Extensions of Chapoton's q-Ehrhart Theory

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Certification of Approval

I certify that I have read Extensions of Chapoton's q-Ehrhart Theory by Thomas Henry Kunze and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirement for the degree Masters of Arts at San Francisco State University.

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Abstract

Given a d-dimensional convex polytope $\mathcal{Q}$ with integral vertices, the number of integer points contained in the dilation $n\mathcal{Q}$ is a polynomial in n, for n a nonnegative integer. We generalize this result by fixing a linear form λ subject to certain conditions and considering weighted sums of form $\sum_{\mathbf{x} \in \mathbb{Z}^d \cap n\mathcal{Q}} q^{\lambda(\mathbf{x})}$. Each weighted sum is the evaluation of a polynomial in $\mathbb{Q}(q)[x]$ at the q-integer $[n]_q$. We now have two variables (q and x) instead of one; we choose this generalization to retain the polynomial structure we encounter in the single-variable case. The main results of this generalization are due to Chapoton [3]. We prove the q-polynomial statement and a related reciprocity result via Brion's theorem, which allows us to encode the lattice points of a polytope using its vertex cones. We then state and prove analogous results for polytopes with rational vertices.

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Table of Contents

List of Figures

Chapter 1

Introduction

This thesis is situated in an area of mathematics called combinatorics, the study of counting. In general, counting is nontrivial, particularly if there are many objects to be counted or if the objects do not follow some predictable pattern. To further complicate matters, as combinatorialists, we are not satisfied by merely counting up the objects in a specific instance (e.g., how many bananas there are in a given bunch). We would like to know if there is some formula that, when given some information about a circumstance, outputs the number of objects to be found in that circumstance (e.g., how many bananas there are in any bunch, provided we know the weight of the bunch). This thesis hinges on the existence of such formulas for a special kind of object (lattice points) considered in a special kind of environment (convex polytopes).

Counting lattice points in polytopes is a fundamental problem in combinatorics. In the 1960s, Eugène Ehrhart proved several results in the study of lattice point enumeration; for $t \in \mathbb{Z}_{>0}$, he showed that the number of lattice points in the tth dilate of a polytope with integral vertices is a polynomial in t. Ehrhart proved a more general version of this theorem for polytopes with rational vertices. In the 2010s, Frédéric Chapoton generalized Ehrhart's ideas by introducing a new variable, q, on top of the dilation variable t. This generalization came as part of a broader study of q-series and partition analysis. Chapoton proved an analogous version of Ehrhart's theorem in this q-context.

The object of this paper is to analyze certain results of Chapoton in this q-Ehrhart theory. In pursuit of this goal, we first familiarize the reader with the central contents of Ehrhart's original theory in sections 2.1–2.2. In section 2.3, we present Brion's theorem, a tool that will allow us to prove certain theorems of Chapoton in new ways; the results of sections 2.1–2.3 are from Beck and Robins [1]. In section 2.4, we give an overview of Chapoton's perspective; we define q-analogues for concepts encountered in the preceding sections. The theory here is from Chapoton [3].

We conclude with our Brion-centered proofs. Our proof of Theorem 3.1 provides a new approach to Theorem 2.46 of [3]: Chapoton's proof involves the notion of *q-Ehrhart series*, while ours relies on the relationship between a polytope and its *vertex cones*. This theorem is a q-anologue of Ehrhart's original statement for lattice polytopes; it asserts that, for a positive integer n and a linear form subject to certain conditions, there is a special polynomial corresponding to a polytope whose evaluation at the q-integer $[n]_q$ yields a weighted sum over the lattice points in the polytope's n^{th} dilation. In proving Theorem 3.7, we give a reciprocity statement that indicates what will happen if a q-integer of form $[-n]_q$ is inputted into the q-Ehrhart polynomials of Theorem 3.1. With Theorem 3.8, we provide a novel generalization of Theorem 3.1 to rational polytopes; similarly, we introduce a new reciprocity statement, Theorem 3.11, which generalizes Theorem 3.7 in the rational case.

Chapter 2

Background

2.1 Basic definitions and theorems.

We begin by defining an important class of objects that will enable us to encode certain characteristics of polytopes.

Definition 2.1. *A **generating function** for a sequence $(a_k)_{k=0}^{\infty}$ of complex numbers has the form*

$$F(z) = \sum_{k \geq 0} a_k z^k.$$

Definition 2.2. *A **convex polytope** is the convex hull of finitely many points in $\mathbb{R}^d$.*

We will use the term "polytope" to refer to a "convex polytope" here.

Definition 2.3. *A **hyperplane** is a set $H \subset \mathbb{R}^d$ such that*

$$H = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$$

for some vector $\mathbf{a} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ *and some* $b \in \mathbb{R}$.

Let H be a hyperplane as above. H is a **supporting hyperplane** of a polytope $\mathcal{P}$ if $\mathcal{P}$ lies entirely on one side of H; that is,

$$\mathcal{P} \subset \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \leq b\} \text{ or } \mathcal{P} \subset \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \geq b\}.$$

A **face** of $\mathcal{P}$ is a set of the form $\mathcal{P} \cap H$, where H is some supporting hyperplane of $\mathcal{P}$. The **dimension** of a face is the dimension of the affine space it spans. A **vertex** of $\mathcal{P}$ is a 0-dimensional face of $\mathcal{P}$. Unless otherwise specified, we work with full-dimensional polytopes; that is, d-dimensional polytopes in $\mathbb{R}^d$.

Definition 2.4. *A* ***vertex description*** *of a polytope* $\mathcal{P}$ *is given by a set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^d$ *such that* $\mathcal{P} = \text{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Definition 2.5. *A* d*-polytope with exactly* $d+1$ *vertices is called a* ***d-simplex****.*

We say a polytope is **integral** (equivalently, that it is a **lattice polytope**) if its vertices are integral. We say a polytope is **rational** if its vertices have rational coordinates. The **denominator** of a rational polytope is the least common multiple of the denominators of its vertex coordinates. We may **dilate** a polytope $\mathcal{P}$ by multiplying each point $p \in \mathcal{P}$ with some number t.

Definition 2.6. *The* ***lattice-point enumerator*** *of the* t^{th} *dilate of a polytope* $\mathcal{P}$ *is defined as*

$$L_{\mathcal{P}}(t) \equiv \#(t\mathcal{P} \cap \mathbb{Z}^d),$$

provided $t \in \mathbb{Z}_{>0}$.

$L_{\mathcal{P}}(t)$ is also known as the **discrete volume** of $\mathcal{P}$.

Example 2.7. *Let $\mathcal{T}$ be the 2-simplex with vertices $(0,0)$, $(1,0)$, and $(0,1)$. We have $L_{\mathcal{T}}(0) = 1, L_{\mathcal{T}}(1) = 3$, and $L_{\mathcal{T}}(2) = 6$. Observing this pattern, one might conjecture that $L_{\mathcal{T}}(t)$ is the formula for the $(t+1)^{th}$ triangular number; indeed, it turns out that $L_{\mathcal{T}}(t) = \frac{(t+1)(t+2)}{2}$. It might seem a happy coincidence that $L_{\mathcal{T}}(t)$ is a polynomial in this case, but we will soon see that $L_{\mathcal{P}}(t)$ is a polynomial in t for any integral polytope.*

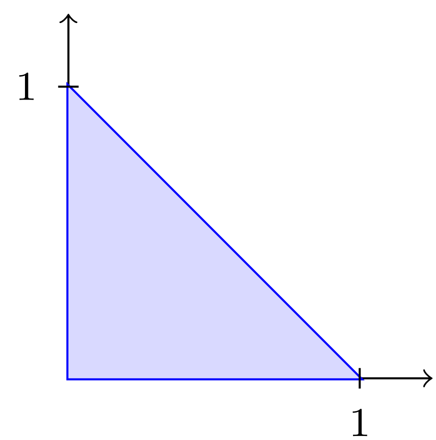

Figure 2.1: Our triangle $\mathcal{T}$.

Remark 2.8. *We note that we have not defined L_P at $t = 0$. It is in fact true that the constant coefficient of the polynomial $L_P(t)$ is 1, but showing this requires an involved approach outside the scope of this paper.*

We now introduce a basic instance of a central definition of this paper.

Definition 2.9. *Let $\mathcal{P}$ be a polytope. Define the generating function*

$$\mathrm{Ehr}_{\mathcal{P}}(z) \equiv 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t.$$

*This generating function is called the **Ehrhart series** of $\mathcal{P}$.*

Example 2.10. *Our triangle $\mathcal{T}$ has the Ehrhart series*

$1 + L_{\mathcal{T}}(1)z + L_{\mathcal{T}}(2)z^2 + \cdots = 1 + 3z + 6z^2 + 10z^3 + \cdots = \frac{1}{(1-z)^3}.$

We will sometimes work with the interiors of polytopes and cones.

Definition 2.11. *The **interior** of a set $S \subseteq \mathbb{R}^d$, denoted S°, is the set of non-boundary points in S; that is, S° is the set of $x \in \mathbb{R}^d$ for which $r > 0$ exists with a d-dimensional open ball $B(x, r) \subseteq S$.*

If $\mathcal{P}$ is a polytope, we define $L^\circ_{\mathcal{P}}(t) \equiv \#(t\mathcal{P}^\circ \cap \mathbb{Z}^d)$ for $t \in \mathbb{Z}_{>0}$.

Theorem 2.12 (Beck [1, Theorem 2.2])**.** *Let $\triangle$ be the convex hull of the $d+1$ points $\mathbf{e}_1, \ldots, \mathbf{e}_d$ and the origin, where $\mathbf{e}_i$ is the unit vector with a 1 in the ith position; $\triangle$ is called the **standard*** $\mathbf{d}$***-simplex**. Then the following hold:*

(a) $L_\triangle(t) = \binom{d+t}{t}$.

(b) Its evaluation at negative integers yields $(-1)^d L_\triangle(-t) = L^\circ_\triangle(t)$.

(c) The Ehrhart series of $\triangle$ is $\mathrm{Ehr}_\triangle(z) = \frac{1}{(1-z)^{d+1}}$.

Example 2.13. *For our triangle $\mathcal{T}$, (a) and (c) immediately hold by Examples 2.7 and 2.10. We convince ourselves of (b): we have $(-1)^d L_{\mathcal{T}}(-t) = \frac{(1-t)(2-t)}{2}$. By inspection,*

$L^\circ_{\mathcal{T}}(1) = 0$, $L^\circ_{\mathcal{T}}(2) = 0$, and $L^\circ_{\mathcal{T}}(3) = 1$ (since this last dilate contains $(1,1)$). Evaluating $\frac{(1-t)(2-t)}{2}$ for $t = 1, 2, 3$ confirms that Theorem 2.12 holds in this instance.[1]

We will soon introduce some nice results for rational polytopes. Of course, we cannot expect things will be as clean here as in the integral case.

Definition 2.14. *A* ***quasipolynomial*** *Q is an expression of form*

$$Q(t) = c_n(t)t^n + \cdots + c_1(t)t + c_0(t),$$

where the $c_0, \ldots, c_n$ are periodic functions in t; i.e., for each $i \in \{0, 1, \ldots, n\}$, there is $p_i \in \mathbb{Z}_{\geq 0}$ such that $c_i(t + p_i) = c_i(t)$ for all $t \in \mathbb{Z}_{>0}$.

Provided $c_n(t) \neq 0$, the **degree** of Q above is n, and the least common multiple of the periods of $c_0, c_1, \ldots, c_n$ is the **period** of Q. One may view quasipolynomials from an alternate perspective: if $Q(t)$ is a quasipolynomial with period p, then there are polynomials $q_0, q_1, \ldots, q_{p-1}$ such that $Q(t) = q_r(t)$ provided $t \equiv r \bmod p$ for $r \in \{0, 1, \ldots, p-1\}$. The polynomials $q_0, q_1, \ldots, q_{p-1}$ are the **constituents** of Q.

Definition 2.15. *A* ***triangulation*** *of a convex d-polytope $\mathcal{P}$ is a finite collection T of d-simplices such that the following hold:*

(a) $\mathcal{P} = \bigcup_{\triangle \in T} \triangle$;

(b) For every $\triangle_1, \triangle_2 \in T$, the intersection $\triangle_1 \cap \triangle_2$ is a face common to both $\triangle_1$ and $\triangle_2$.

A triangulation of $\mathcal{P}$ uses **no new vertices** if every $\triangle \in T$ has vertices belonging only to the vertex set of $\mathcal{P}$.

[1] Assuming $L^\circ_\triangle$ is a quadratic polynomial in t.

Theorem 2.16 (Beck [1, Theorem 3.1])**.** *Every convex polytope can be triangulated using no new vertices.*

Definition 2.17. *A (polyhedral)* **cone** $\mathcal{K} \subseteq \mathbb{R}^d$ *is a set of the form*

$$\mathcal{K} = \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \cdots + \lambda_m \mathbf{w}_m : \lambda_1, \ldots, \lambda_m \geq 0\},$$

where $\mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{R}^d$*.*

The vector $\mathbf{v}$ in the above definition is called an **apex** of $\mathcal{K}$, and the $\mathbf{w}_k$ are called the **generators** of $\mathcal{K}$. The **dimension** of $\mathcal{K}$ is the dimension of the affine space spanned by $\mathcal{K}$. A cone is **rational** if we can choose $\mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{Q}^d$. A **simplicial cone** $\mathcal{K}$ is one of dimension d having exactly d linearly independent generators.

Definition 2.18. *A* ***pointed cone*** $\mathcal{K} \subseteq \mathbb{R}^d$ *is a cone with apex* $\mathbf{v}$ *such that there is a hyperplane* H *so that* $H \cap \mathcal{K} = \{\mathbf{v}\}$*. In particular,* $\mathcal{K} \setminus \{\mathbf{v}\}$ *lies strictly on one side of* H*.*

Naturally, H is a **supporting hyperplane** for a pointed d-cone $\mathcal{K}$ if $\mathcal{K}$ lies entirely on one side of H. The faces of a cone are defined analogously as in the polytope case.

Definition 2.19. *A* d*-cone* $\mathcal{K}$ *can be triangulated with a collection* T *of simplicial* d*-cones if we have the following:*

(a) $\mathcal{K} = \bigcup_{\mathcal{S} \in T} \mathcal{S}$*;*

(b) For every $\mathcal{S}_1, \mathcal{S}_2 \in T$*,* $\mathcal{S}_1 \cap \mathcal{S}_2$ *is a face common to both* $\mathcal{S}_1, \mathcal{S}_2$*.*

We say that $\mathcal{K}$ can be triangulated **using no new generators** if there exists a triangulation T such that the generators of every $\mathcal{S} \in T$ are generators of $\mathcal{K}$.

Theorem 2.20 (Beck [1, Theorem 3.2])**.** *Every pointed cone can be triangulated into simplicial cones using no new generators.*

This statement is simple but powerful. As we will see later on, it enables us to prove a reciprocity theorem that we will use to prove another result for the q-Ehrhart case. We obtain the following corollary.

Corollary 2.21. *Every pointed cone $\mathcal{K}$ has integer-point transform equal to an inclusion-exclusion sum of integer-point transforms of simplicial cones whose generators are a subset of $\mathcal{K}$'s.*

Definition 2.22. *Let $S \subseteq \mathbb{R}^d$ be a rational cone or a polytope. Define*

$$\sigma_S(\mathbf{z}) = \sigma_S(z_1, \ldots, z_d) \equiv \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}},$$

where the notation $\mathbf{z}^{\mathbf{m}}$ signifies $\prod_{1 \le i \le d} z_i^{m_i}$. This generating function is the ***integer-point transform*** *of S.*

The transform σ_S encodes the integer points of S; each corresponds to a Laurent monomial term in σ_S.

Example 2.23. *Our standard 2-simplex $\mathcal{T}$ has integer point transform*

$\sigma_{\mathcal{T}}(z) = z_1^0 z_2^0 + z_1^1 z_2^0 + z_1^0 z_2^1 = 1 + z_1 + z_2.$

Theorem 2.24 (Beck [1, Theorem 3.5])**.** *Suppose $\mathcal{K} = \{\lambda_1 \mathbf{w}_1 + \cdots + \lambda_d \mathbf{w}_d : \lambda_1, \ldots, \lambda_d \ge 0\}$ is a simplicial d-cone with generators $\mathbf{w}_1, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$. Then for $\mathbf{v} \in \mathbb{R}^d$, the integer-point* *transform* $\sigma_{\mathbf{v}+\mathcal{K}}$ *of the shifted cone* $\mathbf{v}+\mathcal{K}$ *is the rational function*

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})\dots(1-\mathbf{z}^{\mathbf{w}_d})},$$

where Π *is the* ***half-open parallelepiped***

$$\Pi \equiv \{\lambda_1\mathbf{w}_1 + \cdots + \lambda_d\mathbf{w}_d : 0 \leq \lambda_1, \dots, \lambda_d < 1\}.$$

This parallelepiped Π *is called the* ***fundamental parallelepiped*** *of* $\mathcal{K}$*.*

From Theorems 2.20 and 2.24, we obtain the following.

Corollary 2.25. *For a pointed cone*

$$\mathcal{K} = \{\mathbf{v} + \lambda_1\mathbf{w}_1 + \cdots + \lambda_m\mathbf{w}_m : \lambda_1, \dots, \lambda_m \geq 0\}$$

with $\mathbf{v} \in \mathbb{R}^d$*,* $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathbb{Z}^d$*, the integer-point transform* $\sigma_{\mathcal{K}}(\mathbf{z})$ *evaluates to a rational function in the coordinates of* $\mathbf{z}$*.*

This result allows us to cleanly express certain integer-point transforms as rational functions; we will later see instances where this equivalence is crucial. We next introduce the first in a long line of significant lattice-point enumeration theorems that we will encounter in this paper.

Theorem 2.26 (Ehrhart's theorem for integral polytopes [4])**.** *If* $\mathcal{P}$ *is an integral convex d-polytope, then the lattice-point enumerator* $L_{\mathcal{P}}(t)$ *is a polynomial in* t *of degree* d*.*

Remark 2.27. *We can now confirm our supposition that* $L_{\mathcal{T}}(t) = \frac{(t+1)(t+2)}{2}$*, since by Ehrhart's theorem,* $L_{\mathcal{T}}(t)$ *must be a polynomial of degree* 2*; it suffices to interpolate using* 3 *points, which is what we did in Example 2.7.*

We call $L_{\mathcal{P}}$ the **Ehrhart polynomial** of $\mathcal{P}$. We will later see more general cases of this important theorem. The following is a generalization to *rational* polytopes.

Theorem 2.28 (Ehrhart's theorem for rational polytopes [4])**.** *If $\mathcal{P}$ is a rational convex d-polytope, then $L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d. Further, the period of $L_{\mathcal{P}}(t)$ divides the denominator of $\mathcal{P}$.*

We conclude this section with an example involving this theorem.

Example 2.29. *Consider the rational polytope $\frac{1}{3}\mathcal{T}$. By Theorem 2.28, we know that $L_{\frac{1}{3}\mathcal{T}}(t)$ will be a quasipolynomial of degree 2 in t with period k dividing 3. If $k = 1$, then $L_{\frac{1}{3}\mathcal{T}}(t)$ would be a quadratic polynomial in t; noting that $L_{\frac{1}{3}\mathcal{T}}(rt) = L_{\mathcal{T}}(t)$ when $r \equiv 0 \bmod 3$, we would have $L_{\frac{1}{3}\mathcal{T}}(t) = \frac{(\frac{1}{3}t+1)(\frac{1}{3}t+2)}{2}$. However, this polynomial gives a non-integral output for $t = 1$. Thus, $k = 3$. We proceed to compute the constituent polynomials p_0, p_1, and p_2. From above, we have*

$$p_0(t) = \frac{(\frac{1}{3}t+1)(\frac{1}{3}t+2)}{2} = \frac{1}{18}t^2 + \frac{1}{2}t + 1.$$

We now interpolate $p_1(t)$: by inspection, $L_{\frac{1}{3}\mathcal{T}}(1) = 1$, $L_{\frac{1}{3}\mathcal{T}}(4) = 3$, and $L_{\frac{1}{3}\mathcal{T}}(7) = 6$. The quadratic that passes through these points is

$$p_1(t) = \frac{1}{18}t^2 + \frac{7}{18}t + \frac{5}{9}.$$

Finally, $L_{\frac{1}{3}\mathcal{T}}(2) = 1$, $L_{\frac{1}{3}\mathcal{T}}(5) = 3$, and $L_{\frac{1}{3}\mathcal{T}}(8) = 6$. The quadratic that passes through these points is

$$p_2(t) = \frac{1}{18}t^2 + \frac{5}{18}t + \frac{2}{9}.$$

Thus, we obtain

$$L_{\frac{1}{3}\mathcal{T}}(t) = \begin{cases} \frac{1}{18}t^2 + \frac{1}{2}t + 1 & \text{if } t \equiv 0 \bmod 3, \\ \frac{1}{18}t^2 + \frac{7}{18}t + \frac{5}{9} & \text{if } t \equiv 1 \bmod 3, \\ \frac{1}{18}t^2 + \frac{5}{18}t + \frac{2}{9} & \text{if } t \equiv 2 \bmod 3. \end{cases}$$

We can also represent this quasipolynomial in the form $\frac{1}{18}t^2 + c_1(t)t + c_0(t)$*, where*

$$c_1(t) = \begin{cases} \frac{1}{2} & \text{if} \quad t \equiv 0 \bmod 3, \\ \frac{7}{18} & \text{if} \quad t \equiv 1 \bmod 3, \\ \frac{5}{18} & \text{if} \quad t \equiv 2 \bmod 3, \end{cases}$$

and

$$c_0(t) = \begin{cases} 1 & \text{if} \quad t \equiv 0 \bmod 3, \\ \frac{5}{9} & \text{if} \quad t \equiv 1 \bmod 3, \\ \frac{2}{9} & \text{if} \quad t \equiv 2 \bmod 3. \end{cases}$$

2.2 Reciprocity.

We introduce some of the main reciprocity theorems. The proofs of most of these are given in [1].

Theorem 2.30 (Ehrhart–Macdonald reciprocity [5])**.** *Suppose $\mathcal{P}$ is a convex rational polytope. Then the evaluation of the quasipolynomial $L_{\mathcal{P}}$ at negative integers yields*

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t).$$

Let p denote the period of $L_{\mathcal{P}}$, and let $L_{\mathcal{P},0}, L_{\mathcal{P},1}, \ldots, L_{\mathcal{P},p-1}$ denote its constituents. In terms of these constituents, the above theorem states that, for integers of the form $kp+r$ for $k \in \mathbb{Z}$ and $r \in \{1, \ldots, p-1\}$, we have

$$L_{\mathcal{P},p-r}(-(kp+r)) = (-1)^d L_{\mathcal{P}^\circ,r}(kp+r),$$

since $-r \equiv (p-r) \bmod p$.

Theorem 2.31 (Stanley reciprocity [6]). *Suppose $\mathcal{K}$ is a rational d-cone with the origin as its apex. Then*

$$\sigma_{\mathcal{K}}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) = (-1)^d \sigma_{\mathcal{K}^\circ}(z_1, \ldots, z_d).$$

We now prove a result that extends upon the above theorem. We will need this more general version when we deal with reciprocity for rational polytopes.

Theorem 2.32 (Generalized Stanley reciprocity). *Suppose $\mathcal{K}$ is a rational d-cone with the origin as its apex, and let $\mathbf{v} \in \mathbb{R}^d$. Then the following equality of rational functions holds:*

$$\sigma_{\mathbf{v}+\mathcal{K}}(z_1, \ldots, z_d) = (-1)^d \sigma_{-\mathbf{v}+\mathcal{K}^\circ}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right).$$

Proof. We can triangulate $\mathcal{K}$ into simplicial d-cones $\mathcal{K}_1, \ldots, \mathcal{K}_n$. By [1, Exercise 3.18], any rational hyperplane $H \subset \mathbb{R}^d$ is some minimal distance away from $\mathbb{Z}^d \setminus H$. Thus, since any pointed cone can be defined by an arrangement of bounding hyperplanes, for each i, the boundary $\partial\mathcal{K}_i$ is some minimal distance from $\mathbb{Z}^d \setminus \partial\mathcal{K}_i$. As there are n cones in the triangulation, $\partial\mathcal{K}$ is some minimal distance away from $\mathbb{Z}^d \setminus (\bigcup_{i=1}^n \partial\mathcal{K}_i)$. Hence, analogous results hold for the translates $\mathbf{v}+\mathcal{K}$ and $-\mathbf{v}+\mathcal{K}$; take $\alpha > 0$ so that every lattice point is either in one of these cones' boundaries or at least α away from both boundaries. Thus, we can choose $\mathbf{w} \in \mathbb{R}^d$ so that $\mathbf{w} = \mathbf{v} + \mathbf{v}'$ for some small $\mathbf{v}' \in \mathcal{K}^\circ$ so that $||\mathbf{w} - \mathbf{v}|| \leq \alpha$ and

$$(\mathbf{v} + \mathcal{K}^\circ) \cap \mathbb{Z}^d = (\mathbf{w} + \mathcal{K}) \cap \mathbb{Z}^d, \tag{2.1}$$

$$(-\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d = (-\mathbf{w} + \mathcal{K}) \cap \mathbb{Z}^d, \tag{2.2}$$

and

$$\partial(\pm\mathbf{w} + \mathcal{K}_i) \cap \mathbb{Z}^d = \emptyset \text{ for each } i \in \{1, \ldots, n\}. \tag{2.3}$$

Hence, there are no lattice points on the boundary of $\pm\mathbf{w} + \mathcal{K}$. By (2.3), for $i \neq j$, the cones $\pm\mathbf{w} + \mathcal{K}_i$ and $\pm\mathbf{w} + \mathcal{K}_j$ share no lattice points, so

$$\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \sigma_{-\mathbf{w}+\mathcal{K}}(\mathbf{z}) = \sum_{i=1}^{n} \sigma_{-\mathbf{w}+\mathcal{K}_i}(\mathbf{z}) \tag{2.4}$$

and

$$\sigma_{\mathbf{v}+\mathcal{K}^\circ}(\mathbf{z}) = \sigma_{\mathbf{w}+\mathcal{K}}(\mathbf{z}) = \sum_{i=1}^{n} \sigma_{\mathbf{w}+\mathcal{K}_i}(\mathbf{z}). \tag{2.5}$$

By [1, Theorem 4.2], each summand $\sigma_{-\mathbf{w}+\mathcal{K}_i}(\mathbf{z})$ on the right-hand side of (2.4) equals $(-1)^d\sigma_{\mathbf{w}+\mathcal{K}_i}(\mathbf{z^{-1}})$ for $i \in \{1, \ldots, n\}$. By comparison with the right-hand side of (2.5), the result follows. □

Definition 2.33. *Let $\mathcal{P}$ be a rational polytope. The **Ehrhart series** of its interior is defined*

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) \equiv \sum_{t\geq 1} L_{\mathcal{P}^\circ}(t) z^t.$$

Theorem 2.34 (Ehrhart reciprocity [4])**.** *Suppose $\mathcal{P}$ is a rational polytope. Then the evaluation of the rational function* $\mathrm{Ehr}_{\mathcal{P}}$ *at $\frac{1}{z}$ yields*

$$\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^{\dim \mathcal{P}+1}\,\mathrm{Ehr}_{\mathcal{P}^\circ}(z).$$

2.3 Brion's theorem.

Definition 2.35. *Let $\mathcal{P}$ be a polytope, and let $\mathcal{F}$ be one of its faces. We define a cone*

$$\mathcal{K}_{\mathcal{F}} \equiv \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x} \in \mathcal{F}, \mathbf{y} \in \mathcal{P}, \lambda \in \mathbb{R}_{\geq 0}\},$$

which is called the ***tangent cone*** *of $\mathcal{F}$.*

Given a vertex $\mathbf{v}$ of $\mathcal{P}$, the cone $\mathcal{K}_{\mathbf{v}}$ is called a **vertex cone**. An oft-used strategy for

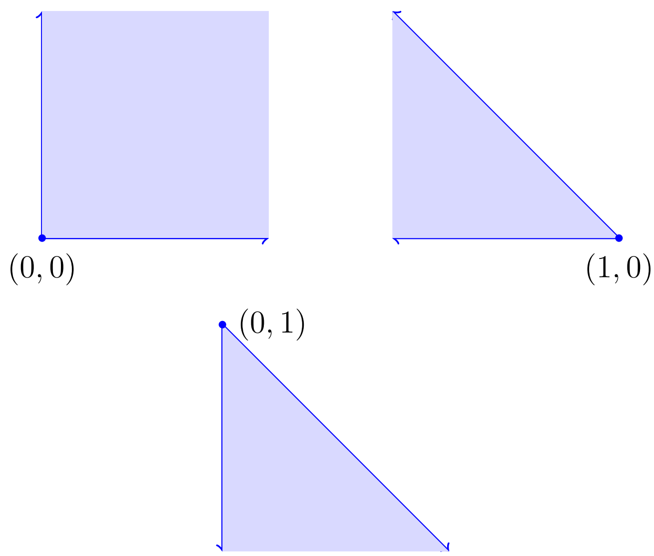

Figure 2.2: The vertex cones of our triangle $\mathcal{T}$ are $\mathcal{K}_{(\mathbf{0},\mathbf{0})}$, $\mathcal{K}_{(\mathbf{1},\mathbf{0})}$, and $\mathcal{K}_{(\mathbf{0},\mathbf{1})}$.

proving results about a polytope $\mathcal{P}$ is to work via its vertex cones. We will employ this method in the critical proofs of this paper. We now possess the necessary tools to discuss Brion's theorem.

Theorem 2.36 (Brion's theorem [2])**.** *Suppose $\mathcal{P}$ is a rational convex polytope. Then we* *have the following identity of rational functions:*

$$\sigma_{\mathcal{P}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z}).$$

A proof of how Theorem 2.28 follows from Theorem 2.36 is given in [1]. We now state a version of Brion's theorem for open polytopes.

Theorem 2.37 (Brion's theorem for open polytopes [2])**.** *Suppose $\mathcal{P}$ is a rational convex polytope. Then we have the following identity of rational functions:*

$$\sigma_{\mathcal{P}^\circ}(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{P}} \sigma_{\mathcal{K}^\circ_{\mathbf{v}}}(\mathbf{z}).$$

A proof of Theorem 2.37 is given in [1, Exercise 11.9].

2.4 The q-Ehrhart theory.

We now segue into the q-analogue of Ehrhart theory. We replace the number of lattice points in a polytope $\mathcal{P}$ with a family of weighted sums, each of which is a polynomial in the indeterminate q.

Definition 2.38. *A **linear form** is a function $\lambda : \mathbb{Q}^d \to \mathbb{Q}$ defined by*

$$\lambda(\mathbf{x}) = \lambda_1 x_1 + \cdots + \lambda_d x_d,$$

where $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$.

Given a d-dimensional polytope $\mathcal{Q}$ (or the interior of a polytope $\mathcal{Q}^\circ$) and a linear form λ, we consider the weighted sum

$$W_\lambda(\mathcal{Q}, q) \equiv \sum_{\mathbf{x} \in \mathcal{Q} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{x})}. \tag{2.6}$$

This sum runs over the lattice points contained in $\mathcal{Q}$. Thus, it is a q-analogue of the number of lattice points in $\mathcal{Q}$, obtained from (2.6) by evaluation at $q = 1$. We note that such sums are not invariant under translation of $\mathcal{Q}$ in general. We require our linear forms λ to satisfy the following conditions:

- **positivity**: for every vertex $\mathbf{x}$ of $\mathcal{Q}$, we have $\lambda(\mathbf{x}) \geq 0$.
- **genericity**: for every edge $\mathbf{x} - \mathbf{y}$ of $\mathcal{Q}$, we have $\lambda(\mathbf{x}) \neq \lambda(\mathbf{y})$.

Remark 2.39. *The domain and codomain of λ are defined so that the phrase "λ satisfies positivity and genericity on a rational polytope" makes sense. We will work with rational polytopes in our generalizations.*

Definition 2.40. *Fix a linear form λ satisfying the positivity and genericity conditions. The* **q-*Ehrhart series*** *of a lattice polytope $\mathcal{Q}$ is defined*

$$\mathrm{Ehr}_{\mathcal{Q},\lambda}(t,q) \equiv \sum_{n \geq 0} W_\lambda(n\mathcal{Q}, q)t^n.$$

Example 2.41. *Consider the line segment from 0 to 1 and the linear form $\lambda = 1$. The q-Ehrhart series is $1 + (1+q)t + (1+q+q^2)t^2 + \cdots = \frac{1}{(1-t)(1-qt)}$.*

Even in simple cases, the q-Ehrhart series can be difficult to compute as a rational function. For more examples, see [3]. A related result follows.

Theorem 2.42 (Chapoton [3, Proposition 1.1])**.** *Suppose λ satisfies the positivity and genericity conditions on $\mathcal{Q}$. Then the q-Ehrhart series $\mathrm{Ehr}_{\mathcal{Q},\lambda}$ is a rational function in t and q. Its denominator is a product without multiplicities of factors $1 - tq^j$ for some integers j with*

$0 \leq j \leq \max_{\mathcal{Q}}(\lambda)$. *The factor with index j appears only if there is a vertex $\mathbf{v}$ of $\mathcal{Q}$ such that $\lambda(\mathbf{v}) = j$.*

We proceed to describe the structure of the q-Ehrhart polynomial.

Definition 2.43. *The **q-integer at t** is defined $[t]_q = \frac{q^t-1}{q-1}$.*

From this definition, we know that $(q-1)[t]_q + 1 = q^t$. Thus, $q^{kt} = ((q-1)[t]_q + 1)^k$. Consequently, we obtain the following.

Lemma 2.44. *q^{kt} is a degree-k polynomial in the q-integer $[t]_q$ with coefficients in $\mathbb{Q}[q]$.*

Let $P_k = ((q-1)[t]_q + 1)^k$ for $k \geq 1$. Let c_j denote the coefficient of $[t]_q^j$ in P_k for $j \in \{0, 1, \ldots, k\}$.

Example 2.45. *P_3 has coefficients $c_3 = (q-1)^3 = q^3 - 3q^2 + 3q - 1, c_2 = 3q^2 - 6q + 3, c_1 = 3q - 3, c_0 = 1$.*

Now, we state one of the main q-Ehrhart theorems.

Theorem 2.46 (Chapoton [3, Theorem 2.1])**.** *Let $\mathcal{Q}$ be a lattice polytope and λ a linear form such that positivity and genericity hold. There is a polynomial $L_{\mathcal{Q},\lambda} \in \mathbb{Q}(q)[x]$ such that for any nonnegative integer n,*

$$L_{\mathcal{Q},\lambda}([n]_q) = W_\lambda(n\mathcal{Q}, q).$$

The degree of $L_{\mathcal{Q},\lambda}$ is $m = \max_{\mathcal{Q}}(\lambda)$. The coefficients of $L_{\mathcal{Q},\lambda}$ have poles only at roots of unity of order less than m.

Example 2.47. *The q-Ehrhart polynomial of our triangle $\mathcal{T}$ and the linear form $\lambda = (1, 2)$ is*

$$L_{\mathcal{T},\lambda}(x) = \frac{q^3x^2}{q+1} + \frac{q(2q+1)x}{q+1} + 1.$$

We will verify this fact in the next section. For now, we note that the evaluation of $L_{\mathcal{T},\lambda}$ at $q = 1$ is $L_{\mathcal{T}}$, the classical Ehrhart polynomial from Example 2.7.

Chapter 3

Main results

We now prove the first part of Theorem 2.46 via Brion's theorem (Theorem 2.36). Our approach is distinct from Chapoton's method involving q-Ehrhart series.

Theorem 3.1. *Let $\mathcal{Q}$ be a lattice polytope and λ a linear form such that positivity and genericity hold. There is a polynomial $L_{\mathcal{Q},\lambda} \in \mathbb{Q}(q)[x]$ such that for any nonnegative integer n,*

$$L_{\mathcal{Q},\lambda}([n]_q) = W_\lambda(n\mathcal{Q}, q).$$

Proof. Let $\mathcal{Q}$ be a d-polytope with integral vertices. Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a linear form satisfying positivity and genericity over $\mathcal{Q}$. Take $t \in \mathbb{Z}_{>0}$. Then

$$W_\lambda(t\mathcal{Q}, q) = \sum_{\mathbf{x} \in t\mathcal{Q} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{x})} = \sigma_{t\mathcal{Q}}(q^{\lambda_1}, \ldots, q^{\lambda_d}).$$

By Brion, we have

$$W_\lambda(t\mathcal{Q}, q) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{Q}} \sigma_{t\mathcal{K}_\mathbf{v}}(q^{\lambda_1}, \ldots, q^{\lambda_d}). \tag{3.1}$$

We write $\mathcal{K}_{\mathbf{v}} = \mathbf{v} + \widetilde{\mathcal{K}}_{\mathbf{v}}$, where $\widetilde{\mathcal{K}}_{\mathbf{v}}$ has apex at the origin. Thus, for each vertex $\mathbf{v}$, we have $t\mathcal{K}_{\mathbf{v}} = t\mathbf{v} + \widetilde{\mathcal{K}}_{\mathbf{v}}$, and so

$$\sigma_{t\mathcal{K}_{\mathbf{v}}}(q^{\lambda_1}, \ldots, q^{\lambda_d}) = q^{\lambda(t\mathbf{v})}\sigma_{\widetilde{\mathcal{K}}_{\mathbf{v}}}(q^{\lambda_1}, \ldots, q^{\lambda_d}). \tag{3.2}$$

Applying Corollary 2.25, each summand on the right-hand side of (3.1) is the product of some $q^{\lambda(t\mathbf{v})} = q^{t\lambda(\mathbf{v})}$ with a fixed rational function independent of t. Applying Lemma 2.44, each $q^{\lambda(t\mathbf{v})}$ is a polynomial in $[t]_q$. Replacing each $[t]_q$ with the indeterminate x on the right-hand side thus yields a polynomial in x. Therefore, the sum $W_\lambda(t\mathcal{Q}, q)$ is the evaluation of the polynomial $L_{\mathcal{Q},\lambda}(x)$ at $[t]_q$. $\square$

Remark 3.2. *We rely on genericity in the proof of Theorem 3.1 when we evaluate the* $\sigma_{\widetilde{\mathcal{K}}_{\mathbf{v}}}$*'s at powers of* q*. When viewed as a rational function, such a transform may have denominator of* 0 *if* λ *is equal on an adjacent pair of vertices. We require genericity in later proofs for the same reason.*

Example 3.3. *Verifying Example 2.47, we compute the* q*-Ehrhart polynomial for our triangle* $\mathcal{T}$ *using the proof of Theorem 3.1 above. Let* $t \in \mathbb{Z}_{>0}$ *and* $\lambda = (1, 2)$*. Applying Brion, we have*

$$W_\lambda(t\mathcal{T}, q) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{T}} \sigma_{t\mathcal{K}_{\mathbf{v}}}(q, q^2).$$

From (3.2) *in the proof, for each vertex* $\mathbf{v}$*, we have two objectives: to express* $q^{\lambda(t\mathbf{v})} = q^{t\lambda(\mathbf{v})}$ *as a polynomial in* $[t]_q$*, and to compute* $\sigma_{\widetilde{\mathcal{K}}_{\mathbf{v}}}(q, q^2)$ *as a rational function in* q*. Employing the* *logic mentioned immediately before Lemma 2.44, we obtain*

$$q^{t\lambda((0,0))} = q^0 = 1,$$

$$q^{t\lambda((1,0))} = q^t = (q-1)[t]_q + 1,$$

$$q^{t\lambda((0,1))} = q^{2t} = ((q-1)[t]_q + 1)^2.$$

To compute the $\sigma_{\widetilde{\mathcal{K}}_{\mathbf{v}}}(q, q^2)$'s, we recall Theorem 2.24:

$$\sigma_{\widetilde{\mathcal{K}}_{(0,0)}}(q, q^2) = \frac{1}{(1-q)(1-q^2)},$$
$$\sigma_{\widetilde{\mathcal{K}}_{(1,0)}}(q, q^2) = \frac{1}{\left(1-\frac{1}{q}\right)(1-q)},$$
$$\sigma_{\widetilde{\mathcal{K}}_{(0,1)}}(q, q^2) = \frac{1}{\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q}\right)}.$$

Thus, replacing $[t]_q$ with x and summing up gives

$$\begin{aligned} L_{\mathcal{T},\lambda}(x) &= \frac{1}{(1-q)(1-q^2)} + \frac{(q-1)x+1}{\left(1-\frac{1}{q}\right)(1-q)} + \frac{((q-1)x+1)^2}{\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q}\right)} \\ &= \frac{q^3x^2}{q+1} + \frac{q(2q+1)x}{q+1} + 1. \end{aligned}$$

Remark 3.4. *Continuing on the theme of Remark 3.2, let us choose a linear form failing genericity on $\mathcal{T}$: for instance, $\lambda = (1, 1)$. We see that the denominator of $\sigma_{\widetilde{\mathcal{K}}_{(1,0)}}(q, q)$ is 0.*

The degree property follows.

Corollary 3.5. *For a given polytope $\mathcal{Q}$ with integral vertices, the degree of its q-Ehrhart polynomial is the maximum of λ on its vertices.*

Proof. This follows immediately from Equation (3.2) in the proof of Theorem 3.1, since $q^{\lambda(t\mathbf{v})} = q^{t\lambda(\mathbf{v})}$; by Lemma 2.44, this is a degree-$\lambda(\mathbf{v})$ polynomial in $[t]_q$. So the degree of the q-Ehrhart polynomial is the maximum of the $\lambda(\mathbf{v})$'s, where $\mathbf{v}$ is a vertex of the polytope. $\square$

We now expand upon the last statement of Theorem 2.46; namely, that the coefficients of $L_{\mathcal{Q},\lambda}$ have poles at roots of unity of order less than $m = \max_{\mathcal{Q}}(\lambda)$. Brion's theorem allows us to infer properties of these roots.

Theorem 3.6. *Let $\mathcal{Q}$ be a polytope with integral vertices, and let λ be a linear form satisfying positivity and genericity on $\mathcal{Q}$. Then any pole of a coefficient of $L_{\mathcal{Q},\lambda}$ is a $|\lambda(g(\mathbf{w}-\mathbf{v}))|^{th}$ root of unity, where $\mathbf{v}$ and $\mathbf{w}$ are adjacent vertices of $\mathcal{Q}$ and $g(\mathbf{w}-\mathbf{v})$ is the primitive vector from $\mathbf{v}$ in the direction of $\mathbf{w}$.*

Proof. Recalling the proof of Theorem 3.1, $L_{\mathcal{Q},\lambda}(t) = \sum_{\mathbf{v}} q^{\lambda(t\mathbf{v})}\sigma_{\widetilde{K}_{\mathbf{v}}}(q^{\lambda_1},\dots,q^{\lambda_d})$. By Corollary 2.21, each of the $\sigma_{\widetilde{K}_{\mathbf{v}}}$'s can be written as an inclusion-exclusion sum of integer-point transforms of simplicial cones using no new generators. Applying Theorem 2.24 to these summands, the reduced denominator of $\sigma_{\widetilde{\mathcal{K}}_{\mathbf{v}}}(q^{\lambda_1},\dots,q^{\lambda_d})$ is of form $\prod_{\mathbf{w}}\left(1-q^{\lambda(\mathbf{w})}\right)$, where the product is taken over primitive generators of $\widetilde{\mathcal{K}}_{\mathbf{v}}$. Thus, this denominator divides $\prod_{\mathbf{w}:\mathbf{v},\mathbf{w}\text{ adjacent}}\left(1-q^{|\lambda(g(\mathbf{w}-\mathbf{v}))|}\right)$ as an element of $\mathbb{Q}(q)$ for each $\mathbf{v}$. (There may be some cancellation when we express $q^{\lambda(t\mathbf{v})}$ as a polynomial in $[t]_q$.) Combining the summands over one denominator gives the result. $\square$

Using Brion and Stanley reciprocity, we provide a new proof of the following theorem given in [3].

Theorem 3.7. *Let $\mathcal{Q}$ be a d-polytope with integral vertices. For every integer $t \in \mathbb{Z}_{>0}$,*

$$L_{\mathcal{Q},\lambda}([-t]_q) = (-1)^d \, W_\lambda \left(t\mathcal{Q}^\circ, \frac{1}{q} \right).$$

Proof. Let $\mathcal{Q}$ be a d-polytope with integral vertices. Set $\mathcal{K}^\circ_{\mathbf{v}} = \mathbf{v} + \widetilde{\mathcal{K}}^\circ_{\mathbf{v}}$. Let $t \in \mathbb{Z}_{>0}$. By Brion's theorem for open polytopes (Theorem 2.37),

$$W_\lambda \left(t\mathcal{Q}^\circ, \frac{1}{q} \right) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{Q}} q^{-t\lambda(\mathbf{v})} \sigma_{\widetilde{\mathcal{K}}^\circ_{\mathbf{v}}}(q^{-\lambda_1}, \ldots, q^{-\lambda_d}).$$

Applying Theorem 2.31 to the right-hand side yields

$$W_\lambda \left(t\mathcal{Q}^\circ, \frac{1}{q} \right) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{Q}} q^{-t\lambda(\mathbf{v})} (-1)^d \sigma_{\widetilde{\mathcal{K}}_{\mathbf{v}}}(q^{\lambda_1}, \ldots, q^{\lambda_d}).$$

From this, and recalling the proof of Theorem 3.1, we obtain

$$W_\lambda \left(t\mathcal{Q}^\circ, \frac{1}{q} \right) = (-1)^d L_{\mathcal{Q},\lambda}([-t]_q),$$

as desired. □

We now consider a more general class of polytopes: those with rational vertices. We show a version of Theorem 3.1 for this kind of polytope.

Theorem 3.8. *Let $\mathcal{Q}$ be a polytope with rational vertices having denominator p and let λ be a linear form such that the positivity and genericity conditions hold for λ on $\mathcal{Q}$. Fix $r \in \{0, 1, \ldots, p-1\}$. Then there is a polynomial $L_{\mathcal{Q},\lambda,r} \in \mathbb{Q}(q)[x]$ such that for any nonnegative integer k,*

$$L_{\mathcal{Q},\lambda,r}([k]_q) = W_\lambda((kp + r)\mathcal{Q}, q).$$

Proof. If $r = 0$, we return to the case handled by Theorem 3.1, so suppose $r > 0$. Let $\mathcal{Q}$ be a polytope with rational vertices having denominator p. Now,

$$W_\lambda((kp+r)\mathcal{Q}, q) = \sum_{\mathbf{x} \in (kp+r)\mathcal{Q} \cap \mathbb{Z}^d} q^{\lambda(\mathbf{x})}.$$

By Brion, we therefore have

$$W_\lambda((kp+r)\mathcal{Q}, q) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{Q}} \sigma_{(kp+r)\mathcal{K}_\mathbf{v}}(q^{\lambda_1}, \ldots, q^{\lambda_d}). \tag{3.3}$$

We write $(kp+r)\mathcal{K}_\mathbf{v} = kp\mathcal{K}_\mathbf{v} + r\mathcal{K}_\mathbf{v}$. Let $kp\mathcal{K}_\mathbf{v} = kp\mathbf{v} + \widetilde{\mathcal{K}}$, where $\widetilde{\mathcal{K}}$ has its apex at the origin. We have $\widetilde{\mathcal{K}} + r\mathcal{K}_\mathbf{v} = \widetilde{\mathcal{K}} + r\mathbf{v} + r\widetilde{\mathcal{K}} = r\mathcal{K}_\mathbf{v}$, so $(kp+r)\mathcal{K}_\mathbf{v} = kp\mathbf{v} + r\mathcal{K}_\mathbf{v}$. We note that $kp\mathbf{v}$ is an integer vector, so that

$$\sigma_{(kp+r)\mathcal{K}_\mathbf{v}}(q^{\lambda_1}, \ldots, q^{\lambda_d}) = q^{\lambda(kp\mathbf{v})}\sigma_{r\mathcal{K}_\mathbf{v}}(q^{\lambda_1}, \ldots, q^{\lambda_d}). \tag{3.4}$$

Applying Corollary 2.25, each summand on the right-hand side of (3.3) is the product of some $q^{\lambda(kp\mathbf{v})} = q^{k\lambda(p\mathbf{v})}$ with a fixed rational function independent of k. We proceed analogously as in the proof of Theorem 3.1: applying Lemma 2.44, each $q^{\lambda(kp\mathbf{v})}$ is a polynomial in $[k]_q$, as p is fixed. Replacing each $[k]_q$ with the indeterminate x on the right-hand side thus yields a polynomial in x. Therefore, $W_\lambda((kp+r)\mathcal{Q}, q)$ is the evaluation of a polynomial in $\mathbb{Q}(q)[x]$ at $[k]_q$. $\square$

Corollary 3.9. *For a given polytope $\mathcal{Q}$ with rational vertices with denominator p and for fixed $r \in \{0, 1, \ldots, p-1\}$, the degree of the polynomial $L_{\mathcal{Q},\lambda,r}$ in Theorem 3.8 is $p \cdot \max_\mathcal{Q}(\lambda)$.*

Proof. Noting that the polynomial in Theorem 3.8 is just the sum of terms involving $q^{k\lambda(p\mathbf{v})}$'s, the proof is analogous to that of Corollary 3.5. $\square$

We state and prove a result akin to Theorem 3.6.

Theorem 3.10. *Let $\mathcal{Q}$ be a d-polytope with rational vertices with denominator p, and let λ be a linear form satisfying positivity and genericity on $\mathcal{Q}$. For fixed $r \in \{1, \dots, p-1\}$, any pole of a coefficient of $L_{\mathcal{Q},\lambda,r}$ is a $|\lambda(g(p(\mathbf{w}-\mathbf{v})))|^{th}$ root of unity, where $\mathbf{v}$ and $\mathbf{w}$ are adjacent vertices of $\mathcal{Q}$ and $g(p(\mathbf{w}-\mathbf{v}))$ is the primitive vector from $\mathbf{v}$ in the direction of $\mathbf{w}$.*

Proof. From the proof of Theorem 3.8, we have $L_{\mathcal{Q},\lambda,r}(t) = \sum_{\mathbf{v}} q^{\lambda(tp\mathbf{v})}\sigma_{r\mathcal{K}_{\mathbf{v}}}\left(q^{\lambda_1}, \dots, q^{\lambda_d}\right)$. We may take the generators of $r\mathcal{K}_{\mathbf{v}}$ to be integer vectors. By Corollary 2.21, each of the $\sigma_{r\mathcal{K}_{\mathbf{v}}}$'s can be written as an inclusion-exclusion sum of integer-point transforms of simplicial cones using no new generators. Applying Theorem 2.24 to these summands, the reduced denominator of $\sigma_{r\mathcal{K}_{\mathbf{v}}}(q^{\lambda_1}, \dots, q^{\lambda_d})$ is of form $\prod_{\mathbf{w}}\left(1 - q^{\lambda(\mathbf{w})}\right)$, where the product is taken over primitive generators of $r\mathcal{K}_{\mathbf{v}}$. We note that $g(r\mathbf{w} - r\mathbf{v}) = g(\mathbf{w}-\mathbf{v})$. Thus, this denominator divides $\prod_{\mathbf{w}:\mathbf{v},\mathbf{w}\text{ adjacent}}\left(1 - q^{|\lambda(g(p(\mathbf{w}-\mathbf{v})))|}\right)$ as an element of $\mathbb{Q}(q)$ for each $\mathbf{v}$. (There may be some cancellation when we express $q^{\lambda(tp\mathbf{v})}$ as a polynomial in $[t]_q$.) Combining the summands over one denominator gives the result. □

We now prove a version of Theorem 3.7 for rational polytopes.

Theorem 3.11. *Let $\mathcal{Q}$ be a polytope with rational vertices having denominator p and let λ be a linear form such that the positivity and genericity conditions hold for λ on $\mathcal{Q}$. Fix $r \in \{0, 1, \dots, p-1\}$. Then for every integer $k \in \mathbb{Z}_{>0}$,*

$$L_{\mathcal{Q},\lambda,r}([-k]_q) = (-1)^d W_\lambda\left((kp-r)\mathcal{Q}^\circ, \frac{1}{q}\right),$$

where d is the dimension of $\mathcal{Q}$.

Proof. If $r = 0$, Theorem 3.7 suffices. Suppose $r \neq 0$ and let $\mathcal{Q}$ be a polytope with rational vertices. Let $k \in \mathbb{Z}_{>0}$. By Brion's theorem for open polytopes,

$$W_\lambda\left((kp-r)\mathcal{Q}^\circ, \frac{1}{q}\right) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{Q}} \sigma_{(kp-r)\mathcal{K}_{\mathbf{v}}^\circ}(q^{-\lambda_1}, \ldots, q^{-\lambda_d}).$$

We write $(kp-r)\mathcal{K}_{\mathbf{v}}^\circ = kp\mathcal{K}_{\mathbf{v}}^\circ - r\mathcal{K}_{\mathbf{v}}^\circ$. Let $kp\mathcal{K}_{\mathbf{v}}^\circ = kp\mathbf{v} + \widetilde{\mathcal{K}}^\circ$, where $\widetilde{\mathcal{K}}^\circ$ has its apex at the origin. We rewrite $-r\mathcal{K}_{\mathbf{v}}^\circ + \widetilde{\mathcal{K}}^\circ$ as $-r\mathbf{v} + \widetilde{\mathcal{K}}_{\mathbf{v}}^\circ$, where $\widetilde{\mathcal{K}}_{\mathbf{v}}^\circ$ has apex at the origin. We note that $kp\mathbf{v}$ is an integer vector, so that

$$\sigma_{(kp-r)\mathcal{K}_{\mathbf{v}}^\circ}(q^{-\lambda_1}, \ldots, q^{-\lambda_d}) = q^{-\lambda(kp\mathbf{v})}\sigma_{-r\mathbf{v}+\widetilde{\mathcal{K}}_{\mathbf{v}}^\circ}(q^{-\lambda_1}, \ldots, q^{-\lambda_d}). \tag{3.5}$$

Thus, we obtain

$$W_\lambda\left((kp-r)\mathcal{Q}^\circ, \frac{1}{q}\right) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{Q}} q^{-\lambda(kp\mathbf{v})}\sigma_{-r\mathbf{v}+\widetilde{\mathcal{K}}_{\mathbf{v}}^\circ}(q^{-\lambda_1}, \ldots, q^{-\lambda_d}).$$

Applying Theorem 2.32 to the right-hand side yields

$$W_\lambda\left((kp-r)\mathcal{Q}^\circ, \frac{1}{q}\right) = \sum_{\mathbf{v} \text{ a vertex of } \mathcal{Q}} q^{-\lambda(kp\mathbf{v})}(-1)^d\sigma_{r\mathbf{v}+\widetilde{\mathcal{K}}_{\mathbf{v}}}(q^{\lambda_1}, \ldots, q^{\lambda_d}).$$

From this, we obtain

$$W_\lambda\left((kp-r)\mathcal{Q}^\circ, \frac{1}{q}\right) = (-1)^d L_{\mathcal{Q},\lambda,r}([-k]_q),$$

as desired. $\square$

We note that the above theorem is a statement about the constituents of the quasipolynomial $L_{\mathcal{Q},\lambda}$, since we fix $r \in \{0, 1, \ldots, p-1\}$, and our choice of r determines the residue class of $(kp + r) \bmod p$.

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