overview of polyhedral geometry, and combinatorial design theory. We also fix notation we will need from group theory and topology. In Chapter 3, we explain how counting the number of distinct t–designs is a geometric problem. In particular, given that the counting function is a quasi–polynomial [13], this leads us to ask about the derivation of a *combinatorial reciprocity theorem*. We will argue that an inequality description for $\mathcal{F}$ is necessary to obtain such a reciprocity theorem. In Chapter 4, we present a way of constructing the interior of a fundamental cell, and we present the main results of the paper. We end with open questions about possible applications for our ideas and suggestions for improving our results.

Notation

We will be making use of the following notation:

- We will denote by $\mathbb{R}^d$ the d–dimensional Euclidean space consisting of column vectors with d coordinates.
- Let $\mathbb{N}$ be the set of positive integers $\{1, 2, 3, \ldots\}$.
- We will reserve the letters n and m for positive integers.
- Let $[n]$ denote the set $\{1, 2, \ldots, n\}$.
- The collection of all possible m–sets of a set X will be referred to as $\binom{X}{m}$.
- When discussing n–subsets of an m–set, where $m \geq n$, we will often write the subsets as strings. Example: Instead of writing

$$\binom{[3]}{2} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\},$$

  we will write

$$\binom{[3]}{2} = \{12, 13, 23\}.$$

Chapter 2

Background

2.1 Polyhedral Geometry

Polytopes and polyhedra are essential objects in our narrative, so we will provide some basic definitions, along with examples.

A subset $A \subseteq \mathbb{R}^d$ is **convex** [22, p. 3] if given arbitrary $x, y \in A$, the line segment joining x and y is also contained in A. Given a finite subset $K = \{v_1, v_2, \ldots, v_n\}$ of $\mathbb{R}^d$, a **convex polytope** $\mathcal{P}$ is the smallest convex set containing K or, more formally [22, p. 3],

$$\mathcal{P} = \bigcap \{K' \subseteq \mathbb{R}^d : K \subseteq K'; K' \text{ is convex}\}.$$

Such an intersection is known as the **convex hull** of K, and it is typically denoted $\mathrm{conv}(K)$. (Convex hulls of arbitrary subsets of $\mathbb{R}^d$ can also be considered. For instance, the convex hull of a circle is a disk.) Therefore, a convex polytope is the convex hull of finitely many points. (Refer to Figure 2.1.) For the remainder of our discussion, we just say "polytope" for short.

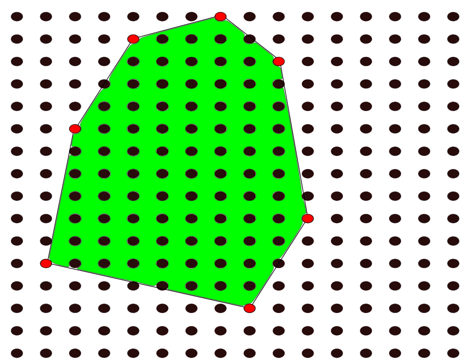

Figure 2.1: A polygon with seven sides.

Polytopes have a notion of dimension. In order to be defined properly, we need some terminology from linear algebra. An **affine subspace** in $\mathbb{R}^d$ is a set of the form $V + x$, where V is a linear subspace of $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Informally, an affine subspace looks like a subspace that has been "shifted."

Given a polytope $\mathcal{P}$, we define the **dimension** [2, p. 26] of $\mathcal{P}$ to be the dimension of the affine space

$$\operatorname{span}(\mathcal{P}) \equiv \{x + \lambda(y - x) : x, y \in \mathcal{P}; \lambda \in \mathbb{R}\}.$$

If the dimension of the polytope is known, say d, then we call $\mathcal{P}$ a d-polytope.

We are now ready to consider some examples of polytopes.

Example 2.1. (Convex) polygons are 2-polytopes.

Example 2.2. A d-simplex [22] is defined to be the convex hull of $d+1$ affinely independent points in some $\mathbb{R}^n$, where $n \geq d$. We say that a finite set of points $X = \{x_1, x_2, \ldots, x_m\}$ of $\mathbb{R}^d$ is **affinely independent** [3] if for any $x \in \mathbb{R}^d$ with $x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m$ an affine linear combination of X (that is, $\sum_{i \in [n]} \lambda_i = 1$), then $\lambda_1, \lambda_2, \ldots, \lambda_m$ can be uniquely determined.

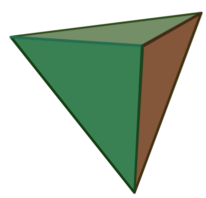

Figure 2.2: A picture of a 3-simplex, also known as a tetrahedron. Image obtained from Wikipedia.

The initial definition for a polytope can be refined. For instance, if A is a finite subset of $\mathbb{Q}^d$, then $\operatorname{conv}(A)$ is called a **rational polytope**. If A is a finite subset of $\mathbb{Z}^d$, then $\operatorname{conv}(A)$ is called an **integral polytope**.

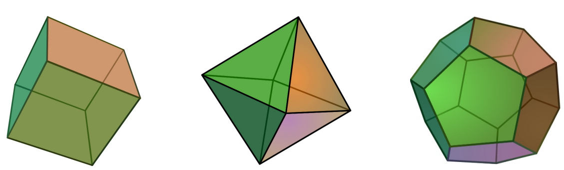

Figure 2.3: These are examples of three-dimensional polytopes: Cube, octahedron and dodecahedron. Images obtained from Wikipedia.

Polytopes can also be defined in terms of halfspaces. First recall that a **hyperplane** is a set of the form $\{x \in \mathbb{R}^d : a \cdot x = b\}$, where $a \in \mathbb{R}^d$, $b \in \mathbb{R}$, and "$a \cdot x$" means we are taking the dot product of the vectors a and x. Then for a **halfspace** $\mathcal{H}$, we replace the equality with an inequality, so $\mathcal{H} = \{x \in \mathbb{R}^d : x \cdot a \geq b\}$ or $\mathcal{H} = \{x \in \mathbb{R}^d : a \cdot x \leq b\}$.

One of the basic results in polytope theory is the equivalence of representing a polytope $\mathcal{P}$ as the convex hull of its vertices and as an intersection of halfspaces. Formally, $\mathcal{P} \subseteq \mathbb{R}^d$ is the convex hull of finitely many points if and only if $\mathcal{P}$ is a bounded intersection of halfspaces [22, p. 29].

There is an important family of geometric objects that, unlike polytopes, are not necessarily bounded. A **polyhedron** $\mathcal{Q}$ is the solution set to a system of linear inequalities. The **hyperplane representation** of $\mathcal{Q}$ provides a compact way of writing such a statement: Let A be a $k \times d$ matrix $A \in \mathbb{R}^{k \times d}$, and $b \in \mathbb{R}^k$. Let A_i denote the ith column of A. Then one can write a system of linear inequalities as $Ax \leq b$, meaning that $A_i \cdot x \leq b_i$. Hence, $\mathcal{Q} = \{x \in \mathbb{R}^d : Ax \leq b\}$ is a polyhedron.

Example 2.3. Let us determine the hyperplane representation of a unit square $\mathcal{P}$ with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$.

Notice that $\mathcal{P}$ is the intersection of four half-spaces: $x \geq 0$, $y \geq 0$, $x \leq 1$ and $y \leq 1$, which can be rewritten as the system of inequalities

$$-x + 0y \leq 0,$$

$$0x - y \leq 0,$$

$$x + 0y \leq 1,$$
$$0x + y \leq 1.$$

We write the coefficients of the variables in the matrix A, which yields

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The vector b will have the corresponding constants to the right-hand side of the inequality, that is, $b = (0, 0, 1, 1)$. Now we see that $\mathcal{P} = \{x \in \mathbb{R}^2 : Ax \leq b\}$.

Later on, we will be using a particular kind of polyhedron in our construction of a fundamental region. Let $Y = \{y_1, y_2, \ldots, y_r\} \subseteq \mathbb{R}^d$ be a finite set of vectors. Then a **cone** $\mathcal{C}$ [22, p. 28] is defined to be

$$\mathcal{C} \equiv \left\{ \sum_{i=1}^{r} \lambda_i y_i : \lambda_i \geq 0 \right\}.$$

A variety of combinatorial data can be associated to polyhedra. One such notion is that of a *face* of a polyhedron:

Definition 2.1 ([22])**.** Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a convex polyhedron. For $c, x \in \mathbb{R}^d$ and $c_0 \in \mathbb{R}$, a linear inequality $cx \leq c_0$ is said to be **valid** for $\mathcal{P}$ if it holds for every

$x \in \mathcal{P}$. A **face** of $\mathcal{P}$ is any set of the form

$$F = \mathcal{P} \cap \{x \in \mathbb{R}^d : cx = c_0\},$$

where $cx \leq c_0$ is a valid inequality for $\mathcal{P}$. The **dimension** of a face is $\dim(\text{aff}(F))$. The *affine hull* [22] of a finite point set $X = \{x_1, x_2, \ldots, x_n\} \subseteq \mathbb{R}^d$, denoted $\text{aff}(X)$, is defined to be

$$\{x \in \mathbb{R}^d : x = \sum_{i \in [d]} \lambda_i x_i; \lambda_j \in \mathbb{R}; \sum_{i \in [d]} \lambda_i = 1\}.$$

One can also consider collections of polyhedra. Specifically, a **polyhedral complex** [22] $\mathcal{C}$ is a finite collection of polyhedra in $\mathbb{R}^d$ such that

- the empty polyhedron is in $\mathcal{C}$,
- if $\mathcal{P} \in \mathcal{C}$, then all the faces of $\mathcal{P}$ are also in $\mathcal{C}$, and
- the intersection $\mathcal{P} \cap \mathcal{Q}$ of two polyhedra $\mathcal{P}, \mathcal{Q} \in \mathcal{C}$ is a face of both $\mathcal{P}$ and $\mathcal{Q}$.

The **dimension** $\dim(\mathcal{C})$ is the largest dimension of a polyhedron in $\mathcal{C}$. We say that $\mathcal{C}$ is **pure** if each of its faces is contained in a face of dimension $\dim(\mathcal{C})$.

2.2 Poset Theory

A **partially ordered set**, or poset, is a pair $(P, \geq)$, where P is a set and $\geq$ is a relation satisfying [12]:

1. (Reflexivity) For any $x \in P$, $x \geq x$.

2. (Transitivity) Given $x, y, z \in P$, if $x \geq y$ and $y \geq z$, then $x \geq z$.

3. (Antisymmetry) For $x, y \in P$, if $x \geq y$ and $y \geq x$, then $x = y$.

Such a relation $\geq$ satisfying the above three properties is called a **partial order**. When the partial order is clear from context, we call P a poset instead of referring to the pair $(P, \geq)$. Additionally, sometimes we write the partial order as $\geq_P$, or any other adequate index, if there is ambiguity.

We say that $x, y \in P$ are **comparable** if either $x \geq y$ or $x \leq y$. Finally, given $x, y \in P$ with $x > y$, we say that x **covers** y if there does not exist $z \in P$ such that $x > z$ and $z > y$. We will write $x \gtrdot y$ to say that x covers y.

Example 2.4. The set of integers $\mathbb{Z}$ forms a poset under the usual ordering. In fact, it is a special kind of poset called a **chain**, which is a poset where any two elements are comparable [12, p. 9].

Partial orders $\leq$ of this kind where any two elements in P are comparable have a name of their own: **Total orders**.

Example 2.5. Let $S = \{\{a, b\}, \{b, c\}, \{c, d\}\}$ be a set, and let $\geq$ be $\subseteq$. In other words, S is a poset where the partial order is reverse set inclusion. Observe that neither $\{a, b\} \subseteq \{c, d\}$ nor $\{c, d\} \subseteq \{a, b\}$ hold. Therefore, $\{a, b\}$ and $\{c, d\}$ are not comparable elements. In fact, no two elements of S are comparable. Posets where no two elements are comparable are called **antichains**.

Let $\geq_1$ and $\geq_2$ be partial orders on a set P. The partial order $\geq_2$ is an **extension** [12, p. 10] of $\geq_1$ if $x \geq_1 y$ implies $x \geq_2 y$. If the partial order $\geq_2$ turns P into a chain, then $\geq_2$ is called a **linear extension** of $\geq_1$.

Sometimes it is convenient to have a pictorial representation of P. For this purpose, one can construct a **Hasse diagram**: Draw a graph of P where vertices are elements of P and a directed edge indicates that x covers y. One can stretch the edges so that x is above y if $x \geq y$.

Example 2.6. Let $P = \{a, b, c, d\}$ be a set with partial order $\geq$ such that the following relations hold: $a \gtrdot b$, $a \gtrdot c$, $b \gtrdot d$ and $c \gtrdot d$. (Refer to Figure 2.4.) Note that a possible linear extension of P is $a \gtrdot b \gtrdot c \gtrdot d$.

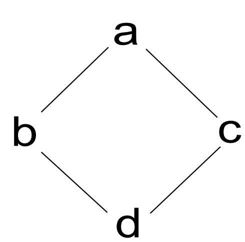

Figure 2.4: The Hasse diagram of the poset from Example 2.6.

Here is one last example of a partial order:

Example 2.7. Let $P_1, P_2, \ldots, P_k$ be posets with partial orders $\leq_{P_1}, \leq_{P_2}, \ldots, \leq_{P_k}$, respectively. Then the **lexicographical order** [15] on $\widetilde{P} \equiv P_1 \times P_2 \times \cdots \times P_k$, denoted $\leq_{\text{lex}}$, is the partial order where given $x, y \in \widetilde{P}$, we have that $x \leq_{\text{lex}} y$ if and only if

$$\begin{gathered} x_1 <_{P_1} y_1, \\ \text{or } x_1 = y_1 \text{ and } x_2 <_{P_2} y_2, \\ \vdots \\ \text{or } x_1 = y_1 \text{ and } x_2 = y_2 \text{ and } \dots \text{ and } x_k \leq_{P_k} y_k. \end{gathered}$$

For instance, setting $P_i = \mathbb{R}$, we have that $P_1 \times P_2 \times \cdots \times P_d = \mathbb{R}^d$ could be ordered lexicographically. In particular, $\mathbb{R}^d$ is totally ordered under the lexicographical order.

2.3 Group Theory

In this brief section, we introduce the ideas and notation from group theory we need to discuss isomorphism types of t–designs and orbits of points in $\mathbb{R}^d$.

Let $X = \{i_1, i_2, \dots, i_d\}$, with $i_1 \succ i_2 \succ \cdots \succ i_d$, be a totally ordered set with d elements, and $G \subseteq S_X$ be a subgroup of the symmetric group on X. Define a **group action** of G on $\mathbb{R}^X$ via

$$\pi \cdot x \equiv (x_{\pi^{-1}(i_1)}, x_{\pi^{-1}(i_2)}, \dots, x_{\pi^{-1}(i_d)}),$$

where $\pi \in G$, and $x = (x_{i_1}, x_{i_2}, \dots, x_{i_d}) \in \mathbb{R}^X$.

The $\boldsymbol{G}$**−orbit of** $\boldsymbol{x}$ is the set

$$G_x \equiv \{\pi \cdot x : \pi \in G\}.$$

We will often write $\mathbb{R}^d$ instead of $\mathbb{R}^X$, with the understanding that $\mathbb{R}^d \cong \mathbb{R}^X$, and that sometimes it is helpful to think of the coordinates of x in RR^d as being indexed by the elements of X.

We can also extend the action to subsets S of X by defining

$$\pi \cdot S \equiv \{\pi \cdot s : s \in S\}.$$

The definition for an action on a collection of subsets of X is identical.

Example 2.8. Let $X = [6]$, $\pi = (1\ 2\ 3\ 4) \in S_X$, and $\mathcal{B} = \{\{1,2,3\},\{3,4,5\},\{1,3,6\}\}$. If we apply π to $\mathcal{B}$, then we have

$$\pi \cdot \mathcal{B} = \{234, 145, 246\}.$$

The number of elements in a finite group G is its **order**, and we denote it by $|G|$. If H is a subgroup of G, then we write the **index** of H in G as $[G : H] \equiv \frac{|G|}{|H|}$.

2.4 t–Designs

Vaguely speaking, a combinatorial design is a collection of sets satisfying certain incidence restrictions. Designs find applications in computer science [6], networking, communications and cryptography [5], as well as statistical design of experiments and combinatorial algorithms [11]. Within mathematics, designs have connections to finite geometry, incidence geometry, graph theory, and group theory [11].

We begin with an example [21] to give a flavor. We will then introduce a definition for the family of combinatorial designs of interest, namely $t-(v,k,\lambda)$ designs.

Example 2.9. Let $S=[6]$. Suppose we want to construct a collection $\mathcal{C}$ of 3–subsets of S with the property that the intersection of any two $A,B\in\mathcal{C}$ has one element. Further, every element of S occurs in three members of $\mathcal{C}$.

Is this possible? Let us take a naive approach to construct $\mathcal{C}$. Let us start with $B_1\equiv\{0,1,2\}$. Since we need intersections among 3–subsets to have one element in common, let us pick $B_2\equiv\{2,x,y\}$, where $x,y\in S$. The intersection $B_1\cap B_2$ already has one element, so x and y must be such that $x,y\notin B_1$. The elements 3 and 4 have this property, so $B_2=\{2,3,4\}$. Since 4 is next to 5, define $B_3\equiv\{2,5,6\}$.

Now we have to be careful because 2 shows up in three 3–subsets we have built. Let us take 1 this time, so $B_4\equiv\{1,x,y\}$. Note that B_4 already has one member in common with B_1; therefore, we need $x,y\notin B_1$, but at the same time we want B_4 to intersect in exactly one element with B_2, and B_3. Notice that if $B_4=\{1,4,6\}$, then B_4 enjoys both properties.

For the next block, let $B_5 \equiv \{1, x, y\}$, where $x, y \in S$. Since $B_5 \cap B_4$ consist of the member 1, we require that $x, y \notin \{4, 6\}$, so 3 and 5 are the only elements of S possible. (We cannot pick 2 because $\{1, 2, y\}$ has two elements in common with B_1.) Thus, we have that $B_5 = \{1, 3, 5\}$.

Our collection so far of B_i's satisfies the desired intersection property. However, 0, 3, 4, 5, and 6 show up in only two of the B_i's. We need them to appear three times, and this can be fixed. Let $B_6 \equiv \{0, \alpha, \beta\}$ and $B_7 \equiv \{0, \gamma, \delta\}$:

- Observe that $\{0, \alpha, \beta\}$ already intersects with B_1, so let $\alpha = 3$. Since $\{0, 3, \beta\}$ intersects with B_1, B_2, B_5, and we cannot pick 1, 2, 4 nor 5, so the only option left is $\beta = 6$. Hence, $B_6 = \{0, 3, 6\}$;

- By a similar line of reasoning, we find that $B_7 = \{0, 4, 5\}$.

Notice that every element of S shows up three times in our collection of B_i's; therefore, the desired collection of 3–subsets of S is

$$\mathcal{C} \equiv \{012, 035, 046, 136, 145, 234, 256\}.$$

Even though $\mathcal{C}$ is an example of a combinatorial design, we will be focusing on examples like Kirkman's Schoolgirl Problem in Chapter 1:

Definition 2.2 ([13])**.** Let $v, k, t \in \mathbb{N}$, where $0 < t < k < v$, and X be a v–set. A $\boldsymbol{t - (v, k, \lambda)}$ **design**, or t–design, is a collection of k–subsets (**blocks**) of X such that any t–subset of X intersects with the blocks exactly λ times. We usually write $\mathcal{D} = (X, \mathcal{B})$ to refer to a t-design, where $\mathcal{B}$ is the collection of blocks.

Example 2.10. The solution to Kirkman's Schoolgirl Problem given in Chapter 1 is an example of a combinatorial design, where $v = 15$, $k = 3$, $t = 2$, and $\lambda = 1$.

Example 2.11. Let $X = [7]$, and $\lambda = 1$. Let $v = 7$, $k = 3$, and $t = 2$. Then

$$\mathcal{B} = \{124, 235, 346, 457, 156, 267, 137\}$$

is a 2–design [11, p. 595].

Example 2.12. ([21]) Let $G = (V, E)$ be a graph with vertex set V and edge set E. The number of edges containing a vertex is called its *degree*. A graph G is **regular** if the degree of every vertex is the same. Therefore, a regular graph is a $1 - (|V|, 2, \lambda)$ design, where λ the degree of every vertex. (Note that $X = V$, and $\mathcal{B}$ consists of the edges of the graph, so $k = 2$ and $t = 1$.)

When $t = 2$, we say that the t–design is a **balanced incomplete block design**, or BIBD [21]. It turns out that 2–designs are of interest in experimental designs in statistics [21, p. 265].

A natural question to ask is when two t–designs are the same. Let $\mathcal{D}_1 = (X, \mathcal{B}_1)$ and $\mathcal{D}_2 = (X, \mathcal{B}_2)$ be t–designs. We say that $\mathcal{D}_1$ is **isomorphic** to $\mathcal{D}_2$ if there is $\pi \in S_X$ such that $\mathcal{B}_2 = \{\pi \cdot b : b \in \mathcal{B}_1\}$. Intuitively, two t–designs are isomorphic if we can change members in each block consistently. Here is an example:

Example 2.13. Consider $\mathcal{D}$ as in Example 2.11. Suppose that for every $b \in \mathcal{B}$, we switch 1 by 2, yielding

$$\mathcal{B}' = \{124, 135, 346, 457, 256, 167, 237\}.$$

Notice that such a relabeling can be thought of as the result of applying the permutation $\pi = (1\ 2)$ to every block in $\mathcal{B}$. In other words, since

$$\pi\cdot\mathcal{B} = \{\pi\cdot\{1,2,4\}, \pi\cdot\{2,3,5\}, \{3,4,6\}, \pi\cdot\{4,5,7\}, \pi\cdot\{1,5,6\}, \pi\cdot\{2,6,7\}, \pi\cdot\{1,3,7\}\},$$

it follows that

$$\pi \cdot \mathcal{B} = \{\{1,2,4\}, \{1,3,5\}, \{3,4,6\}, \{4,5,7\}, \{2,5,6\}, \{1,6,7\}, \{2,3,7\}\} = \mathcal{B}',$$

so $\mathcal{B}$ is isomorphic to $\mathcal{B}'$.

The set G of all permutations respecting the structure of a t–design can be described algebraically. Before presenting this description, we introduce one piece of notation. Let $v, k \in \mathbb{N}$, where $v > k$, and X be a v–set. We denote a permutation on $\binom{X}{k}$ by

$$(A_1, A_2, \ldots, A_d),$$

where $d = \binom{v}{k}$, and each $A_i \in \binom{X}{k}$ appears exactly once in the sequence.

Although the following fact appears in [13], we include a proof for the sake of completeness.

Proposition 2.1. *Let $v, k \in \mathbb{N}$, where $v > k$, X be a v-set, and $d = \binom{v}{k}$. Let G be the subset of permutations from S_d that are induced by $\pi \in S_X$ acting on $\binom{X}{k}$ via $\pi \cdot A \equiv \{\pi \cdot x : x \in A\}$, where $A \in (A_1, A_2, \ldots, A_d)$. We will denote by $\widetilde{\pi}$ the permutation of $(A_1, A_2, \ldots, A_d)$ induced by π. Then $G \cong S_X$.*

Proof. The proof consists of three parts. First, we show that the mapping $\pi \mapsto \widetilde{\pi}$ is well defined. Then we argue that it is a bijection from S_X to G, and we end the proof by arguing that S_X and G have the same group structure.

Let $A \in (A_1, A_2, \ldots, A_d)$, and π and σ be in S_X. Since $\pi(x) = \sigma(x)$ for all $x \in X$, it follows that $\pi \cdot A = \sigma \cdot A$. Therefore, $\widetilde{\pi} = \widetilde{\sigma}$, so the map taking π to $\widetilde{\pi}$ is well defined.

To argue surjectivity of the map, notice that $|G| \leq |S_X|$ because every permutation π in S_X induces a permutation $\widetilde{\pi}$. For injectivity, let $x \in X$ be such that $\pi(x) \neq \sigma(x)$, where $\pi, \sigma \in S_X$. Since σ is a bijection, $\pi(x) = \sigma(x')$ for some $x' \in X$. By assumption, $k < v$, so we can pick $A \in (A_1, A_2, \ldots, A_d)$ such that $x \in A$ and $x' \notin A$. It follows, then, that $\pi(x) \in \pi \cdot A$ and $\pi(x) \notin \sigma \cdot A$. Hence, $\widetilde{\pi} \neq \widetilde{\sigma}$. Since the map $\pi \mapsto \widetilde{\pi}$ is injective, we have that $|S_X| \leq |G|$, so we conclude that $|G| = |S_X|$.

Now we verify that G and S_X are isomorphic. First, we prove that for all $\pi, \sigma \in S_X$, the permutation $\pi\sigma$ induces $\widetilde{\pi\sigma}$.

Let $A \in (A_1, A_2, \ldots, A_d)$. Observe that

$$(\pi\sigma) \cdot A = \pi \cdot (\sigma \cdot A) = \pi \cdot \{\sigma(x) : x \in A\} = \{\pi(\sigma(x)) : x \in A\} = \{\pi\sigma(x) : x \in A\},$$

which shows that $\pi\sigma$ induces $\widetilde{\pi\sigma}$. In particular, G is a subgroup of S_d. Next, look at the permutations of $\binom{X}{k}$ induced by generators of S_v. Specifically, consider $g_i \mapsto \widetilde{g_i}$, where $i \in \{1, 2, \ldots, v-1\}$ and $g_i = (i\ i+1)$. It is immediate that the $\widetilde{g_i}$ satisfy the same relations as the transpositions by which they are induced (for the remainder of proof, let $A \in (A_1, A_2, \ldots, A_d)$ be given):

1. Let $r \in [v]$ be given. Then

$$g_i^2 \cdot A = \{g_i^2(x) : x \in A\} = \{x : x \in A\} = A,$$

which implies that $\widetilde{g_i^2}$ is equal to the multiplicative identity in $S_{\binom{X}{k}}$.

2. Let $r, s \in [v-1]$, where $|r - s| \geq 1$. Then

$$(g_r g_s) \cdot A = (g_s g_r) \cdot A,$$

so $\widetilde{g_r}\widetilde{g_s} = \widetilde{g_s}\widetilde{g_r}$.

3. Let $r \in [v-1]$. Then

$$(g_r g_{r+1} g_r) \cdot A = (g_{r+1} g_r g_{r+1}) \cdot A,$$

so $\widetilde{g}_r\widetilde{g}_{r+1}\widetilde{g}_r = \widetilde{g}_{r+1}\widetilde{g}_r\widetilde{g}_{r+1}$.

We conclude that $G \cong S_v$, as claimed. □

We summarize the observation that G is a group in the following definition:

Definition 2.3. Let $v, k, t \in \mathbb{N}$, where $0 < t < k < v$, and $\lambda \in \mathbb{N}$. Let X be a v-set, and $\mathcal{D}$ be a t–design. Let $\mathrm{Aut}(\mathcal{D})$ be the subset of permutations from S_d, where $d = \binom{v}{k}$, that are induced by $\pi \in S_X$ acting on $\binom{X}{k}$ as in Proposition 2.1. We call $\mathrm{Aut}(\mathcal{D})$ the **automorphism group** of $\mathcal{D}$.

2.5 Topology

Given $x = (x_1, x_2, \ldots, x_d)$ in $\mathbb{R}^d$, we define the **Euclidean metric** on $\mathbb{R}^d$, namely [15]

$$|x - y| \equiv \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_d - y_d)^2}.$$

We denote by $(x^{(n)})_{n\in\mathbb{N}}$ a sequence of points in $\mathbb{R}^d$.

Below we present an example of a continuous function we will encounter repeatedly in the next chapter.

Lemma 2.2. *Let G be a subgroup of S_d acting on $\mathbb{R}^d$ by permuting coordinates, and let $\pi \in G$. Then π is continuous on $\mathbb{R}^d$.*

Proof. Let $(x^{(n)})_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$ converging to x. Therefore, for any $i \in [d]$, we have $(x^{(n)})_i \to x_i$ as $n \to \infty$. Moreover, for any $n \in \mathbb{N}$, since

$\pi \cdot (x^{(n)})_i = (x^{(n)})_{\pi^{-1}(i)}$, it follows that $\pi \cdot (x^{(n)})_i$ converges to $x_{\pi^{-1}(i)}$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \pi \cdot x^{(n)} = \pi \cdot x.$$

□

We also provide a lemma about sequences that we will need in Chapter 4. Some arguments in the lemma are similar to ideas used in the proof for our result concerning strong purity of the fundamental cell we constructed.

Lemma 2.3. *Let $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. Then there exists a sequence $(\alpha^{(n)})_{n\in\mathbb{N}}$ satisfying the following three properties:*

1. *As $n \to \infty$, we have $\alpha^{(n)} \to x$;*
2. *For every $n \in \mathbb{N}$, $\alpha_i^{(n)} \neq \alpha_j^{(n)}$ whenever $i \neq j$;*
3. *For every $n, m \in \mathbb{N}$, if $\alpha_i^{(n)} > \alpha_j^{(n)}$, then $\alpha_i^{(m)} > \alpha_j^{(m)}$.*

Proof. Define the set

$$S = \{|x_i - x_j| : i, j \in [v]; x_i \neq x_j\}.$$

We claim that the sequence $\alpha^{(n)} = (x_1 + \epsilon_n(1), x_2 + \epsilon_n(2), \dots, x_d + \epsilon_n(d))$ has the desired properties, where $\epsilon_n(i)$ is defined depending on the following two cases:

- Suppose that $S = \emptyset$, which implies that all coordinates of x are equal. Let $\epsilon_n(i) = i/n$. Since the sequence $(\alpha^{(n)})_{n\in\mathbb{N}}$ clearly satisfies items (1) and (2), we will verify the third one only:

  Let $n \in \mathbb{N}$ and assume $\alpha_i^{(n)} > \alpha_j^{(n)}$, so $\alpha_i^{(n)} - \alpha_j^{(n)} > 0$. Observe that

$$\alpha_i^{(n)} - \alpha_j^{(n)} = x_i + \frac{i}{n} - x_j - \frac{j}{n} > 0.$$

  Since $x_i = x_j$, we have that

$$\alpha_i^{(n)} - \alpha_j^{(n)} = \frac{i-j}{n} > 0,$$

  so $i - j > 0$. Therefore, for any $m \in \mathbb{N}$, have $\frac{i-j}{m} > 0$, which implies that $\alpha_i^{(m)} - \alpha_j^{(m)} > 0$.

- Now assume that $S \neq \emptyset$. Let $\epsilon_n(i) = i/(N+n)$, where $N \in \mathbb{N}$ is such that $\frac{d}{N} < \min S$.

  1. It is clear that $\alpha^{(n)} \to x$ as $n \to \infty$;

2. We have that

$$\begin{aligned} |\alpha_i^{(n)} - \alpha_j^{(n)}| &= \left| x_i + \frac{i}{N+n} - x_j - \frac{j}{N+n} \right| \\ &\geq |x_i - x_j| - \left| \frac{i-j}{N+n} \right| \\ &> \frac{d}{N} - \frac{|i-j|}{N+n} \\ &> 0, \end{aligned}$$

   assuming $x_i \neq x_j$, so $\alpha_i^{(n)} \neq \alpha_j^{(n)}$.

3. Let $n \in \mathbb{N}$ and suppose $\alpha_i^{(n)} - \alpha_j^{(n)} > 0$. Similar to the previous case, we have that

$$\alpha_i^{(n)} - \alpha_j^{(n)} > 0 \Longrightarrow x_i - x_j > -\frac{i-j}{N+n}.$$

   It must be the case that $x_i - x_j \geq 0$. (Otherwise, since $|x_i - x_j| > d/N$, it follows that $x_i - x_j < -\frac{d}{N}$; however, $-\frac{d}{N} < -\frac{i-j}{N+n} < x_i - x_j$, which is a contradiction.) Therefore, we have that

$$x_i - x_j > \frac{d}{N} > -\frac{i-j}{N+m} > 0 \Longrightarrow \alpha_i^{(m)} - \alpha_j^{(m)} > 0,$$

   as claimed.

□

Let $K \subseteq \mathbb{R}^d$. The **interior** of K, denoted K°, is the union of all open sets contained in K [15]. The **closure** of K, denoted $\overline{K}$, is the intersection of every closed set containing K [15]. The **boundary** of K, which we write as ∂K, is the set $\overline{K} \cap (\overline{\mathbb{R}^d - K})$ [15].

Example 2.14. Let $\mathcal{P}$ be a polyhedron (see Section 2.1 for the definition of polyhedron). Then $\mathcal{P}$ is closed.

In connection with polyhedra, we define the **relative interior** of a polyhedron $\mathcal{P}$ to be the interior of $\mathcal{P}$ with respect to its affine hull $\operatorname{aff}(\mathcal{P})$ in which $\mathcal{P}$ is full–dimensional [22, p. 60].

Chapter 3

Motivation

After some preliminaries, we introduce Lisoněk's idea [13] that motivated our research question. In Section 3.3, we will explain what our research question is, and how we plan on tackling it.

3.1 Ehrhart Theory

Let $\mathcal{P} = \{x \in \mathbb{R}^d : Ax \geq b\}$, where $A \in \mathbb{Z}^{k \times d}$ and $b \in \mathbb{Z}^d$, be an integral d–polytope. We denote the number of lattice points in the t^{th} dilate of $\mathcal{P}$ by $L_{\mathcal{P}}(t) \equiv |t\mathcal{P} \cap \mathbb{Z}^d|$, where $t\mathcal{P} = \{x \in \mathbb{R}^d : Ax \geq tb\}$ and $t \in \mathbb{N}$. In this section, we provide a brief introduction to Ehrhart theory, which is concerned with the study of $L_{\mathcal{P}}$.

We exhibit several known results that hold for rational polytopes. The notion of a quasi–polynomial will arise throughout.

Definition 3.1. A **quasi–polynomial** Q is a function of the form [2, p. 44]

$$Q(t) = c_n(t)t^n + c_{n-1}(t)t^{n-1} + \cdots + c_1(t)t + c_0(t),$$

where the c_i are periodic functions in t. We call n the **degree** of Q, and the least common period among the c_i is called the **period** of Q. Equivalently, there exist $k \in \mathbb{N}$ and polynomials $p_0, p_1, \ldots, p_{k-1}$ with the property that

$$Q(t) = \begin{cases} p_0(t), & \text{if } t \equiv 0 \mod k, \\ p_1(t), & \text{if } t \equiv 1 \mod k, \\ \quad\vdots & \\ p_{k-1}(t), & \text{if } t \equiv k-1 \mod k. \end{cases}$$

The p_i are known as the **constituents** of Q.

The following theorem is one of the basic results in Ehrhart theory [9]:

Theorem 3.1 (Ehrhart's theorem)**.** *Let $\mathcal{P}$ be a rational convex d–polytope. Then $L_{\mathcal{P}}$ is a quasi–polynomial in t of degree d, and its period divides the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$.*

Definition 3.1 can be extended. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Suppose there exist $N \in \mathbb{N}$ and polynomials $p_0, ..., p_{s-1}$ such that, for all $n \geq N$, $a_n = p_i(n)$, where $i \equiv n$ (mod s). In this case, we say that $(a_n)_{n\in\mathbb{N}}$ is **quasi-polynomial in *n*** [13]. Notice that according to Definition 3.1, a quasi–polynomial Q is also a quasi–polynomial in the extended sense, where $N = 1$.

Hereinafter, whenever we say "quasi–polynomial," we will be referring to the extended definition and not Definition 3.1.

Examples of applications of lattice–point enumeration can be found in computer science (resource allocation when compiling code with nested loops [7, p. 5]), representation theory (counting lattice points of the Gelfand–Tsetlin polytope [7, p. 7]), commutative algebra (computing the Hilbert series of certain monomial algebras and finding Gröbner bases of toric ideals [7, p. 8]), statistics (counting contingency tables with integral entries [7, p. 5]), and optimization (counting the number of maximal integer-valued flows on a graph [7, p. 4]). The number of integer points in a permutohedron equals the number of forests on n labeled vertices [16].

3.2 Combinatorial Reciprocity Theorems

Combinatorial reciprocity theorems are ubiquitous in enumerative combinatorics. We have the following setup:

- We are given a combinatorial family, and we want to enumerate some combinatorial data.
- Let f be the function counting such combinatorial data. Suppose that the domain of f is $\mathbb{N}$.

- The counting function f is a polynomial, or quasi-polynomial.

- We want to evaluate f at negative integers.

When there is a combinatorial interpretation for such an evaluation of f at negative integers, we say that we have a **combinatorial reciprocity theorem**.

For our purposes, the most relevant example of a reciprocity theorem is *Ehrhart–Macdonald Reciprocity* [14]:

Theorem 3.2. *Let $\mathcal{P}$ be a rational convex d–polytope. Let $\mathcal{P}^\circ$ denote the interior of $\mathcal{P}$. Then the following algebraic identity holds:*

$$L_{\mathcal{P}}(-t) = (-1)^d L_{\mathcal{P}^\circ}(t).$$

Concrete applications of reciprocity theorems to answer enumeration questions arise in counting the number of regions in hyperplane arrangements, P–partitions of a poset P, and acyclic orientations of graphs [1]. On the other hand, reciprocity theorems have also been used to prove structural results about certain combinatorial objects. Instances of these applications can be found in the study of abstract manifolds (imposing linear constraints on the number of $(i+1)$–sets contained in a finite simplicial complex [17, p. 18]), posets (establishing the length of maximal chains by using the order polynomial of the poset [17, p. 7]), and a generalization of the Dehn–Sommerville equations to arbitrary convex polytopes [17, p. 11].

3.3 Lisonek's Method for Enumerating t–Designs

In Section 2.4, we saw what it means for two t–designs to be the same. Lisoněk [13, p. 623] showed what kind of counting function arises when t, v and k are fixed, and we try to count the isomorphism types of t–designs with respect to λ:

Theorem 3.3. *Let $t, v, k \in \mathbb{N}$, with $0 < t < k < v$. Then the function counting the number of different $t - (v, k, \lambda)$ designs is a quasi-polynomial in λ, where $\lambda \in \mathbb{N}$.*

While we will not show the full proof, here is what we are interested in. The proof concludes that the lattice points of the rational polytope

$$\lambda\mathcal{P} \equiv \{x \in \mathbb{R}^d : Ax = \lambda 1; x \geq 0\}$$

correspond to t–designs, where $A \in \mathbb{Z}^{k \times d}$ is a carefully constructed matrix, and 1 is the vector whose coordinates are 1. Notice that the parameter λ, which is associated to the design, is the dilation factor of $\mathcal{P}$. To construct the matrix A, suppose $\binom{X}{t}$ and $\binom{X}{k}$ have a fixed ordering, where X is a v–set. Then A is a matrix whose rows and columns are indexed by t–subsets of X and k–subsets of X, respectively, such that $a_{ij} = 1$ if the ith t–subset is contained in the jth k–subset. Otherwise, $a_{ij} = 0$. The matrix A is known as an *incidence matrix* [21]. Finally, notice that lattice points in $\mathcal{P}$ could have coordinates greater than 1 —this would be the case if a particular kind of block shows up more than once in the corresponding design.

Example 3.1. Let $v = 4$, $t = 1$, and $k = 2$. (So X is the 4–set [4].) Let

P be the poset consisting of the set $\binom{X}{k}$ ordered by the reverse lexicographical order. Therefore, for instance, in P we have that $\{1,2\} \succ \{1,3\}$. Let $\mathcal{B}_1 = \{\{1,2\},\{1,2\},\{3,4\},\{3,4\}\}$. Then $\mathcal{B}$ is a $1-(4,2,2)$ design, and it is the lattice point $(2,0,0,0,0,2)$ in $2\mathcal{P}$.

Two lattice points x and y in $\mathcal{P}$ correspond to isomorphic combinatorial designs if and only if there exists $\pi \in \mathrm{Aut}(\mathcal{D})$ such that $\pi \cdot x = y$. In particular, for all integer points x in $\mathcal{P}$, we have that $G_x \subseteq \mathcal{P}$. Therefore, it is enough to enumerate the number of G–orbits in $\mathcal{P} \cap \mathbb{Z}^d$ in order to count the number of isomorphism types of t–designs. Due to another result in [13], boundedness of $\mathcal{P}$ yields quasi–polynomiality of the function counting the number of G–orbits.

Since the function counting the G–orbits is a quasi–polynomial, one can ask what a combinatorial reciprocity theorem could possibly say. However, as we will see in the next chapter, it is not possible to apply Ehrhart–Macdonald Reciprocity directly. What we do have is the following generalization [17, p. 218] of Theorem 3.2: Let Δ denote the set of facets of a rational d–polytope $\mathcal{P}$, and let T be a subset of Δ. Define

$$B = \bigcup_{F \in T} F,$$

and

$$B' = \bigcup_{F \in \Delta - T} F.$$

Assume that B is homeomorphic to $(d-1)$–manifold. Then

$$L_{\mathcal{P}\setminus B}(\lambda) = (-1)^d L_{\mathcal{P}\setminus B'}(-\lambda).$$

It would be desirable to construct $\mathcal{R}$ such the hypotheses of the generalization of Theorem 3.2 hold.

One line of attack to find $\mathcal{R}$ is by defining a system of linear inequalities. If such a system exists, then $\mathcal{R}$ is amenable to methods from Ehrhart theory. Furthermore, one can proceed to search for an explicit hyperplane representation of $\mathcal{R}$. This representation is necessary to get an insight as to how t–designs "live" in $\mathcal{P}$, thereby helping us determine what kinds of t–designs $L_{\mathcal{R}^\circ}(\lambda)$ is counting.

Instead of focusing on the problem at the level of polytopes, we will focus on finding a geometric object slightly more general than a polyhedral complex. Specifically, we will construct a region $\mathcal{F}$ such that for *all* x in $\mathbb{R}^d$, we have that $\mathcal{F}$ contains a representative from the orbit G_x, and with no repetitions. If we are able to find such a region $\mathcal{F}$ for $\mathbb{R}^d$, then $\mathcal{F} \cap \mathcal{P}$ will be a suitable region $\mathcal{R}$. Therefore, if we are able to find an inequality description for $\mathcal{F}$, we would be able to derive a combinatorial reciprocity theorem for t–designs.

We call the region $\mathcal{F}$ a *fundamental domain*, and it will occupy us in the next chapter. We will pick the set of lexicographically largest elements in their G–orbit and show that the resulting object has desirable combinatorial properties. We also exhibit an inequality description for the interior of $\mathcal{F}$.

Chapter 4

Construction of a Fundamental Cell under the Action of Aut($\mathcal{D}$)

We just introduced a result by Lisoněk [13] that states what kind of function arises when we try to count isomorphism types of t–designs. In the proof of the result, a polytope $\mathcal{P}$ is constructed with the property that its lattice points correspond to t–designs. Two integer points $x, y \in \mathcal{P}$ correspond to the same t–design if and only if there exists $\pi \in \mathrm{Aut}(\mathcal{D})$ such that $\pi \cdot x = y$. Then it suffices to count lattice points in a region $\mathcal{R}$ containing exactly one representative from every Aut($\mathcal{D}$)–orbit, for all $x \in \mathcal{P} \cap \mathbb{Z}$, and no repetitions.

Recall that in Section 2.3, we introduced a notion of group action. Let $X = \{i_1, i_2, \ldots, i_d\}$, with $i_1 \succ i_2 \succ \cdots \succ i_d$, be a totally ordered set with d elements. Throughout our discussion, whenever we say that $G \subseteq S_X$ acts on $\mathbb{R}^X$, we mean that for any $\pi \in G$, we have that

$$\pi \cdot x \equiv (x_{\pi^{-1}(i_1)}, x_{\pi^{-1}(i_2)}, \ldots, x_{\pi^{-1}(i_d)}),$$

where $x = (x_{i_1}, x_{i_2}, \ldots, x_{i_d}) \in \mathbb{R}^X$.

In this chapter, we inquire about an explicit inequality description for $\mathcal{R}$. In order to go about this task, we construct a convex set $\mathcal{F}$ in $\mathbb{R}^d$ such that for all $x \in \mathbb{R}^d$, we have that $\mathcal{F}$ contains one element of G_x. More precisely:

Definition 4.1. Let $\mathcal{F} \subseteq \mathbb{R}^d$ be convex, and let G be a finite group acting on $\mathbb{R}^d$ by permuting coordinates. Suppose that for every $x \in \mathbb{R}^d$, the equality $|G_x \cap \mathcal{F}| = 1$ holds. Then we call $\mathcal{F}$ a **fundamental cell**. (We will sometimes also use the terms "fundamental region," and "fundamental domain" for the same concept.)

First, we prove some general results about the geometric structure of $\mathcal{F}$. Then we proceed to find $\mathcal{F}$ when $\mathrm{Aut}(\mathcal{D})$ acts on $\mathbb{R}^d$ in the case when $\mathcal{D}$ is a $t-(v,2,\lambda)$ design. We will exhibit a system of linear inequalities for the interior of $\mathcal{F}_{\max}$.

4.1 Existence of Fundamental Cells

An alternative way of defining a fundamental region is as a subset of $\mathbb{R}^d$ satisfying the following two properties:

1. For every $x \in \mathcal{F}$ and $\pi \in G \backslash \{e\}$, where e denotes the identity of G, we have that $\pi \cdot x \notin \mathcal{F}$;

2. For every $x \in \mathbb{R}^X$, there exists $\pi \in G$ such that $\pi \cdot x \in \mathcal{F}$.

We will refer to the first property as ***G*-uniqueness**, and the latter one as ***G*-existence**.

We introduce a partition of $\mathbb{R}^d$ we will use often:

Definition 4.2 ([18])**.** For $i, j \in [d]$, with $i \neq j$, let $H_{ij} = \{x \in \mathbb{R}^d : x_i - x_j = 0; 1 \leq i < j \leq d\}$. The **braid arrangement**, denoted $\mathcal{B}_d$, on $\mathbb{R}^d$ is the set

$$\mathcal{B}_d = \{H_{ij} : 1 \leq i < j \leq n\}.$$

The braid arrangement $\mathcal{B}_d$ induces a collection of cones, and their relative interiors form a partition of $\mathbb{R}^d$ [22, p. 192]. Let $\mathcal{T}_d$ denote the collection of these regions.

Definition 4.3. Let F be a subset of $\mathcal{T}_d$. For any z in the relative interior of F, define

$$\begin{aligned} I_z &\equiv \{(i, j) \in [d]^2 : i \neq j; z_i - z_j > 0\}, \\ E_z &\equiv \{(i, j) \in [d]^2 : i \neq j; z_i - z_j = 0\}. \end{aligned}$$

If all pairs of coordinates fall in either I_z or E_z, then we call F a **face** of $\mathcal{T}_d$, in which case we say that F is a face of the braid arrangement.

Lemma 4.1. *Let $G \subseteq S_d$ be a group acting on $\mathbb{R}^d$. Let $\mathcal{T}_d$ be the partition of $\mathbb{R}^d$ induced by the braid arrangement $\mathcal{B}_d$. Let F be a face of $\mathcal{T}_d$, and x be in the relative interior of F, where x is a lexicographically largest element in its G-orbit. Let y be in the relative interior of F be such that $x \neq y$. Then y is lexicographically maximal in its G-orbit.*

Proof. Assume the opposite. Let $\pi \in G$ be such that $y < \pi \cdot y$, and $\alpha \in [d]$ be the minimal index for which $y_\alpha < y_{\pi^{-1}(\alpha)}$ holds. Therefore, if there is $i \in [d]$ with $1 \leq i < \alpha$, then it follows that $y_i = y_{\pi^{-1}(i)}$.

Since x is in the relative interior of F, it follows that the indices of the coordinates of x agree with I_y and E_y. Therefore, we have that $x_\alpha < x_{\pi^{-1}(\alpha)}$. Furthermore, if $i \in [d]$ with $1 \leq i < \alpha$, then it follows that $x_i = x_{\pi^{-1}(i)}$, which implies that $x < \pi \cdot x$. However, $x < \pi \cdot x$ contradicts x being lexicographically maximal in its G-orbit.

Therefore, y is lexicographically maximal in its G-orbit. $\square$

In light of Lemma 4.1, one might be inclined to collect, for all x in $\mathbb{R}^d$, the lexicographically maximal elements in their G-orbits.

Definition 4.4. Let K be a finite subset of a totally ordered set T (possibly infinite). If $K \neq \emptyset$, then we define the **minimum** of K to be the smallest element in K, that is,

$$\min K \equiv \{x \in K : x \leq y; y \in K\}.$$

Furthermore, the **maximum** of K is the largest element in K, i.e.,

$$\max K \equiv \{x \in K : x \geq y; y \in K\}.$$

We define the set of lexicographically largest elements in $\mathbb{R}^d$ in their respective G–orbits as

$$\mathcal{F}_{\max} \equiv \{x \in \mathbb{R}^d : x \geq \pi \cdot x; \pi \in G\}.$$

Recall that in Section 2.1 we introduced the notion of a polyhedral complex. A slightly more general notion is that of a **partial polyhedral complex**, which is a subset of the faces of a polyhedral complex that is not necessarily closed under taking faces. We will use the term "complex" interchangeably with the term "partial polyhedral complex." We view each face of a complex as relatively open.

Example 4.1. The subset $\mathcal{P}$ of a triangle T shown in Figure 4.1, which consists of the maximal 2–dimensional face of T and a vertex, is a 2–dimensional partial polyhedral complex. The dashed lines mean that $\mathcal{P}$ does not contain any of the edges of T.

In the following proposition, we collect the observations that $\mathcal{F}_{\max}$ is a fundamental domain, and that $\mathcal{F}_{\max}$ is a subcomplex of $\mathcal{T}_d$.

Proposition 4.2. *Let $G \subseteq S_d$ be a group acting on $\mathbb{R}^d$. Then $\mathcal{F}_{\max}$ is a subcomplex of $\mathcal{T}_d$. Furthermore, $\mathcal{F}_{\max}$ is a fundamental cell.*

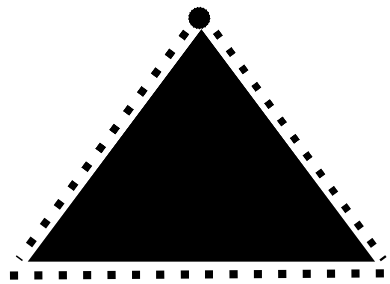

Figure 4.1: Example of a partial polyhedral complex.

Proof. It is immediate that $\mathcal{F}_{\max}$ is a fundamental cell. (For any $x \in \mathbb{R}^d$, we can find the largest element in G_x.)

To argue that $\mathcal{F}_{\max}$ is a subcomplex of the braid arrangement, observe that Lemma 4.1 tells us that if F is a face of $\mathcal{T}_d$, and x is in both F and $\mathcal{F}_{\max}$, then $F \subseteq \mathcal{F}_{\max}$. Thus, faces of $\mathcal{F}_{\max}$ are the faces of $\mathcal{T}_d$ containing a point that is lexicographically maximal in its G-orbit. □

We call $\mathcal{F}_{\max}$ the **lex–maximal fundamental cell**. Note that an identical argument would work if we chose the set of lexicographically minimal elements in their respective G-orbit.

Corollary 4.3. *Let $G \subseteq S_d$ act on $\mathbb{R}^d$. Define*

$$\mathcal{F}_{\min} \equiv \{x \in \mathbb{R}^d : x \leq \pi \cdot x; \pi \in G\}.$$

Then $\mathcal{F}_{\min}$ is a fundamental cell, and it is a subcomplex of the partition induced by the braid arrangement $\mathcal{B}_d$.

We observe that $\mathcal{F}_{\max}$ is not necessarily a polyhedral complex in general because $\mathcal{F}_{\max}$ need not contain all the points in its topological closure.

Example 4.2. Let $G = \mathrm{Aut}(\mathcal{D})$, where $\mathcal{D}$ is a t-design with parameter $k = 2$ (so $t = 1$; refer to Section 2.4 for the definition of $\mathrm{Aut}(\mathcal{D})$). As a consequence of results in the next section, we have that

$$\mathcal{C} = \{x \in \mathbb{R}^d : x_1 > x_2; x_2 > x_3; x_2 > x_4; x_2 > x_5; x_1 > x_6\}$$

is the interior of a fundamental cell under the action of G. Consider the images of the points below under certain permutations in G:

$$\begin{aligned}
z_1 = (6,6,5,4,3,2) &\xrightarrow{(1\,2)(5\,6)} (6,6,5,4,2,3),\\
z_2 = (6,5,5,4,3,2) &\xrightarrow{(2\,3)(4\,5)} (6,5,5,3,4,2),\\
z_3 = (6,5,4,5,3,2) &\xrightarrow{(2\,4)(3\,5)} (6,5,3,5,4,2),\\
z_4 = (6,5,4,3,5,2) &\xrightarrow{(2\,5)(3\,4)} (6,5,3,4,5,2).
\end{aligned}$$

All the z_i's are in $\mathcal{F}_{\max}$. However, even though the points to which the z_i's are mapped are not points in $\mathcal{F}_{\max}$, they are in the topological closure of $\mathcal{C}$, namely

$$\overline{\mathcal{C}} = \{x \in \mathbb{R}^d : x_1 \geq x_2; x_2 \geq x_3; x_2 \geq x_4; x_2 \geq x_5; x_1 \geq x_6\}.$$

Therefore, $\mathcal{F}_{\max}$ is not a topologically closed set, so it is not a polyhedral complex in the sense presented in Section 2.1.

There is a connection between ordered set partitions and faces of the braid arrangement, which now we make explicit. A **partition** [19] of a finite set Z is a collection $\Pi = \{B_1, B_2, \ldots, B_l\}$ of subsets of Z such that no B_i is empty; $B_i \cap B_j = \emptyset$ when $i \neq j$; and $B_1 \cup B_2 \cup \cdots \cup B_l = Z$. Sometimes the B_i are referred to as the *blocks* of the partition. We call a partition of Z an **ordered set partition** (also known as *ordered partition*) when the blocks are ordered linearly, and we denote it by $B_1|B_2|\ldots|B_l$, where we say that if $k < k'$, then $B_k > B'_k$. We denote the collection of all ordered set partitions on Z by Part_Z.

Definition 4.5. Let X be a v–set, and let L be a poset consisting of all the elements in $\binom{X}{2}$, ordered in the reverse lexicographical ordering. Let $Y \equiv \{x_i : i \in \binom{X}{2}\}$, where Y inherits the ordering on $\mathbb{R}$, and $l \equiv |Y|$.

Then we define the **partition map** $\psi : \mathbb{R}^L \to \mathrm{Part}_{\binom{X}{2}}$ to be $x \mapsto B_1|B_2|\ldots|B_l$, where

$$B_j \equiv \{i \in \binom{X}{2} : x_i = y_j; y_j \in Y\}.$$

We say that two partitions $B_1|B_2|\ldots|B_l$ and $B'_1|B'_2|\ldots|B'_{l'}$ are **equivalent** when $l = l'$, and for all $i \in [l]$, we have that $B_i = B'_i$.

Note that the following two statements are true:

- Let X be a v–set, Φ be a partition on $\binom{X}{2}$, and $d \equiv \binom{v}{2}$. Then $\psi^{-1}(\Phi)$ is the relative interior of a face of the braid arrangement B_d. To see why this is the case, note that $\psi^{-1}(\Phi) \subseteq \mathbb{R}^d$ consists of all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ such that $x_i = x_j$ if $i, j \in B_k$, for some $k \in [l]$. We have that $x_i > x_j$ if $x_i \in B_k$ and $x_j \in B_{k'}$, where $k < k'$. Since the B_i are blocks in a partition of $\binom{X}{2}$, we have that

$$\psi^{-1}(\Phi) = \{x \in \mathbb{R}^d : x_i = x_j; (i,j) \in B_k\} \cup \{x \in \mathbb{R}^d : x_i > x_j; i \in B_k; j \in B_{k'}; k < k'\},$$

which defines the relative interior of a face of the braid arrangement.

- Note, as a consequence, that there is a bijection between faces of the braid triangulation and ordered set partitions. Specifically, the bijection is given by taking any pair $(i, j) \in \binom{X}{2}$ and the rules

$$(i,j) \in B_k \longleftrightarrow (i,j) \in E_x,$$

and

$$i \in B_k, j \in B_{k'}, k < k' \longleftrightarrow (i,j) \in I_x.$$

Since the above correspondences hold for all x in F, we conclude that F corresponds to a unique partition Φ of $\binom{X}{2}$.

- Another consequence of the first observation is that ψ is constant on faces of the braid arrangement.

To finalize the discussion concerning the connection between points in $\mathbb{R}^d$, and ordered set partitions on $\binom{X}{2}$, note that

$$\pi\psi = \psi\widetilde{\pi},$$

with the understanding that π acts on the blocks of a partition, whereas $\widetilde{\pi}$ acts on points in $\mathbb{R}^d$ by permuting coordinates. To see why the above identity is true, we provide a "diagram chasing" argument. We will use the same symbols as we used in Definition 4.5.

Let $x \in \mathbb{R}^L$. We have that

$$\pi \cdot \psi(x) = \pi \cdot B_1 | \pi \cdot B_2 | \ldots | \pi \cdot B_l,$$

and

$$\widetilde{\pi} \cdot x = (x_{\pi^{-1} \cdot i_1}, x_{\pi^{-1} \cdot i_2}, \ldots, x_{\pi^{-1} \cdot i_d}).$$

Notice that $\pi \cdot Y = Y$, so $l = l'$. Furthermore, we have that the blocks of $\psi(\widetilde{\pi} \cdot x)$ are of the form

$$B_j = \left\{ \pi \cdot i \in \binom{X}{2} : x_{\pi^{-1} \cdot i} = y_j \right\}.$$

In particular, $B'_j = B_j$.

We will now present one of the main results of the manuscript. Before doing so, we need the following notion.

Definition 4.6. We say that a d–dimensional partial polyhedral complex $\mathcal{P}$ in $\mathbb{R}^d$ is **strongly pure** if for any k-dimensional face σ of $\mathcal{P}$, where $0 \leq k < d$, we have that σ is a face of a $(k+1)$-dimensional face of $\mathcal{P}$.

We note that the notions of purity (defined in Section 2.1) and strong purity are the same in polyhedra. However, they need not agree for partial polyhedral complexes. In Example 4.1, for instance, we have that $\mathcal{P}$ is pure because the vertex (the only non–maximal face in $\mathcal{P}$) is a face of the relative interior of the triangle. However, $\mathcal{P}$ is not strongly pure because a vertex is a 0–dimensional face of T, while the relative interior of T is 2–dimensional.

Theorem 4.4. *Let $G \subseteq \mathbb{R}^d$ act on $\mathbb{R}^d$. Then the boundary of $\mathcal{F}_{\max}$ is strongly pure.*

Proof. Before providing the proof for Theorem 4.4, we give an outline. We will pick an arbitrary point x in a k-dimensional face σ of $\mathcal{F}_{\max}$, where $1 \leq k < d$. Since the dimension of σ is less than d, we know that at least one pair of coordinates of x must be equal. Among all the possible coordinates that are equal, we choose a suitable index I and add a small enough number $\epsilon_n(I)$, where $\epsilon_n(I) \to 0$ as $n \to \infty$. We will show by contradiction that the terms of the resulting sequence $x^{(n)}$ are lexicographically maximal in their G–orbit. Since the $x^{(n)}$ are in a $(k+1)$–dimensional face σ', and $x^{(n)}$ converges to x, it follows that σ' has σ as a face.

Let σ be a k-dimensional face of $\mathcal{F}_{\max}$, where $0 \leq k < d$, and $x = (x_1, x_2, \ldots, x_d)$ be in the relative interior of σ.

We introduce a multiset consisting of coordinates of x. Specifically, define

$$K \equiv \bigcup_{\substack{i,j \in P \\ i \neq j}} \{x_i : x_i = x_j\}.$$

We know that $K \neq \emptyset$ because σ is a non-trivial face of $\mathcal{F}_{\max}$ —so there is at least one pair of coordinates of x that are equal.

Let $I \equiv \min\{i \in [d] : x_i = \max K\}$. Define $r \equiv \min S$, where

$$S \equiv \{|x_i - x_j| : i \neq j; x_i \neq x_j\},$$

and let $N \in \mathbb{N}$ be such that $\frac{d}{N} < r$ when $S \neq \emptyset$. (If $S = \emptyset$, then all the coordinates of x are equal. Therefore, the sequence $x^{(n)} = (x_1 + \frac{1}{n}, x_2, \ldots, x_d)$ has all the desired properties. First, it is clear that $x^{(n)}$ is lexicographically maximal in its G-orbit. For every $n \in \mathbb{N}$, we have that $x^{(n)}$ is a point in a $(k+1)$-dimensional face. Further, $x^{(n)}$ converges to x as $n \to \infty$.)

We claim that for every $n \in \mathbb{N}$, the sequence

$$x^{(n)} = (x_1 + \epsilon_n(1), x_2 + \epsilon_n(2), \ldots, x_d + \epsilon_n(d)),$$

where $\epsilon_n(i) = \frac{i}{N+n}$ if $i = I$, and $\epsilon_n(i) = 0$ otherwise, is lexicographically maximal in its G-orbit. Observe that $(x^{(n)})_i = x_i$ as long as $i \neq I$. If $i = I$, then $(x^{(n)})_i > x_i$. As an additional observation, note for any $n \in \mathbb{N}$, we have that $(x^{(n)})_I \neq (x^{(n)})_i$ for any $i \neq I$.

We will prove by contradiction that $x^{(n)} \in \mathcal{F}_{\max}$. Let $\pi \in G$ and $n \in \mathbb{N}$ be such that $x^{(n)} < \pi \cdot x^{(n)}$. Let $\alpha \in [d]$ be the minimal index with the property that $(x^{(n)})_\alpha < (x^{(n)})_{\pi^{-1}(\alpha)}$. Note that if i is such that $1 \leq i < \alpha$, then $(x^{(n)})_i = (x^{(n)})_{\pi^{-1}(i)}$. We argue that this setup implies that $x < \pi \cdot x$.

We have two cases:

1. Suppose that $\alpha = I$. Then

$$(x^{(n)})_I < (x^{(n)})_{\pi^{-1}(I)} \implies x_I + \frac{I}{N+n} < x_{\pi^{-1}(I)}.$$

   To see why the second inequality is true, note that if α were a fixed point of π^{-1}, then $(x^{(n)})_\alpha < (x^{(n)})_{\pi^{-1}(\alpha)}$, contradicting $(x^{(n)})_\alpha < (x^{(n)})_{\pi^{-1}(\alpha)}$. Thus, $\pi^{-1}(I) \neq I$, implying that $x_I + \frac{I}{N+n} < x_{\pi^{-1}(I)}$. From this last inequality, we have that $x_I < x_{\pi^{-1}(I)}$ because if $x_I = x_{\pi^{-1}(I)}$, then it follows that $x_I + \frac{I}{N+n}$ is greater than $x_{\pi^{-1}(I)}$, which is inconsistent with the inequality $(x^{(n)})_I < (x^{(n)})_{\pi^{-1}(I)}$.

   On the other hand, if there is $i \in [d]$ with $1 \leq i < I$, then the equality $x_i = (x^{(n)})_i$ holds because $i \neq I$. Furthermore, since $i < \alpha$, it follows that $(x^{(n)})_i = (x^{(n)})_{\pi^{-1}(i)}$. To summarize,

$$x_i = (x^{(n)})_i = (x^{(n)})_{\pi^{-1}(i)} = x_{\pi^{-1}(i)}.$$

Hence, $x < \pi \cdot x$.

2. Now assume that $\alpha \neq I$. Note that we only need to consider the case when $\alpha < I$ because otherwise,

$$(x^{(n)})_I = (x^{(n)})_{\pi^{-1}(I)} \implies x_{\pi^{-1}(I)} = x_I + \frac{I}{N+n},$$

which is impossible due to how we chose N.

Since $\alpha \neq I$, we know that $(x^{(n)})_\alpha = x_\alpha$, so $x_\alpha < (x^{(n)})_{\pi^{-1}(\alpha)}$. Now we consider two separate subcases:

- Suppose $\pi^{-1}(\alpha) = I$. By assumption, $(x^{(n)})_\alpha < (x^{(n)})_{\pi^{-1}(\alpha)}$, which implies

$$x_\alpha < x_I + \frac{I}{N+n} \implies x_\alpha < x_{\pi^{-1}(\alpha)},$$

where the second step follows by our definition of N. For any i with $1 \leq i < \alpha$, we have

$$x_i = (x^{(n)})_i = (x^{(n)})_{\pi^{-1}(i)} = x_{\pi^{-1}(i)},$$

because $\pi^{-1}(i) \neq I$. Thus, $x < \pi \cdot x$.

- Assume now that $\pi^{-1}(\alpha) \neq I$. Since $x_\alpha < (x^{(n)})_{\pi^{-1}(\alpha)}$, it follows that

$x_\alpha < x_{\pi^{-1}(\alpha)}$. For any $i \in [d]$ with $1 \leq i < \alpha$, we see that

$$x_i = (x^{(n)})_i = (x^{(n)})_{\pi^{-1}(i)} = x_{\pi^{-1}(i)},$$

where the last equality is obtained by observing that if $\pi^{-1}(i) = I$, then $x_i = x_I + \frac{I}{N+n}$, which is not possible. Since $x_i = x_{\pi^{-1}(i)}$ for all $i < \alpha$, we infer that $x < \pi \cdot x$.

Therefore, in either case we reach a contradiction.

Since the $x^{(n)}$ are in the relative interior of the same face, say σ', we conclude that for all $n \in \mathbb{N}$, we have that $x^{(n)}$ is lexicographically maximal in its G–orbit by Lemma 4.1. Thus, σ is a face of σ', so $\mathcal{F}_{\max}$ is strongly pure. □

4.2 Fundamental Cells when $\mathcal{D}$ is a t–Design with $k = 2$

First, we introduce a poset that will enable us to exhibit an inequality description for the interior of the lex–maximal fundamental cell $\mathcal{F}_{\max}$. For the remainder of the discussion, let G denote the automorphism group of a t–design $\mathcal{D}$ with $k = 2$. One reason why the case $k = 2$ is interesting is due to Theorem 4.9, where we are able to exhibit an explicit system of linear inequalities for the interior of $\mathcal{F}_{\max}$. Knowledge of these inequalities could lead to a function for counting the t–designs for which the lattice points in $\mathcal{F}_{\max}^{\circ} \cap \mathcal{P}$ correspond, where $\mathcal{P}$ is the polytope associated to

t–designs. (Refer to Section 3.3 for an explanation about how $\mathcal{P}$ and t–designs are related.) Furthermore, knowledge of the interior of $\mathcal{F}_{\max}$ might be a first step in deriving a system of inequalities for the entire complex.

Definition 4.7. Let $X = \{1, 2, \ldots, v\}$, and $d = \binom{v}{2}$. Let A_1 be the antichain consisting of all 2–subsets of X that do not contain 1 nor 2; A_2 be the antichain consisting of all 2–subsets of X that contain 2, but do not contain 1. The collection B consists of those 2–subsets containing a 1 and it is ordered by the reverse lexicographical order. Define P to be the poset on 2–subsets of X with the following relations:

- For any $Z \in \binom{X}{2}$, we have $\{1, 2\} \succeq Z$;
- For all $Y \in A_1$, the relation $\{1, 3\} \succ Y$ holds;
- For $\{1, i\}, \{1, j\} \in B$, we have that $\{1, i\} \succ \{1, j\}$ when $i < j$.

We call P the **interior poset**. (Refer to Figure 4.2.)

Remark. For the remainder of the paper, we will reserve the letter P to refer to an interior poset.

We now introduce a notion that will link interior posets with fundamental cells:

Definition 4.8 ([20])**.** Suppose P is an interior poset with d elements. The **order**

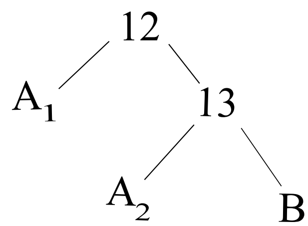

Figure 4.2: Hasse diagram of P in Definition 4.7.

cone of P, denoted $\mathcal{C}(P)$, is the set

$$\mathcal{C}(P) \equiv \bigcap_{\substack{i,j \in P \\ i \gtrdot j}} \{x \in \mathbb{R}^P : x_i \geq x_j\},$$

where $i \gtrdot j$ means that i covers j. Equivalently, $\mathcal{C}(P)$ is the set of all order-preserving maps $f : P \to \mathbb{R}$, where we identify every possible f with points $x = (x_1, x_2, \ldots, x_d)$ in $\mathbb{R}^P$ such that $x_i > x_j$ if and only if $f(i) > f(j)$.

The next two results are intended to show why working with P is helpful in constructing a fundamental cell. First, we introduce a definition.

Definition 4.9. Let P be an interior poset and $\pi \in S_X$. We say that π is **order-preserving with respect to *P*** if for every $\alpha, \beta \in P$ with $\alpha \succ \beta$, we have that either $\pi \cdot \alpha \succ \pi \cdot \beta$, or $\{\pi \cdot \alpha, \pi \cdot \beta\}$ is an antichain.

Proposition 4.5. *Let P be an interior poset. Then $S_X \backslash \{e\}$ has no order-preserving permutations with respect to P.*

Proof. Suppose $\pi(1) = 2$ and $\pi(2) = 1$, or $\pi(1) = 1$ and $\pi(2) = 2$. (Otherwise, $\pi \cdot \{1,2\} \neq \{1,2\}$, so π would not be order-preserving.)

1. Note that $\pi \cdot \{1,3\} = \{2, \pi(3)\}$ and $\pi \cdot \{2,i\} = \{1, \pi(i)\}$. Since $\pi(3) \neq 1$, then $\pi \cdot \{1,3\} \in B$. There exists $i \in [v]$ such that $\pi(i) = 3$, then $\pi \cdot \{2,i\} \succ \pi \cdot \{1,3\}$, which is a contradiction.

2. Let $i > j$ such that $\pi(i) < \pi(j)$. Although $\{1,i\} \prec \{1,j\}$, we have that $\pi \cdot \{1,i\} \not\prec \pi \cdot \{1,j\}$.

Thus, no π in $S_X \setminus \{e\}$ is order-preserving. □

Proposition 4.6. *For every $\pi \in S_X \setminus \{e\}$, π is not order-preserving with respect to P if and only if $\mathcal{C}(P)^\circ$ has the G-uniqueness property.*

Proof. Forward implication: Let $\pi \in S_X$ be a permutation that does not preserve the order in P, and let $x \in \mathcal{C}(P)$. Let $i, j \in P$ be such that $i \succ j$, and $\pi \cdot i \succ \pi \cdot j$. If $i \succ j$, then $x_i > x_j$, so for $\pi \cdot x$, we have that $x_{\pi^{-1}(i)} < x_{\pi^{-1}(j)}$. Thus, $\pi \cdot x$ is not in $\mathcal{C}(P)$.

Backward implication: We prove the contrapositive. Take the longest chain in P, and take $\alpha \gtrdot \beta$ such that α and β are mapped non-trivially by $\pi \in G$. Since π is order-preserving, we have that $\pi \cdot \{\alpha, \beta\}$ is an antichain in P, or $\pi \cdot \alpha \succ \pi \cdot \beta$. Hence, if $x \in \mathcal{C}(P)$ satisfies $x_\alpha > x_\beta$, then $x_{\pi^{-1}(\alpha)} > x_{\pi^{-1}(\beta)}$ will be a valid inequality in $\mathcal{C}(P)$, so $\pi \cdot x \in \mathcal{C}(P)$. □

Given this evidence that studying P could be helpful, we proceed to show that the covering relations in P enable us to describe the interior of $\mathcal{F}_{\max}$ in Theorem 4.9. Specifically, we will show that the interior of $\mathcal{C}(P)$ is the interior of the lex–maximal fundamental cell. To go about the proof of this claim, we establish a result concerning linear extensions of P.

Theorem 4.7. *Let P be an interior poset, and X a v–set. Let $v, k \in \mathbb{N}$, where $v < k$. Define $\mathrm{Perm}\binom{X}{k}$ to be the set of all permutations on $\binom{X}{k}$. Let*

$$G\backslash\backslash\mathrm{Perm}\binom{X}{k} \equiv \left\{\mathcal{O}_{\mathfrak{S}} : \mathfrak{S} \in \mathrm{Perm}\binom{X}{k}\right\},$$

that is, $G\backslash\backslash\mathrm{Perm}\binom{X}{k}$ is the set of all G–orbits of permutations on $\binom{X}{k}$. Then there exists a unique permutation in every G–orbit $\mathcal{O}_{\mathfrak{S}} \in G\backslash\backslash\mathrm{Perm}\binom{X}{k}$ that is a linear extension of P.

Proof. Existence: Let $T = (T_1, T_2, \ldots, T_d)$ be a permutation on $\binom{X}{k}$. We will construct a $\pi \in G$ such that $\pi \cdot T$ is a linear extension of P.

1. If $T_1 \neq \{1, 2\}$, then let $p \in S_X$ be such that $p \cdot T_1 = \{1, 2\}$.

2. Let $\omega \equiv \min\{i \in [d] \setminus \{1\} : 1 \in T_i\}$ and $\tau \equiv \min\{i \in [d] \setminus \{2\} : 2 \in T_i\}$. There are two cases to consider:

   - $\omega < \tau$: Then let $q = (j\ 3)$, so that $q \cdot T_\omega = \{1, 3\}$, where $j \in T_\omega$.
   - $\tau < \omega$: Then let $q = (1\ 2)(3\ h)$, so that $q \cdot T_\tau = \{1, 3\}$, where $h \in T_\tau$.

Notice that in either case, the resulting permutation $q \cdot T$ has the property that $\{1,2\}$ is the minimum, and $\{1,3\} > S$, where $S \in A_2$.

3. If $\{\{1,2\},\{1,3\},\ldots,\{1,v\}\}$ is a subposet of T, then we are done. Otherwise, we can find a permutation $r \in S_X$ such that $r \cdot i_1 = \{1,2\} \prec r \cdot i_2 \prec \cdots \prec r \cdot i_v = \{1,v\}$, where $i_j \in B$.

We conclude that $(rqp) \cdot T$ is a linear extension of P.

Uniqueness: Let L be a linear extension of P. Let $p \in S_X \setminus \{e\}$, where e denotes the identity element in S_X. Without loss of generality, assume $p \cdot \{1,2\} = \{1,2\}$. (Otherwise, the resulting permutation $p \cdot L$ no longer has $\{1,2\}$ in the first coordinate, so $p \cdot L$ cannot possibly be a linear extension of P.) We have two cases to consider:

1. Suppose $p(1) = 1$ and $p(2) = 2$. Since L is a linear extension of P, then $\{1,2\} \succ \{1,3\} \succ \cdots \succ \{1,v\}$ is a subposet of L. Since p is a non-trivial permutation, p cannot respect the ordering on this subposet, so $p \cdot L$ is not a linear extension of P.

2. Suppose p has the cycle (1 2). Observe that $\pi \cdot \{1,3\} = \{2,\pi(3)\}$ and $\pi \cdot \{2,j\} = \{1,3\}$, for some $j \in \{2,3,\ldots,v-1\}$. Since $\{1,3\} \succ \{2,j\}$ is a covering relation in P, it follows that

$$\pi \cdot \{1,3\} = \{2,\pi(3)\} \not\succ \pi \cdot \{2,j\} = \{1,3\}$$

because $\pi(3) \neq 1$.

We conclude that for any $\pi \in S_X \setminus \{e\}$, we have that $\pi \cdot L$ is not a linear extension of P. □

Remark. Let $Y = \mathrm{Perm}\binom{X}{2}$. Notice that by Burnside's lemma [8],

$$|G\backslash\backslash Y| = \frac{1}{G}\sum_{\pi \in G} |\mathrm{Fix}(\pi)| = \frac{\binom{v}{2}!}{v!},$$

where $\mathrm{Fix}(\pi)$ denotes the set of elements in Y fixed by π. Therefore, $[S_d : G]$ is the number of linear extensions of P.

Now we can prove that $\mathcal{C}(P)$ contains all the points we need.

Proposition 4.8. *The order cone $\mathcal{C}(P)$ has the G-existence property.*

Proof. Let x be a point in $\mathbb{R}^d$. Define $(\alpha^{(n)})_{n\in\mathbb{N}}$ to be a sequence such that $\alpha^{(n)}$ converges to x, and if $\alpha_i^{(n)} > \alpha_j^{(n)}$, then $\alpha_i^{(m)} > \alpha_j^{(m)}$ for all $m \in \mathbb{N}$. We know such a sequence exists by Lemma 2.3. By Theorem 4.7, there exists a unique $\pi \in G$ such that for every $n \in \mathbb{N}$, it follows that $\pi \cdot \alpha^{(n)} \in \mathcal{C}(P)^{\circ}$. Since π is continuous (Lemma 2.2), we conclude that $\pi \cdot x \in \mathcal{C}(P)$. □

Consequently, we can write explicitly the inequality description of the interior for $\mathcal{C}(P)$:

Theorem 4.9. *The interior poset P determines the interior of the lex–maximal* *fundamental domain. In particular,*

$$\mathcal{C}(P)^\circ = \bigcap_{\substack{i,j \in P \\ i \gtrdot j}} \{x \in \mathbb{R}^d : x_i > x_j\}$$

defines the interior of a fundamental region under the action of G.

Proof. In order to prove the claim, we will argue that

$$\mathcal{C}(P)^\circ = \mathcal{F}_{\max}^\circ.$$

Let $x \in \mathcal{C}(P)^\circ$. Since $\mathcal{C}(P)^\circ$ has the G–uniqueness property, for all $\pi \in G \backslash \{e\}$ we have that $\pi \cdot x \notin \mathcal{C}(P)^\circ$, meaning that at least one of the defining inequalities of $\mathcal{C}(P)^\circ$ is violated. By construction, $x = (x_{i_1}, x_{i_2}, \ldots, x_{i_d})$ in $\mathbb{R}^d$ is such that $x_{i_k} > x_{i'_k}$ if $i_k \succ i_{k'}$. Therefore, if $\pi \cdot x \notin \mathcal{C}(P)^\circ$ for all $\pi \in G \backslash \{e\}$, then x is lexicographically maximal in its G–orbit. Hence, $x \in \mathcal{F}_{\max}$. Since $\mathcal{C}(P)^\circ$ is open, it follows that $\mathcal{C}(P)^\circ \subseteq \mathcal{F}_{\max}^\circ$.

To argue that $\mathcal{F}_{\max}^\circ \subseteq \mathcal{C}(P)^\circ$, notice firstly that $\mathcal{F}_{\max}^\circ \subseteq \mathcal{C}(P)$ because if $x \notin \mathcal{C}(P)$, then x violates at least one of the relations in P. Then, by Proposition 4.8, let $\pi \in G$ be such that $\pi \cdot x \in \mathcal{C}(P)$. In particular, $\pi \cdot x > x$, contradicting that x is lexicographically largest in its G–orbit.

Since $\mathcal{C}(P) = \mathcal{C}(P)^\circ \sqcup \partial\mathcal{C}(P)$, it suffices to show that if $x \in \mathcal{F}_{\max}^\circ$, then $x \notin \partial\mathcal{C}(P)$. Suppose that $x \in \mathcal{F}_{\max}^\circ$, and let $N \subseteq \mathcal{F}_{\max}$ be an open set containing x. We will prove by contradiction that $x \notin \partial\mathcal{C}(P)$.

Assume that $x \in \partial\mathcal{C}(P)$, and let $y \in N$ be such that $y \notin \mathcal{C}(P)$. However, $y \notin \mathcal{C}(P)$ contradicts $\mathcal{F}^{\circ}_{\max} \subseteq \mathcal{C}(P)$. Therefore, $\mathcal{F}^{\circ}_{\max} \subseteq \mathcal{C}(P)^{\circ}$, and we conclude that $\mathcal{F}^{\circ}_{\max} = \mathcal{C}(P)^{\circ}$. $\square$

Chapter 5

Concluding Remarks and Future Outlook

Here are some questions and remarks of interest for further research and to improve on the results discussed in this paper.

- Notice that since $S_{\binom{X}{2}} \cong S_{\binom{X}{v-2}}$, it follows that the lex–maximal fundamental region $\mathcal{F}_{\max}$ we constructed when $k = 2$ will also work when $k = v - 2$. As a result, we can consider t-designs with parameters $0 < t < v - 2 < v$, where $v \geq 4$. In particular, one may be able to formulate a combinatorial reciprocity theorem for a restricted class of BIBDs.

- Methods used in the proof of Theorem 4.4 could be employed to shed further light on the combinatorial structure of $\mathcal{F}_{\max}$, such as determining whether it is partitionable or shellable. (These two latter notions would need to be extended to fit our framework of partial polyhedral complexes.) Structural results of this kind could yield a reciprocity theorem for the function counting the number of isomorphism types of t–designs.

- What is an inequality description for the boundary of $\mathcal{F}_{\max}$?

The proof of Theorem 4.7 gives an algorithm for transforming a point x in a full–dimensional face of the braid arrangement into x', a representative in the same G–orbit that is contained in the lex–maximal fundamental cell $\mathcal{F}_{\max}$. This algorithm runs in polynomial time, so it can be used to verify whether two points represent isomorphic t–designs.

Observe that if we had such an algorithm for all points in $\mathbb{R}^d$, then this algorithm would solve the graph isomorphism problem because 2–dimensional faces of the braid arrangement have points that correspond to graphs. However, since the graph isomorphism problem is not expected to be polynomial–time computable, an inequality description of $\mathcal{F}_{\max}$ will be significantly more complex than our description of its interior in terms of order cones of posets.

Finally, recall that Chapter 4 deals with $t-(v,k,\lambda)$ designs with $k=2$, so it is natural to ask how one can find interior posets when $k>2$ —and, consequently, the interior of a fundamental cell. It might be fruitful to implement an algorithm that could yield examples of such regions. To this end, one could to find a subgraph of the **permutohedron** in $\mathbb{R}^d$, which is the $(d-1)$–polytope

$$\Pi_{d-1} \equiv \text{conv}(\{\pi \cdot (1,2,\ldots,d) : \pi \in G\}.$$

Here is an idea for the algorithm:

1. Take the lexicographically maximal vertex of Π_{d-1}, say v_0;

2. Find the neighbors of v_0;

3. Determine what neighbors are in the same orbit (it is possible that some of these neighbors are in the same orbit as v_0);

4. For each of the neighbors of v_0, apply Steps 1 and 2, making sure that the new vertices are representatives from orbits that we have not "visited" before.

We have included Sage code that could be part of such an implementation. Part of this code is intended to carry out Step 2, which is possible due to a result of Gaiha and Gupta [10].

Appendix: Sage Code

Methods to generate the symmetric group induced by permutations on X.

```
def turnSubsetsIntoLists(S):

    for i in IntegerRange(Integer(len(S))):

        S[i] = list(S[i])

    return S


def turnSubsetEltsIntoStrings(S):

    for i in IntegerRange(Integer(len(S))):

        for j in IntegerRange(Integer(len(S[i]))):

            S[i][j] = str(S[i][j])

    return S


def concatenateElts(S):

    for i in IntegerRange(Integer( len(S) )):
```

```
        foo = ''
        for j in IntegerRange(Integer( len(S[i]) )):
            foo += S[i][j]
        S[i] = foo
    return S

def actOnSubsets(S, p):
    if type(S[0]) == Integer:
        return list(Set(map(p,S)))
    else:
        return map(lambda x: actOnSubsets(x,p), S)
```

Given a vertex v_0 of the permutohedron, the following methods find the neighbors of v_0, and they return a set of vertices where no two are in the same orbit. The vertices are lexicographically maximal in their G–orbits.

```
def neighbors(x):
    L2 = []
    pi = Permutation(x)

    for j in range(d-1):

        L1 = []
```

```
        for i in range(d):

            if (pi.action(initial)[i] != j+1)
            & (pi.action(initial)[i] != j+2):
                L1.append(pi.action(initial)[i])
            elif pi.action(initial)[i] == j+2:
                L1.append(j+1)
            elif pi.action(initial)[i] == j+1:
                L1.append(j+2)

        L2.append(L1)

    return L2

def sameOrbits(x):
    temp1 = neighbors(x)
    temp2 = neighbors(x)
    print temp1
    transps = []
    b = len(temp1)
```

```
    print b
    l = []

    for i in range(b):
        for j in range(i+1, b):
            if ( Permutation(temp1[i])*Permutation(
             temp2[j]).inverse() in G) & (not(temp2[j] in l) ):
                l.append(temp2[j])
                transps.append((Permutation(temp1[i])
                    *Permutation(temp2[j]).inverse()).to_cycles())

    for i in range(len(l)):
        temp1.remove(l[i])

    return [temp1, transps]
```

Just as in Theorem 4.7, it may be helpful to partition certain sequences on $\binom{X}{k}$ into orbits. Here is a method that accomplishes this goal.

```
v = 6 #number of elements of X

Aut = PermutationGroup([[(2,3),(4,5)],[(2,4),(3,5)],[(1,2),(5,6)]])
```

```
GPerm1 = list(CartesianProduct(*([sets]*v)))
GPerm2 = list(CartesianProduct(*([sets]*v)))
L2 = []

L2.append([Permutation(Aut[0]).action(GPerm1[0])])

for i in GPerm1:
    L1 = []
    flag = True

    for j in Aut:
        temp = Permutation(j).action(i)

        if any(temp in k for k in L2):
            flag = False
            break
        elif not(temp in L1):
            L1.append(temp)

    if flag:
        L2.append(L1)
```

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