Problems of Enumeration and Realizability
on Matroids, Simplicial Complexes, and Graphs

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Contents

Abstract iii

Acknowledgments iv

Chapter 1. Introduction 1

1.1. Matroids 2

1.2. Simplicial complexes 5

1.3. Graphs 9

Chapter 2. h-Vectors of Small Matroid Complexes 18

2.1. Rank-2 matroids 18

2.2. Corank-2 matroids 19

2.3. Rank-3 matroids 20

2.4. Matroids on at most nine elements 23

2.5. Further questions and directions 25

Chapter 3. Polytopal Embeddings of Cayley Graphs 27

3.1. Not all Cayley graphs are polyhedral 27

3.2. Two families of polyhedral Cayley graphs 32

3.3. Further questions and directions 43

Chapter 4. Flows on Simplicial Complexes 44

4.1. Structure of the boundary matrices of simplicial complexes 44

4.2. The number of nowhere-zero $\mathbb{Z}_q$-flows on a simplicial complex 48

4.3. The period of the flow quasipolynomial 53

4.4. Flows on triangulations of manifolds 57

4.5. Further questions and directions 58

Bibliography 60

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Problems of Enumeration and Realizability on Matroids, Simplicial Complexes, and Graphs

Abstract

This thesis explores several problems on the realizability and structural enumeration of geometric and combinatorial objects. After providing an overview of the thesis and some of the relevant background material in Chapter 1, we consider in Chapter 2 a conjecture of Stanley on the h-vectors of matroid complexes. We use the geometric structure of these objects to verify the conjecture in the case that the matroid corank is at most two, and provide new, simple proofs for the case when the matroid rank is at most three. We discuss an implementation based on simulated annealing and Barvinok-type methods to verify the conjecture for all matroids on at most nine elements using computers.

In Chapter 3, we study the geometry of Cayley graphs, in particular the embeddability of Cayley graphs as the 1-dimensional skeletons of convex polytopes. We find an example of a Cayley graph for which no such embedding exists, and provide an extension of Maschke's classification of planar groups with a new proof that emphasizes the connectivity and associated actions of the Cayley graphs and uses polyhedral techniques such as Steinitz's theorem. We further study the groups of symmetry of regular, convex polytopes and recall the Wythoff construction, which gives a polytope with 1-skeleton equal to the Cayley graph of the associated symmetry group.

Finally, in Chapter 4 we define a higher-dimensional extension of the graph-theoretic notion of nowhere-zero $\mathbb{Z}_q$-flows, and begin a systematic study of the enumerative and structural qualities of flows on simplicial complexes. We extend Tutte's result for the enumeration of $\mathbb{Z}_q$-flows on graphs to simplicial complexes, and find examples of complexes that, unlike graphs, do not admit a polynomial flow enumeration function. In light of work by Dey, Hirani, and Krishnamoorthy, we study the boundary matrices of a subfamily of simplicial complexes, and consider possible bounds for the period of their flow quasipolynomials.

At the end of each chapter, we present open questions and future directions related to each of the research topics.

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CHAPTER 1

Introduction

What does a combinatorial space look like? How is it put together? How many are there? These fundamental questions, and others like them, give rise to two important themes in geometric combinatorics: realizability and enumeration. Research in these areas has led to a plethora of theorems that connect the geometric and combinatorial properties of objects such as matroids, simplicial complexes, and graphs. Famous results include the following theorems. These theorems, and their extensions in the papers of Maschke [**Mas96**], Stanley [**Sta77**], Tutte [**Tut47**], and too many others to list, provided particular inspiration for work and results presented in this thesis.

Theorem 1.0.1 ([**Kur30**]). *A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5, the complete graph on five vertices, or $K_{3,3}$, the complete bipartite graph with parts of size three.*

Theorem 1.0.2 ([**Ehr62**]). *Let $\mathbb{P}$ be the convex hull of finitely many points in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$. Then, the lattice-point counting function*

$$\mathrm{Ehr}_{\mathbb{P}}(t) := \# \left(t\mathbb{P} \cap \mathbb{Z}^d \right)$$

is a polynomial in positive integer variable t.

The goal of this thesis is to gain a deeper understanding of the structure and combinatorics of matroids, simplicial complexes, and graphs by exploring them from the perspectives of realizability and enumerative combinatorics. Jointly with Matthias Beck, Jesús De Loera, and Steven Klee [**BK12, DLKK12**], we study (1) structural quantities of simplicial complexes derived from matroids; (2) embeddings of Cayley graphs as 1-skeletons of d-polytopes; and (3) combinatorial quantities of simplicial complexes. In the next sections, we present the background and motivation for our work, and state our main results. In the chapters following this introduction, we provide further details and proofs.

1.1. Matroids

We begin with an object prominent in combinatorics: the *matroid* [**Oxl92, Wel76, Whi92**]. There are many ways to define a matroid; we give the (perhaps) most intuitive here. In this thesis, all matroids are *finite*: for every M, $E(M)$ is a finite set.

Definition 1.1.1. *A* matroid $M = (E(M), \mathcal{I}(M))$ *consists of a* ground set $E(M)$ *and a family of subsets* $\mathcal{I}(M) \subseteq 2^{E(M)}$ *called* independent sets *such that*

(1) $\emptyset \in \mathcal{I}$;

(2) if $I \in \mathcal{I}$ *and* $J \subset I$, *then* $J \in \mathcal{I}$; *and*

(3) if $I, J \in \mathcal{I}$, *and* $|J| < |I|$, *then there exists some* $e \in I \setminus J$ *such that* $J \cup \{e\} \in \mathcal{I}$.

A *basis* of M is a maximal independent set under inclusion – by (3) above, all bases will have the same cardinality. The *rank* of a subset $S \subseteq E(M)$ is the size of a largest independent set $A \subseteq S$; in particular, the *rank* of M is the cardinality of a basis. A *loop* is a singleton $\{e\} \notin \mathcal{I}(M)$, and a *coloop* is an element that is contained in every basis. If M is a loopless matroid, elements $e, e' \in E(M)$ are pairwise *parallel* if $\{e, e'\} \notin \mathcal{I}(M)$. The *parallelism classes* of M are maximal subsets $E_1, \dots, E_t \subseteq E(M)$ with the property that all elements in each set E_i are parallel. It can be easily checked that if $\{e_{i_1}, \dots, e_{i_k}\} \in \mathcal{I}(M)$ with $e_{i_j} \in E_j$, then $\{e'_{i_1}, \dots, e'_{i_k}\} \in \mathcal{I}(M)$ for any choice of $e'_{i_j} \in E_j$. To see this, it is enough to show that if $\{e_{i_1}, \dots, e_{i_j}, \dots, e_{i_k}\}$ is an independent set with $e_{i_l} \in E_{i_l}$ for all $1 \leq l \leq k$, then $\{e_{i_1}, \dots, e'_{i_j}, \dots, e_{i_k}\}$ is also independent, where e'_{i_j} is any element in E_{i_j}. We know $\{e'_{i_j}, e_{i_1}\}$ is independent, as they are in separate parallelism classes. Therefore, there exists $e_{i_m} \neq e_{i_j} \in \{e_{i_1}, \dots, e_{i_j}, \dots, e_{i_k}\}$ such that $\{e'_{i_j}, e_{i_1}, e_{i_m}\}$ is independent (by (3) above). We can iteratively build the set in this way, and preserve the independence, until we have $\{e_{i_1}, \dots, e'_{i_j}, \dots, e_{i_k}\}$. Alternatively, the parallelism classes of M are maximal rank-one subsets of $E(M)$.

Given a matroid M on the ground set $E(M)$ with bases $\mathcal{B}(M)$, we define its *dual matroid*, M^*, to be the matroid on $E(M)$ whose bases are $\mathcal{B}(M^*) = \{E \setminus B : B \in \mathcal{B}(M)\}$. In this context, $\{e\}$ is a coloop in M if $\{e\}$ is a loop in M^*.

There are many objects that can be thought of very naturally in terms of matroids. For instance, any matrix or any graph gives a matroid: for the former, elements of $E(M)$ are the columns of the matrix, and $\mathcal{I}(M)$ consists of the independent subsets of columns. For the latter, elements of $E(M)$ are the edges of the graph, and $\mathcal{I}(M)$ consists of the subforests of the graph.

Matroids were in fact originally conceived as a generalization of the linear independences given by the columns of a matrix or the edges of a graph.

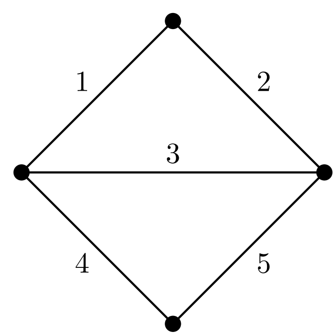
EXAMPLE 1.1.2. *The matroids given by the objects in Figure 1.1 are all equivalent.*

$$\begin{aligned} M = (E(M), \mathcal{I}(M)) \quad = \quad & (\{1,2,3,4,5\}, \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \\ & \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{1,2,4\}, \\ & \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}\}) \end{aligned}$$

(A)

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$
$$\begin{pmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(B)



(C)

FIGURE 1.1. Three equivalent matroids with different presentations.

Matroids, and in particular a subclass of matroids called *orientable* (see [**BLVS$^+$99, Rei05**] for general information on oriented matroids), have been extremely useful in fields such as linear programming and optimization, and have received a great deal of attention in the areas of enumeration and realizability. We consider two aspects of these areas of research in the following sections.

1.1.1. Matroids and enumeration: the Tutte polynomial. One way in which counting problems concerning matroids arise is through the *Tutte polynomial*, an invariant defined on the class of matroids. As with matroids themselves, the Tutte polynomial has several formulations, we give two here. First, the Tutte polynomial of a matroid $M = (E(M), \mathcal{I}(M))$ can be defined as

$$T_M(x,y) := \sum_{S \subseteq E(M)} (x-1)^{\mathrm{rk}(M)-\mathrm{rk}(S)}(y-1)^{|S|-\mathrm{rk}(S)}.$$

There is also a recursive definition for $T_M(x,y)$. Before giving this version, we recall that the *deletion* of an element $e \in E(M)$ from a matroid M is the matroid M_{-e} with ground set

$E(M_{-e}) = E(M) \setminus \{e\}$, and independent sets $\mathcal{I}(M_{-e}) = \{I \in \mathcal{I}(M) : e \notin I\}$. Then, the *contraction* of a matroid M about an element e is the matroid $M_{/e}$ with ground set $E(M_{/e}) = E(M) \setminus \{e\}$ and independent sets $\mathcal{I}(M_{/e}) = \{I \in \mathcal{I}(M) : e \notin I \text{ and } I \cup \{e\} \in \mathcal{I}(M)\}$. Note that the class of matroids is closed under the operations of deletion and contraction. Given these, we have the following alternative definition of the Tutte polynomial:

$$T_M(x,y) = \begin{cases} x & \text{if } M = (\{e\}, \{\emptyset, \{e\}\}), \text{ i.e. } M \text{ is a coloop,} \\ y & \text{if } M = (\{e\}, \{\emptyset\}), \text{ i.e. } M \text{ is a loop,} \\ T_{M-e}(x,y) + T_{M/e}(x,y) & \text{if } e \text{ is neither a loop nor a coloop,} \\ xT_{M-e}(x,y) & \text{if } e \text{ is a coloop, and} \\ yT_{M-e}(x,y) & \text{if } e \text{ is a loop.} \end{cases}$$

Some evaluations of the Tutte polynomial have known interpretations. For instance, $T_M(1,1)$ counts the number of bases of M (see [**Whi92**, Chapter 6] for an extensive list of interpretations). Often, as is the case with our example, these evaluations correspond to enumerations of structural characteristics of the matroid, and specializations of the Tutte polynomial to univariate polynomials give enumeration functions in some cases. We discuss several particular specializations in Section 1.3.2.

1.1.2. Matroids and realizations: matrices, arrangements, and polytopes. Often, we want to realize matroids as matrices over a field. For instance, the matroid in Example 1.1.2 is *regular* — it can be realized over any field. We can also think of realizations in terms of *hyperplane arrangements* (i.e., a collection of subspaces of codimension one in $\mathbb{R}^n$) or *convex polytopes* (that is, convex hulls of finitely many points) with defining coordinates in a particular field. There has been extensive research on this question, particularly on orientable matroids, see [**Bok08, RG96, BS86, BS87, BS89, FMM13, KP11, Mnë85, Sey79**] for just a few of the many results. When a matroid is realizable, the Tutte polynomial becomes particularly meaningful for some evaluations, see [**Jae89, Rei99, Whi92**].

In some sense, any matroid can be realized geometrically; matroids make up a subfamily of the objects we discuss next: simplicial complexes.

1.2. Simplicial complexes

Simplicial complexes exemplify the bridge between geometry and combinatorics. We recall their definition:

Definition 1.2.1. *An* (abstract) simplicial complex Δ *on a* vertex set V *is a set of subsets of* V*. These subsets are called the* faces *of* Δ*, and we require that*

(1) for all $v \in V$*,* $\{v\} \in \Delta$*, and*
(2) for all $F \in \Delta$*, if* $G \subseteq F$*, then* $G \in \Delta$*.*

The *dimension* of a face F is $\dim(F) = |F| - 1$, and the *dimension* of Δ is $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$. A simplicial complex is *pure* if all *maximal faces* (that is, faces that are not properly contained in any other face) have the same cardinality. In this case, a maximal face is called a *facet*, and a *ridge* is a face of codimension one. In this thesis, we will work only with pure and finite (i.e., $|\Delta| < \infty$)simplicial complexes.

In addition to the combinatorial definition, we have geometric realizations of every simplicial complex as collections of simplices (i.e., vertices, line segments, triangles, and their higher dimensional analogues). For more background on simplicial complexes, see [**Sta96**, Chapter 0.3] or [**Hat02**, Chapter 2]. Simplicial complexes arise in many areas: they are useful as computational examples and test cases, they can encode combinatorial problems and information, and they have connections to fields such as optimization and linear programming. The fact that simplicial complexes appear in so many contexts has been extremely fruitful for both combinatorics and geometry, allowing us to compare complexes that arise from different combinatorial problems, understand combinatorial families of complexes by developing intuition about their geometry, and more [**Gra02, HKW03, KS10, San12, LLS08**]. It is therefore of interest to gain a deep understanding of the structural aspects of simplicial complexes.

One structural quantity of a simplicial complexes that is very natural to measure is the number of faces of each dimension. This inspires the following definition:

Definition 1.2.2. *The* f-vector *of a* $(d-1)$*-dimensional simplicial complex* Δ *is a vector* $f(\Delta) := (f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$*, where* $f_i = |\{F \in \Delta : \dim(F) = i\}|$ *and* $f_{-1} = 1$*. The* $f_i(\Delta)$*s are called the* f-numbers *of* Δ*.*

Oftentimes, it is more convenient to study the *h-vector* $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ of a simplicial complex Δ, whose entries are defined by the relation

$$\sum_{j=0}^{d} h_j(\Delta)\lambda^j = \sum_{i=0}^{d} f_{i-1}(\Delta)\lambda^i(1-\lambda)^{d-i}. \tag{1.1}$$

See [**Sta96**] for more on h-vectors and the combinatorics of simplicial complexes. Given that f- and h-vectors are defined for all simplicial complexes, it is natural to ask whether we can describe exactly the possible f- and h-vectors of these objects. Schützenberger [**Sch59**], and later Kruskal [**Kru63**] and Katona [**Kat68**], provided the characterization of the f-vectors (and hence, the h-vectors) of simplicial complexes. We give one necessary definition, then state their theorem.

Definition 1.2.3. *Given two integers $k, i > 0$, write*

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. Define

$$k^{(i)} = \binom{n_i}{i+1} + \binom{n_{i-1}}{i} + \cdots + \binom{n_j}{j+1}.$$

Theorem 1.2.4 ([**Kat68, Kru63, Sch59**]). *A vector $(1, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}^{d+1}$ is the f-vector of some $(d-1)$-dimensional simplicial complex Δ if and only if*

$$0 < f_{i+1} \leq f_i^{(i+1)}, \quad 0 \leq i \leq d-2.$$

Certain subclasses of simplicial complexes have more specific f- and h-vector characterizations, such as Cohen–Macaulay complexes (see [**Mac26, Sta96**] for more on Cohen–Macaulay complexes). However, many subclasses remain uncharacterized, such as flag complexes [**Sta96**] and, the focus of the next section, *independence complexes of matroids.*

1.2.1. Independence complexes of matroids. Matroids make up a highly-structured subfamily of simplicial complexes. In particular, the independent sets of a matroid M form the face set of a simplicial complex Δ called the *independence complex* of M (see [**Whi92**, Chapter 7]). In Chapter 2 we explore in greater detail certain structural quantities of the independence complexes of small matroids. In that chapter, we will consider only loopless matroids: since the loops of a matroid are not seen by the independence complex, no generality will be lost in doing so.

As the independence complex of a matroid is a simplicial complex, we can define the f- and h-vector of this object. In Chapter 2 and in the following sections, we will think of the f- and h-vectors of a matroid as follows. If M is a matroid of rank d, the *f-vector* of M, $f(M) := (f_{-1}(M), \ldots, f_{d-1}(M))$, is given by $f_{i-1}(M) := |\{A \in \mathcal{I}(M) : |A| = i\}|$ for $0 \leq i \leq d$. (As a face of dimension i has cardinality $i+1$, this definition is equivalent to the one for the simplicial complex that is the independence complex of M.) The h-vector $h(M)$ is defined with respect to $f(M)$ in the same way as above (equation (1.1)).

It should not be expected that the h-numbers of an arbitrary simplicial complex are nonnegative. For instance, the simplicial complex in Figure 1.2 has an h-vector with a negative entry.

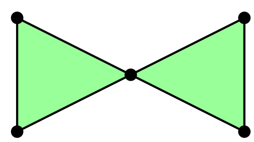


FIGURE 1.2. This simplicial complex has h-vector $(h_0, h_1, h_2, h_3) = (1, 2, -1, 0)$.

However, the h-numbers of a matroid M may be interpreted combinatorially in terms of certain invariants of M, and are therefore nonnegative. Fix a total ordering $\{e_1 < e_2 < \ldots < e_n\}$ on $E(M)$. Given a basis $B \in \mathcal{I}(M)$, an element $e_j \in B$ is *internally passive in* B if there is some $e_i \in E(M) \setminus B$ such that $e_i < e_j$ and $(B \setminus e_j) \cup e_i$ is a basis of M. Dually, $e_j \in E(M) \setminus B$ is *externally passive in* B if there is an element $e_i \in B$ such that $e_i < e_j$ and $(B \setminus e_i) \cup e_j$ is a basis. (Alternatively, e_j is externally passive in B if it is internally passive in $E(M) \setminus B$ in M^*.) It is well known [**Whi92**, Equation (7.12)] that

$$\sum_{j=0}^{d} h_j(M)\lambda^j = \sum_{B \in \mathcal{B}(M)} \lambda^{\mathrm{ip}(B)}, \tag{1.2}$$

where $\mathrm{ip}(B)$ counts the number of internally passive elements in B. This proves that the h-numbers of a matroid complex are nonnegative. Alternatively,

$$\sum_{j=0}^{d} h_j(M)\lambda^j = \sum_{B \in \mathcal{B}(M^*)} \lambda^{\mathrm{ep}(B)}, \tag{1.3}$$

where $\mathrm{ep}(B)$ counts the number of externally passive elements in B. Since the f-numbers (and hence the h-numbers) of a matroid depend only on its independent sets, equations (1.2) and (1.3) hold for *any* ordering of the ground set of M. It is worth remarking that the h-polynomial above is actually a specialization of the Tutte polynomial of the corresponding matroid (see [**Whi92**]). For further information on independence complexes and h-vectors, we refer the reader to the books of Oxley [**Oxl92**], White [**Whi92**], and Stanley [**Sta96**].

1.2.2. $\mathcal{O}$-Sequences and Stanley's conjecture. In order to state our results and the problem to which they pertain, we need several more definitions.

DEFINITION 1.2.5. *An order ideal $\mathcal{O}$ is a family of monomials (say of degree at most r) in a finite number of variables with the property that if $\mu \in \mathcal{O}$ and $\nu|\mu$, then $\nu \in \mathcal{O}$.*

Let $\mathcal{O}_i$ denote the collection of monomials in $\mathcal{O}$ of degree i. Let $F_i(\mathcal{O}) := |\mathcal{O}_i|$ and $F(\mathcal{O}) = (F_0(\mathcal{O}), F_1(\mathcal{O}), \ldots, F_r(\mathcal{O}))$. We say that $\mathcal{O}$ is *pure* if all of its maximal monomials (under divisibility) have the same degree. A vector $\mathbf{h} = (h_0, \ldots, h_d)$ is a *pure O-sequence* if there is a pure order ideal $\mathcal{O}$ such that $\mathbf{h} = F(\mathcal{O})$.

A longstanding conjecture of Stanley [**Sta77**] suggests that matroid h-vectors are highly structured.

CONJECTURE 1.2.6. *For any matroid M, $h(M)$ is a pure O-sequence.*

Conjecture 1.2.6 is known to hold for several families of matroid complexes, such as paving matroids [**MNRIVF12**], cographic matroids [**Mer01**], cotransversal matroids [**Oh10**], lattice path matroids [**Sch10**], and matroids of rank at most three [**TSZ10, Sto09**]. The purpose of Chapter 2 is to present proofs of the following theorems, originally published in [**DLKK12**].

THEOREM 2.1.1. *Let M be a matroid of rank 2. Then $h(M)$ is a pure O-sequence.*

THEOREM 2.2.1. *Let M be a matroid of rank 2. Then $h(M^*)$ is a pure O-sequence.*

THEOREM 2.3.1. *Let M be a loopless matroid of rank $d \geq 3$. Then the vector $(1, h_1(M), h_2(M), h_3(M))$ is a pure O-sequence.*

THEOREM 2.4.1. *Let M be a matroid on at most nine elements. Then $h(M)$ is a pure O-sequence.*

While Stanley's conjecture was already known to hold for matroids of rank two [**Sto09**] and rank three [**TSZ10**], we use the geometry of the independence complexes of matroids of small rank to provide much simpler shorter proofs in these cases. Our results show that any counterexample to Stanley's conjecture must have at least ten elements, rank at least four, and corank at least three.

Chapter 2 will use several ideas from the theory of multicomplexes and monomial ideals. Although a general classification of matroid h-vectors or pure O-sequences seems to be an incredibly difficult problem, some properties are known and will be used in the proofs in that chapter. In particular:

THEOREM 1.2.7. [**BC92, Cha97, Hib89**] *Let* $\mathbf{h} = (h_0, h_1, \ldots, h_d)$ *be a matroid* h*-vector or a pure* O*-sequence with* $h_d \neq 0$*. Then*

(1) $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor}$,

(2) $h_i \leq h_{d-i}$ *for all* $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$*, and*

(3) for all $0 \leq s \leq d$ *and* $\alpha \geq 1$*, we have*

$$\sum_{i=0}^{s} (-\alpha)^{s-i} h_i \geq 0. \tag{1.4}$$

Inequality (1.4) is known as the Brown–Colbourn inequality [**BC92**, Theorem 3.1].

1.3. Graphs

Graphs, like simplicial complexes, are particularly ubiquitous objects, and have been useful because of their ability to encode and represent data from a variety of contexts and problems (for instance [**ALS12, BMS11, Dij59, ES12, ER59, FF62, Fra11**], though this is just a small part of the body of work on graphs). Most commonly, a *graph* $G = (V, E)$ is a set of vertices V together with a set of edges E between vertices. In this thesis, graphs may have *loops* (an edge with endpoints that are the same vertex) and *parallel edges* (multiple edges between the same pair of vertices). If a graph is *simple* — no loops or parallel edges — then it can also be defined as a 1-dimensional simplicial complex. The dual nature of graphs as geometric and combinatorial objects has been very fruitful, and plays a large role in theorems and proofs about them.

On the geometric side, some of the many questions asked about graphs concern their embeddability in surfaces of varying genus or as the 1-dimensional skeletons of higher-dimensional objects. For instance, recall Kuratowski's theorem (Theorem 1.0.1), which characterizes all graphs that can be embedded in the surface of the 2-sphere. In Chapter 3 we look at the particular case of *Cayley graphs*, a subfamily of graphs that we introduce in the next section.

1.3.1. Polytopal embeddings of Cayley graphs. Given a group Γ and a set of generators and relations Λ of Γ, it is well-known [**Cay78, Mas96**] that we can construct a directed, edge-colored graph $C(\Gamma, \Lambda)$ in the following way. The elements of Γ are the vertices of the graph, and there is a directed edge of color h from g_1 to g_2 if there exists a generator $h \in \Lambda$ such that $g_1 h = g_2$. This graph is the *Cayley color graph* of (Γ, Λ), denoted $C(\Gamma, \Lambda)$. If we forget the colors and directions of the edges of a Cayley color graph $C(\Gamma, \Lambda)$, we obtain the *Cayley graph* of (Γ, Λ), denoted $G(\Gamma, \Lambda)$. In this thesis, we will work only with Cayley (color) graphs of finite groups, though there exists a theory of infinite Cayley graphs as well.

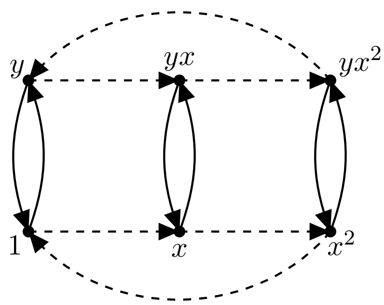
Example 1.3.1. *Say we have a presentation of the group $\mathbb{Z}_3 \times \mathbb{Z}_2$:*

$$(\Gamma, \Lambda) = \langle x, y \mid xy = yx,\ x^3 = y^2 = 1 \rangle.$$

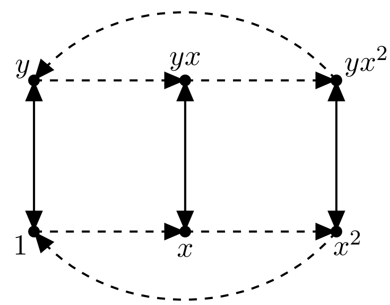
Then, the Cayley color graph is given by Figure 1.3a, where dashed lines indicate multiplication on the right by x, and solid lines indicate multiplication on the right by y. Typically, when we have elements of order two, we collapse the edges corresponding to these elements, as in Figure 1.3b. Finally, forgetting about the colors and directions of the edges gives us the Cayley graph, as in Figure 1.3c.

It is immediate to recover the group from a Cayley color graph: all the basic information of the group is contained in the graph. In particular, the relations of the group are the cycles of the graph, and a word in the group is a walk on the graph. In the rest of this section and in Chapter 3 we restrict ourselves to Cayley graphs that come from *minimal* presentations of the group, namely presentations in which no generator in Λ can be expressed in terms of the remaining generators.

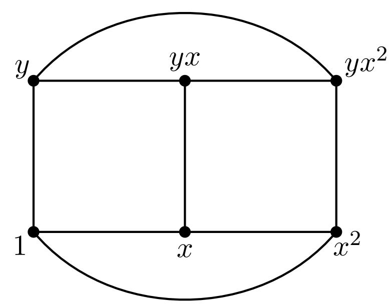
The geometric properties of groups have been studied for a long time in various contexts. First, graph theorists have looked at the embeddability of Cayley graphs on surfaces. For every graph G, we can find an orientable surface S of minimal genus such that G has an embedding in S. The *genus* of G is then the genus of S. The genus of a group Γ, $\gamma(\Gamma)$, is the minimal genus among the genera of all possible Cayley graphs of Γ. The classification of the groups of



(A) The Cayley color graph $C(\Gamma, \Lambda)$, as given in Example 1.3.1.



(B) The simplified Cayley color graph $C(\Gamma, \Lambda)$, as given in Example 1.3.1.



(C) The Cayley graph $G(\Gamma, \Lambda)$, as given in Example 1.3.1.

FIGURE 1.3

a given genus has been completed for genus zero [**Mas96**], one [**Pro78**], and two [**Tuc84**], but it remains an open problem for genus greater than two [**Whi84**]. Further, the embeddings of (mainly infinite) Cayley graphs have been the subject of research investigations by geometric group theorists (see [**AC04**]). The combinatorial representation theory of finite groups has been a third point of intersection for convex geometry and group theory. In this case, polytopes arise as the convex hulls of images of finite groups under fixed real representations. Of particular interest are *permutation polytopes* — these are polytopes arising from a subgroup H of the symmetric group S_n, where the representation of H is obtained by restricting a permutation representation of S_n (see e.g., [**BHNP09, GP06**] and references therein). In what follows, we will use the terms "1-skeleton of a polytope" and "graph of a polytope" interchangeably. The focus of Chapter 3 is the question: *when is a Cayley graph the graph of a d-dimensional convex polytope?* We show that this is not always the case, but give a few instances in which the Cayley graph allows for these "convex polyhedral" embeddings:

Theorem 3.1.1. *The Cayley graph of a minimal presentation of the quaternion group cannot be embedded as the graph of a convex polytope of any dimension.*

Theorem 3.2.12. *Let* Γ *be a finite group with a minimal set of generators and relations* Λ*. The associated Cayley graph is the graph of a* 3*-dimensional polytope if and only if* Γ *is a finite group of isometries in* 3*-dimensional space.*

1.3.2. Graph polynomials and their extensions. As mentioned above, any graph G can be interpreted as a matroid, and the Tutte polynomial of a graphic matroid is particularly interesting. Several specializations have been studied extensively, including the chromatic polynomial $\chi_G(q)$, the reliability polynomial $R_G(p)$, and the flow polynomial $\phi_G(q)$. In terms of the Tutte polynomial of $M(G)$, the matroid given by G, we have:

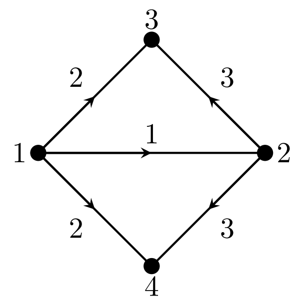
$$\begin{aligned}
\chi_G(q) &= (-1)^{|V|-k(G)} q^{k(G)} T_{M(G)}(1-q, 0), \\
R_G(p) &= (1-p)^{|V|-k(G)} p^{|E|-|V|+k(G)} T_{M(G)}\left(1, \frac{1}{p}\right), \text{ and} \\
\phi_G(q) &= (-1)^{|E|+|V|+k(G)} T_{M(G)}(0, 1-q),
\end{aligned}$$

where E is the edge-set of G, V is the vertex-set of G, and $k(G)$ is the number of connected components of G. These specializations have particular meaning in the case of graphs: for positive integers q, $\chi_G(q)$ counts the number of proper q-colorings of G (see [**Whi92**, Chapter 6]), and for negative q we have additional interpretations (see [**Sta73, BZ06a**]). For $0 \leq p \leq 1$, the polynomial $R_G(p)$ gives the probability that a network G will "fail" — that is, become disconnected – given that each edge is removed with probability p. Interest in understanding the reliability polynomial inspired the study of h-vectors of matroid independence complexes, which we advance in Chapter 2, because

$$R_G(p) = p^{|V|-1} \left(\sum_{i=0}^{|E|-|V|+1} h_i (1-p)^i \right),$$

where $(h_0, \ldots, h_{|E|-|V|+1})$ is the h-vector of $M(G)$.

Lastly, for positive integers q, $\phi_G(q)$ counts the number of nowhere-zero $\mathbb{Z}_q$-flows on the graph (see [**Whi92**, Chapter 6]). To define a flow on a graph G, we first give an initial orientation to its edges (this orientation is arbitrary, but fixed). Then, a $\mathbb{Z}_q$*-flow* on G is an assignment of values from $\mathbb{Z}_q$ to each edge such that modulo q, the sum of values entering each node is equal to the sum of values leaving it. If none of the edges receive zero weight, the $\mathbb{Z}_q$-flow is *nowhere-zero*. In terms of the signed incidence matrix M of G, a $\mathbb{Z}_q$-flow is an element of the kernel of M mod q. See Figure 1.4 for an example of a nowhere-zero $\mathbb{Z}_5$-flow.



(A) An oriented graph with a nowhere-zero $\mathbb{Z}_5$-flow.

	12	**13**	**14**	**23**	**24**
1	−1	−1	−1	0	0
2	1	0	0	−1	−1
3	0	1	0	1	0
4	0	0	1	0	1

(B) The signed incidence matrix corresponding to the graph on the left.

FIGURE 1.4. $(1, 2, 2, 3, 3)^T$ is a nowhere-zero $\mathbb{Z}_5$-flow and an element of the kernel (mod 5) of the incidence matrix.

Flows on graphs were first studied by Tutte ([**Tut47, Tut48**]; see also [**BSST75**] and [**Tut76**]) in the context of Kirchoff's electrical circuit laws, and Tutte [**Tut47**] first proved the following:

THEOREM 1.3.2. *The number of nowhere-zero $\mathbb{Z}_q$-flows on a graph, $\phi_G(q)$ is a polynomial in q.*

Since that time, a great deal of work (see, for instance, [**BZ06b, Jae88, Sey95, Whi92**]) has been done, and applications found in network and information theory, optimization, and other fields.

In order to extend the definition of a flow to a larger class of objects, we make use of the definition of a graph as a one-dimensional simplicial complex. Let Δ be a pure simplicial complex of dimension d with vertex set $V = \{v_0, \ldots, v_n\}$. Assign an ordering to V so that $v_0 < v_1 < \cdots < v_n$. Then, any r-dimensional face of Δ can be written (with respect to this ordering) as $[v_{i_0} \cdots v_{i_r}]$. We then have the following definition:

DEFINITION 1.3.3. *The* boundary map ∂ *on the simplicial chains of* Δ *is defined as*

$$\partial[v_{i_0} \cdots v_{i_r}] = \sum_{j=0}^{r} (-1)^j [v_{i_0} \cdots \widehat{v_{i_j}} \cdots v_{i_r}].$$

Often, it is convenient to think of the map in terms of a matrix.

Definition 1.3.4. *The* boundary matrix $\partial\Delta$ *of a simplicial complex* Δ *is the matrix whose rows correspond to the ridges and columns to the facets of* Δ*. The entries of* $\partial\Delta$ *are* ± 1 *or* 0*, depending on the sign of the ridge in the boundary of the facet.*

In the case of a graph, the facets are the edges, and the ridges are the vertices, and, under the natural ordering of the vertices, the boundary matrix is identical to the signed incidence matrix of the graph. For more background on the boundary map and boundary matrices, see [**Hat02**, Chapter 2].

Now, since a flow on a graph (1-dimensional simplicial complex) is an element of the kernel mod q of the signed incidence (boundary) matrix, we have a natural way to extend the notion of a flow on a graph to a flow on a simplicial complex.

Definition 1.3.5. *A* $\mathbb{Z}_q$-flow *on a pure simplicial complex* Δ *is an element of the kernel of* $\partial\Delta$ *mod* q*. A* nowhere-zero $\mathbb{Z}_q$-flow *is a* $\mathbb{Z}_q$*-flow with no entries equal to zero mod* q*.*

The idea of flows on simplicial complexes has previously been explored, for example by Nevo [**Nev08**], who proved the existence of a nowhere-zero $\mathbb{Z}$-flow on all doubly Cohen–Macaulay complexes.

Example 1.3.6. *Consider the surface of a tetrahedron, with vertices* $V = \{1, 2, 3, 4\}$ *given the natural ordering. Its boundary matrix* $\partial\Delta$ *is:*

	124	**134**	**234**	**123**
14	-1	-1	0	0
24	1	0	-1	0
34	0	1	1	0
12	1	0	0	1
13	0	1	0	-1
23	0	0	1	1

One instance of a nowhere-zero $\mathbb{Z}_q$*-flow on* Δ *is* $(x_{123}, x_{124}, x_{134}, x_{234})^T = (1, q-1, 1, q-1)^T$.

A subclass of simplicial complexes of particular interest to us are those that are *convex ear decomposable*, originally defined by Chari [**Cha97**].

Definition 1.3.7. *A* convex ear decomposition *of a pure rank-d simplicial complex* Δ *is an ordered sequence* $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ *(the* ears*) of pure rank-d subcomplexes of* Δ *such that*

(1) Σ_1 *is the boundary complex of a simplicial d-polytope, while for each* $i = 2, \ldots, n$, Σ_i *is a* $(d-1)$*-ball which is a (proper) sub-complex of the boundary complex of a simplicial d-polytope, and*

(2) for $i \geq 2$, $\Sigma_i \cap \left(\bigcup_{j=1}^{i-1} \Sigma_j\right) = \partial \Sigma_i$.

Convex ear decomposable (henceforth abbreviated as CED) simplicial complexes are a useful subclass of simplicial complexes as they are homologically wedges of spheres, they are built from readily understandable pieces, and they are conducive to inductive arguments. We may further specialize to *PS-ear decomposable* simplicial complexes. ("PS" represents the fact that products of simplices and boundaries of simplices are crucially involved in this definition. See [**Cha97**].)

Definition 1.3.8. PS-ear decomposable *simplicial complexes are pure, rank d CED complexes that satisfy the following additional requirements:*

(1) Σ_1 *is a* PS-$(d-1)$-sphere, *i.e., the direct product of boundaries of simplices, and*

(2) $\Sigma_2, \ldots, \Sigma_n$ *are* PS-$(d-1)$-balls, *i.e., direct products of a simplex and a PS-sphere.*

We will abbreviate PS-ear decomposable as PSED. One subfamily of simplicial complexes that is PSED is the family of matroid independence complexes. Again, see [**Cha97**] for more details.

1.3.3. Ehrhart quasipolynomials and Ehrhart's theorem. Before stating our results, which are based on work first published in [**BK12**], we recall a few more definitions.

Definition 1.3.9. *A* rational polytope $\mathbb{P} \subset \mathbb{R}^d$ *is:*

(1) the convex hull of finitely many points in $\mathbb{Q}^d$*; or equivalently,*

(2) a set of the form $\left\{x \in \mathbb{R}^d : Ax \leq b\right\}$, *where A is an integral matrix and b is an integral vector.*

For more information on polytopes, see [**Zie95**]. We will also make use of the following:

Definition 1.3.10. *A function* $q(t)$ *is a* quasipolynomial *in the integer variable t if there exist polynomials* $p_0(t), p_1(t), \ldots, p_{k-1}(t)$ *such that*

$$q(t) = p_j(t) \qquad \text{if} \qquad t \equiv j \bmod k\,.$$

In this case, the minimal such k is the period *of $q(t)$, and the polynomials $p_0(t), p_1(t), \ldots, p_{k-1}(t)$ are its* constituents*.*

We have the following generalization of Theorem 1.0.2:

THEOREM 1.3.11 ([**Ehr62**]). *Let $\mathbb{P}$ be a rational polytope. Then, the lattice-point counting function*

$$\mathrm{Ehr}_{\mathbb{P}}(t) := \# \left(t\mathbb{P} \cap \mathbb{Z}^d \right)$$

is a quasipolynomial in the positive integer variable t.

For more background on Ehrhart theory and quasipolynomials, see [**BS13, BR07, Ehr62**].

We mentioned previously that $\phi_G(q)$, which counts the number of nowhere-zero $\mathbb{Z}_q$-flows of a graph, is a polynomial in q. Our first main result is an extension of Tutte's polynomiality result for graphs:

THEOREM 4.2.5. *The number $\phi_\Delta(q)$ of nowhere-zero $\mathbb{Z}_q$-flows on Δ is a quasipolynomial in q. Furthermore, there exists a polynomial $p(x)$ such that $\phi_\Delta(k) = p(k)$ for all integers k that are relatively prime to the period of $\phi_\Delta(q)$. In addition, there are examples where the periodicity of the quasipolynomial is strictly larger than one.*

In other words, the flow quasipolynomial of a simplicial complex does *not* always reduce to a polynomial. We were able to specify the constituent polynomial $p(x)$ mentioned above by proving the following:

THEOREM 4.2.3. *Let q be a sufficiently large prime number, and let Δ be a simplicial complex of dimension d. Then the number $\phi_\Delta(q)$ of nowhere-zero $\mathbb{Z}_q$-flows on Δ is a polynomial in q of degree $\beta_d(\Delta) = \dim_{\mathbb{Q}}(\widetilde{H}_d(\Delta, \mathbb{Q}))$.*

For the case where the simplicial complex triangulates a manifold, we prove more.

PROPOSITION 4.4.1. *Let Δ be a triangulation of a manifold. Then*

$$\phi_\Delta(q) = \begin{cases} 0 & \text{if } \Delta \text{ has boundary,} \\ q-1 & \text{if } \Delta \text{ is without boundary, } \mathbb{Z}\text{-orientable,} \\ 0 & \text{if } \Delta \text{ is without boundary, non-}\mathbb{Z}\text{-orientable, and } q \text{ even,} \\ 1 & \text{if } \Delta \text{ is without boundary, non-}\mathbb{Z}\text{-orientable, and } q \text{ odd.} \end{cases}$$

We conclude each chapter with future directions and further questions stemming from our work. In particular, we propose problems in three areas. First, we ask how our geometric techniques in Chapter 2 can be extended to further classes of matroids and related objects. Second, we propose exploring additional polytopal constructions and Cayley graphs in order to better understand 1-skeletons of d-dimensional polytopes. Finally, we propose exploring the relationship between embeddability of simplicial complexes and their Ehrhart quasipolynomials, as well as combinatorial reciprocity theorems for the flow quasipolynomial.

CHAPTER 2

h-Vectors of Small Matroid Complexes

2.1. Rank-2 matroids

In this section, we prove Conjecture 1.2.6 in the case of matroids of rank 2. Let M be a loopless matroid of rank 2. The independence complex of M is a complete multipartite graph whose partite sets $E_1, \dots, E_t$ are the parallelism classes of M. Let $s_i := |E_i|$. Choose one representative $e_i \in E_i$ from each parallelism class of M so that the simplification of M is a complete graph on $\{e_1, \dots, e_t\}$, and let $\widetilde{E}_i = E_i \setminus e_i$. We can then write

$$\begin{aligned} f_0(M) &= \sum_{i=1}^{t}(s_i - 1) + t \\ \text{and} \quad f_1(M) &= \sum_{1 \le i < j \le t}(s_i - 1)(s_j - 1) + (t-1)\sum_{i=1}^{t}(s_i - 1) + \binom{t}{2}, \end{aligned}$$

and hence

$$\begin{aligned} h_1(M) &= \sum_{i=1}^{t}(s_i - 1) + (t-2) \\ \text{and} \quad h_2(M) &= \sum_{1 \le i < j \le t}(s_i - 1)(s_j - 1) + (t-2)\sum_{i=1}^{t}(s_i - 1) + \binom{t-1}{2}. \end{aligned}$$

Consider the pure O-sequence $\mathcal{O} = (1, \mathcal{O}_1, \mathcal{O}_2)$ with

$$\begin{aligned} \mathcal{O}_1 &= \{x_1, \dots, x_{t-2}\} \cup \{x_e : e \in \widetilde{E}_i, 1 \le i \le t\} \\ \mathcal{O}_2 &= \{x_e x_{e'} : e \in \widetilde{E}_i, e' \in \widetilde{E}_j, 1 \le i < j \le t\} \\ &\quad \cup \{x_i x_e : e \in \widetilde{E}_j, 1 \le i \le t-2, 1 \le j \le t\} \\ &\quad \cup \{\text{degree-2 monomials in } x_1, \dots, x_{t-2}\}. \end{aligned}$$

We see that $h(M) = F(\mathcal{O})$, which proves the following theorem.

Theorem 2.1.1. *Let M be a matroid of rank 2. Then $h(M)$ is a pure O-sequence.*

2.2. Corank-2 matroids

In this section, we prove Conjecture 1.2.6 for corank-2 matroids.

THEOREM 2.2.1. *Let M be a matroid of rank 2. Then $h(M^*)$ is a pure O-sequence.*

PROOF. As before, let $E_1, \ldots, E_t$ denote the parallelism classes of M. Impose a total order on the ground set $E(M)$ so that $v_i < v_j$ for all $v_i \in E_k$ and $v_j \in E_\ell$ with $1 \leq k < \ell \leq t$.

For each basis $B = \{v_i, v_j\}$ of M with $v_i \in E_k$, $v_j \in E_\ell$, and $k < \ell$, let

$$\begin{aligned} a_1(B) &:= \#\{i' > i : v_{i'} \in E_k \cup \cdots \cup E_{\ell-1}\} \\ \text{and} \quad a_2(B) &:= \#\{j' > j : v_{j'} \in E_\ell \cup \cdots \cup E_t\}, \end{aligned}$$

and set $\mu_B := x_1^{a_1(B)} x_2^{a_2(B)}$. We claim that $\mathcal{O} := \{\mu_B : B \in \mathcal{B}(M)\}$ is a pure order ideal and that $F(\mathcal{O}) = h(M^*)$.

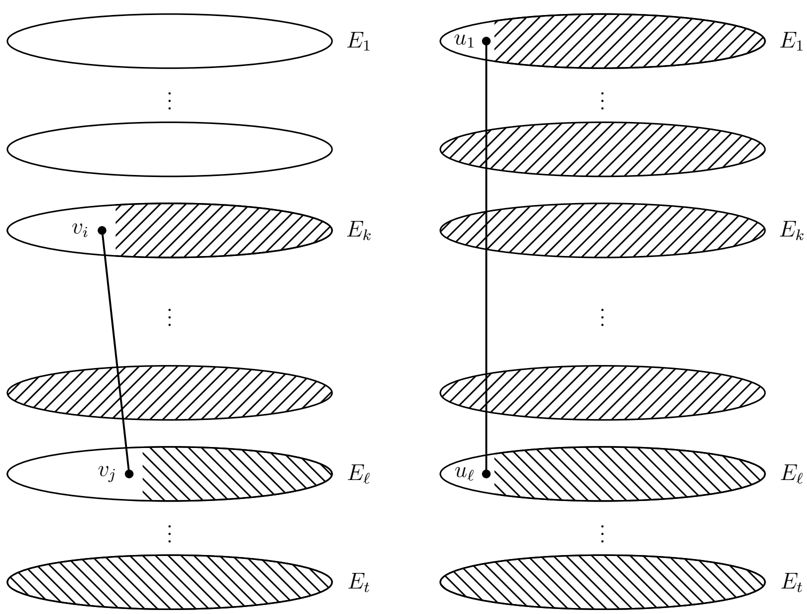


FIGURE 2.1. The bases $B = \{v_i, v_j\}$ (left) and $\widetilde{B} = \{u_1, u_\ell\}$ (right) with their externally passive elements shaded.

We see that $a_1(B)$ counts the number of elements $v \in E(M) \setminus B$ that are externally passive in B for which $v_i < v < v_j$ (shown in Figure 2.1 (left) shaded with lines of slope 1); and $a_2(B)$ counts the number of elements $v \in E(M) \setminus B$ that are externally passive in B for which $v_j < v \leq v_n$ (shown in Figure 2.1 (left) shaded with lines of slope -1). Since $a_1(B) + a_2(B)$ counts the number of externally passive elements in B, Equation (1.3) shows that $h(M^*) = F(\mathcal{O})$.

To see that $\mathcal{O}$ is an order ideal, we need only show that if $\nu | \mu_B$ and $\deg(\nu) = \deg(\mu_B) - 1$, then $\nu \in \mathcal{O}$. Let $B = \{v_i, v_j\}$ as before. If $a_1(B) > 0$, consider $B' = \{v_{i+1}, v_j\} \in \mathcal{I}(M)$. Clearly $a_1(B') = a_1(B) - 1$ and $a_2(B') = a_2(B)$ so that $\mu_{B'} \in \mathcal{O}$ and $\deg(\mu_{B'}) = \deg(\mu_B) - 1$. If $a_2(B) > 0$, we must consider two possible cases. If $v_{j+1} \in E_\ell$, then consider $B'' = \{v_i, v_{j+1}\} \in \mathcal{I}(M)$. Again $a_1(B'') = a_1(B)$ and $a_2(B'') = a_2(B) - 1$ so that $\mu_{B''} = x_1^{a_1(B)} x_2^{a_2(B)-1}$. On the other hand, if $v_{j+1} \in E_{\ell+1}$, then $v_{j-a_1(B)} \in E_{k'}$ for some $k' \leq \ell$, and so $B''' = \{v_{j-a_1(B)}, v_{j+1}\} \in \mathcal{I}(M)$. Again we see that $\mu_{B'''} = x_1^{a_1(B)} x_2^{a_2(B)-1}$. This establishes that $\mathcal{O}$ is an order ideal.

Finally, we must show that $\mathcal{O}$ is pure. For each $1 \leq i \leq t$, let u_i denote the smallest element of E_i. For any basis $B = \{v_i, v_j\}$ as above, let $\widetilde{B} = \{u_1, u_\ell\}$. As Figure 2.1 (right) indicates, $a_1(B) \leq a_1(\widetilde{B})$ and $a_2(B) \leq a_2(\widetilde{B})$, and hence $\mu_B | \mu_{\widetilde{B}}$. Moreover, $\deg(\mu_{\widetilde{B}}) = |E_1| + \cdots + |E_t| - 2$, and hence each such monomial $\mu_{\widetilde{B}}$ has the same degree. □

The techniques used to prove Theorem 2.2.1 can be extended to prove that $h(M^*)$ is a pure $\mathcal{O}$-sequence for any matroid M whose simplification is a uniform matroid. However, these techniques may not be used to prove Stanley's conjecture for the Fano matroid (see [**Oxl92**]), thus these techniques cannot be extended to corank 3.

2.3. Rank-3 matroids

Our goal for this section is to give a simple, short, geometric-combinatorial proof of the following theorem, first proved in [**TSZ10**] for the case that $d = 3$ using the language of commutative algebra.

THEOREM 2.3.1. *Let M be a loopless matroid of rank $d \geq 3$. Then the vector $(1, h_1(M), h_2(M), h_3(M))$ is a pure O-sequence.*

We need the following lemmas for our proof of Theorem 2.3.1.

LEMMA 2.3.2. *Let* $s_1, \ldots, s_t$ *be positive integers, and let* $d \geq 3$. *Then the vector* $\mathbf{h} = (1, h_1, h_2, h_3)$ *with*

$$
\begin{aligned}
h_1 &= \sum_{i=1}^{t}(s_i - 1) + (t-d), \\
h_2 &= \sum_{1 \leq i < j \leq t}(s_i-1)(s_j-1) + (t-d)\sum_{i=1}^{t}(s_i-1) + \binom{t-d+1}{2}, \\
h_3 &= \sum_{1 \leq i < j < k \leq t}(s_i-1)(s_j-1)(s_k-1) + (t-d)\sum_{1 \leq i < j \leq t}(s_i-1)(s_j-1) \\
&\quad + \binom{t-d+1}{2}\sum_{i=1}^{t}(s_i-1) + \binom{t-d+2}{3},
\end{aligned}
$$

is a pure O-sequence.

PROOF. Consider disjoint sets $\widetilde{E}_1, \ldots, \widetilde{E}_t$ with $|\widetilde{E}_i| = s_i - 1$ for all i. We will construct a pure order ideal $\mathcal{O}$ with $F(\mathcal{O}) = \mathbf{h}$ whose degree-one terms are

$$\mathcal{O}_1 = \{x_1, \ldots, x_{t-d}\} \cup \{x_e : e \in \widetilde{E}_i\}_{i=1}^{t}.$$

We explicitly construct such an order ideal by setting

$$
\begin{aligned}
\mathcal{O}_2 &= \{x_e x_{e'} : e \in \widetilde{E}_i, e' \in \widetilde{E}_j, 1 \leq i < j \leq t\} \\
&\quad \cup \{x_j x_e : e \in \widetilde{E}_i, 1 \leq i \leq t, 1 \leq j \leq t-d\} \\
&\quad \cup \{\text{all degree 2 monomials in } x_1, \ldots, x_{t-d}\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{O}_3 &= \{x_e x_{e'} x_{e''} : e \in \widetilde{E}_i, e' \in \widetilde{E}_j, e'' \in \widetilde{E}_k, 1 \leq i < j < k \leq t\} \\
&\quad \cup \{x_k x_e x_{e'} : e \in \widetilde{E}_i, e' \in \widetilde{E}_j, 1 \leq k \leq t-d, 1 \leq i < j \leq t\} \\
&\quad \cup \{x_j x_k x_e : e \in \widetilde{E}_i, 1 \leq j < k \leq t-d, 1 \leq i \leq t\} \\
&\quad \cup \{x_j^2 x_e : e \in \widetilde{E}_i, 1 \leq i \leq t, 1 \leq j \leq t-d\} \\
&\quad \cup \{\text{all degree 3 monomials in } x_1, \ldots, x_{t-d}\}.
\end{aligned}
$$

□

LEMMA 2.3.3. *For any positive integers $s_1, \ldots, s_t$, the vector $\mathbf{h}' = (1, h_1, h_2, h_3)$ with*

$$\begin{aligned} h_1 &= \sum_{i=1}^{t}(s_i - 1) + (t-d), \\ h_2 &= \sum_{1\le i<j\le t}(s_i-1)(s_j-1) + (t-d)\sum_{i=1}^{t}(s_i-1) + \binom{t-d+1}{2}, \\ h_3 &= \sum_{1\le i<j\le t}(s_i-1)(s_j-1) + (t-d-1)\sum_{i=1}^{t}(s_i-1) + \binom{t-d}{2} + 1, \end{aligned}$$

is a pure O-sequence.

PROOF. As in the proof of Lemma 2.3.2, let $\widetilde{E}_1, \ldots, \widetilde{E}_t$ be disjoint sets with $|\widetilde{E}_i| = s_i - 1$. Recall the order ideal $\mathcal{O}$ constructed in the proof of Lemma 2.3.2. We will construct a pure order ideal $\widetilde{\mathcal{O}}$ with $F(\widetilde{\mathcal{O}}) = \mathbf{h}'$ such that $\widetilde{\mathcal{O}}_1 = \mathcal{O}_1$, $\widetilde{\mathcal{O}}_2 = \mathcal{O}_2$, and $\widetilde{\mathcal{O}}_3 \subseteq \mathcal{O}_3$. We set

$$\begin{aligned} \widetilde{\mathcal{O}}_3 &= \{x_1 x_e x'_e : e \in \widetilde{E}_i, e' \in \widetilde{E}_j, 1 \le i < j \le t\} \\ &\quad \cup \{x_j^2 x_e : e \in \widetilde{E}_i, 1 \le i \le t, 2 \le j \le t-d\} \\ &\quad \cup \{x_i^2 x_j : 1 \le i < j \le t-d\} \cup \{\mu_0\}, \end{aligned}$$

where μ_0 is a monomial defined as follows: if $\widetilde{E}_1 \cup \cdots \cup \widetilde{E}_t$ is nonempty, choose some $e_0 \in \widetilde{E}_1 \cup \cdots \cup \widetilde{E}_t$ and set $\mu_0 = x_1^2 x_{e_0}$. Otherwise, set $\mu_0 = x_1^3$. This distinction in the monomial μ_0 is necessary for handling the cases in which $|\widetilde{E}_1 \cup \cdots \cup \widetilde{E}_t| \le 1$. □

We now prove Theorem 2.3.1.

PROOF. Let $E_1, \ldots, E_t \subseteq E(M)$ denote the parallelism classes of M, and set $s_i := |E_i|$. Choose one representative e_i from each class E_i, and let $W = \{e_1, \ldots, e_t\}$. Observe that $\Delta := M|_W$ is a simple matroid of rank d. Let $\widetilde{E}_i = E_i \setminus e_i$, and notice that for any choice of

$\widetilde{e}_{i_j} \in E_{i_j}, \{\widetilde{e}_{i_1}, \ldots, \widetilde{e}_{i_k}\} \in \mathcal{I}(M)$ if and only if $\{e_{i_1}, \ldots, e_{i_k}\} \in \Delta$. Thus

$$\begin{aligned}
f_0(M) &= \sum_{i=1}^{t} s_i \text{ and hence} \\
h_1(M) &= \sum_{i=1}^{t}(s_i - 1) + (t-d); \\
f_1(M) &= \sum_{1\le i<j\le t} s_i s_j \\
&= \sum_{1\le i<j\le t} (s_i-1)(s_j-1) + (t-1)\sum_{i=1}^{t}(s_i-1) + \binom{t}{2} \text{ and hence} \\
h_2(M) &= \sum_{1\le i<j\le t} (s_i-1)(s_j-1) + (t-d)\sum_{i=1}^{t}(s_i-1) + \binom{t-d+1}{2}; \\
f_2(M) &\le \sum_{1\le i<j<k\le t} s_i s_j s_k \text{ and hence} \\
h_3(M) &\le \sum_{1\le i<j<k\le t} (s_i-1)(s_j-1)(s_k-1) + (t-d)\sum_{1\le i<j\le t}(s_i-1)(s_j-1) \\
&\quad + \binom{t-d+1}{2}\sum_{i=1}^{t}(s_i-1) + \binom{t-d+2}{3}.
\end{aligned}$$

On the other hand, by the Brown–Colbourn inequality (1.4),

$$\begin{aligned}
h_3(M) &\ge h_2(M) - h_1(M) + h_0(M) \\
&= \sum_{1\le i<j\le t} (s_i-1)(s_j-1) + (t-d-1)\sum_{i=1}^{t}(s_i-1) + \binom{t-d}{2} + 1.
\end{aligned}$$

We construct a pure order ideal $\mathcal{O}'$ with $F(\mathcal{O}') = h(M)$ as follows. Following the notation used in Lemmas 2.3.2 and 2.3.3, we set $\mathcal{O}'_1 = \mathcal{O}_1$; $\mathcal{O}'_2 = \mathcal{O}_2$, and choose $\widetilde{\mathcal{O}}_3 \subseteq \mathcal{O}'_3 \subseteq \mathcal{O}_3$ with $|\mathcal{O}'_3| = h_3(M)$.

□

2.4. Matroids on at most nine elements

This part of the chapter is experimental and is crucially based on the data provided by Dillon Mayhew and Gordon Royle. They constructed a database of all 385,369 matroids on at most nine elements [**MR08**]. We used this data to generate a list of all possible h-vectors of matroid complexes on at most nine elements and then checked to see if they matched an element in a list of all possible pure O-sequences for particular rank and corank, giving us the following:

THEOREM 2.4.1. *Let M be a matroid on at most nine elements. Then $h(M)$ is a pure O-sequence.*

To generate the O-sequences, we used a combination of `Perl` and `Maple` code available at `www.math.ucdavis.edu/~ykemper/matroids.html`. On that webpage, we have recorded the monomials that generate the pure order ideal corresponding to each of the matroid h-vectors.

Given a loopless, coloopless matroid M of rank d on n elements, we searched for a pure O-sequence $\mathcal{O}$ with $h(M) = F(\mathcal{O})$ in the following way: we know that $h_d(M)$ counts the number of top-degree monomials in $\mathcal{O}$, and $h_1(M) = n - d$ counts the number of variables (degree-one terms) in $\mathcal{O}$. By sampling the space of monomials of degree d on $n - d$ variables, we can generate thousands of pure O-sequences that are candidates to be h-vectors of matroid complexes. Of course, because of the tremendous restrictions that the basis exchange axioms place on matroids, and hence also on their h-vectors, we often generated pure O-sequences that were not matroid h-vectors. For example $(1, 5, 15, 27, 22)$ and $(1, 5, 15, 27, 35)$ are both valid pure O-sequences that were generated, but the only h-vectors of matroid complexes of rank four with initial value $(1, 5, 15, 27, *)$ are

$$(1, 5, 15, 27, 0), (1, 5, 15, 27, 19), (1, 5, 15, 27, 20), (1, 5, 15, 27, 21), (1, 5, 15, 27, 24),$$

$$(1, 5, 15, 27, 25), (1, 5, 15, 27, 26), (1, 5, 15, 27, 27), (1, 5, 15, 27, 30), (1, 5, 15, 27, 36).$$

The key idea of our software to generate O-sequences is that $m = h_d(M)$ provides us with the size of a monomial set to be sampled in a given number of variables $k = h_1(M)$. Specifically, we started with an initial set of m monomials within the simplex $\{(x_1, x_2, \ldots, x_k) : \sum_i x_i = d,\ x_i \geq 0\}$, then calculated the corresponding pure O-sequence by counting the number of monomials of each degree less than or equal to d that divide one or more of the initial monomials. One approach we used to generate large numbers of O-sequences was to sample randomly within the lattice points of this simplex. Another was to perform "mutation" operations based on the idea that within the simplex, all lattice points are connected by the vectors $e_i - e_j$ of the root system A_n. We could therefore move "locally" from one pure order ideal to the next. In addition, we partially adapted a simulated annealing-type method (that is, the algorithm was

Rank/Corank	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	–
2	1	2	4	6	8	12	17	20	–	–
3	1	3	9	22	49	101	196	–	–	–
4	1	4	18	67	244	816	–	–	–	–
5	1	5	31	186	1132	–	–	–	–	–
6	1	6	51	489	–	–	–	–	–	–
7	1	7	79	–	–	–	–	–	–	–
8	1	8	–	–	–	–	–	–	–	–
9	1	–	–	–	–	–	–	–	–	–

TABLE 2.1. Number of distinct matroid h-vectors for particular rank and corank.

designed to evolve as it was iterated) to search for particular h-vectors (program labeled `Boxy`) not found in our random sampling. `Boxy` is also quite useful for computing the O-sequence of a family of monomials given the top-degree monomials of that family. For example, by entering $[[0,0,0,5],[0,0,2,3],[1,3,0,1]]$, one can obtain the corresponding O-sequence $(1,4,7,7,6,3)$.

The data we present on the website is grouped by rank and corank. The largest groups are concentrated around rank four and corank five. We have decided not to include monomials in the cases of rank one, two, and three, and corank one and two because they are consequences of theorems presented earlier. Note that we have not listed monomials for matroids with coloops: a matroid having j coloops has an h-vector with j zeros at the end, and the non-zero entries are equivalent to the h-vector of the same matroid with all coloops contracted. Since this new matroid also has a ground set of at most nine elements, a family of monomials has been provided for it elsewhere in the table, or the matroid satisfies the conditions of one of the proven cases. The total number distinct matroid h-vectors (including h-vectors corresponding to matroids with coloops) and the total number of matroids per rank and corank are listed in Tables 2.1 and 2.2. When the rank plus corank is greater than nine, we have no information on the quantities of matroids or distinct h-vectors, and have indicated this with the symbol '–.'

2.5. Further questions and directions

Though the results we present above cannot be extended directly to higher rank and corank, we feel that a geometric viewpoint is valuable in verifying Stanley's conjecture for further classes of matroids, and propose studying more carefully how the geometry of the independence complexes or their duals affects their combinatorics for these cases. In addition, as mentioned

Rank/Corank	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	1	1	1	1	1	1	1	1
1	9	8	7	6	5	4	3	2	1	–
2	8	14	24	30	40	42	42	29	–	–
3	7	18	45	100	210	434	950	–	–	–
4	6	20	72	255	1664	189274	–	–	–	–
5	5	20	93	576	189889	–	–	–	–	–
6	4	18	102	1217	–	–	–	–	–	–
7	3	14	79	–	–	–	–	–	–	–
8	2	8	–	–	–	–	–	–	–	–
9	1	–	–	–	–	–	–	–	–	–

TABLE 2.2. Total number of matroids, for particular rank and corank.

in the introduction, matroid complexes are one family that satisfies the definition of a PS-ear decomposable simplicial complex (see [**Cha97**]). A PS-ear decomposable simplicial complex is not only conducive to arguments by induction, by definition it satisfies the operations of deletion and contraction. Can the extra structure given by a PS-ear decomposition enable us to attack Stanley's conjecture in a new and effective way?

A further class – also a matroid construction – on which to study the problem of characterizing f- and h-vectors (or individually, necessary and sufficient conditions) is *matroid polytopes.*

DEFINITION 2.5.1. *Let M be a matroid on n elements. Given a basis $B \subseteq \{1, \ldots, n\}$ of M, the* indicator vector *of B is*

$$\mathbf{e}_B := \sum_{i \in B} \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the standard i^{th} unit vector in $\mathbb{R}^n$. Then, the matroid polytope *P_M is the convex hull of the set of indicator vectors of the bases of M.*

For more on matroid polytopes, see [**Zie95**]. There has been extensive work already on the f-vectors of convex polytopes (see for instance [**BL93, KK95**]), and some of these results have been extended to matroid polytopes. In general, exploring f-vectors and various related parameters, such as fatness and complexity [**Zie02**], is difficult because of the lack of examples; matroid polytopes however may be constructed inductively, and several algorithmic methods exist for their construction [**BBG09**].

CHAPTER 3

Polytopal Embeddings of Cayley Graphs

3.1. Not all Cayley graphs are polyhedral

The purpose of this section is to prove the following theorem, and more generally, to show that not all Cayley graphs can be embedded as the graphs of convex polytopes.

THEOREM 3.1.1. *The Cayley graph of a minimal presentation of the quaternion group cannot be embedded as the graph of a convex polytope of any dimension.*

To prove this theorem, we study in detail the possible presentations of the quaternion group, denoted Q_8. There are many presentations of Q_8, however, the only *minimal* presentations are those of the form:

$$\Lambda = \langle x, y \mid x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle,$$

where, if $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and $\{\pm i, \pm j, \pm k\}$ are the elements of order four, $x, y \in \{\pm i, \pm j, \pm k\}$, and $x \neq -y$. (Since any non-inverse pair in $\{\pm i, \pm j, \pm k\}$ generates the entire group, and we need at least two elements to generate the group, adding further generators would create redundancies.) All presentations of this type have the same Cayley graph, but for concreteness we let

$$\Lambda = \langle i, j \mid i^4 = 1, i^2 = j^2, j^{-1}ij = i^{-1} \rangle.$$

Figure 3.1 gives the Cayley color graph $C(Q_8, \Lambda)$ of this presentation. This graph is non-planar [**Mas96**], but has a toroidal embedding, as seen in Figure 3.2. The genus of $\gamma(Q_8)$ Q_8 is thus one.

We prove Theorem 3.1.1 by contradiction.

PROOF. Suppose that $G(Q_8, \Lambda)$ is the graph of some convex polytope P. We notice two things:

(1) $G(Q_8, \Lambda)$ is 4-connected, thus $\dim(P) \leq 4$ (see [**Bal61**]). Further, as $G(Q_8, \Lambda)$ is non-planar, $\dim(P) > 3$. We see that $\dim(P) = 4$.

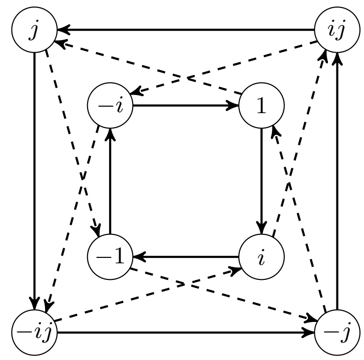


FIGURE 3.1. The Cayley color graph $C(Q_8, \Lambda)$. Dashed lines represent multiplication on the right by j, and solid lines represent multiplication on the right by i.

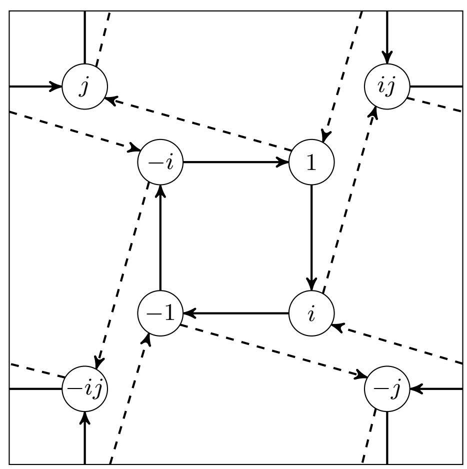


FIGURE 3.2. The Cayley color graph $C(Q_8, \Lambda)$, embedded on a torus.

(2) Every vertex of $G(Q_8, \Lambda)$ is of the same degree, thus P is simple. Blind and Mani [**BML87**] show that if P is a simple polytope, then its graph $G(P)$ determines the entire combinatorial structure of P.

Kalai [**Kal88**] gave a simpler construction for a simple polytope P given its graph $G(P)$. Later, Joswig [**Jos00**] generalized Kalai's methods to non-simple polytopes. We will show it is impossible to complete this construction for $G(Q_8, \Lambda)$, henceforth abbreviated as G. As in Kalai's construction, we consider the set of all acyclic orientations of G in order to find the "good" ones. Good acyclic orientations are given as follows. Let O be an acyclic orientation of G, and h_k^O be the number of vertices with in-degree k with respect to O. Define

$$f^O := h_0^O + 2h_1^O + 4h_2^O + 8h_3^O + 16h_4^O.$$

For all orientations, $f^O \geq f$, the number of faces of P. An orientation O is good if and only if $f^O = f$. Of course, we do not know what f *is*, but if G is indeed the graph of a simple polytope, a good orientation must exist. Therefore, we must find the minimum f^O among all acyclic orientations O of G. In particular, we will start with a well-chosen orientation O and corresponding f^O, and show that we cannot do better. To this end, consider the orientation of G given in Figure 3.3.

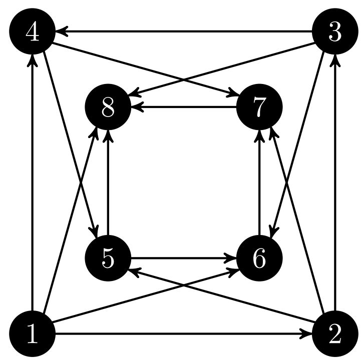


FIGURE 3.3. An orientation of $G(Q_8, \Lambda)$, given by the natural ordering of the vertex labels.

We have the in-degrees and corresponding vertices from the ordering in Figure 3.3 in the following chart:

In-Degree	**Vertices**
0	1
1	2,3
2	4,5
3	6,7
4	8

And for this orientation, we have:

$$f^O = 1 \cdot 1 + 2 \cdot 2 + 4 \cdot 2 + 8 \cdot 2 + 16 \cdot 1 = 45.$$

To see that this is the smallest possible f^O, first note that we have the equalities

$$0 \cdot h_0^O + 1 \cdot h_1^O + 2 \cdot h_2^O + 3 \cdot h_3^O + 4 \cdot h_4^O = 16, \text{ and} \tag{3.1}$$

$$h_0^O + h_1^O + h_2^O + h_3^O + h_4^O = 8. \tag{3.2}$$

Moreover, any good orientation O has $f^O \leq 45$, as good orientations have minimal values for f^O. Further, we must have a largest vertex with respect to any total ordering, thus $h_4^O \geq 1$. Assume for now that $h_4^O = 1$. Then, we are able to derive the system consisting of

$$h_0^O + h_1^O + h_2^O + h_3^O = 7 \text{ and} \tag{3.3}$$

$$h_1^O + 2h_2^O + 3h_3^O = 12. \tag{3.4}$$

From these equations and the fact that $f^O \leq 45$, we have the inequality

$$h_2^O + 4h_3^O \leq 10. \tag{3.5}$$

Note also that $h_0^O \geq 1$ and that the in-degree of the vertex labeled 2 is at most one. Thus,

$$\begin{aligned}(h_2^O, h_3^O) \in \{&(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),\\ &(2,2),(3,0),(3,1),(4,0),(4,1),(5,0),(5,1)\}.\end{aligned}$$

All cases except $(h_2^O, h_3^O) = (2,2)$ and $(h_2^O, h_3^O) = (4,1)$ may be eliminated based on equalities (3.3) and (3.4). The vector$(h_2^O, h_3^O) = (2,2)$ corresponds to the case given by the ordering in Figure 3.3. The vector $(h_2^O, h_3^O) = (4,1)$ means that we have $h_0^O = h_1^O = h_3^O = h_4^O = 1$, and $h_2^O = 4$. In this case, the vertices labeled 1 and 2 must be adjacent (if not, $h_0^O > 1$). Further, we must have the edges $(1,3)$ and $(2,3)$, as the in-degree of vertex 3 is at most two, and $h_0^O = h_1^O = 1$. However, this indicates that we have a cycle of length three, when there are none in G, a contradiction. Therefore if $h_4^O = 1$, the minimal f^O is given by $(h_0^O, h_1^O, h_2^O, h_3^O, h_4^O) = (1,2,2,2,1)$.

Now, suppose that $h_4^O = 2$. We need not address the case $h_4^O = 3$, as such an orientation would have $f^O \geq 48$. Simple algebra with

$$h_0^O + 2h_1^O + 4h_2^O + 8h_3^O + 16 \cdot 2 \leq 45, \tag{3.6}$$

$$h_1^O + 2h_2^O + 3h_3^O + 4 \cdot 2 = 16, \text{ and} \tag{3.7}$$

$$h_0^O + h_1^O + h_2^O + h_3^O + 2 = 8 \tag{3.8}$$

shows that $h_2^O + 4h_3^O \leq -1$, a contradiction. Therefore, if G is the graph of a simple polytope P, the orientation pictured above is a good one, and the number of faces of P is 45.

The next question is to find the faces. We have the following from Kalai's method to recover the polytope from its graph:

LEMMA 3.1.2. *An induced, connected, k-regular subgraph H of G is the graph of a k-face of $P(G)$ if and only if its vertices are initial with respect to some good acyclic orientation O of G.*

In the above, $P(G)$ refers to the polytope given by a graph G. Recall that a subgraph H of an oriented graph G is *initial* if there are no edges in G directed into H. Let f_i be the number of faces of dimension i. Then,

$$\sum_{i=0}^{4} f_i = 45,$$

and $(f_0, f_1, f_2, f_3, f_4) = (8, 16, ?, ?, 1)$, so $f_2 + f_3 = 20$. However, using the vertex labeling from above, the set of 2- and 3-faces include (but are not limited to):

2-faces	1234, 1256, 1278, 1458, 1568, 1467, 2358, 2367, 2567, 3456, 3478, 3678, 4578, 5678...
3-faces	123456, 123478, 123458, 123467, 125678, 145678, 235678, 345678...

This is not an exhaustive list of all faces of dimensions two and three, and it already includes more than 20 faces. We see that G cannot be the graph of a polytope, and no minimal presentation of Q_8 can be embedded as the graph of a convex polytope. □

REMARK 3.1.3. *The condition of minimality is an important one. If we take the presentation of Q_8:*

$$\Lambda = \langle i, j, k, -1 | (-1)^2 = 1,\ i^2 = j^2 = k^2 = ijk = -1 \rangle,$$

then $G(Q_8, \Lambda)$ is the complete graph on eight vertices, which can be embedded as the graph of the 8-simplex.

Note that Q_8 is also the *smallest* example of a group whose Cayley graph cannot be embedded as the graph of a convex polytope of any dimension. All other groups of order less than or equal to eight are planar.

3.2. Two families of polyhedral Cayley graphs

In this section, we study two families of groups whose Cayley graphs can be embedded as the graphs of convex polytopes. In the first subsection, we study groups of symmetries of convex, regular polytopes, and consider the Coxeter complex [**Hum90**] and a related construction of Wythoff [**Wij18**]. In the second, we consider a result of Maschke [**Mas96**], and extend his work with a new proof method using graph connectivity and polyhedral techniques.

3.2.1. Groups of symmetries. In this section, we will discuss embeddings of the *reflection presentations* of the groups of symmetry of convex, regular polytopes, and our discussion will lead to a construction that gives the polytopes whose graphs are equal to the Cayley graphs of the reflection presentations of these groups. We begin with a review of the necessary background; our exposition follows that of Humphreys [**Hum90**].

Given a (real) Euclidean space V with a positive definite, symmetric, bilinear form (λ, μ), a *reflection* s_α is a linear operator on V that sends a vector $\alpha \in V$ to $-\alpha$ while fixing the hyperplane H_α orthogonal to α. We can describe the image of any vector $\lambda \in V$ under s_α by

$$s_\alpha \lambda = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha.$$

In some cases, the set of reflections generates a *finite* group, and one such family is the groups of symmetries of convex, regular polytopes. For instance, the symmetric group S_n is the group of symmetries of the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$. The reflections $s_{\alpha_{ij}}$ correspond to the vectors $\alpha_{ij} = \varepsilon_i - \varepsilon_j$, where $\varepsilon_1, \ldots, \varepsilon_n$ are the standard basis vectors of $\mathbb{R}^n$. (We can think of $s_{\alpha_{ij}}$ as a transposition that permutes the i^{th} and j^{th} vertices.)

Let W be a finite reflection group acting on the Euclidean space V. The vectors α corresponding to the reflections s_α make up a set of vectors Φ called a *root system*. More generally, a *root system* in V is a set of vectors Φ such that

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$, and

(R2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

In some cases, the root systems are *crystallographic*, meaning that they satisfy an additional requirement:

(R3) $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Typically, we start with a root system Φ and then study the group generated by the reflections, but given a group of reflections, we can also construct the root system. For crystallographic root systems, the basic strategy is as follows. First, we pick a suitable lattice L in $\mathbb{R}^n$, and define Φ to be the set of all vectors in L having one or two prescribed lengths (for a proof that this is a sufficient number of lengths, see [**Hum90**]). Then, we verify that all scalars $2\frac{(\alpha,\beta)}{(\beta,\beta)}$ are integers. It then follows that the reflections s_α with respect to $\alpha \in \Phi$ stabilize L and permute Φ as required. See [**Bou68**] for more details. We list in Figure 3.4 the convex, regular polytopes (for dimension $d > 1$), their associated symmetry groups, and the root systems that generate these groups. In Figure 3.4, ε_i is the i^{th} standard basis vector of $\mathbb{R}^n$ (where n depends on the dimension of the polytope). H_3 and H_4, below, are *not* crystallographic, and are constructed in a different way than described above, see [**Hum90, Ste99**] for specific details. In these cases, let:

$$\begin{aligned}
a &= 2\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{2}, \\
b &= a^2 = a+1, \\
\alpha_1 &= \varepsilon_1 + \varepsilon_2, \\
\alpha_2 &= \frac{1}{2}(1-a)(\varepsilon_1+\varepsilon_2+\varepsilon_3) + \frac{1}{2}(1+a)\,\varepsilon_4, \\
\alpha_3 &= \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3) + \left(a - \frac{1}{2}\right)\varepsilon_4, \\
\alpha_4 &= \frac{1}{2}a(\varepsilon_1+\varepsilon_2+\varepsilon_3) + \left(\frac{1}{2}a - 1\right)\varepsilon_4, \text{ and} \\
\mathcal{D}_4 &= \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le 4\}.
\end{aligned}$$

Note that polytopes that are polar duals of each other have the same symmetry groups. While each group of symmetry is completely determined by its root system, we notice that the root systems are also extremely large and can be difficult to work with. Fortunately, every root system Φ has a subset Π called a *simple system*.

Polytope	**Group of Symmetry**	**Root System**
m-gon	D_m (or $I_2(m)$)	$\left\{(\cos\left(\frac{k\pi}{m}\right), \sin\left(\frac{k\pi}{m}\right)) : 0 \le k \le 2m\right\}$
3-simplex	S_4 (or A_3)	$\{\varepsilon_i - \varepsilon_j : 1 \le i \ne j \le 4\}$
cube/octahedron	B_3	$\{\pm\,\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le 3\} \cup$ $\{\pm\,\varepsilon_i : 1 \le i \le 3\}$
dodecahedron/icosahedron	H_3	$\{\varepsilon_i : 1 \le i \le 3\} \cup$ $\{\pm\,\varepsilon_1 \pm a\,\varepsilon_2 \pm b\,\varepsilon_3\} \cup$ $\{\pm\,\varepsilon_2 \pm a\,\varepsilon_3 \pm b\,\varepsilon_1\} \cup$ $\{\pm\,\varepsilon_3 \pm a\,\varepsilon_1 \pm b\,\varepsilon_2\}$
4-simplex	S_5 (or A_4)	$\{\varepsilon_i - \varepsilon_j : 1 \le i \ne j \le 5\}$
4-hypercube/4-cross-polytope	B_4	$\{\pm\,\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le 4\} \cup$ $\{\pm\,\varepsilon_i : 1 \le i \le 4\}$
24-cell	F_4	$\{\pm\,\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le 4\} \cup$ $\{\pm\,\varepsilon_i : 1 \le i \le 4\} \cup$ $\left\{\frac{1}{2}(\pm\,\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\right\}$
120-cell/600-cell	H_4	$\mathcal{D}_4 \cup D_4\alpha_2 \cup D_4\alpha_3 \cup D_4\alpha_4$
n-simplex	S_{n+1} (or A_n)	$\{\varepsilon_i - \varepsilon_j : 1 \le i \ne j \le n+1\}$
n-cube/n-orthoplex	B_n	$\{\pm\,\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\} \cup$ $\{\pm\,\varepsilon_i : 1 \le i \le n\}$

FIGURE 3.4. The groups of symmetry of regular, convex polytopes and their associated root systems. In the above, the Weyl group D_4 acts on α_i as an index-two subgroup of the group of all signed permutations of the coordinates. Se [**Ste99**].

DEFINITION 3.2.1. *A subset* Π *of a root system* Φ *is a* simple system *if* Π *is a vector space basis for the* $\mathbb{R}$*-span of* Φ *in* V *and if each* $\alpha \in \Phi$ *is a linear combination of* Π *with coefficients all of the same sign, with respect to a total ordering on the real vector space* V*. The elements of* Π *are* simple roots.

In other words, the simple system is enough to generate the entire root system. Moreover, given a reflection group W that is generated by a root system Φ, we have the following theorem:

THEOREM 3.2.2. [**Hum90**, Theorem 1.9] *Fix a simple system* Π *in* Φ*. Then* W *is generated by the set* $S := \{s_\alpha : \alpha \in \Pi\}$*, subject only to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$$

for all $\alpha, \beta \in \Pi$*, where* $m(\alpha,\beta)$ *denotes the order of the product in* W*.*

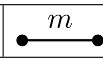
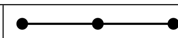
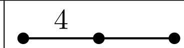
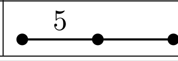
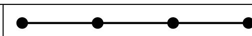
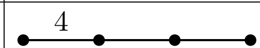
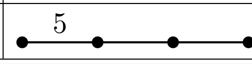
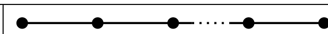
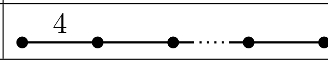
Polytope	**Group of Symmetry**	**Diagram**
m-gon	D_m (or $I_2(m)$)	
3-simplex	S_4 (or A_3)	
cube/octahedron	BC_3	
dodecahedron/icosahedron	H_3	
4-simplex	S_5 (or A_4)	
4-hypercube/4-cross-polytope	BC_4	
24-cell	F_4	
120-cell/600-cell	H_4	
n-simplex	S_{n+1} (or A_n)	
n-cube/n-orthoplex	BC_n	

FIGURE 3.5. The groups of symmetry of regular, convex polytopes and their associated reflection presentations.

Note that the $m(\alpha, \beta)$ can be determined from the inner products (α, α), (β, β), and (α, β). When we speak of a *reflection presentation* of a finite reflection group W, we mean the presentation given by a simple system of the root system of W. Often, we depict this presentation with a *Coxeter diagram*, a collection of nodes (corresponding to the $\alpha \in \Pi$) and labeled edges between the nodes. The edges correspond to the $m(\alpha, \beta)$ in the following way. If $m(\alpha, \beta) = 2$, then there is no edge between node α and node β. If $m(\alpha, \beta) = 3$, there is an *unlabeled* edge between α and β. If $m(\alpha, \beta) \geq 4$, then there is an edge between α and β, labeled with $m(\alpha, \beta)$. We give the polytopes, groups of symmetry, and associated Coxeter diagrams in Figure 3.5.

EXAMPLE 3.2.3. *A simple system for S_4 is made up of the vectors*

$$\begin{aligned} \sigma_1 &= \varepsilon_1 - \varepsilon_2, \\ \sigma_2 &= \varepsilon_2 - \varepsilon_3, \ \textit{and} \\ \sigma_3 &= \varepsilon_3 - \varepsilon_4 \,. \end{aligned}$$

Then, the corresponding reflection presentation for S_4 is

$$\Pi = \langle \sigma_1, \sigma_2, \sigma_3 \mid (\sigma_1\sigma_2)^3 = (\sigma_2\sigma_3)^3 = (\sigma_1\sigma_3)^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \rangle. \tag{3.9}$$

See Figure 3.8 for the Cayley color graph, $C(S_4, \Pi)$.

An important geometric notion associated with each finite reflection group W is its *fundamental domain.*

DEFINITION 3.2.4. *Let W be a finite reflection group in a vector space V with a total ordering, and let Π be a simple system of W with respect to this ordering. Then, the* fundamental domain D *is given by*

$$D := \{\lambda \in V : (\lambda, \alpha) \geq 0 \textit{ for all } \alpha \in \Pi\}.$$

D is a closed, convex cone, and it gets its name from the fact that it represents the fundamental domain for the action of W on V — that is, each vector $\lambda \in V$ is conjugate under W to precisely one point in D (for a proof of this, see [**Hum90**, Theorem 1.12]). We can partition D into faces in the following way:

$$C_I := \{\lambda \in D : (\lambda, \alpha) = 0 \text{ for all } \alpha \in \Pi_I,\ (\lambda, \alpha) > 0 \text{ for all } \alpha \in \Pi \setminus \Pi_I\},$$

where S is the set of simple reflections of Δ, and I is any subset of S (including the empty set). Define also

$$\mathcal{C} := \{wC_I : I \subseteq S \text{ and } w \in W\}.$$

If Π spans V, then $\mathcal{C}$ partitions V, and we call $\mathcal{C}$ the *Coxeter complex* of W. For more on the Coxeter complex, see [**Hum90**]. In addition, when we intersect the elements of $\mathcal{C}$ with the unit sphere in V, we have a simplicial decomposition of the sphere; we denote $S^n \cap \mathcal{C}$ as $\mathcal{W}$. In the cases of groups of symmetries W of convex, regular polytopes, Π spans $\mathbb{R}^n$, and we have $\mathcal{C}$ and $\mathcal{W}$ defined for all the groups we are considering.

The process of partitioning the sphere with reflections of the fundamental domain is also known as the *Wythoff construction* (see [**Cox73, DDSS08, Max89, MP95, Wij18**]). Notice that each facet of $\mathcal{C}$ corresponds to an element $w \in W$, and that two facets share a ridge if and only if they differ by a simple reflection — that is, one of the generators of W. It is then clear that the polar dual of $\mathcal{W}$ has 1-skeleton equivalent to the Cayley graph of the reflection presentation of the group of symmetries. We therefore have an embedding of $G(W, \Pi)$ as the graph of a convex n-polytope.

EXAMPLE 3.2.5. *As an example, we consider the case of S_4. The fundamental domain of S_4 (intersected with the 2-sphere), with respect to the presentation in* (3.9)*, is shown in Figure 3.6. Figure 3.7 then shows $\mathcal{W}$ for the case of S_4 with presentation* (3.9)*. Taking the polar dual* *of this complex gives a convex, simple polytope with* $G(S_4, \Pi)$ *as its graph, drawn in the plane in Figure 3.8.*

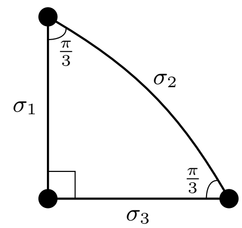


FIGURE 3.6. The fundamental domain of S_4 (intersected with the 2-sphere).

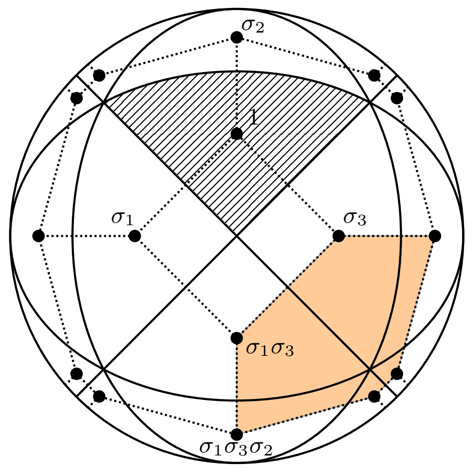


FIGURE 3.7. The polytope constructed using the Wythoff construction with the fundamental domain of S_4. The fundamental domain is shaded with lines, and several region labels are included. A face of the resulting polytope is shaded, matching that in Figure 3.8.

3.2.2. Cayley graphs of 3-dimensional convex polytopes. In 1896, Maschke [**Mas96**] provided a classification of all possible finite planar Cayley graphs, i.e., those that can be drawn on the plane without non-vertex intersections of the edges. Maschke's list relates finite planar groups with subgroups of symmetries of the Platonic solids. In this section, we make use of results of Steinitz [**Ste22**] and Mani [**Man71**] to reconsider the classification question for planar graphs, and provide an alternative method and an extension to Maschke's proof.

Intuitively, for each planar Cayley graph we will show there exists a "rigid" 3-dimensional convex polytopal model with the property that the symmetries of this model realize the group of automorphisms of the associated Cayley graph. In [**Ste22**], Steinitz proved that a graph is the 1-skeleton of a polyhedron if and only if it is 3-connected and planar (for a nice proof see

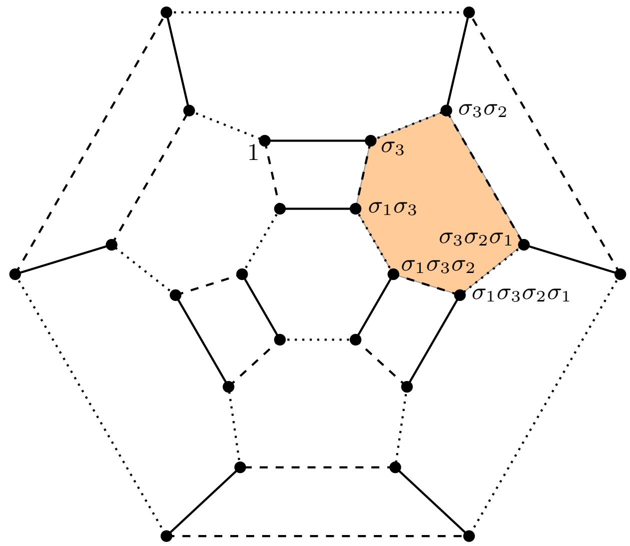


FIGURE 3.8. The Cayley color graph of S_4 with the presentation in (3.9). Solid lines represent multiplication by σ_1, dotted lines represent multiplication by σ_2, and dashed lines represent multiplication by σ_3.

[**Grü03**]). A result of Mani [**Man71**] extends Steinitz's theorem in a convenient way: every 3-connected, planar graph G is the 1-skeleton of a polyhedron P such that every automorphism of G is induced by a symmetry of P. Our main goal is to show that if a Cayley graph of the group Γ embeds in the 2-dimensional sphere, then it acts on the sphere by isometries.

We first give some definitions, well known from graph theory.

DEFINITION 3.2.6. *A* separator *of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that $G \setminus S$ is disconnected and has at least two non-empty subgraphs called* components. *A k*-separator *is a separator of cardinality k. A graph is k*-connected *if there exist no separators of cardinality less than k.*

DEFINITION 3.2.7. *An* automorphism *φ of a Cayley color graph $C(\Gamma, \Lambda)$ is a permutation of its vertices such that for all pairs of vertices g_1 and g_2, and generators $h \in \Gamma$, $\varphi(g_1)h = \varphi(g_2)$ if and only if $g_1 h = g_2$.*

It is well known that the group of automorphisms of a Cayley color graph corresponding to a group Γ is isomorphic to Γ. A Cayley graph has Γ as a subgroup of its automorphism group, and Γ acts transitively on the vertices of the graph. We will show that any planar Cayley graph is either 3-connected or a cycle; we make use of the transitive action of Γ to prove this. First, we recall the notion of a *quadrant*.

DEFINITION 3.2.8. *A graph G is* vertex-transitive *if for any two vertices v_1 and v_2 there exists an automorphism φ of G such that $\varphi(v_1) = \varphi(v_2)$.*

DEFINITION 3.2.9. *Let A and B be two separators of the graph G. Suppose A separates G into the components A_1 and A_2, and B separates G into B_1 and B_2. The* quadrant Q_{ij} *is given:*

$$Q_{ij} := (A_i \cap B) \cup (B_j \cap A) \cup (A \cap B).$$

For a visualization of this concept, we refer to Figure 3.9. These ideas, including the following remark, go back to Neumann–Lara [**NL89**].

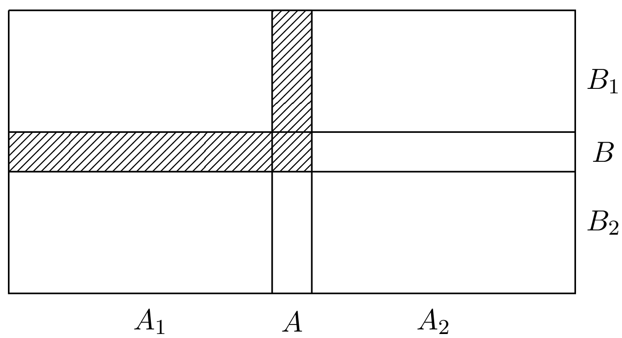


FIGURE 3.9. The graph G, represented by the largest rectangle, with A and B drawn as orthogonal strips. Q_{11} has been shaded.

REMARK 3.2.10. *Let A and B be two separators of the same connected graph G. If $A_i \cap B_j$ is non-empty then the quadrant Q_{ij} is a separator.*

PROOF. The proof is simple. Assume without loss of generality that $i = j = 1$. Further, assume $A_1 \cap B_1$ is non-empty (see Figure 3.9). Then there are no edges connecting $A_1 \cap B_1$ and B_2 because B is a separator. Similarly, there are no edges connecting $A_1 \cap B_1$ and A_2. Thus, Q_{11} separates $A_1 \cap B_1$ and $A_2 \cup B_2$. □

Now we are able to prove the following result:

PROPOSITION 3.2.11. *Let G be a connected, vertex-transitive graph with minimum degree at least two. Then G is a cycle or a 3-connected graph.*

PROOF. First, note that G must be at least 2-connected, as it is vertex-transitive with minimum degree at least two. Second, we assume that $G \neq K_3$ (the proposition holds in this case, but there are not enough vertices for our argument). Suppose G is not 3-connected. Let $A = \{x, y\}$ be a 2-separator with components A_1 and A_2 such that A_1 is minimal among all possible components of 2-separators in G. Since G is vertex-transitive, there exists an automorphism f such that $f(x)$ belongs to A_1. The image set of A under f, $f(A) = B$, is another separator of the graph. We claim that the elements $x, y, f(x)$, and $f(y)$ are arranged as in Figure 3.10 (up to swapping x and y).

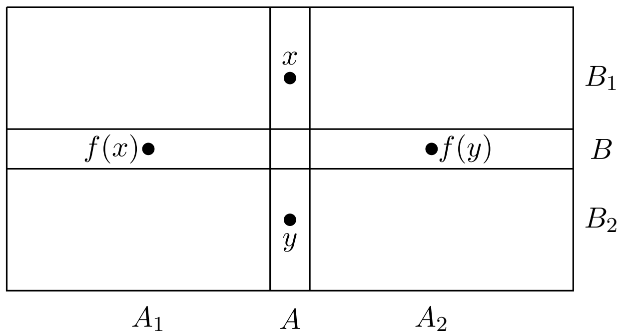


FIGURE 3.10

To show this, we must consider seven cases. First, suppose $x \in B$. Then, $y \notin B$; otherwise, we have that $f(A) = B = \{f(x), f(y)\} = \{x, y\}$, or $A = B$, which contradicts the fact that $f(x) \in A_1$. If $y \notin B$, then $y \in B_1$ or B_2. Without loss of generality we suppose $y \in B_2$. In this case $f(y) = x$ and we have the situation pictured in Figure 3.11.

Then, it must be that $A_2 \cap B_1 = \emptyset$; otherwise, by the remark above, $x = f(y)$ would be a 1-separator (impossible as G is at least 2-connected). Further, $A_1 \cap B_1 = \emptyset$, else $\{x, f(x)\}$ would separate G into two components, one of which would be a strict subset of A_1, a contradiction.

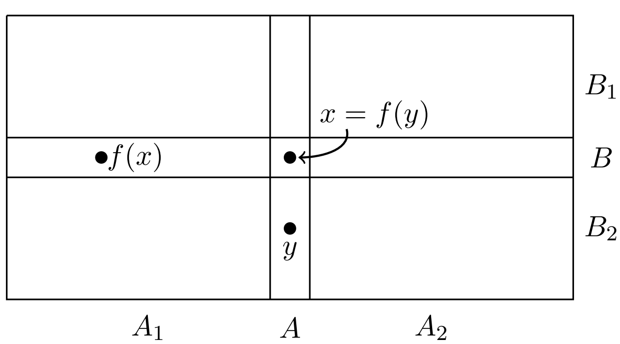


FIGURE 3.11

However, then $B_1 = \emptyset$, contradicting the fact that B is a separator. Therefore, x must be in B_1 or B_2.

Now, we claim that x and y are separated by B. Suppose not. Then, we have one of the situations pictured in Figures 3.12, 3.13, and 3.14.

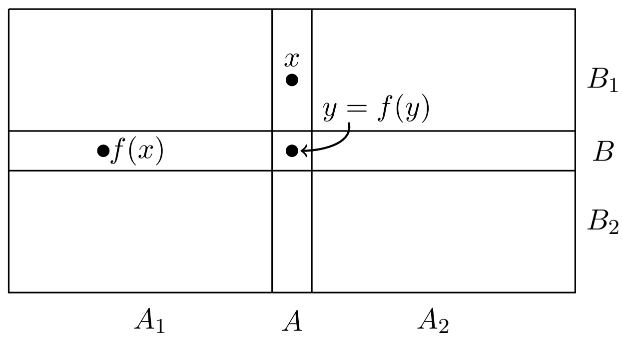


FIGURE 3.12

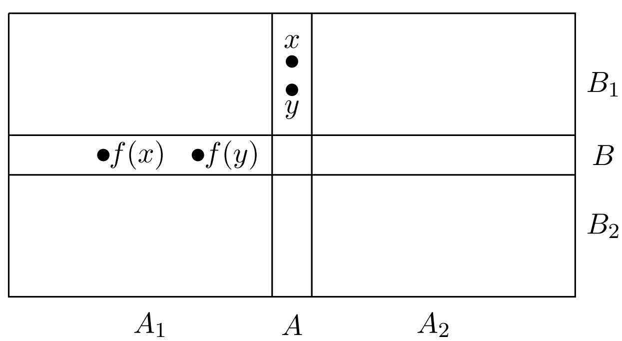


FIGURE 3.13

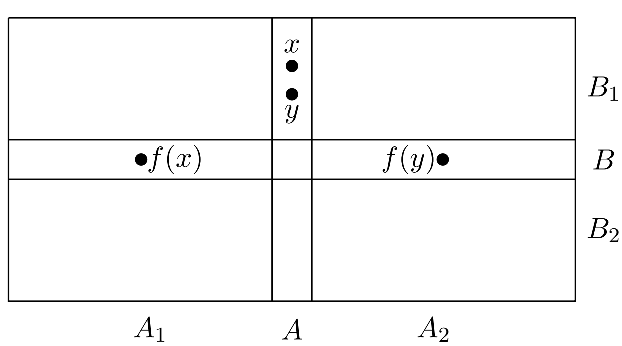


FIGURE 3.14

In Figure 3.12, we see that $A_2 \cap B_2$ must be empty (otherwise, y would be a 1-separator), and $A_1 \cap B_2$ must be empty (otherwise, we would have a separator with a component strictly smaller than A_1). This implies that B is not a separator, a contradiction. In Figures 3.13 and 3.14, similar arguments may be used. Therefore, x and y must be separated by B. Finally, we claim that $f(y)$ is a point in A_2. Suppose that $f(y) \in A_1$, as shown in Figure 3.15.

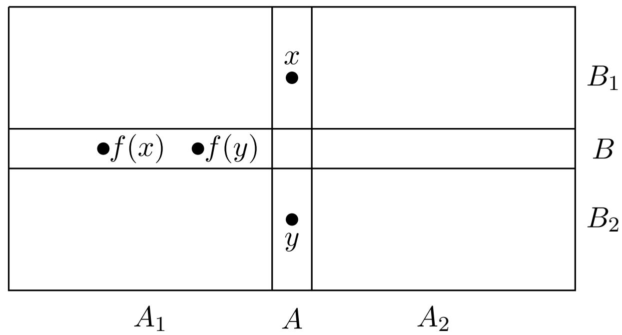


FIGURE 3.15

It follows from our remark that $A_2 \cap B_2$ and $A_2 \cap B_1$ are empty. Otherwise the quadrants Q_{21} and Q_{22} would be separators of G, but these quadrants have only one point, thus this point is a 1-separator of the graph. Thus, A_2 is empty, contradicting the fact that A is a separator. Further, $f(y)$ is not in A, otherwise $f(x) = x$ or $f(x) = y$, contradicting the fact that B separates x and y. Therefore, $f(y) \in A_2$, and we must have the arrangement in Figure 3.10.

In this case, $A_1 \cap B_1$ and $A_1 \cap B_2$ must be empty. Otherwise, the quadrants Q_{11} and Q_{12} would be separators, but removing either quadrant would leave a component with fewer vertices than A_1. Therefore, $f(x)$ is the only vertex in A_1, and it must be adjacent to both x and y, as every vertex has degree at least two. $f(x)$ is thus a vertex of degree exactly two, which implies that G is regular of degree two. Since G is connected, G must be a cycle. Finally, Steinitz's theorem says that a planar graph is the graph of a 3-polytope if and only if it is 3-connected. Proposition 3.2.11 then follows. □

Now we can prove the main theorem of this section.

THEOREM 3.2.12. *Let Γ be a finite group with a minimal set of generators and relations Λ. The associated Cayley graph is the graph of a 3-dimensional polytope if and only if Γ is a finite group of isometries in 3-dimensional space.*

Proof. $G(\Gamma, \Lambda)$ satisfies the hypotheses of Proposition 3.2.11. If $G(\Gamma, \Lambda)$ is a cycle, then Γ is a dihedral group or a cyclic group. Both groups have an isometric action on the sphere as symmetries of an n-dihedron. In the case that $G(\Gamma, \Lambda)$ is 3-connected, Mani's result [**Man71**] gives the required action. See [**Tuc83**] for a different view of the theorem. □

3.3. Further questions and directions

We have found one example of a Cayley graph that does not appear as the graph of a convex d-polytope. It is therefore natural to ask for further examples of groups whose Cayley graphs do not embed in this way.

Question 3.3.1. *Are there (infinite) families of groups whose minimal presentations cannot be embedded as the graphs of convex d-polytopes?*

One possible family is the set of *generalized quaternion groups*, of which Q_8 is the smallest member. More generally, we propose to explore group-theoretic characterizations of non-embeddability.

Question 3.3.2. *Can we use group theory to characterize the embeddability of Cayley graphs? Can we characterize subgroups that in some sense "block" the embedding of the Cayley graphs? Or can we show that there exist no such subgroups?*

We can also approach the problem of embedding Cayley graphs from a graph-theoretic standpoint.

Question 3.3.3. *Are there forbidden minor characterizations for the embeddability of Cayley graphs, as in the case of Kuratowski's theorem (Theorem 1.0.1)?*

Cayley graphs have many special properties that could be used in answering this question. For example, Cayley graphs are vertex-transitive, and could thus only be the graphs of simple polytopes. Studying these questions is additionally of interest for the purpose of better understanding the 1-skeletons of convex polytopes.

The Coxeter complex and the Wythoff construction described above are further sources of questions.

Question 3.3.4. *Can we design other constructions that give d-polytopes with graphs equal to Cayley graphs?*

CHAPTER 4

Flows on Simplicial Complexes

4.1. Structure of the boundary matrices of simplicial complexes

To begin, we examine the structure of the boundary matrix $\partial\Delta$ of a simplicial complex Δ, given a certain ordering of the rows and columns. Before stating our first lemma, we recall a few definitions.

DEFINITION 4.1.1. *Let Δ be a simplicial complex, and let F be a face of Δ. Then, the* link *of F in Δ, denoted $\mathrm{lk}_\Delta(F)$, is given by*

$$\mathrm{lk}_\Delta(F) := \{G \in \Delta : G \cap F = \emptyset,\ G \cup F \in \Delta\}.$$

DEFINITION 4.1.2. *Let Δ be a simplicial complex, and let F be a face of Δ. Then, the* deletion *of F from Δ, denoted $\Delta - F$, is given by*

$$\Delta - F := \{G \in \Delta : G \cap F = \emptyset\}.$$

DEFINITION 4.1.3. *A* simplicial cone *is a pure simplicial complex Δ with a vertex v such that v is contained in every facet of Δ.*

Note that the class of simplicial complexes is closed under the operations of taking the link of a face and deleting a face. We can now state the following.

LEMMA 4.1.4. *Let Δ be a simplicial complex of dimension d that is* not *a cone, with ordering $v_1 < v_2 < \cdots < v_n$ on the vertices. Arrange the rows (indexed by the ridges of Δ) and columns (indexed by the facets of Δ) of $\partial\Delta$ in the following way:*

(1) Let all ridges containing the vertex v_n come first, and order them lexicographically, according to the ordering on the vertices of Δ.
(2) Let all ridges in $\mathrm{lk}_\Delta(v_n)$ be next, ordered lexicographically.
(3) Order the remaining ridges lexicographically.
(4) Let all facets containing v_n come first, and order them lexicographically.

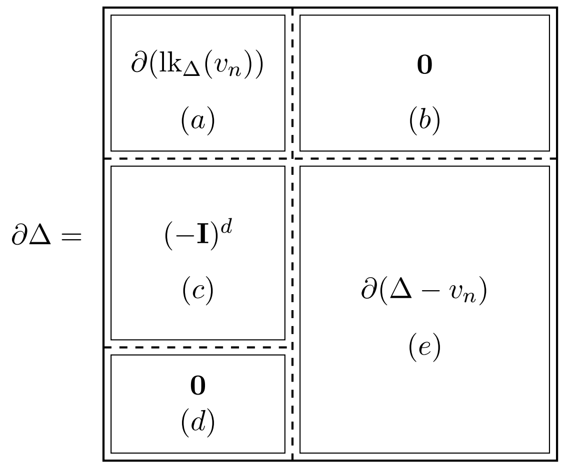


FIGURE 4.1. The matrix structure described in Lemma 4.1.4.

(5) Order the remaining facets (those that do not contain v_n) lexicographically.

Under these conditions, the boundary matrix takes the form given by Figure 4.1.

PROOF. We will verify the matrix blocks individually.

(a) This submatrix is equivalent to the boundary matrix of $\mathrm{lk}_\Delta(v_n)$. Intuitively, the facets of the link of a vertex are the faces $F \setminus v_n$, where F is a facet of Δ containing v_n. Similarly, the ridges of $\mathrm{lk}_\Delta(v_n)$ are the ridges $R \setminus v_n$, where R is a ridge of Δ containing v_n. These ridges R and facets F are precisely the rows and columns of (a). Then, since we are removing the lexicographically largest element from each ridge and facet, we do not affect the signs of ridges in the boundary map, and the entries of this submatrix are thus identical to those of $\partial(\mathrm{lk}_\Delta(v_n))$.

(b) This block contains only zeros as the rows all contain v_n, but the columns do not.

(c) This block is $(-\mathbf{I})^d$, where d is the dimension of Δ. The ridges of this region are precisely the facets of this region with v_n removed, and we have ordered them both lexicographically. The sign depends on the dimension of Δ as we are removing the d^{th} element, so the sign of each ridge is $(-1)^d$.

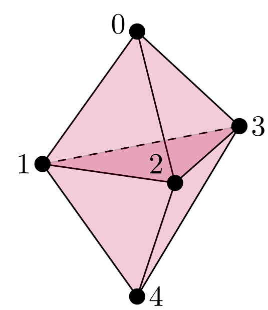
(d) This block is $\mathbf{0}$ because we have already listed all ridges that contain v_n or are in the link of v_n. Therefore, these ridges cannot be contained in any facet containing v_n. We can also think of this block as containing the ridges that have a vertex v_i that is parallel to v_n, so there exists no facet containing v_i and v_n.

(e) This block corresponds to the boundary matrix of the deletion of v_n in Δ, $\partial(\Delta - v_n)$. The facets of $\Delta - v_n$ are the facets of Δ that do not contain v_n, and the ridges of $\Delta - v_n$ are the ridges of Δ that do not contain v_n. This corresponds precisely to the rows and columns of (e), and since we do not affect the parity of the vertices in the facets, the signs remain the same. □

See Figure 4.2 for an example of Lemma 4.1.4.

	124	**134**	**234**	**012**	**013**	**023**	**123**
14	−1	−1	0	0	0	0	0
24	1	0	−1	0	0	0	0
34	0	1	1	0	0	0	0
12	1	0	0	0	1	0	−1
13	0	1	0	0	1	0	−1
23	0	0	1	0	0	1	1
01	0	0	0	1	1	0	0
02	0	0	0	−1	0	1	0
03	0	0	0	0	−1	−1	0

(A) Boundary matrix of 4.2b, with regions (a) – (e) blocked off.



(B) A triangular bipyramid on five vertices.

FIGURE 4.2

4.1.1. Flows on simplicial cones. If Δ is a cone over a vertex v, then we have a variation of Lemma 4.1.4. Note that if Δ is a cone over v, then every ridge either contains v, or is in the link of v. Moreover, by definition, there are no facets that do not contain v.

LEMMA 4.1.5. *Let Δ be a pure simplicial complex that is a cone over a vertex v, and order the vertices $v_0, v_1, \ldots, v_n$ so that $v = v_n$ is the largest. Arrange the rows (indexed by ridges of Δ) and columns (indexed by facets of Δ) of $\partial\Delta$ in the following way:*

(1) Let all ridges containing the vertex v_n come first, and order them lexicographically.

(2) Order the remaining ridges lexicographically (these are all the ridges in $\mathrm{lk}_\Delta(v_n)$).

(3) Order the facets lexicographically (all contain v_n).

Under these conditions, the boundary matrix takes the form given by Figure 4.3.

PROOF. Let Δ be a pure simplicial cone over a vertex v and order the vertices $v_0, v_1, \ldots, v_n$ so that $v = v_n$ is the largest. In the boundary matrix described in the statement of the lemma, we only have two regions (there are two types of ridges and one type of facet): region (i) in

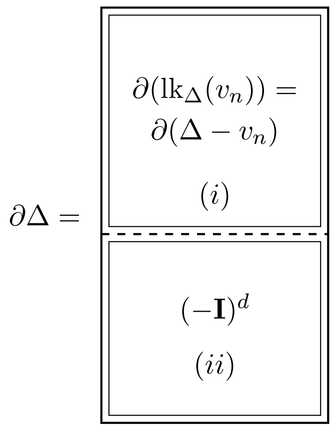


FIGURE 4.3. The boundary matrix of a simplicial cone, as described in Lemma 4.1.5.

Figure 4.3 corresponds to region (a) in Figure 4.2, and region (ii) corresponds to region (c). The proofs that region (i) is equivalent to $\partial(\mathrm{lk}_\Delta(v_n))$ and that region (ii) is $(-\mathbf{I})^d$ are identical to the proofs for regions (a) and (c); we omit them here. We need only show that $\Delta - v_n = \mathrm{lk}_\Delta(v_n)$, but this is immediate from their definitions:

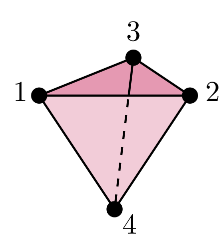
$$\begin{aligned} \Delta - v_n &= \{F \in \Delta : v_n \notin F\} \\ &= \{F \in \Delta : F \cap v_n = \emptyset\} \\ &= \{F \in \Delta : F \cap v_n = \emptyset,\ F \cup v_n \in \Delta\} \\ &= \mathrm{lk}_\Delta(v_n). \end{aligned}$$

The third equality follows because v_n is in every maximal face, thus, if $v_n \notin F \in \Delta$, then $(F \cup v_n) \in \Delta$. □

See Figure 4.4 for an example of this matrix and corresponding simplicial cone.

	124	**134**	**234**
14	−1	−1	0
24	1	0	−1
34	0	1	1
12	1	0	0
13	0	1	0
14	0	0	1

(A) Boundary matrix of 4.4b, with regions (i) and (ii) blocked off.



(B) A triangular cone on four vertices.

FIGURE 4.4

LEMMA 4.1.6. *Let Δ be a pure simplicial complex that is a cone over a vertex v. Then Δ has no nontrivial $\mathbb{Z}_q$-flows.*

PROOF. Order the vertices $v_0, v_1, \ldots, v_n$ so that $v = v_n$ is the largest. Since Δ is a cone over v_n, we have a boundary matrix of the form in Figure 4.3. Notice that $(-\mathbf{I})^d$ extends through all columns. Therefore, $\ker(\partial\Delta) \subseteq \ker((-\mathbf{I})^d) = \{\mathbf{0}\}$, and there exist only trivial flows. □

4.2. The number of nowhere-zero $\mathbb{Z}_q$-flows on a simplicial complex

To study the question of counting nowhere-zero flows, we will use matroid theory, in particular a generalization of the Tutte polynomial given in [**OW79**], called a *Tutte-Grothendieck invariant.* This generalization allows the recursive definition to include coefficients in the case that e is neither a loop nor a coloop.

DEFINITION 4.2.1. *A function on the class of matroids is a* generalized Tutte–Grothendieck invariant *if it is a function f from the class of matroids to a field $\mathbb{F}$ such that for all matroids M and N, $f(M) = f(N)$ whenever $M \cong N$, and for all $e \in E(M)$,*

$$f(M) = \begin{cases} \sigma f(M_{-e}) + \tau f(M_{/e}) & \text{if } e \text{ is neither a loop nor a coloop, and} \\ f(M(e))f(M_{-e}) & \text{otherwise,} \end{cases}$$

where $M(e)$ is the matroid consisting of the single element e, and σ and τ are nonzero elements of $\mathbb{F}$.

For more background on Tutte–Grothendieck invariants, see [**Whi92**, Chapter 6]. In the following, we will denote by C the matroid consisting of a single coloop and by L the matroid consisting of a single loop. In order to relate the invariant we find in the proof of Theorem 4.2.3 to the Tutte polynomial, we will use the following fact [**OW79**, Theorem 6.2.6]:

THEOREM 4.2.2. *Let σ and τ be nonzero elements of a field $\mathbb{F}$. Then there is a unique function t' from the class of matroids into the polynomial ring $\mathbb{F}[x, y]$ having the following properties:*

(i) *$t'_C(x, y) = x$ and $t'_L(x, y) = y$.*

(ii) *If e is an element of the matroid M and e is neither a loop nor a coloop, then*

$$t'_M(x, y) = \sigma\, t'_{M-e}(x, y) + \tau\, t'_{M/e}(x, y).$$

(iii) *If e is a loop or a coloop of the matroid M, then* $t'_M(x,y) = t'_{M(e)}(x,y)\, t'_{M-e}(x,y)$.

Furthermore, this function t' is given by $t'_M(x,y) = \sigma^{|E|-\mathrm{rk}(E)}\tau^{\mathrm{rk}(E)}\, T_M(\frac{x}{\tau}, \frac{y}{\sigma})$.

Finally, recall that any matrix may be realized as a matroid M by taking $E(M)$ to be the list of columns, and $\mathcal{I}(M)$ to be the linearly independent subsets of columns. If $y \in \mathbb{Z}_q^{|E(M)|}$, then the *support* of y is $\mathrm{supp}(y) := \{e \in E(M) : y_e \neq 0\}$. We will now prove the following:

THEOREM 4.2.3. *Let q be a sufficiently large prime number, and let Δ be a simplicial complex of dimension d. Then the number $\phi_\Delta(q)$ of nowhere-zero $\mathbb{Z}_q$-flows on Δ is a polynomial in q of degree $\beta_d(\Delta) = \dim_{\mathbb{Q}}(\widetilde{H}_d(\Delta, \mathbb{Q}))$.*

PROOF. Let Δ be a pure simplicial complex, and $\partial\Delta$ be the boundary matrix associated with Δ. Let M be the matroid given by the columns of $\partial\Delta$, and denote the ground set as E. For convenience, and by a slight abuse of notation, we will use M to denote both the matroid and the matrix representing it.

First, we claim that

$$T_M(0, 1-q) = |\{y \in \ker(M) \bmod q : \mathrm{supp}(y) = E\}|$$

for any prime q that is sufficiently large. We show this by proving that the function

$$g_M(q) := |\{y \in \ker(M) \bmod q : \mathrm{supp}(y) = E\}|$$

is a generalized Tutte-Grothendieck invariant with $\sigma = -1$ and $\tau = 1$. The matrix for the single coloop C is a single (linearly independent) vector, thus $g_C(q) = 0$. The matrix for the single loop L is the zero vector, so $g_L(q) = q - 1$. Thus $g_M(q)$ is well-defined if $|E| = 1$.

Assume g is well-defined for $|E| < n$, and let $|E| = n$. Let $e \in E$, and suppose that e is neither a loop nor a coloop. In the case of a matroid corresponding to a vector configuration (as is the case with the boundary matrix), contraction models quotients: let V be the vector space given by E (the columns of the boundary matrix), and let $\pi : V \to V/V_e$ be the canonical quotient map, where V_e is the vector space spanned by the column vector e. Then, the contracted matroid M/e is the matroid associated with the vector configuration $\{\pi(v)\}_{v \in E\setminus e}$ in the quotient space V/V_e. Let

$$W := \{y \in \ker(M) : \operatorname{supp}(y) = E\},$$
$$X := \{y \in \ker(M - e) : \operatorname{supp}(y) = E \setminus e\}, \text{ and}$$
$$Z := \{y \in \ker(M/e) : \operatorname{supp}(y) = E \setminus e\}.$$

We see that $W \cap X = \emptyset$. By linearity of π, we have that $W \cup X \subset Z$. Now, suppose we have $y \in Z$. This is a linear combination of all vectors in $E \setminus e$ that equals a scalar $\alpha \in \mathbb{Z}_q$ times e (that is, a linear combination equivalent to zero in the quotient space). If $\alpha \neq 0$, then $(y, \alpha) \in W$. If $\alpha = 0$, $y \in X$. Therefore

$$g_M(q) = g_{M/e}(q) - g_{M-e}(q)\,.$$

If e is a loop (a column of zeros), then e may be assigned any of the $q - 1$ nonzero values of $\mathbb{Z}_q$. Thus, for loops

$$\begin{aligned} g_M(q) &= (q-1)\, g_{M-e}(q) \\ &= g_L(q)\, g_{M-e}(q)\,. \end{aligned}$$

Finally, suppose e is a coloop. Then every maximally independent set of columns contains e, and we can use row operations to rewrite M so that the column corresponding to e has precisely one nonzero element, and this element is also the only nonzero element in its row. Therefore the entry of a vector $y \in \ker(M)$ corresponding to e must be 0 for all $y \in \ker(M)$, and

$$\begin{aligned} g_M(q) &= 0 \cdot g_{M-e}(q) \\ &= g_I(q)\, g_{M-e}(q)\,. \end{aligned}$$

Thus $g_M(q)$ is well-defined and a generalized Tutte–Grothendieck invariant. We showed the cases for $|E| = 1$ above, so we see that by Theorem 4.2.2

$$g_M(q) = t'_M(0, q-1) = (-1)^{|E| - \operatorname{rk}(E)} T_M(0, 1-q),$$

and so

$$T_M(0, 1-q) = |\{y \in \ker(M) \bmod q : \operatorname{supp}(y) = E\}| .$$

It follows that the number of nowhere-zero $\mathbb{Z}_q$-flows on a simplicial complex Δ is equal to $T_M(0, 1-q)$ and hence is a polynomial in q. Using the definition of the Tutte polynomial, we see that the degree of this polynomial, in terms of the matroid, is $|E| - \operatorname{rk}(M)$. From linear algebra, we know that

$$|E| = \dim(\operatorname{rk}(M)) + \dim(\ker(M)) = \operatorname{rk}(M) + \dim(\ker(M)) ,$$

so $|E| - \operatorname{rk}(M) = \dim(\ker(M))$. But $\dim(\ker(M))$ is just the dimension of the top rational homology of Δ, denoted $\dim_{\mathbb{Q}}(\widetilde{H}_d(\Delta; \mathbb{Q}))$. By definition, this is $\beta_d(\Delta)$, where $d = \dim(\Delta)$. □

REMARK 4.2.4. *We require that q be prime as otherwise V and V_e would be modules, rather than vector spaces. Requiring that q be sufficiently large ensures that the matrices reduce correctly over $\mathbb{F}_q$; that is, we require that the linear (in)dependencies of $\partial\Delta$ are the same over $\mathbb{Q}$ as they are over $\mathbb{F}_q$. For a simplicial complex of dimension d, a sufficient bound would be q greater than $(d+1)^{\frac{d+1}{2}}$, or q greater than the absolute value of the determinant of every submatrix, though these are not necessarily tight.*

4.2.1. Quasipolynomiality of $\phi_\Delta(q)$. In this section, we prove the following:

THEOREM 4.2.5. *The number $\phi_\Delta(q)$ of nowhere-zero $\mathbb{Z}_q$-flows on Δ is a quasipolynomial in q. Furthermore, there exists a polynomial $p(x)$ such that $\phi_\Delta(k) = p(k)$ for all integers k that are relatively prime to the period of $\phi_\Delta(q)$. In addition, there are examples where the periodicity of the quasipolynomial is strictly larger than one.*

PROOF. Recall that rational polytopes are sets of the form $\left\{x \in \mathbb{R}^d : Ax \leq b\right\}$ for some integral matrix A and integral vector b. Then, by definition, the flow counting function $\phi_\Delta(q)$ counts the integer lattice points $x = (x_1, x_2, \ldots, x_n)$ (where n is the number of facets of Δ) that satisfy

$$0 < x_j < q \qquad \text{and} \qquad \partial\Delta(x) = mq \ \text{ for some } \ m \in \mathbb{Z} .$$

(Note that we only need to consider finitely many m.) Therefore, to compute $\phi_\Delta(q)$, we count the lattice points in a collection of rational polytopes. By Ehrhart's theorem (Theorem 1.3.11) $\phi_\Delta(q)$ is a sum of Ehrhart quasipolynomials and thus also a quasipolynomial in q. Now, suppose that $\phi_\Delta(q)$ has period p. By Dirichlet's theorem, there exist infinitely many primes of the form $j + kp$ for $\gcd(j, p) = 1$ and $k \in \mathbb{Z}_{\geq 0}$, and thus $\phi_\Delta(j + kp)$ agrees with the polynomial found in Theorem 4.2.3 for those j with $\gcd(j, p) = 1$. □

We remark that a setup similar to the one in our proof was used in [**BS12**] to study flow polynomials of graphs from the perspective of Ehrhart theory.

Given that the number of nowhere-zero $\mathbb{Z}_q$-flows on a graph is a polynomial in q, it is natural to ask whether the same is true for all simplicial complexes. Interestingly, this is *not* always the case. Consider the case of the Klein bottle, K. We have the following top homologies for K:

$$H_2(K, \mathbb{Z}_q) = \begin{cases} 0 & \text{if } q \text{ is odd, and} \\ \mathbb{Z}_2 & \text{if } q \text{ is even.} \end{cases}$$

Therefore, the number of nowhere-zero $\mathbb{Z}_q$-flows on K is given by a quasipolynomial of period 2:

$$\phi_K(q) = \begin{cases} 0 & \text{if } q \text{ is odd, and} \\ 1 & \text{if } q \text{ is even.} \end{cases}$$

4.2.2. Calculating $\phi_\Delta(q)$: an example. Consider the triangular bipyramid Δ with vertex set $\{0, 1, 2, 3, 4\}$ and facets $\{012, 013, 023, 123, 124, 134, 234\}$. Then $\partial\Delta$ is given by Figure 4.2a. The kernel for this matrix is generated by

$$\{(1, -1, 1, 0, 0, 0, -1)^T, (0, 0, 0, -1, 1, -1, 1)^T\}.$$

Thus, any nowhere-zero $\mathbb{Z}_q$-flow will have the form $(a, -a, a, -b, b, -b, b - a)^T$ where $a, b \in \mathbb{Z}_q$ and $a \neq b$. We have $q - 1$ nonzero choices for a, and $q - 2$ nonzero choices for b. We see that $\phi_\Delta(q) = (q-1)(q-2)$. For primes $p > 1$ (one being the maximum over all absolute values of all subdeterminants of $\partial\Delta$), we may also compute $\phi_\Delta(q) = T_M(0, 1 - q)$, where M is the matroid given by ∂M, as

$$\begin{aligned} T_M(0, 1 - q) &= \sum_{S \subseteq E} (-1)^{\text{rk}(M) - \text{rk}(S)} (-q)^{|S| - \text{rk}(S)} \\ &= q^2 - 3q + 2 \\ &= (q - 1)(q - 2). \end{aligned}$$

4.3. The period of the flow quasipolynomial

Tutte's polynomiality result for the number of nowhere-zero flows on graphs also follows from the fact that the boundary matrix of every graph is totally unimodular, that is, every subdeterminant is $-1, 0$, or 1. The total unimodularity of the boundary matrix guarantees that the flow polytope has integral vertices (see for instance [**BS13**, Chapter 10.1]), which in turn implies that the period of the Ehrhart quasipolynomial is one.

With the example of the Klein bottle above, we have already established that the flow quasipolynomial does in some cases have period strictly greater than one. It is natural, then, to ask if there are subclasses of simplicial complexes which have flow quasipolynomials with period *equal* to one. In particular, we hoped to use the following result of Dey, Hirani, and Krishnamoorthy to establish total unimodularity of boundary matrices for certain subclasses of simplicial complexes.

THEOREM 4.3.1. [**DHK11**, Theorem 5.2] *For a finite simplicial complex Δ of dimension greater than $d-1$, the d-dimensional boundary matrix $\partial\Delta$ is totally unimodular if and only if $H_{d-1}(L, L_0)$ is torsion-free for all pure subcomplexes L_0, L in Δ of dimensions $d-1$ and d respectively, where $L_0 \subset L$.*

(For more information on homology and torsion, see [**Hat02**, Chapter 2].) In other words, we can rephrase our original combinatorial question as a topological question, which allows us to use the language and tools of topology to address the total unimodularity of boundary matrices. With this in mind, we examined the case of CED complexes. Interestingly, we found examples demonstrating that CED complexes do not in general have totally unimodular boundary matrices, nor do they have flow quasipolynomials with period necessarily equal to one. We give two examples. First, we take the triangulated rectangular bipyramid given in Figure 4.5.

Consider the submatrix of the boundary matrix that corresponds to facets $\{124, 126, 236, 346, 345, 145\}$ and ridges $\{12, 26, 36, 34, 45, 14\}$ of the simplicial complex in Figure 4.5, written out in Figure 4.6.

The determinant of this submatrix is two — hence the boundary matrix is *not* totally unimodular. In the language of [**DHK11**], the facets in this subcomplex correspond to a *Möbius band* embedded in the simplicial complex. There are two other items of note: first, this

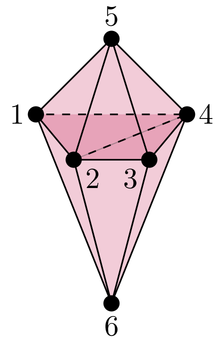


FIGURE 4.5. A triangulated rectangular bipyramid on six vertices.

	124	**126**	**236**	**346**	**345**	**145**
12	1	1	0	0	0	0
26	0	1	0	−1	0	0
36	0	0	0	1	0	−1
34	0	0	0	0	1	1
45	0	0	1	0	1	0
14	−1	0	1	0	0	0

FIGURE 4.6. The submatrix corresponding to facets $\{124, 126, 236, 346, 345, 145\}$ and ridges $\{12, 26, 36, 34, 45, 14\}$ of the simplicial complex in Figure 4.5.

complex is in fact a matroid complex, and thus PS-ear decomposable; we see that this subclass of CED complexes is also not necessarily totally unimodular. Second, the flow quasipolynomial of this simplicial complex has period equal to one ($\phi_\Delta(q) = (q-1)(q-2)$), even though $\partial\Delta$ is not totally unimodular. However, this does not always happen. For example, consider the CED complex in Figure 4.7, originally proposed by Felix Breuer. Using code written by Jeremy Martin (available at `http://www.math.ku.edu/~jmartin/sourcecode/`), we find that

$$\phi_\Delta(q) = q^3 - 7q^2 + 15q - 8 - \gcd(2, q).$$

Observe that this simplicial complex is 2-dimensional and cannot be embedded in 3-space. We will remark further upon this in Section 4.5.

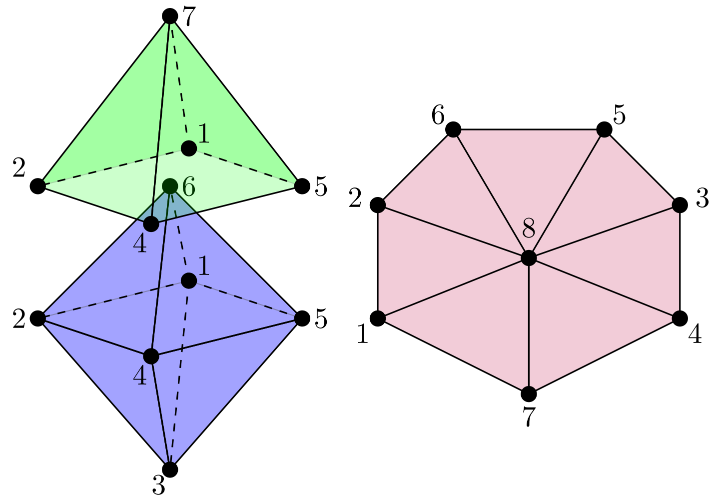


FIGURE 4.7. A CED complex that is not totally unimodular and has period equal to 2. Vertices with the same labels are identified, as are edges between two identified vertices. The boundary matrix is given in Figure 4.8.

	123	**126**	**127**	**128**	**135**	**156**	**157**	**178**	**234**	**246**	**247**	**268**	**345**	**348**	**358**	**456**	**457**	**478**	**568**
12	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	−1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	−1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
16	0	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	−1	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0
23	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	−1	1	1	0	0	0	0	0	0	0	0
26	0	1	0	0	0	0	0	0	0	−1	0	1	0	0	0	0	0	0	0
27	0	0	1	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0
28	0	0	0	1	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0
35	0	0	0	0	1	0	0	0	0	0	0	0	−1	0	1	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0
46	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	−1	0	0	0
47	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	−1	1	0
48	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1	0
56	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1
57	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0
58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1
68	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
78	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0

FIGURE 4.8. The boundary matrix of the CED simplicial complex in Figure 4.7.

4.4. Flows on triangulations of manifolds

In this section, we prove the following:

PROPOSITION 4.4.1. *Let* Δ *be a triangulation of a manifold. Then*

$$\phi_\Delta(q) = \begin{cases} 0 & \text{if } \Delta \text{ has boundary,} \\ q-1 & \text{if } \Delta \text{ is without boundary, } \mathbb{Z}\text{-orientable,} \\ 0 & \text{if } \Delta \text{ is without boundary, non-}\mathbb{Z}\text{-orientable, and } q \text{ is even,} \\ 1 & \text{if } \Delta \text{ is without boundary, non-}\mathbb{Z}\text{-orientable, and } q \text{ is odd.} \end{cases}$$

PROOF. Consider a pure simplicial complex that is a triangulation of a connected manifold. First, if the manifold has boundary, then there exists at least one ridge that belongs to only one facet. Since this corresponds to a row in its boundary matrix with precisely one nonzero entry, any vector in the kernel must have a zero in the coordinate corresponding to the facet containing this ridge. Therefore, manifolds with boundary do not admit nowhere-zero flows.

If a triangulated manifold N is without boundary, then every ridge belongs to precisely two facets. This corresponds to every row of ∂N having exactly two nonzero entries. Therefore, since our manifold is connected, in a valid flow the assigned value of any facet is equal or opposite mod q to the assigned value of any other facet. It follows that every flow that is somewhere zero must in fact be trivial.

It is known (see, for instance [**Hat02**, Chapter 3.3, Theorem 3.26]) that the top homology of a closed, connected, and $\mathbb{Z}$-orientable manifold N of dimension n is

$$\widetilde{H}_n(N, \mathbb{Z}) \cong \mathbb{Z}.$$

In terms of boundary matrices, this means that the kernel of the boundary matrix has rank one, since there are no simplices of dimension higher than n. By our comment above, we see that the non-trivial elements of the kernel must be nowhere-zero, and all entries are $\pm a \in \mathbb{Z}$. It is then easy to see that triangulations of closed, connected, orientable manifolds have precisely $q-1$ nowhere-zero $\mathbb{Z}_q$-flows.

If M is non-orientable, connected, and of dimension n, then we know (again, see [**Hat02**]) that the top homology is

$$\widetilde{H}_n(M,\Gamma) \cong \begin{cases} 0 & \text{if } \Gamma = \mathbb{Z}, \\ 0 & \text{if } \Gamma = \mathbb{Z}_{2k+1}, \text{ and} \\ \mathbb{Z}_2 & \text{if } \Gamma = \mathbb{Z}_2. \end{cases}$$

Therefore, if q is odd, we have no non-zero $\mathbb{Z}_q$-flows. However, as every manifold is orientable over $\mathbb{Z}_2$, and since every row of the boundary matrix has precisely two nonzero entries, the vector with all entries equal to k is a nowhere-zero $\mathbb{Z}_{2k}$-flow on M (in fact, the only one). □

4.5. Further questions and directions

We have shown above that not all simplicial complexes have flow quasipolynomials that are true polynomials. However, it is still natural to ask for which complexes $\phi_\Delta(q)$ *is* a polynomial. Early experiments suggest that matroid and shifted complexes may be examples of such subclasses. Moreover, for the particular case of CED complexes, we propose the following questions:

Question 4.5.1. *What topological/geometric conditions on a simplicial complex Δ are* necessary *for $\phi_\Delta(q)$ to have period equal to one?*

Question 4.5.2. *What topological/geometric conditions on a simplicial complex Δ are* sufficient *for $\phi_\Delta(q)$ to have period equal to one?*

We also propose to find topological properties of simplicial complexes that guarantee that the flow quasipolynomial has period strictly *greater* than one. One possible condition – inspired by [**DHK11**], the Klein bottle example, and our second CED example – involves the ability to embed the complex in space of a particular dimension. Both the Klein bottle and the complex in Figure 4.7 are 2-dimensional simplicial complexes, and cannot be embedded in $\mathbb{R}^3$ without improper intersections. Is it more generally true that if a d-dimensional (CED) simplicial complex Δ cannot be embedded in $\mathbb{R}^{d+1}$, then the period of $\phi_\Delta(q)$ is greater than one?

Further, in the examples in this thesis, as well as in all examples computed outside of this work, the period of $\phi_\Delta(q)$ was at most two. We therefore propose to explore:

Question 4.5.3. *Is there a bound on the period of $\phi_\Delta(q)$ for general simplicial complexes Δ, or for subfamilies of simplicial complexes? Can we construct examples of simplicial complexes whose flow quasipolynomials have period* greater *than two?*

In addition, we see that certain constructions, such as the disjoint union of simplicial complexes, preserve the polynomiality of the flow function. We propose to further explore the question of preserving polynomiality, and specify operations that do so. Moreover, can we use these operations to construct infinite (and interesting) families of simplicial complexes with polynomial flow functions?

Many graph polynomials satisfy *combinatorial reciprocity* theorems, i.e., they have an (a priori non-obvious) interpretation when evaluated at negative integers. The classical example is Stanley's reciprocity theorem connecting the chromatic polynomial of a graph to acyclic orientations [**Sta74**]; the reciprocity theorem for flow polynomials is much younger and was found by Breuer and Sanyal [**BS12**], starting with a geometric setup not unlike that of our proof of Theorem 4.2.5. Beck, Breuer, Godkin, and Martin [**BBGM12**] further explore combinatorial reciprocity for simplicial complexes and cell complexes in general. Their paper includes several open problems related to flows on simplicial complexes. We restate two of particular interest:

QUESTION 4.5.4 ([**BBGM12**]). *Kook, Reiner, and Stanton* [**KRS99**] *gave a formula for the Tutte polynomial of a matroid as a convolution of tension and flow polynomials. Breuer and Sanyal* [**BS12**] *used the Kook–Reiner–Stanton formula, together with reciprocity results to give a general combinatorial interpretation of the values of the Tutte polynomial of a graph G at positive integers; see also* [**Rei99**] *and* [**Bre09**, Theorem 3.11.7]*. Do these results generalize to simplicial complexes whose tension and flow functions are polynomials?*

See [**Whi92**] for background on tension polynomials of graphs, and [**BBGM12**] for background on tension polynomials of simplicial complexes.

QUESTION 4.5.5 ([**BBGM12**]). *In the case of enumeration polynomials of graphs, the geometric setup has proven extremely useful for establishing bounds on the coefficients of the chromatic polynomial (see* [**HS08**]*) and the tension and flow polynomials, in particular in the modular case (see* [**BD11**]*). Moreover these geometric constructions are closely related to Steingrímsson's coloring complex* [**BD10, Ste01**]*. Can these methods be extended to the case of counting quasipolynomials defined in terms of cell complexes? In particular, what are good bounds on the coefficients of the flow quasipolynomial?*

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