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Linear Diophantine Systems: Partition Analysis and Polyhedral Geometry

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Abstract

The main topic of this thesis is the algorithmic treatment of two problems related to linear Diophantine systems. Namely, the first one is counting and the second one is listing the non-negative integer solutions of a linear system of equations/inequalities.

The general problem of solving polynomial Diophantine equations does not admit an algorithmic solution. In this thesis we restrict to linear systems, so that we can treat them algorithmically. In 1915, MacMahon, in his seminal work "Combinatory Analysis", introduced a method called partition analysis in order to attack combinatorial problems subject to linear Diophantine systems. Following that line, in the beginning of this century Andrews, Paule and Riese published fully algorithmic versions of partition analysis, powered by symbolic computation. Parallel to that, the last decades saw significant progress in the geometric theory of lattice-point enumeration, starting with Ehrhart in the 60's, leading to important theoretical results concerning generating functions of the lattice points in polytopes and polyhedra. On the algorithmic side of polyhedral geometry, Barvinok developed the first polynomial-time algorithm in fixed dimension able to count lattice points in polytopes.

The main goal of the thesis is to connect these two lines of research (partition analysis and polyhedral geometry) and combine tools from both sides in order to construct better algorithms.

The first part of the thesis is an overview of conic semigroups from both a geometric and an algebraic viewpoint. This gives a connection between polyhedra, cones and their generating functions to generating functions of solutions of linear Diophantine systems. Next, we provide a geometric interpretation of Elliott Reduction and Omega2 (two implementations of partition analysis). The partial-fraction decompositions used in these two partition-analysis methods are interpreted as decompositions of cones. With this insight and employing tools from polyhedral geometry, such as Brion's theorem and Barvinok's algorithm, we propose a new algorithm for the evaluation of the $\Omega_{\geq}$ operator, the central tool in MacMahon's work on partition analysis. This gives an implementation of partition analysis heavily based on the geometric understanding of the method. Finally, a classification of linear Diophantine systems is given, in order to systematically treat the algorithmic solution of linear Diophantine systems, especially the difference between parametric and non-parametric problems.

Zusammenfassung

Das Hauptaugenmerk dieser Arbeit liegt auf der algorithmischen Lösung zweier Probleme, welche im Zusammenhang mit linearen diophantischen Systemen auftreten. Das erste Problem ist die Bestimmung der Anzahl der nicht-negativen, ganzzahligen Lösungen linearer Systeme von (Un-)Gleichungen, das zweite die Auflistung ebendieser Lösungen.

Die Lösung von polynomiellen diophantischen Gleichungen ist nicht ohne Einschränkungen algorithmisch handhabbar. Aus diesem Grund beschränken wir uns in der vorliegenden Arbeit auf lineare Systeme. 1915 führte MacMahon in seinem Werk "Combinatory Analysis" eine Methode namens Partition Analysis ein, um kombinatorische Probleme zu lösen, in denen lineare diophantische Systeme auftreten. Darauf aufbauend, jedoch unter weitreichender Miteinbeziehung von Techniken des symbolischen Rechnens, veröffentlichten Andrews, Paule und Riese zu Beginn dieses Jahrhunderts algorithmische Varianten der Partition Analysis. Parallel dazu gab es in den letzten Jahrzehnten erhebliche Fortschritte in der geometrischen Theorie zur Aufzählung von Gittervektoren, beginnend mit Erhart in den 70er Jahren, welche zu wichtigen theoretischen Erkenntnissen bezüglich erzeugender Funktionen von Gittervektoren in Polytopen und Polyedern führten. Einen wesentlichen Beitrag in der algorithmischen Entwicklung leistete Barvinok mit der Formulierung des ersten Algorithmus, der die Aufzählung von Gittervektoren in Polytopen ermöglicht und bei fester Dimension eine polynomielle Laufzeit aufweist.

Das Ziel dieser Arbeit is es, beide Forschungsstränge -Partition Analysis und Polyeder Geometrie- zu verbinden und die Werkzeuge aus beiden Theorien zu neuen, besseren Algorithmen zu kombinieren. Der erste Teil gibt einen Überblick über konische Halbgruppen aus zweierlei Blickwinkeln, dem geometrischen und dem algebraischen. Dies ermöglicht es uns, die Verbindung von Polyedern, Kegeln und ihrer erzeugenden Funktionen zu den erzeugenden Funktionen der Lösungen von linearen diophantischen Systemen herzustellen. Darauf folgend behandeln wir geometrische Interpretationen von Elliotts Reduction und Omega2, zweier Realisierungen von Partition Analysis. Die Partialbruchzerlegungen, welche in diesen beiden Methoden auftreten, werden als Zerlegungen von Kegeln interpretiert. Zusammen mit geeigneten Hilfsmitteln aus der Polyeder Geometrie, wie etwa dem Satz von Brion und Barvinoks Algorithmus, führt dies zur Formulierung eines neuen Algorithmus zur Anwendung des Omega-Operators, dem zentralen Werkzeug in MacMahons Arbeit an Partition Analysis. Wir erhalten daraus eine algorithmische Realisierung von Partition Analysis, die entscheidend auf dem geometrischen Verständnis der Methode basiert. Abschließend wird eine Klassifikation von linearen diophantischen Systemen eingeführt, um systematisch deren algorithmische Lösung aufarbeiten zu können. Speziell wird dabei auf die Unterscheidung zwischen parametrischen und nicht-parametrischen Problemen geachtet.

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbststndig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe. Die vorliegende Dissertation ist mit dem elektronisch bermittelten Textdokument identisch.

Zafeirakis Zafeirakopoulos

Acknowledgments

Writing a thesis proved to be harder than what I thought. But, now I know, it would have been impossible without the help of the people who supported me during the last 3-4 years.

First I want to thank my advisor Prof. Peter Paule for his support in a range of issues spanning scientific and everyday life. The two most important things I am grateful to him for are his suggestion for my PhD topic and his support for my participation in the AIM SQuaRE for "Partition Theory and Polyhedral Geometry". When I first mentioned to him my interests, Prof. Paule successfully predicted that the topic of this thesis would be for me exciting enough so that even passing through frustrating and exhausting periods, today I am still as interested as the first day.

Concerning the SQuaRE meetings in the American Institute of Mathematics, I cannot exaggerate about how helpful they were for me. The three weeks of these meetings were undoubtedly among the most intensive learning and working experiences I ever had. Each one of the participants - Prof. Matthias Beck, Prof. Ben Brown, Prof. Matthias Köppe and Prof. Carla Savage - helped me in a different way to understand parts of the beauty lying in the intersection of algebraic combinatorics, polyhedral geometry, number theory and the algorithmic methods involved.

Prof. Beck acted as a host during my five months stay in San Francisco visiting San Francisco State University in the context of a Marshall Plan Scholarship. Except for being excellent in his formal duty as a host (advising me during that period), Matt and Tendai acted as real hosts, making my stay in San Francisco a very positive experience both in academic and social terms.

During my stay in San Francisco, I met Dr. Felix Breuer with whom I spent a good part of my time there discussing about interesting topics, not always concerning mathematics. During our mathematical discussions emerged a number of topics that hopefully I will have the pleasure to pursue in the future, as well as one chapter of this thesis. Landing in a city as a stranger always gives a strange feeling, but Felix and Chambui contributed in making this feeling negligible.

The first time I landed in a new city was in 2009. The task of making this move easier fell on the hands of Veronika, Flavia, Burcin and Ionela. To a large extend, I owe the fact that I am here to them and the great job they did in making Linz a warm place (which is not easy in any sense, especially after living 27 years in Athens), when I first arrived.

PhD studies is like an extended summer camp. And I don't just mean exhausting but fun. You meet people you like and then they leave, or you leave. You do not know when you will see each other again, possibly next summer in a conference. But, if you are fortunate enough, you make some friends. I was fortunate to meet Max, Hamid, Jakob, Cevo, Nebiye and Madalina, who I do consider as friends.

Distance makes communication between friends harder, but you know that you care about them and you hope they do the same. Although I love Athens as a city, after taking the time to think about what I really miss, it is the people. It is the friends I know for half my life. A couple of them for more than that. For me, the most frustrating thing about living abroad was not being there in the little moments you never care about, except if you are not there.

Giorgos, Dimitris and NN, being themselves in a position similar to mine, distracted me efficiently through these years with pointless but enjoyable discussions. Elias proved to be always a valuable source of information and support. I also want to thank my colleagues at RISC and especially Manuel, Matteo, Ralf and Veronika for the very useful discussions and directions they gave me on various mathematical topics. At this point I can say that the bridge at RISC is the place where I made the most friends and also the most work done these last years.

Finally, I want to thank my family. My parents supported me in all possible ways during my studies in Greece and keep supporting me until today. I do owe them more than I can possibly offer. My brother is still in Athens, in a stay-behind role. I owe to him my luxury of being able to leave in order to pursue my goals. Achieving one goal only asks for the next one. And for me that is to be able to support Ilke for the rest of my life, the way she supported me during these years. Good that she is an endless source of happiness and inspiration for me.

Contents

Introduction

Some History

One of the earliest recorded uses of geometry as a counting tool is the notion of figurate numbers. Ancient Greeks used pebbles (χαλίxι, where the word calculus comes from), in order to do arithmetic. For example, an arrangement of pebbles would be used to calculate triangular numbers. In general figurate numbers were popular in ancient times.

Today we mostly care about squares and some special cases (like pentagonal numbers in Euler's celebrated theorem). Among the mathematicians that were interested in figurate numbers was, of course, Diophantus who wrote a book on polygonal numbers (see [25]). After an interesting turn of events, one of the most prominent methods for the solution of linear Diophantine systems relies on generalizing figurate numbers.

This generalization bears the name of Eugene Ehrhart who made the first major contributions to what is today called Ehrhart theory. The goal is to compute and investigate properties of functions enumerating the lattice points in polytopes. In dimension 2 and for the simple regular polygons this coincides with the concept of figurate numbers, but naturally we are interested in higher-dimensional polytopes and polytopes with more complicated geometry.

Diophantus in his masterpiece "Arithmetica" dealt with the solution of equations [25]. But in the time of Diophantus, a couple of things were essentially different than in modern mathematics:

- No notion of zero existed.
- Fractions were not treated as numbers (Diophantus was the first to do so).
- There was no notation for arithmetic.

"Arithmetica" consisted of 13 books, out of which only six survived and possibly another four through Arab translations (found recently), dealing with the solution of 130 equations. On one hand his work is important because it is the oldest account we have for indefinite equations (equations with more than one solutions). More importantly though, Diophantus introduced a primitive notational system for (what later was called) algebra.

For our purposes, the essential part of his work is his view on what is the solution of an equation. He considered equations with positive rational coefficients whose solutions are positive rationals. Following this path, we consider a Diophantine problem to have integer coefficients and non-negative integer solutions.

In 1463 the German mathematician Regiomontanus, traveling to Italy, he came across a copy of Arithmetica. He considered it to be an important work when he reported to a friend of his about it. He intended to do the translation, but he could only find six out of the thirteen books that Diophantus mentioned in his introduction, thus he did not proceed [14].

Although the most famous marginal note to be found in a copy of Arithmetica is by Fermat (his last theorem), there is another one which is very interesting. The Byzantine scholar Chortasmenos notes "Thy soul, Diophantus, be with Satan because of the difficulty of your problems" (funny enough next to what came to be known as Fermat's last theorem).

This last comment, combined with the note that Diophantus did not have a general method (after solving the 100th problem, you still have no clue how to attack the 101st) is important for us. Of course, for non-linear Diophantine problems one cannot expect a general method (Hilbert's 10th problem). But we present algorithmic solutions (developed during the last century) for linear Diophantine systems and examine their connections.

Contributions

The main contribution of this thesis is in the direction of connecting partition analysis, a method for solving linear Diophantine systems, with polyhedral geometry. The analytic viewpoint of traditional partition analysis methods (due to Elliott, MacMahon, Andrews, Paule and Riese) is interpreted in the context of polyhedral geometry. Through this interpretation we are allowed to use tools from geometry in order to enhance the algorithmic procedures and the understanding of partition analysis. More precisely, the main contributions are:

Geometric and algebraic interpretation of partition analysis

- In Theorem 3.2 a geometric interpretation of the generalized partial fraction decomposition employed in Omega2 is given.
- We define geometric objects related to the crude generating function and provide a geometric version of the $\Omega_{\geq}$ operator.
- An algebraic version of $\Omega_{\geq}$ is given based on gradings of algebraic structures and a variant of Hilbert series.

Algorithmic contributions to partition analysis

- A new algorithm for the evaluation of the $\Omega_{\geq}$ operator, following the traditional paradigm of recursive elimination of λ variables (see Section 4.1).
- A new algorithm, motivated by the geometric understanding of the action of $\Omega_{\geq}$, that performs simultaneous elimination of all λ variables (see Section 4.2).

A second goal of the thesis, especially given the interest in algorithmic solutions, is the classification of linear Diophantine problems. In the literature the notion of a linear Diophantine problem is not consistent and depends on the motivation of the author.

Finally, we note that the main reason why we believe it is important to explore such connections, although in both contexts (partition analysis and polyhedral geometry) there are already methods to attack the problem, is that knowledge transfer from one area to another often helps to develop algorithmic tools that are more efficient.

Chapter 1

Geometry, Algebra and Generating Functions

In this chapter we present a basic introduction to geometric, algebraic and generating function related concepts that will be used in later chapters. This introduction is not meant to be complete but rather a recall of basic notions mostly to set up notation. In Section 1.1, some geometric notions and the discrete analog of cones are presented. For a detailed introduction see [17, 11, 38]. In Section 1.2 a hierarchy of algebraic structures related to formal power series is presented first and then an introduction to generating functions is given. Finally, in Section 1.3 we present the relation of formal power series and generating functions via polyhedral geometry and give a short description of more advanced geometric tools we will use later.

We note that in this chapter we will use boldface fonts for vectors and the multi-index notation $\mathbf{x}^a = x_1^{a_1} x_2^{a_2} \ldots, x_d^{a_d}$.

1.1 Polyhedra, Cones and Semigroups

In this section we give a short introduction to polyhedra and polytopes. We define the fundamental objects of polyhedral geometry and provide terminology and notation that will be used later. For a more detailed introduction to polyhedral geometry see [17, 38] and references therein.

All of the theory developed in this section takes place in some Euclidean space. For simplicity of notation and clarity, we agree this to be $\mathbb{R}^d$ for some $d \in \mathbb{N}$. We fix d to denote the dimension of the ambient space $\mathbb{R}^d$.

In Section 1.1.1 we will introduce polyhedra and in Section 1.1.2 we discuss about polytopes, while in Section 1.1.3 we will present definitions and notation related to polyhedral cones. In Section 1.1.4 we introduce semigroups and their connection to polyhedral cones.

1.1.1 Polyhedra

There are many different ways to introduce the notion of polyhedra in literature [11, 17, 38, 12]. According to our intuition, polyhedra are geometric objects with "flat sides". Thus, it is expected that we will resort to some sort of linear relations defining polyhedra. In order to make this more precise we will use linear functionals.

Definition 1.1 (Linear Functional)
Given a real vector space $\mathbb{R}^d$, a linear functional is a map $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$f(au + bv) = af(u) + bf(v)$$

for all $u, v \in \mathbb{R}^d$ and $a, b \in \mathbb{R}$. □

Observe that any linear equation/inequality in d variables $x_1, x_2, \ldots, x_d$, i.e.

$$\sum_{i=1}^{d} a_i x_i \geq b \text{ for } a_1, a_2, \ldots, a_d, b \in \mathbb{R}$$

can be expressed as $f(\mathbf{x}) \geq b$ in terms of a linear functional

$$f(\mathbf{x}) = (a_1, a_2, \ldots, a_d)^T \mathbf{x}.$$

For $\mathbf{a} \in \mathbb{R}^d$, we denote by $f_a(\mathbf{x})$ the linear functional $f_a(\mathbf{x}) = (a_1, a_2, \ldots, a_d)^T \mathbf{x}$.

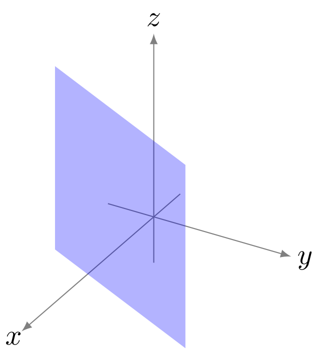
Having a concise and formal way to express linear relations, the next step is to define the solution space of a linear equation/inequality. Given $\mathbf{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, the solution space of $f_a(\mathbf{x}) = b$ is called an affine hyperplane and is denoted by $H_{a,b}$, i.e.,

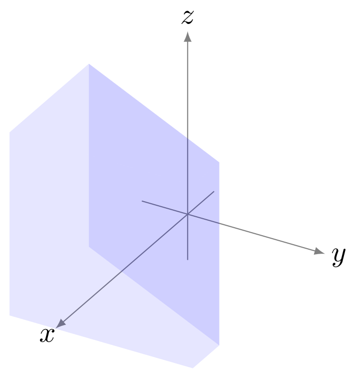
$$H_{a,b} = \{\mathbf{x} \in \mathbb{R}^d : f_a(\mathbf{x}) = b\}.$$

On the other hand, given $\mathbf{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, the solution space of the inequality $f_a(\mathbf{x}) \geq b$ is called an affine halfspace and is denoted by $\mathcal{H}^+_{a,b}$, i.e.,

$$\mathcal{H}^+_{a,b} = \{\mathbf{x} \in \mathbb{R}^d : f_a(\mathbf{x}) \geq b\} \text{ for some } b \in \mathbb{R}.$$

Naturally, a second half-space is given by $\mathcal{H}^-_{a,b} = \{\mathbf{x} \in \mathbb{R}^d : f_a(\mathbf{x}) \leq b\}$. We note that the hyperplane $H_{a,b}$ is the intersection of the two halfspaces $\mathcal{H}^+_{a,b}$ and $\mathcal{H}^-_{a,b}$, or, equivalently, the hyperplane divides the Euclidean space into two halfspaces.





A hyperplane and a halfspace it defines.

Let's now consider (finite) intersections of halfspaces. Given two linear functionals f_{a_1} and f_{a_2} for some $\mathbf{a_1}, \mathbf{a_2} \in \mathbb{R}^d$, and two scalars $b_1, b_2 \in \mathbb{R}$, we consider the intersection of the two halfspaces

$$\mathcal{H}^+_{a_1,b_1} \cap \mathcal{H}^+_{a_2,b_2} = \left\{\mathbf{x} \in \mathbb{R}^d : f_{a_1}(\mathbf{x}) \geq b_1, f_{a_2}(\mathbf{x}) \geq b_2\right\}$$

For simplicity, we omit the linear functional notation and use matrix notation, as it is customary for systems of linear inequalities. In other words, given a set of vectors $\mathbf{a_i} \in \mathbb{R}^d$ and scalars $b_i \in \mathbb{R}$ for $i \in [m]$, we write

$$\bigcap_{i \in [m]} \mathcal{H}^+_{a_i,b_i} = \left\{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \geq b\right\}$$

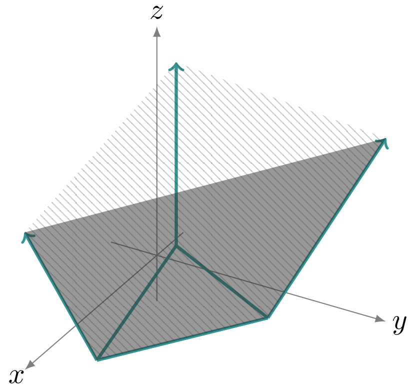
where A is the matrix with $\mathbf{a_i}$ as its i-th row and $b = (b_1, b_2, \ldots, b_m)^T$.
With the above notation and terminology we can proceed with the definition of the first fundamental object in polyhedral geometry.

Definition 1.2 (Polyhedron)
A polyhedron is the intersection of finitely many affine halfspaces in $\mathbb{R}^d$. More precisely, given $A \in \mathbb{R}^{m \times d}$ and $\mathbf{b} \in \mathbb{R}^m$, then the polyhedron $\mathcal{P}_{A,b}$ is the subset of $\mathbb{R}^d$

$$\mathcal{P}_{A,b} = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \geq b\}.$$

□

An important subclass of polyhedra, in which we will restrict for the rest of the thesis, is that of rational polyhedra. In the definition of $\mathcal{P}_{A,b}$ we assume $A \in \mathbb{R}^{m \times d}$ and $\mathbf{b} \in \mathbb{R}^m$. If $A' \in \mathbb{Z}^{m \times d}$ and $b' \in \mathbb{Z}^m$ can be chosen such that $\mathcal{P}_{A,b} = \mathcal{P}_{A',b'}$, then the polyhedron $\mathcal{P}_{A,b}$ is called **rational**. For brevity, if no characterization is given, then we assume a polyhedron is rational.



A polyhedron

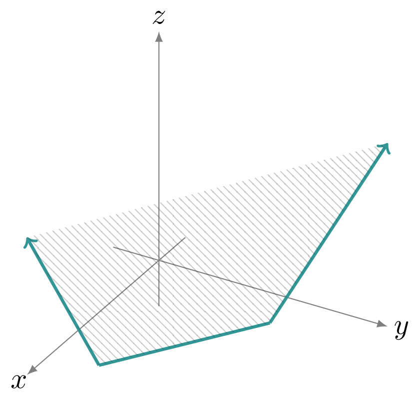
In the beginning of the section we fixed d to denote the dimension of the ambient Euclidean space. Moreover, in the definitions of halfspaces and polyhedra we assumed that these are subsets of $\mathbb{R}^d$. Nevertheless, the dimension of a polyhedron is not necessarily equal to the dimension of the ambient space as we shall see. In order to define the notion of the dimension of a polyhedron, we will use the affine hull of a set. Given a real vector space $\mathbb{R}^d$ and a set $S \subset \mathbb{R}^d$, then $\operatorname{affhull}_{\mathbb{R}^d} S$ is the set of all affine combinations of elements in S, i.e.,

$$\operatorname{affhull}_{\mathbb{R}^d} S = \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{s_i} \;\middle|\; n \in \mathbb{N}^*, \sum_{i=1}^{n} \alpha_i = 1, a_i \in \mathbb{R} \right\}.$$

Let $\mathcal{P}_{A,b}$ be a polyhedron in the ambient Euclidean space $\mathbb{R}^d$. The **dimension of** $\mathcal{P}_{A,b}$, denoted by $\dim \mathcal{P}_{A,b}$ is defined to be the dimension of the affine hull of $\mathcal{P}_{A,b}$, i.e.,

$$\dim \mathcal{P}_{A,b} = \dim_{\mathbb{R}^d} \left(\operatorname{affhull}_{\mathbb{R}^d} \mathcal{P}_{A,b} \right).$$

Observe that $\dim \mathcal{P}_{A,b} \leq d$ in general. If $\dim \mathcal{P}_{A,b} = d$, then we say that $\mathcal{P}_{A,b}$ is **full dimensional**. A k-polyhedron is a polyhedron of dimension k.

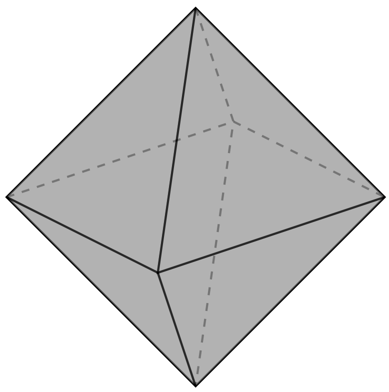


A non full-dimensional polyhedron. The ambient space dimension is 3*, while the dimension of the polyhedron is* 2*.*

1.1.2 Polytopes

Although we will mostly deal with polyhedra, a very important object, especially when it comes to lattice-point enumeration, is the polytope. Among the numerous equivalent definitions of a polytope, we chose the following:

Definition 1.3 (Polytope)
A bounded polyhedron is called a polytope. □



A 3*-polytope.*

One of the fundamental properties of polytopes, which we will also use in later chapters, is that a polytope has two equivalent representations, the H-representation and the V-representation. H-representation stands for halfspace representation and it is the set of halfspaces whose intersection is the polytope, i.e., the description we used in the definition. V-representation stands for vertex representation. A polytope is the convex hull of its vertices, thus it can be described by a finite set of points in $\mathbb{R}^d$. For a detailed proof of this non-trivial fact see Appendix A in [17].

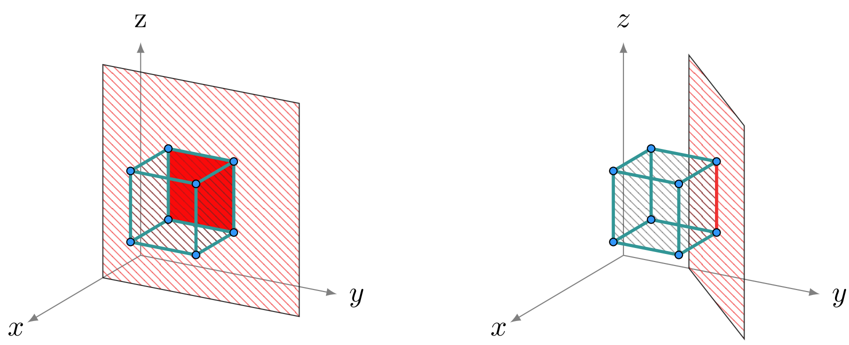
We note that the convex hull of a set of k points $S \in \mathbb{R}^d$ is the smallest convex set containing S, or alternatively, the set of all convex combinations of elements from S, i.e.,

$$\left\{ \sum_{i=1}^{k} \alpha_i \mathbf{s_i} \;\middle|\; \mathbf{s_i} \in S, \sum_{i=1}^{k} \alpha_i = 1, \alpha \in \mathbb{R} \right\}.$$

In order to establish terminology and to define the notion of vertex used above, we define the notions of supporting hyperplanes and faces of polyhedra and polytopes. A hyperplane $H \in \mathbb{R}^d$ is said to support a set $S \subseteq \mathbb{R}^d$ if

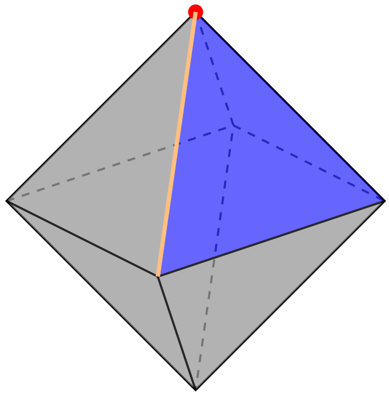
- S is contained in one of the two closed halfspaces determined by H,
- there exists $x \in S$ such that $x \in H$.

In other words, a supporting hyperplane H for the polyhedron P is a hyperplane that intersects P and leaves P entirely on one side.



Supporting hyperplanes for a cube in 3D

With the use of supporting hyperplanes, we can define now the faces of a polyhedron or polytope P in $\mathbb{R}^d$. A face F of P is the intersection of P with a supporting hyperplane S. The dimension of a face F is the dimension of $\operatorname{affhull}_{\mathbb{R}^d} F$. A face of dimension k is called a k-face. In particular a 0-face is called vertex, a 1-face is called edge or extreme ray and a $(\dim P - 1)$-face is called facet. We note that a face of a polyhedron (resp. polytope) is again a polyhedron (resp. polytope) and that the definition of face dimension is compatible with the the definition of the dimension of a polyhedron.



A facet, a 1-face and a vertex of a 3-polytope.

Simplices are polytopes of very simple structure and are used as building blocks in polyhedral geometry. We first define the standard simplex in dimension d.

Definition 1.4 (Standard Simplex)
The standard d-simplex is the subset of $\mathbb{R}^d$ defined by

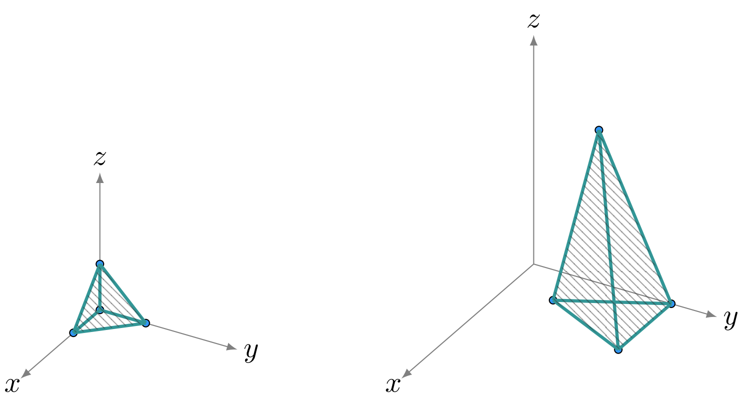
$$\Delta^d = \left\{ (x_1, x_2 \dots, x_d) \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{d} x_i \leq 1 \text{ and } x_i \geq 0 \ \forall i \right\}$$

Equivalently, the standard d-simplex Δ^d is the convex hull of the $d+1$ points:

$$e_0 = (0,0,0,\dots,0), e_1 = (1,0,0,\dots,0), e_2 = (0,1,0,\dots,0), \dots e_d = (0,0,0\dots,1).$$

□

In general a d-simplex is defined to be the convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$.



The standard 3-simplex and the simplex defined by $(1,1,0), (2,3,0), (0,3,0)$ *and* $(1,2,4)$.

1.1.3 Cones

In polyhedral geometry, the main object is the polyhedron but the main tool is the polyhedral cone. It is usual to use cones in order to compute with polyhedra and polytopes or in general to investigate their properties. A fundamental relation is given by Brion's theorem (see Theorem 1.3), connecting polyhedra and cones. Cones are polyhedra of a special type. They are finite intersections of linear halfspaces.

Definition 1.5 (Polyhedral Cone)
Given a set of linear functionals $f_i(\mathbf{x}) : \mathbb{R}^d \mapsto \mathbb{R}$ for $i \in [k]$, we define the polyhedral cone

$$C = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; f_i(\mathbf{x}) \geq 0 \text{ for all } i \in [k] \right\}$$

□

For the definition of a cone we used inequalities, complying with the definition of polyhedron. In what follows though, we will most often define cones via their generators. We first define the conic hull of a set of vectors.

Definition 1.6
Given a set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\} \in \mathbb{R}^d$, its conic hull is

$$\operatorname{co}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}) = \left\{ \sum_{i=1}^{k} r_i \mathbf{v_i} \;\middle|\; r_i \in \mathbb{R}, r_i \geq 0 \right\}.$$

□

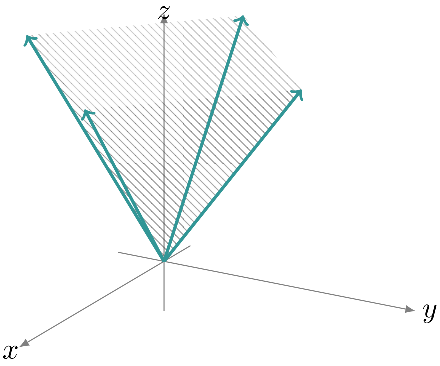
The following lemma certifies that a cone can be defined via a finite set of generators.

Lemma 1. *A polyhedral cone is the conic hull of its (finitely many) extreme rays.* □

Now, given $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k} \in \mathbb{R}^d$, we define the cone generated by $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k}$ as

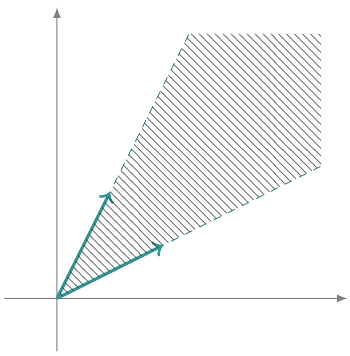
$$\mathcal{C}_{\mathbb{R}}\left(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k}\right) = \operatorname{co}\left(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k}\right) = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; x = \sum_{i=1}^{k} \ell_i \mathbf{a_i}, \ell_i \geq 0, \ell_i \in \mathbb{R} \right\}.$$

A cone will be called rational if there exists a set of generators for the cone such that the generators contain only rational coordinates. As with polyhedra, since all our cones are rational, we omit the characterization.



A polyhedral cone.

Among cones, there are two special types that have a central role in computational polyhedral geometry. These are simplicial and unimodular cones. A cone in $\mathbb{R}^d$ is called **simplicial** if and only if it is generated by linearly independent vectors in $\mathbb{R}^d$. We note that although usually the definition of a simplicial cone asserts d linearly independent generators, we do not enforce this restriction. The reason is that often our cones, and polyhedra in general, will not be full dimensional.



A simplicial cone.

The connection to simplices is more than nominal, in particular a section by a hyperplane (not containing the cone apex) of a simplicial k-cone is a $(k-1)$-simplex. For the rest of the section, except if stated differently, our cones are simplicial.

An object encoding all the information contained in a cone is the **fundamental parallelepiped** of the cone.

Definition 1.7 (Fundamental Parallelepiped)
Given a simplicial cone $C = \mathcal{C}_{\mathbb{R}}\left(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_d}\right) \in \mathbb{R}^d$, we define its fundamental parallelepiped as

$$\Pi_{\mathbb{R}}(C) = \left\{ \sum_{i=1}^{d} k_i \mathbf{a_i} \;\middle|\; k_i \in [0,1) \right\}$$

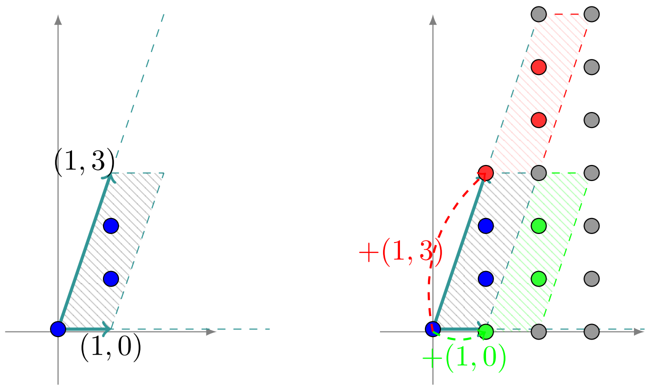
and the set of lattice points in the fundamental parallelepiped as

$$\Pi_{\mathbb{Z}^d}(C) = \Pi_{\mathbb{R}}(C) \cap \mathbb{Z}^d$$

□

Note that we assume as working lattice the standard lattice $\mathbb{Z}^d$.

In the figure below we can see that the cone is spanned by copies of its fundamental parallelepiped. This is a general fact, i.e., any simplicial cone is spanned by copies of its fundamental parallelepiped.



The fundamental parallelepiped of the cone generated by $(1, 0)$ *and* $(1, 3)$.

A cone C is called **unimodular** if and only if $\Pi_{\mathbb{Z}^d}(C) = \{0\}$. An equivalent definition is that the matrix of cone generators is a unimodular matrix, i.e., it has determinant equal to ± 1. The following lemma gives a condition for a non full-dimensional cone to be unimodular.

Lemma 2. *Let* $C = \mathcal{C}_{\mathbb{R}}(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$ *in* $\mathbb{R}^d$*, for* $n \leq d$*. Let* $G = [\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}]$*, the matrix with* $\mathbf{a_i}$ *as columns. If* G *contains a full rank square* $n \times n$ *submatrix with determinant* ± 1*, then* C *is unimodular.* □

Proof. Let I be an index set such that $[G_i]_{i \in I}$ is a full rank square $n \times n$ submatrix with determinant ± 1, where G_i is the i-th row of G. Without loss of generality rearrange the coordinate system such that I are the first n coordinates. Let C_n be the (orthogonal) projection of C into R^n. Then C_n is unimodular. Since projection maps lattice points to lattice points, the only way that C is not unimodular is that $\Pi(C)$ contains lattice points that all project to the origin. In that case, there would be a generator of the form $(0, 0, \ldots, 0, \mathbf{a_{n+1}}, \mathbf{a_{n+2}}, \ldots, \mathbf{a_m})$, which contradicts the assumption that $[G_i]_{i \in I}$ is full rank. □

All cones we have seen until now are **pointed cones**, i.e., they contain a vertex. According to the definition of cone, this is not necessary. A cone may contain a line or higher dimensional spaces. It is easy to see though, that a cone is pointed if and only if it does not contain a line. The unique vertex of a pointed cone is called the **apex of the cone**. Observe that simplicial cones are always pointed.

1.1.4 Semigroups

A semigroup is a very basic algebraic structure, essentially expressing that a set has a well behaved operation. The semigroup operation will be called addition and will be denoted by $+$.

Definition 1.8 (Semigroup)
A semigroup is a set S together with a binary operation $+$ such that

- For all $a, b \in S$ we have $a + b \in S$.
- For all $a, b, c \in S$, we have $(a + b) + c = a + (b + c)$.

□

If a semigroup has an identity element, i.e., there exists e in S such that for all a in S we have $e + a = a + e = a$, then it is called a **monoid**. The standard example of a monoid is the set of natural numbers $\mathbb{N}$.

Let S be a semigroup with operation $+$. A set $A \subset S$ is called a subsemigroup of S if it is closed under the operation $+$, i.e., for all a and b in A we have that $a + b$ is in A.

An important notion connecting the algebraic and geometric points of view is that of an affine semigroup. We note that all semigroups we consider in what follows are affine (except if the contrary is explicitly stated) due to Theorem 1.1. For brevity we may omit the characterization affine later.

Definition 1.9 (Affine semigroup [20])
A finitely generated semigroup that is isomorphic to a subsemigroup of $\mathbb{Z}^d$ for some d is called affine semigroup. We will call rank(S) or dimension of S the number d, i.e., the least dimension such that the semigroup S is a subsemigroup of $\mathbb{Z}^d$. □

The following theorem by Gordan, in the frame of invariant theory, shows the connection between cones and semigroups.

Theorem 1.1 (Gordan's Lemma, [33])
If $C \subset \mathbb{R}^d$ is a rational cone and G a subgroup of $\mathbb{Z}^d$, then $C \cap G$ is an affine semigroup. □

Gordan's Lemma allows us to think of lattice points in cones as semigroups. We define, the other way around, the cone of a semigroup.

Definition 1.10 (Cone of S)
Given a subsemigroup S of $\mathbb{Z}^d$, let $\mathcal{C}(S)$ be the smallest cone in $\mathbb{R}^d$ containing S. □

Given a semigroup S, a set $\{g_1, g_2, \ldots, g_n\} \subset S$ is called a generating set for S if and only if for all a in S there exist $\ell_1, \ell_2, \ldots, \ell_n$ in $\mathbb{N}$ such that $a = \sum_{i=0}^{n} \ell_i g_i$. We say that the semigroup S is generated by $G \subset S$ if G is a generating set for S. Note that we do not set any kind of requirements about the elements of the generating set.

We already used the term lattice points in an intuitive way. Formally lattice points are elements of a lattice and lattice is a group isomorphic to $\mathbb{Z}^d$ (see [20]]). If no other lattice is specified, then one should assume that lattice means $\mathbb{Z}^d$.

We next investigate semigroups that have special structural properties. Let S be the intersection of a rational cone with a subgroup of $\mathbb{Z}^d$ (due to Gordan's Lemma the intersection is a semigroup). An important question is whether the semigroup generated by the generators of the cone is equal to S. In order to formalize the question we need the notions of integral element and integral closure.

Definition 1.11 (Integral element, [20])
Given a lattice $\mathcal{L}$ and a subsemigroup S of $\mathcal{L}$, an element $x \in \mathcal{L}$ is called integral over S if $cx \in S$ for some $c \in \mathbb{N}$. □

Definition 1.12 (Integral Closure)
Given a lattice $\mathcal{L}$ and a semigroup S of $\mathcal{L}$, the set of all elements of $\mathcal{L}$ that are integral over S is called the integral closure of S in $\mathcal{L}$ and is denoted by $\bar{S}_L$. □

A semigroup S is called integrally closed over (or saturated in) a lattice $\mathcal{L}$ if $S = \bar{S}_{\mathcal{L}}$. Thus, the question above becomes "is S saturated?". When we say that S is saturated, without specifying the lattice, we mean that it is saturated in $\mathbb{Z}^d$. Note that most authors would mean that the semigroup is saturated in its group of differences $\mathbb{Z}S$, rather than $\mathbb{Z}^d$.

The following proposition provides the connection between a saturated semigroup and its cone.

Proposition 1 (2.1.1. in [20])**.** *Let S be an affine semigroup of the lattice $\mathcal{L}$ generated by $g_1, g_2, \ldots, g_n$. Then*

- $\bar{S}_{\mathcal{L}} = \mathcal{L} \cap \mathcal{C}_{\mathbb{R}}(g_1, g_2, \ldots, g_n)$
- $\exists s_1, s_2, \ldots, s_k$ *such that* $\bar{S}_{\mathcal{L}} = \bigcup_{i=1}^{k} s_i + S$
- $\bar{S}_{\mathcal{L}}$ *is an affine semigroup.*

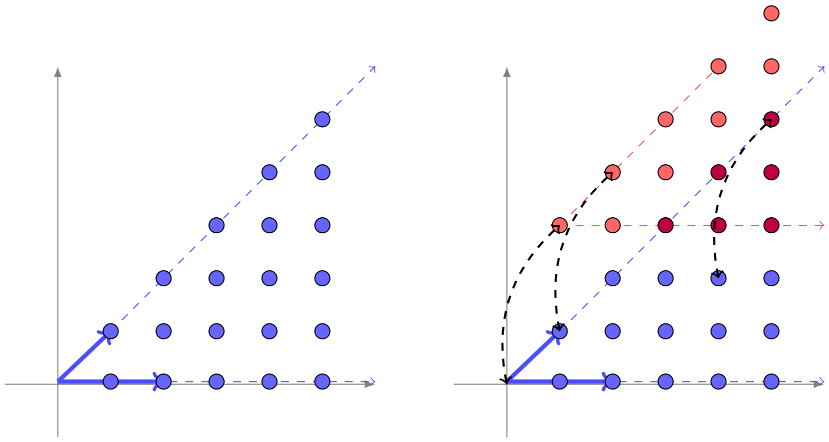
□

Now it is clear that there are three objects we could think of as cones, namely the polyhedral cone, the lattice points contained in the polyhedral cone and the semigroup generated by the generators of the cone. This already shows that we need some notation to distinguish these cases, but before establishing it, we will examine cones whose apices are not the origin and cones that have open faces.

Let $C \in \mathbb{R}^d$ be the cone $\mathcal{C}_{\mathbb{R}}(g_1, g_2, \ldots, g_n)$ and let $C' \in \mathbb{R}^d$ be the cone C translated by the vector $q \in \mathbb{R}^d$. Then we have a bijection $\phi : \mathbb{R}^d \to \mathbb{R}^d$ given by $\phi(x) = q + x$, where addition is the vector space addition, such that $C' = \phi(C)$. As long as q is in the lattice $\mathcal{L}$, then lattice points are mapped to lattice points. Although ϕ is a semigroup homomorphism, it is not a monoid homomorphism. This is why we prefer to view the monoids coming from cones with apex at the origin as semigroups and not monoids. It is important to note that any unimodular transformation, i.e., a linear transformation given by a square matrix with determinant ± 1, preserves the lattice point structure. [1] Thus

[1] for translation we have to homogenize in order to have a square matrix with determinant ± 1 giving the transformation. Otherwise we can allow transformations given by a unimodular matrix plus a translation.

there exists a bijection between the lattice points of the original and the transformed cone. This fact will be used later in order to enumerate lattice points in the fundamental parallelepiped of a cone.

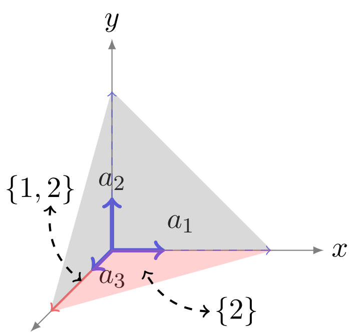


Translation of cones.

Assume C is a simplicial cone and let $[k] = I \subset \mathbb{N}$ be the index set for its generators. Then each face of C can be identified with a set $I_F \subseteq I$, since a face has the form

$$\left\{ x \in \mathbb{R}^d \;\middle|\; x = \sum_{i=1}^{k} \ell_i a_i, \ell_i \in \mathbb{R}, \ell_i > 0 \text{ for } i \notin I_F, \ell_j = 0 \text{ for } j \in I_F \right\}.$$

Essentially, this means that the face is generated by a subset of the generators. Note that I_F contains the indices of the generators of the simplicial cone C that are not used to generate F. A facet is identified with the index of the single generator not used to generate it.



Half-open cones

Given a cone C, removing from C the points lying on a facet F of C, we obtain a cone C' that has the same faces as C except for the faces included in F. From the previous discussion, for each facet we need only to mention the generator corresponding to that facet. We will use a $0-1$ vector o to express the openess of a cone, i.e., the openess vector has an entry of 1 in the position k if the facet corresponding to the k-th generator is open. If $I=[k]$ then the cone is open and if $\emptyset \neq I \subset [k]$ then the cone is half-open.

The above discussion motivates an updated definition of polyhedral cones and related notation.

Definition 1.13
Given $a_1, a_2, \ldots, a_k, q \in \mathbb{Z}^d$ and an index set $I \subseteq [k]$, we define:

- the (real) cone generated by $a_1, a_2, \ldots, a_k$ at q as

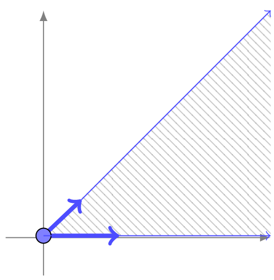
$$\mathcal{C}_{\mathbb{R}}^{I}\left(a_1, a_2, \ldots, a_k; q\right)=\left\{x \in \mathbb{R}^d : x=q+\sum_{i=1}^{k} \ell_i a_i, \ell_i \geq 0, \ell_i \in \mathbb{R}, \ell_j>0 \text{ for } j \in I\right\}$$

- the semigroup generated by $a_1, a_2, \ldots, a_k$ at q as

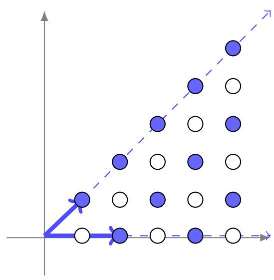
$$\mathcal{C}_{\mathbb{Z}}^{I}\left(a_1, a_2, \ldots, a_k; q\right)=\left\{x \in \mathbb{R}^d : x=q+\sum_{i=1}^{k} \ell_i a_i, \ell_i \geq 0, \ell_i \in \mathbb{Z}, \ell_j>0 \text{ for } j \in I\right\}$$

- given a lattice $\mathcal{L}$, the saturated semigroup generated by $a_1, a_2, \ldots, a_k$ at q as

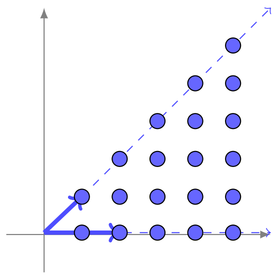
$$\bar{\mathcal{C}}_{\mathbb{R}, \mathcal{L}}^{I}\left(a_1, a_2, \ldots, a_k; q\right)=\mathcal{C}_{\mathbb{R}}^{I}\left(a_1, a_2, \ldots, a_k; q\right) \cap \mathcal{L}$$



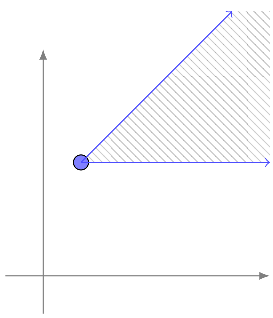
$\mathcal{C}_{\mathbb{R}}\left((1,1),(2,0)\right)$



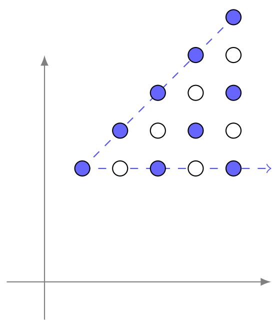
$\mathcal{C}_{\mathbb{Z}}\left((1,1),(2,0)\right)$



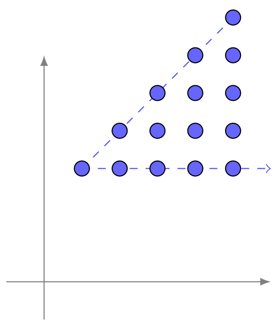
$\bar{\mathcal{C}}_{\mathbb{R},\mathbb{Z}^d}\left((1,1),(2,0)\right)$



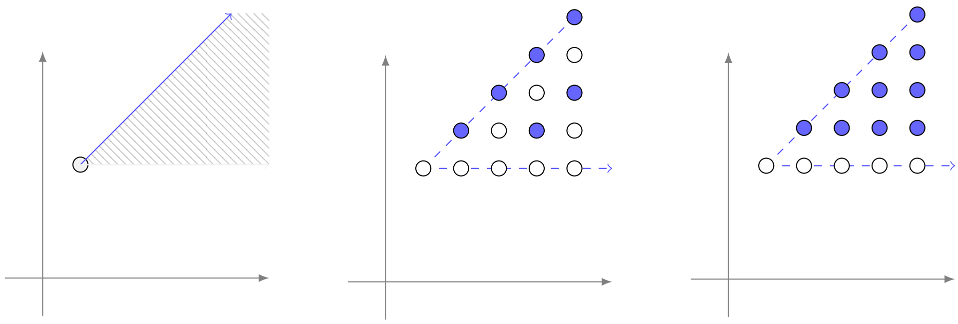
$\mathcal{C}_{\mathbb{R}}\left((1,1),(2,0);(1,3)\right)$



$\mathcal{C}_{\mathbb{Z}}\left((1,1),(2,0);(1,3)\right)$



$\bar{\mathcal{C}}_{\mathbb{R},\mathbb{Z}^d}\left((1,1),(2,0);1,3)\right)$



$\mathcal{C}_{\mathbb{R}}^{\{1\}}((1,1),(2,0);(1,3))$ $\mathcal{C}_{\mathbb{Z}}^{\{1\}}((1,1),(2,0);(1,3))$ $\bar{\mathcal{C}}_{\mathbb{R},\mathbb{Z}^d}^{\{1\}}((1,1),(2,0);(1,3))$

Illustration of the definitions.

□

An important tool in polyhedral geometry is triangulation. A polyhedral object with complicated geometry can be decomposed into simpler ones. The building block is the simplex. Since we are interested mostly in cones, we present the definition for the triangulation of a cone.

Definition 1.14 (Triangulation of a cone)
We define a triangulation of a real cone $C = \mathcal{C}_{\mathbb{R}}(g_1, g_2, \ldots, g_k)$ as a finite collection of simplicial cones $\Gamma = \{C_1, C_2, \ldots, C_t\}$ such that:

- $\bigcup C_i = C$,
- If $C' \in \Gamma$ then every face of C' is in Γ,
- $C_i \cap C_j$ is a common face of C_i and C_j.

□

The following proposition says that we can triangulate a cone without introducing new rays.

Proposition 2 (see [17])**.** *A pointed convex polyhedral cone C admits a triangulation Γ whose 1-dimensional cones are the extreme rays of C.* □

A concept that we will use heavily in later chapters is that of the vertex cone.

Definition 1.15 (Tangent and Feasible Cone)
Let $P \subseteq \mathbb{R}^d$ be a polyhedron and v a vertex of P, the *tangent cone* $\mathcal{K}_v$ and the *feasible cone* $\mathcal{F}_v$ of P at v are defined by

$$\begin{aligned} \mathcal{F}_v &:= \{u \mid \exists \delta > 0 : v + \delta u \in P\}, \\ \mathcal{K}_v &:= v + \mathcal{F}_v. \end{aligned}$$

□

We will call $\mathcal{K}_v$ a vertex cone.

1.2 Formal Power Series and Generating Functions

1.2.1 Formal Power Series

Starting from the univariate polynomial ring, we will climb up an hierarchy of algebraic structures that are related to polyhedral geometry and partition analysis. For a more detailed exposition see [34, 37] and references therein.

Let $\mathbb{K}$ be a field, that we can assume is the field of complex numbers. As usual, we denote by $\mathbb{K}[z]$ the polynomial ring in the variable z. We will generalize polynomials in four ways, namely by considering fractions of polynomials, allowing an infinite number of terms, allowing negative exponents and considering more than one variables.

The polynomial ring $\mathbb{K}[z]$ contains no zero-divisors, thus it is an integral domain. Since $\mathbb{K}[z]$ is an integral domain, we can define its fraction field, i.e., the field of univariate rational functions, denoted by $\mathbb{K}(z)$. A polynomial can be considered as a formal sum

$$\sum_{i\in I} a_i z^i \text{ for some finite index set } I \subseteq \mathbb{N} \text{ and } a_i \in \mathbb{K}.$$

If we allow as index sets infinite subsets of the natural numbers, then we obtain formal powerseries

$$a(z) = \sum_{i=0}^{\infty} a_i z^i \text{ for } a_i \in \mathbb{K}.$$

Another way to see formal powerseries is as sequences. Given an infinite sequence $A = [a_1, a_2, \ldots]$ with elements from $\mathbb{K}$ and a formal variable z, then $a(z)$ represents A. The usual definition of formal powerseries is as sequences. Define the following two operations of addition and (Cauchy) multiplication for two formal powerseries $a(z)$ and $b(z)$:

$$a(z) + b(z) = \sum_{i=0}^{\infty} (a_i + b_i)\, z^i,$$

$$a(z)b(z) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) z^i.$$

The ring of formal powerseries is the set

$$\mathbb{K}[\![z]\!] = \left\{ \sum_{i=0}^{\infty} a_i z^i \;\middle|\; a_i \in \mathbb{K} \right\}$$

equipped with the above defined addition and multiplication. Note that the ring of polynomials is a subring of the ring of formal powerseries. If we allow for polynomials containing negative exponents (but only finitely many terms), then we obtain the ring of Laurent polynomials. We denote this ring by $\mathbb{K}\left[z, z^{-1}\right]$ or $\mathbb{K}\left[z^{\pm 1}\right]$. Next, we construct the polynomial ring $\mathbb{K}[x, y]$ as $(\mathbb{K}[y])\,[x]$, i.e., the ring of polynomials in x with coefficients in the ring $\mathbb{K}[y]$. We define recursively the polynomial ring in d variables, denoted by

$\mathbb{K}[z_1, z_2, \ldots, z_d]$. The fraction field of $\mathbb{K}[z_1, z_2, \ldots, z_d]$, which is the multivariate version of the rational function field, is denoted by $\mathbb{K}(z_1, z_2, \ldots, z_d)$. As in the univariate case, allowing for negative exponents, we obtain the multivariate Laurent polynomial ring, denoted by $\mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_d^{\pm 1}]$. Like we did with polynomials, we can construct the ring $\mathbb{K}[[x, y]]$ of formal powerseries in x and y as the ring of formal powerseries in x with coefficients in $\mathbb{K}[[y]]$. A ring that we will use extensively in the next chapters is that of multivariate formal powerseries in d variables, defined recursively and denoted by $\mathbb{K}[[z_1, z_2, \ldots, z_d]]$.

An important property of the ring of formal powerseries is given by the following

Lemma 3 (see [27])**.** *If D is an integral domain then $D[[z]]$ is an integral domain as well.* □

Now, let us define the field of univariate Laurent series, which can be thought of either as the fraction field of the ring of univariate formal powerseries or as Laurent polynomials with finitely many terms of negative exponent but possibly infinitely many terms of positive exponent. More formally

Definition 1.16 (Univariate formal Laurent series)
The set of formal expressions

$$\mathbb{K}((z)) = \left\{ \sum_{k \in \mathbb{Z}} a_k z^k \;\middle|\; a_k \in \mathbb{K}, a_k = 0 \text{ for all but finitely many negative values of } k \right\}$$

equipped with the usual addition and (Cauchy) multiplication is the field of univariate formal Laurent series. □

The multivariate equivalent is much harder to deal with. In particular, there are many candidates as appropriate generalizations. We will explore some of them.

In [37], Xin presents two generalizations of formal Laurent series. The first and more straightforward is that of iterated Laurent series.

Definition 1.17 (Iterated Laurent series, Section 2.1 in [37])
The field of iterated Laurent series in k variables is defined recursively as

$$\mathbb{K}\langle\langle z_1, z_2, \ldots, z_k \rangle\rangle = \mathbb{K}\langle\langle z_1, z_2, \ldots, z_{k-1} \rangle\rangle((z_k))$$

where $\mathbb{K}\langle\langle z_1 \rangle\rangle = ((z_1))$. □

Next he presents Malcev-Neuman series, which allow the most general setting among the ones presented here.

Definition 1.18 (Malcev-Neuman series, Theorem 3-1.6 in [37])
Let G be a totally ordered monoid and R a commutative ring with unit. A formal series η on G has the form

$$\eta = \sum_{g \in G} a_g g$$

where $a_g \in R$ and g is regarded as a symbol. The support of η is defined to be the set $\{g \in G \mid a_g \neq 0\}$. A Malcev-Neumann series is a formal series on G that has a well-ordered support. We define $R_w[G]$ to be the set of all such MN-series. Then $R_w[G]$ is a ring. □

In [34], Aparicio-Monforte and Kauers introduce three algebraic structures related to multivariate formal Laurent series. They use polyhedral cones and the notion of a compatible additive ordering for their definitions.

Definition 1.19 (Compatible additive ordering [34])
Given a pointed simplicial cone $C \in \mathbb{R}^d$ with apex at the origin, a total order $\leq$ on $\mathbb{Z}^d$ is called additive if for all $a, b, c \in \mathbb{Z}^d$ we have $a \leq b \to a + c \leq b + c$. It is called compatible with the cone C if $0 \leq k$ for all $k \in C$. □

First they define the ring of multivariate formal Laurent series supported in a cone.

Definition 1.20 (Multivariate Laurent series from cones [34])
Given a pointed cone $C \in \mathbb{R}^d$, the set

$$\mathbb{K}_C \left[\!\left[z_1, z_2, \ldots, z_d \right]\!\right] = \left\{ \sum_{k \in C} a_k z^k \;\middle|\; a_k \in \mathbb{K} \right\}$$

equipped with the usual addition and Cauchy multiplication forms a ring . □

A larger ring is obtained if we fix an ordering and consider the collection $\mathcal{C}$ of all cones that are compatible with this ordering $\leq$. Then

$$\mathbb{K}_{\leq} \left[\!\left[z_1, z_2, \ldots, z_d \right]\!\right] = \bigcup_{C \in \mathcal{C}} \mathbb{K}_C \left[\!\left[z_1, z_2, \ldots, z_d \right]\!\right]$$

is a ring.

Finally, Aparicio-Monforte and Kauers construct a field of multivariate formal Laurent series by taking the union of translates of $\mathbb{K}_{\leq} \left[\!\left[z_1, z_2, \ldots, z_d \right]\!\right]$ to all lattice points.

$$\mathbb{K}_{\leq} (\!(z_1, z_2, \ldots, z_d)\!) = \bigcup_{e \in \mathbb{Z}^d} z^e \mathbb{K}_{\leq} \left[\!\left[z_1, z_2, \ldots, z_d \right]\!\right].$$

The lattice points of a pointed cone form a well-ordered set. Moreover, given the additive compatible ordering, required by the construction of $\mathbb{K}_{\leq} [\![z_1, z_2, \ldots, z_d]\!]$, the support of each object in $\mathbb{K}_{\leq} [\![z_1, z_2, \ldots, z_d]\!]$ is well-ordered (and 0 is the minimum). Finally, in the construction of $\mathbb{K}_{\leq} (\!(z_1, z_2, \ldots, z_d)\!)$ the supports appearing in $\mathbb{K}_{\leq} [\![z_1, z_2, \ldots, z_d]\!]$ may be translated, but this does not affect the fact that they are well-ordered. This means that these three constructions are examples of Malcev-Neuman series, as noted in [34].

Now we will put into the picture of the algebraic structures the sets of power series and rational functions occurring in the geometric context. The generating function for the lattice points of a rational simplicial polyhedral cone is a rational function, see next section.

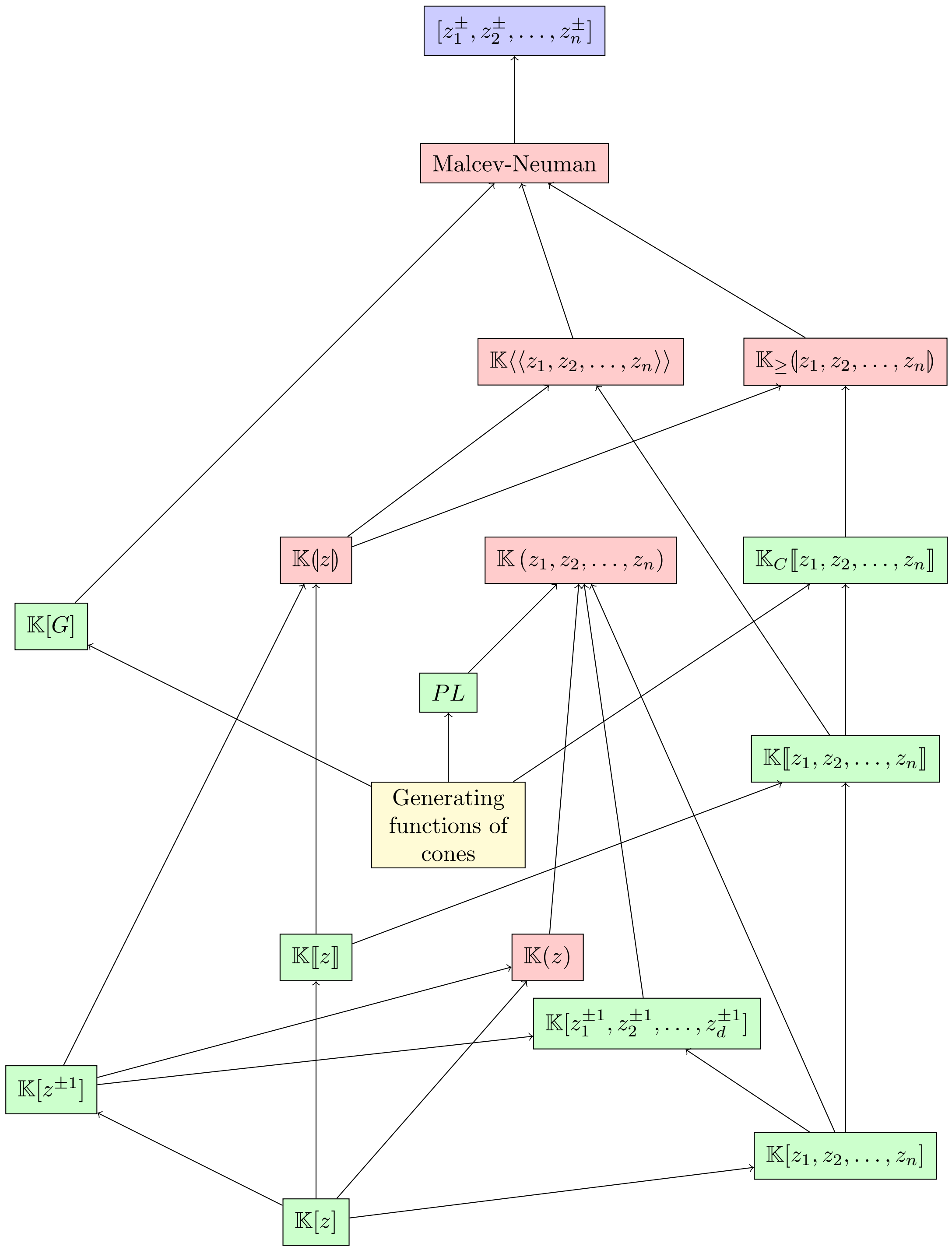
Definition 1.21 (Polyhedral Laurent Series [16])
The space PL of polyhedral Laurent series is the $\mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_d^{\pm 1}]$-submodule of $\mathbb{K}[[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_d^{\pm 1}]]$ generated by the set of formal series

$$\left\{ \sum_{s \in \left(C \cap \mathbb{Z}^d\right)} z^s \;\middle|\; C \text{ is a simplicial rational cone} \right\}.$$

□

Since any cone can be triangulated by using only simplicial cones, we have that PL contains the formal power series expressions of the generating functions of all rational cones. Due to the vertex cone decomposition of polyhedra (see Section 1.3.2), PL also contains the generating functions of all polyhedra.

The following diagram presents the relations of the structures discussed above.



Relations of algebraic structures related to polynomials, formal power series and rational functions.

1.2.2 Generating Functions

One of the most important tools in combinatorics and number theory, when dealing with infinite sequences, is that of generating functions. The hope is that a generating function, although encoding full information of an infinite object, has a short (or at least finite) representation. We will restrict the definitions presented here in a way that covers our use of generating functions without introducing generality that reduces clarity. For a detailed introduction see [27, 36].

The generating function is a tool for representing infinite sequences in a handy way. A sequence is a map from $\mathbb{N}$ to a set of values. We will assume that our values come from a field $\mathbb{K}$, but we note that for the most part of the thesis, the only possible values are 0 and ± 1. The usual notation for a sequence is $A = [a_0, a_1, a_2, \ldots] = (a_i)_{i \in \mathbb{N}}$. We define the generating function of A as the formal sum

$$\Phi_A = \sum_{i=0}^{\infty} a_i z^i$$

This is a formal powerseries in $\mathbb{K}[\![z]\!]$.

If we want to compute the generating function of a set S, subset of the natural numbers, we can use the above definition by considering the sequence $A = [a_0, a_1, a_2, \ldots]$, where $a_i = 1$ if $i \in S$ and $a_i = 0$ otherwise. More formally, we consider the indicator function of the set S, defined as

$$[S](x) = \begin{cases} 1 & : x \in S, \\ 0 & : x \notin S. \end{cases}$$

and then $A = ([S](i))_{i \in \mathbb{N}}$. We will use the notation Φ_S for the generating function of the set S, where $\Phi_S = \Phi_A$ for $A = ([S](i))_{i \in \mathbb{N}}$. Naturally, the first example is the natural numbers:

$$\Phi_{\mathbb{N}}(z) = \sum_{i \in \mathbb{N}} [\mathbb{N}](i) z^i = \sum_{i \in \mathbb{N}} z^i = \frac{1}{(1-z)}.$$

A slight variation is the sequence of even natural numbers. Then we have

$$\Phi_{2\mathbb{N}}(z) = \sum_{i \in \mathbb{N}} [2\mathbb{N}](i) z^i = \sum_{j \in \mathbb{N}} z^{2j} = \frac{1}{(1-z^2)}.$$

Although these two examples look too simple, it is their multivariate versions that cover the majority of the cases we are interested in.

With the above definition we can use generating functions to deal with subsets of the natural numbers, but this is not sufficient for our purposes. We need to be able to describe sets containing negative integers. Thus, we extend the definition for subsets of $\mathbb{Z}$. Given a set $S \subseteq \mathbb{Z}$, we define

$$\Phi_S = \sum_{i \in S} z^i.$$

This is not a formal powerseries anymore and depending on the set S it may not be a formal Laurent series either.

Although univariate generating functions are very useful for counting, for our purposes multivariate ones are essential. The generalization is straightforward.

Definition 1.22 (Generating Function)
Given a set $S \subset \mathbb{Z}^d$, we define the generating function of S as the formal sum

$$\Phi_S = \sum_{i \in \mathbb{Z}^d} [S](i) z^i.$$

□

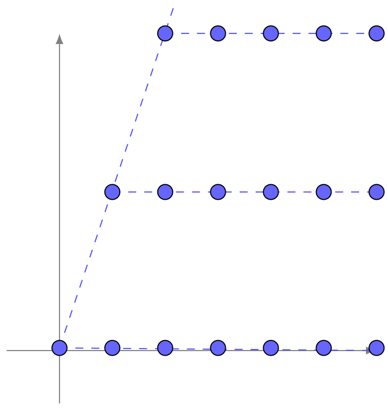
The formal sum representation of a set is no more than syntactic sugar. The representation of a set of numbers or of its formal sum representation have the same (potentially) infinite number of terms. As already stated, the expectation is to have a more condensed representation. A glimpse on that was given in the example of the generating function of the natural numbers, through the use of the geometric series expansion formula. We proceed now more formally and setting the necessary notation, defining what a rational generating function is.

Definition 1.23 (Rational Generating Function)
Let $S \subset \mathbb{Z}^d$. If there exists a rational function in $\mathbb{K}(z_1, z_2, \ldots, z_d)$, which has a series expansion equal to Φ_S, then we denote that rational function by $\rho_S(z_1, z_2, \ldots, z_d)$ and call it a rational generating function of S. □

Here one should note that we make no assumptions on the form of the rational function (e.g. reduced, the denominator has a specific form etc.). This means that a formal powerseries (or formal Laurent series) may have more than one rational function forms. In the other way, the same rational function can have more than one series expansions. In other words, although the formal sum generating function of a set is a well defined object, the rational generating function of a set is subject to more parameters. It is not well defined (yet), but extremely useful. We will heavily use rational generating functions and in Section 1.3 we will explain why we are on safe ground.

1.2.3 Generating Functions for Semigroups

In this thesis we deal with generating functions related to polyhedra. For this, it is sufficient to compute with generating functions of cones. There are two types of semigroups that are of interest for us, the discrete semigroup generated by a set of integer vectors and the corresponding saturated semigroup. We first compute the generating function of the semigroup $S = \mathcal{C}_{\mathbb{Z}}((1,3),(1,0))$.



$\mathcal{C}_{\mathbb{Z}}(\mathbb{N})((1,3),(1,0))$.

The generating function as a formal power series is

$$\Phi_S(\mathbf{z}) = \sum_{\mathbf{i}\in\{k(1,3)+\ell(1,0)|k,\ell\in\mathbb{N}\}} \mathbf{z}^{\mathbf{i}} = \sum_{k,\ell\in\mathbb{N}} \mathbf{z}^{k(1,3)+\ell(1,0)} = \left(\sum_{k\in\mathbb{N}} \mathbf{z}^{k(1,3)}\right)\left(\sum_{\ell\in\mathbb{N}} \mathbf{z}^{\ell(1,0)}\right).$$

Thus, by the geometric series expansion formula we have

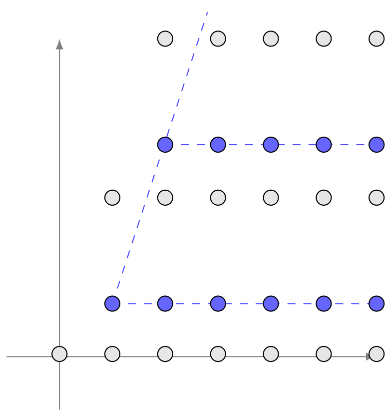
$$\rho_S(\mathbf{z}) = \left(\frac{1}{1-\mathbf{z}^{(1,3)}}\right)\left(\frac{1}{1-\mathbf{z}^{(1,0)}}\right) = \frac{1}{(1-z_1z_2^3)(1-z_1)}.$$

In general it is easy to see that if $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are linearly independent, then for the semigroup $S = \mathcal{C}_{\mathbb{Z}}(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$ we have

$$\rho_S(\mathbf{z}) = \frac{1}{(1-\mathbf{z}^{\mathbf{a_1}})(1-\mathbf{z}^{\mathbf{a_2}})\cdots(1-\mathbf{z}^{\mathbf{a_n}})}.$$

This is true because in the expansion of $\frac{1}{(1-\mathbf{z}^{\mathbf{a_1}})(1-\mathbf{z}^{\mathbf{a_2}})\cdots(1-\mathbf{z}^{\mathbf{a_n}})}$ we get as exponents all non-negative integer combinations of the exponents in the denominator.

Observe that if we translate the semigroup to a lattice point, then the structure does not change at all.



$\mathcal{C}_{\mathbb{Z}}(\mathbb{N})((1,3),(1,0);(1,1))$.

There is a bijection between $\mathcal{C}_{\mathbb{Z}}((1,3),(1,0))$ and $\mathcal{C}_{\mathbb{Z}}((1,3),(1,0;(1,1))$, given by $f(\mathbf{s}) = \mathbf{s} + (1,1)$. Translating this to generating functions we have that

$$\begin{aligned}
\Phi_{\mathcal{C}_{\mathbb{Z}}((1,3),(1,0);(1,1))}(\mathbf{z}) &= \sum_{\mathbf{i} \in \{k(1,3)+\ell(1,0)+(1,1) | k,\ell \in \mathbb{N}\}} \mathbf{z}^{\mathbf{i}} \\
&= \sum_{k,\ell \in \mathbb{N}} \mathbf{z}^{k(1,3)+\ell(1,0)+(1,1)} \\
&= \mathbf{z}^{(1,1)} \left(\sum_{k \in \mathbb{N}} \mathbf{z}^{k(1,3)} \right) \left(\sum_{\ell \in \mathbb{N}} \mathbf{z}^{\ell(1,0)} \right).
\end{aligned}$$

Now it is clear that the generating function for $\mathcal{C}_{\mathbb{Z}}((1,3),(1,0;(1,1))$ is rational , namely

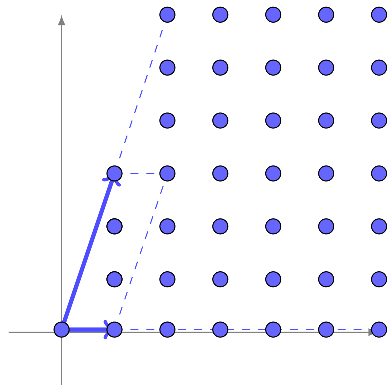
$$\rho_{\mathcal{C}_{\mathbb{Z}}((1,3),(1,0);(1,1))}(\mathbf{z}) = \mathbf{z}^{(1,1)} \left(\frac{1}{1-\mathbf{z}^{(1,3)}} \right) \left(\frac{1}{1-\mathbf{z}^{(1,0)}} \right) = \frac{z_1 z_2}{\left(1 - z_1 z_2^3\right)\left(1 - z_1\right)}.$$

In general for a semigroup $S = \mathcal{C}_{\mathbb{Z}}(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}; \mathbf{q})$ we have

$$\rho_S(\mathbf{z}) = \frac{\mathbf{z}^{\mathbf{q}}}{(1-\mathbf{z}^{\mathbf{a_1}})(1-\mathbf{z}^{\mathbf{a_2}})\cdots(1-\mathbf{z}^{\mathbf{a_n}})}.$$

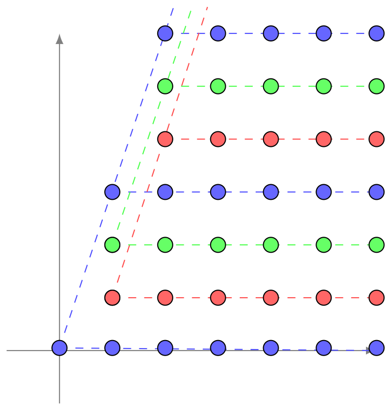
Saturated Semigroup

Let's consider the saturated semigroup $\bar{S} = \bar{\mathcal{C}}_{\mathbb{R},\mathbb{Z}^d}((1,3),(1,0))$, and compute its generating function as a formal powerseries $\Phi_{\bar{S}} = \sum_{s \in \bar{S}} \mathbf{z}^s$.



$\bar{\mathcal{C}}_{\mathbb{R},\mathbb{Z}^d}((1,3),(1,0))$.

We observe in the following figure that $\bar{S}$ can be partitioned in three sets. The blue points are reachable starting from the origin and using only the cone generators. The red points are reachable by using only the cone generators if we start from the point $(1,1)$. And similarly for the green ones if we start from $(1,2)$.



The saturated semigroup separated in three subsets.

This provides a direct sum decomposition

$$\begin{aligned}\bar{\mathcal{C}}_{\mathbb{R},\mathbb{Z}^d}\left((1,3),(1,0)\right) &= \mathcal{C}_{\mathbb{Z}}\left((1,3),(1,0)\right)\\ &\oplus \mathcal{C}_{\mathbb{Z}}\left((1,3),(1,0);(1,1)\right)\\ &\oplus \mathcal{C}_{\mathbb{Z}}\left((1,3),(1,0);(1,2)\right).\end{aligned}$$

In general we have the following relation:

$$\bar{\mathcal{C}}_{\mathbb{R},\mathbb{Z}^d}\left(\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_n}\right) = \oplus_{\mathbf{q}\in\Pi_{\mathbb{Z}^d}\left(\mathcal{C}_{\mathbb{R}}(\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_n})\right)}\mathcal{C}_{\mathbb{Z}}\left(\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_n};\mathbf{q}\right).$$

If the apex q of the cone of the saturated semigroup is not the origin, we have to translate every lattice point by q, thus multiply the generating function by $\mathbf{z^q}$.
From the relation above and the form of the rational generating function of a discrete semigroup $\mathcal{C}_{\mathbb{Z}}\left(\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_n};\mathbf{q}\right)$ we immediately deduce the lemma:

Lemma 4. *For a saturated semigroup* $S = \bar{\mathcal{C}}_{\mathbb{R},\mathbb{Z}^d}\left(\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_n};q\right)$ *we have*

$$\rho_S(\mathbf{z}) = \frac{z^q\sum_{\alpha\in\Pi_{\mathbb{Z}^d}\left(\mathcal{C}_{\mathbb{R}}(\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_n})\right)}\mathbf{z}^{\alpha}}{\left(1-\mathbf{z^{a_1}}\right)\left(1-\mathbf{z^{a_2}}\right)\cdots\left(1-\mathbf{z^{a_n}}\right)}.$$

□

Generating Functions of Open Cones

Given a simplicial cone

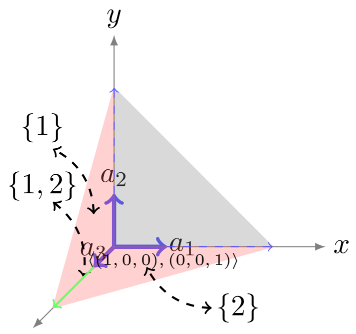
$$\mathcal{C}^I_{\mathbb{R}}\left(\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_k};\mathbf{q}\right) = \left\{x\in\mathbb{R}^d : x = q+\sum_{i=1}^{k}\ell_i\mathbf{a}_i, \ell_i\in\mathbb{N}, \ell_j>0 \text{ for } j\in I\right\}$$

we can apply the inclusion exclusion principle to get an expression of the open cone as a signed sum of closed cones

$$\mathcal{C}^I_{\mathbb{R}}\left(\mathbf{a_1},\mathbf{a_2},\ldots,\mathbf{a_k};\mathbf{q}\right) = \sum_{M\subset I}(-1)^{|M|}\mathcal{C}_{\mathbb{R}}\left(\left(\mathbf{a_i}\right)_{i\in I\setminus M};\mathbf{q}\right).$$

Example 1. *Given the half-open cone* $\mathcal{C}_{\mathbb{R}}^{\{1,2\}}\left((1,0,0),(0,1,0),(0,0,1)\right)$ *we have*

$$\begin{aligned}\mathcal{C}_{\mathbb{R}}^{\{1,2\}}\left((1,0,0),(0,1,0),(0,0,1)\right) &= \mathcal{C}_{\mathbb{R}}\left((1,0,0),(0,1,0),(0,0,1)\right)\\ &\quad - \left(\mathcal{C}_{\mathbb{R}}\left((1,0,0),(0,0,1)\right) + \mathcal{C}_{\mathbb{R}}\left((0,1,0),(0,0,1)\right)\right)\\ &\quad + \mathcal{C}_{\mathbb{R}}\left((0,0,1)\right).\end{aligned}$$



Inclusion-exclusion for $\mathcal{C}_{\mathbb{R}}^{\{1,2\}}\left((1,0,0),(0,1,0),(0,0,1);0\right)$.

□

1.3 Formal Series, Rational Functions and Geometry

In this section we review some facts from polyhedral geometry and generating functions of polyhedra. The main goal is to arrive to a consistent correspondence between rational generating functions for polyhedra and formal powerseries. This is done through the definition of expansion directions. In the end of the section we present two fundamental theorems from polyhedral geometry for the decomposition of rational generating functions and shortly review Barvinok's algorithm.

1.3.1 Indicator and rational generating functions

We start by the definition of the space of indicator functions of rational polyhedra.

Definition 1.24 (from [11])
The real vector space spanned by indicator functions $[P]$ of rational polyhedra $P \in \mathbb{R}^d$ is called the algebra of rational polyhedra in $\mathbb{R}^d$ and is denoted by $\mathcal{P}_{\mathbb{Q}}\left(\mathbb{R}^d\right)$. □

Theorem 3.3 in [11] is essential for the connection of geometry with rational functions.

Theorem 1.2 (VIII.3.3 in [11])
There exists a map

$$\tau : \mathcal{P}_{\mathbb{Q}}\left(\mathbb{R}^d\right) \to \mathbb{C}\left(z_1, z_2, \ldots, z_d\right)$$

such that

- τ is a linear transformation (valuation).
- If $P \in \mathbb{R}^d$ is a rational polyhedron without lines, then $\tau[P] = \rho_P(\mathbf{z})$ such that $\rho_{P\cap\mathbb{Z}^d}(\mathbf{z}) = \Phi_{P\cap\mathbb{Z}^d}(\mathbf{z})$, provided that $\Phi_{P\cap\mathbb{Z}^d}(\mathbf{z})$ converges absolutely.
- For a function $g \in \mathcal{P}_{\mathbb{Q}}\left(\mathbb{R}^d\right)$ and $\mathbf{q} \in \mathbb{Z}^d$, let $h(\mathbf{z}) = g(\mathbf{z} - \mathbf{q})$ be a shift of g. Then $\tau h = \mathbf{z}^{\mathbf{q}}\tau g$.
- If $P \in \mathbb{R}^d$ is a rational polyhedron containing a line, then $\tau[P] \equiv 0$.

□

Let us see how this map works in an example.

Example 2. *Let $d = 1$. We consider the polyhedra:*

- $P_+ = \mathbb{R}_+$ *(including the origin)*
- $P_- = \mathbb{R}_-$ *(including the origin)*
- $P = \mathbb{R}$ *(the real line)*
- $P_0 = \{0\}$

The associated generating functions (converging in appropriate regions) are:

- $\Phi_{P_+\cap\mathbb{Z}}(z) = \displaystyle\sum_{m\in P_+\cap\mathbb{Z}} z^m = \sum_{m=0}^{\infty} z^m = \frac{1}{1-z} = \rho_{P_+\cap\mathbb{Z}}(z)$

- $\Phi_{P_-\cap\mathbb{Z}}(z) = \displaystyle\sum_{m\in P_-\cap\mathbb{Z}} z^m = \sum_{m=-\infty}^{0} z^m = \frac{1}{1-z^{-1}} = \rho_{P_-\cap\mathbb{Z}}(z)$

- $\Phi_{P_0\cap\mathbb{Z}}(z) = \displaystyle\sum_{m\in P_0\cap\mathbb{Z}} z^m = x^0 = 1 = \rho_{P_0\cap\mathbb{Z}}(z)$

From Theorem 1.2 we have that:

- $\tau[P_+] = \frac{1}{1-z}$
- $\tau[P_-] = \frac{1}{1-z^{-1}}$
- $\tau[P_0] = 1$

By inclusion-exclusion we have

$$[P] = [P_+] + [P_-] - [P_0]$$

Thus one expects that

$$0 = \tau[P] = \tau[P_+] + \tau[P_-] - \tau[P_0] = \frac{1}{1-z} + \frac{1}{1-z^{-1}} - 1$$

which is indeed the case. In other words

$$\rho_{P\cap\mathbb{Z}}(z) = \rho_{P_+\cap\mathbb{Z}}(z) + \rho_{P_-\cap\mathbb{Z}}(z) - \rho_{P_0\cap\mathbb{Z}}(z).$$

□

Example 3. *Compute the integer points of $P = [k, n] \subset \mathbb{R}$ for $k < n$. Let $P_1 = [-\infty, n]$ and $P_2 = [k, \infty]$. Then*

$$[P] = [P_1] + [P_2] - [\mathbb{R}]$$

and

- $\Phi_{P_1\cap\mathbb{Z}}(z) = \sum_{m=k}^{\infty} z^m = \frac{z^k}{1-z} \to \tau[P_1] = \frac{z^k}{1-z} = \rho_{P_1\cap\mathbb{Z}}(z)$
- $\Phi_{P_2\cap\mathbb{Z}}(z) = \sum_{m=-\infty}^{n} z^m = \frac{z^n}{1-z^{-1}} \to \tau[P_2] = \frac{z^n}{1-z^{-1}} = \rho_{P_2\cap\mathbb{Z}}(z)$
- $\tau[\mathbb{R}] = 0 = \rho_{\mathbb{R}\cap\mathbb{Z}}(z)$

Thus

$$\rho_{P\cap\mathbb{Z}} = \frac{z^k}{1-z} + \frac{z^n}{1-z^{-1}} = \frac{z^k - z^{n+1}}{1-z}$$

This is expected because $\dfrac{z^k}{1-z}$ *is the generating function of the ray starting at* k*, while* $\dfrac{z^{n+1}}{1-z}$ *is the generating function of the ray starting at* $n+1$*. Subtracting the second from the first we obtain the segment* P*.*

If we evaluate $\rho_{P\cap\mathbb{Z}}(z)$ *at* $z=1$ *(using de l'Hospital's rule) we get* $n-k+1$*, the number of lattice points in* P*.* $\square$

Theorem 1.2 provides a correspondence between different representations of generating functions of polyhedra.

Given an indicator function $g : \mathbb{R}^d \to \{0,1\}$, $g \in \mathcal{P}_{\mathbb{Q}}\left(\mathbb{R}^d\right)$, we consider its restriction $g' : \mathbb{Z}^d \to \{0,1\}$. We denote by $\mathcal{P}_{\mathbb{Q}}\left(\mathbb{Z}^d\right)$ the set of restricted indicator functions.

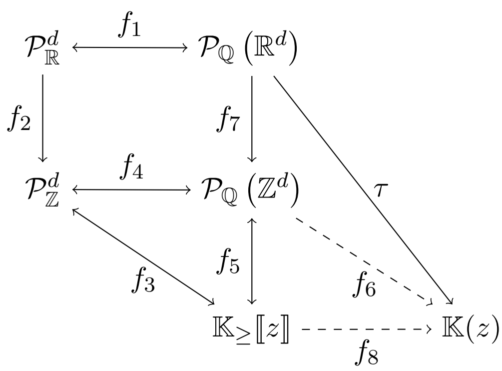


Figure 1.1: Polyhedra, indicator functions, Laurent and rational generating functions.

The arrows in Figure 1.1 have the following meaning:

f_1 Bijection between polyhedra in $\mathbb{R}^d$ and their indicator functions.

f_2 Map a polyhedron $P \in \mathbb{R}^d$ to $P \cap \mathbb{Z}^d$.

f_3 Bijection between sets of lattice points of polyhedra and their Laurent series generating functions.

f_4 Bijection between lattice points in polyhedra and their restricted indicator functions.

f_5 Bijection between restricted indicator function of polyhedra and their Laurent series generating functions.

f_6 Map the restricted indicator function of a polyhedron to a rational function.

f_7 Restriction.

f_8 Map a Laurent series with support in a polyhedron to a rational function.

In order to be able to map multivariate Laurent series to rational functions we restrict to series in $\mathbb{K}_{\geq}[[z]]$, i.e., we assume that the series have support in a pointed cone [34]. The most important maps are f_8 and f_6, but unfortunately they are not bijective. We will first see why and then fix this bijection.

When trying to establish the connection between formal powerseries, rational functions and polyhedral geometry we encounter problems due to the one-to-many relation between rational functions and their Laurent expansions.

Example 4 ([23])**.** *Consider the finite set $S = \{0, 1, 2, \ldots, a\} \subset \mathbb{N}$. Its rational generating function is the truncated geometric series*

$$\rho_S(z) = \frac{1 - z^{a+1}}{1 - z}$$

Now we can rewrite this expression as

$$\begin{aligned}\rho_S(z) &= \frac{1}{(1-z)} - \frac{z^{a+1}}{(1-z)} \\ &= \frac{1}{(1-z)} + \frac{z^a}{(1-z^{-1})}\end{aligned}$$

and we can expand each summand using the geometric series expansion formula.

$$\begin{aligned}\rho_S(z) &= \frac{1}{(1-z)} + \frac{z^a}{(1-z^{-1})} \\ &= \left(z^0 + z^1 + z^2 + \ldots\right) + \left(z^a + z^{a-1} + z^{a-2} + \ldots\right) \\ &= 2\Phi_S(z) + \sum_{i=a+1}^{\infty} z^i + \sum_{i=-\infty}^{-1} z^i.\end{aligned}$$

If we interpret each of the summands geometrically we end up with the required segment counted twice and the rest of the line counted once, which is not what we expected. This is due to the fact that the power series expansions around 0 *and* ∞ *of the two summands, viewed as analytic functions, have disjoint regions of convergence. In particular we have:*

$$\frac{1}{(1-z)} = \begin{cases} z^0 + z^1 + z^2 + \cdots & \text{for} |z| < 1, \\ -z^{-1} - z^{-2} - z^{-3} \cdots & \text{for} |z| > 1 \end{cases}$$

and

$$\frac{z^a}{1-z^{-1}} = \begin{cases} -z^{a+1} - z^{a+2} - z^{a+3} \cdots & \text{for} |z| < 1, \\ z^a + z^{a-1} + z^{a-2} \cdots & \text{for} |z| > 1. \end{cases}$$

Thus, only the choices where the region of convergence coincide seem natural:

$$\begin{aligned}\rho_S(z) &= \frac{1}{(1-z)} + \frac{z^a}{(1-z^{-1})} \\ &= \left(z^0 + z^1 + z^2 + \cdots\right) + \left(-z^{a+1} - z^{a+2} - z^{a+3} \cdots\right) \qquad = \Phi_S(z)\end{aligned}$$

or

$$\begin{aligned}\rho_S(z) &= \frac{1}{(1-z)} + \frac{z^a}{(1-z^{-1})} \\ &= \left(-z^{-1} - z^{-2} - z^{-3} \cdots\right) + \left(z^a + z^{a-1} + z^{a-2} \cdots\right) \qquad = \Phi_S(z).\end{aligned}$$

□

Moving from the rational generating function of a set to its geometric representation, we are fixing an expansion direction. This is implicitly done through choosing the Laurent expansion we will interpret geometrically. In Example 4 essentially we pick a direction (either towards positive infinity of negative infinity) and consistently take series expansions with respect to this direction. Even if each of the summands have a perfectly meaningful expansion with respect to any direction, when we want to make sense out of their sum (or simultaneously consider their geometry) we need a consistent choice. The following example shows how this is important for our purposes.

Example 5. *Consider the rational function* $f(z) = \frac{z^3}{(1-z)}$. *One can take either the "forward" or the "backward" expansion of* f *as Laurent power series corresponding to expansions around* 0 *or* ∞.

$$\frac{z^3}{(1-z)} = \begin{cases} z^3 + z^4 + z^5 + z^6 + \cdots & = F_1 \\ -z^2 - z^1 - z^0 - z^{-1} - \cdots & = F_2 \end{cases}$$

If we denote by F_i' *the power series having only the terms of* F_i *that have non-negative exponents then*

$$F_1' = z^3 + z^4 + z^5 + z^6 + \cdots = \Phi_{[3,\infty]}(z)$$

and

$$F_2' = -z^2 - z^1 - z^0 = -\Phi_{[0,2]}(z)$$

Observe that the two resulting rational functions $\rho_{[3,\infty]}(z)$ *and* $-\rho_{[0,2]}(z)$ *are different, one is a polynomial and the other is not, although* F_1 *and* F_2 *are expansions of the same rational function.* □

The operation of keeping only the terms with non-negative exponents is a special kind of taking intersections in terms of geometry. Moreover, it is an essential part of the computation of $\Omega_{\geq}$ in a geometric way (see Section 3.1).

Now we will define the notion of forward expansion direction more formally. For $\mathbf{b_i} \in \mathbb{R}^n$, let $B = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ be an ordered basis of $\mathbb{R}^n$ and

$$H_B = \left\{ \sum_i \ell_i \mathbf{b_i} : \ell_i \in \mathbb{R}, \text{if } j \in [n] \text{ is minimal such that } \ell_j \neq 0 \text{ then } \ell_j > 0 \right\} \cup \{0\}.$$

We need the notion of the recession cone in order to define directions.

Definition 1.25 (Recession Cone)
Given a polyhedron P, its recession cone is defined as

$$\operatorname{rec}(P) = \{y \in \mathbb{R}^n : x + ty \in P \text{ for all } x \in P \text{ and for all } t \geq 0\}$$

□

In order for a Laurent series with support S to be well defined we require that there exists a polyhedron P such that $S \subset P$ and $\operatorname{rec}(P) \subset H_B$, for some ordered basis B. We say that H_B defines the "forward directions", in the following sense.

Definition 1.26 (Forward Direction)
For any ordered basis B of $\mathbb{R}^n$ we call $v \in \mathbb{R}^n$ forward if and only if $v \in H_B$. □

Note that for every $v \in \mathbb{Z}^n - \{0\}$ either v or $-v$ is forward. Now we can fix the problem we had in the diagram about correspondence of generating functions.

Lemma 5. *Fix an ordered basis. If Laurent series expansions are taken, consistently, only with respect to forward directions, then there is a bijection between Laurent series generating functions of lattice points of polyhedra and the respective rational generating functions.* □

From now on, we will use series expansions using only forward directions.

Example 6. *Let $P_1 = [k, \infty]$ and $P_2 = [-\infty, n]$. We want to compute $\Phi_{P_1+P_2}(z)$ and $\rho_{P_1+P_2}(z)$.*

The forward directions are given by the basis $B = \{1\}$ of $\mathbb{R}$ and thus H_B contains all vectors that have positive first coordinate, i.e., $v \in \mathbb{R}$ with $v > 0$. In other words, we accept any Laurent series that are (possibly) infinite towards ∞, but not towards $-\infty$.

We have

$$\Phi_{P_1\cap\mathbb{Z}}(z) = \sum_{m=k}^{\infty} z^m = \frac{z^k}{1-z} \Rightarrow \tau[P_1] = \frac{z^k}{1-z} = \rho_{P_1\cap\mathbb{Z}}(z)$$

but for the second cone, the expansion should be consistent, i.e., towards ∞. Thus we will use $P_3 = [n+1, \infty]$ and the fact that $P_2 = \mathbb{R} - P_3$. We know that $[P_2] = [\mathbb{R}] - [P_3]$ since $P_2 \cap P_3 = \emptyset$. From Theorem 1.2 we know that $\tau[\mathbb{R}] = 0$, thus $[P_2] = -[P_3]$. This implies that in the computation of the rational generating function we can ignore $\mathbb{R}$.

$$\Phi_{P_2\cap\mathbb{Z}}(z) = \Phi_{\mathbb{R}\cap\mathbb{Z}}(z) - \Phi_{P_3\cap\mathbb{Z}}(z) = \sum_{m=-\infty}^{\infty} z^m - \sum_{m=n+1}^{\infty} z^m$$

$$\sim -\sum_{m=n+1}^{\infty} z^m = -\frac{z^{n+1}}{1-z} \Rightarrow \tau[P_2] = -\frac{z^{n+1}}{1-z} = \rho_{P_2\cap\mathbb{Z}}(z).$$

$\sim$ means that the two Laurent series correspond to polyhedra with the same rational function. Geometrically this means to "flip" the ray from pointing to $-\infty$ to pointing to ∞.

Thus

$$\rho_{P\cap\mathbb{Z}}(z) = \frac{z^k}{1-z} - \frac{z^{n+1}}{1-z} = \frac{z^k - z^{n+1}}{1-z}.$$

□

The reason why this phenomenon is not a problem for the classical $\Omega_{\geq}$ algorithms (see Section 3.1) is that by the construction of the crude generating function, we make a choice of expansion directions for all λ variables. In particular, when converting the multisum expression into a rational function, we chose to consider only "forward" expansions for all variables (where forward is defined with respect to the standard basis).

1.3.2 Vertex cone decompositions

We present two theorems that provide formulas for writing the rational generating function of a polyhedron as a sum of rational generating functions of vertex cones. The main difference between the two formulas is that in the Lawrence-Varchenko formula all cones involved are considered as forward expansions, while in Brion's formula this is not true.

As a matter of fact, Brion's formula holds true in the indicator functions level moding out lines, which is equivalent to changing the direction of a ray. When reading the formula in the rational function level though, this is not visible since the generating functions of cones obtained by flipping directions are different as Laurent series but the same as rational functions.

Theorem 1.3 (Brion, 4.5 in [11])
Let $P \subset \mathbb{R}^d$ be a rational polyhedron and $\mathcal{K}_v$ be the tangent cone of P at vertex v. Then

$$[P] = \sum_{v \text{ vertex of } P} [\mathcal{K}_v] \text{ mod polyhedra containing lines}$$

□

Observe that from a general inclusion exclusion formula for indicator functions of tangent cones, if we mod out polyhedra containing lines, then only the vertex cones survive. As a corollary, applying the map of Theorem1.2, we obtain

Proposition 3 (see [17]). *If P is a rational polyhedron, then*

$$\rho_P(\mathbf{z}) = \sum_{v \text{ vertex of } P} \rho_{\mathcal{K}_v}(\mathbf{z})$$

□

A similar theorem is given by Lawrence and Varchenko (independently), see [16]. Assume P is a simple polytope in $\mathbb{R}^d$ and fix a direction $\xi \in \mathbb{R}^d$ that is not perpendicular to any edge of P. Denote by $E_v^+(\xi)$ the set of edge directions w at the vertex v such that $w \cdot \xi > 0$ and $E_v^-(\xi)$ the set of edge directions w such that $w \cdot \xi < 0$. We define the vertex cones at v by flipping the direction of the edges that belong in $E_v^-(\xi)$. More precisely

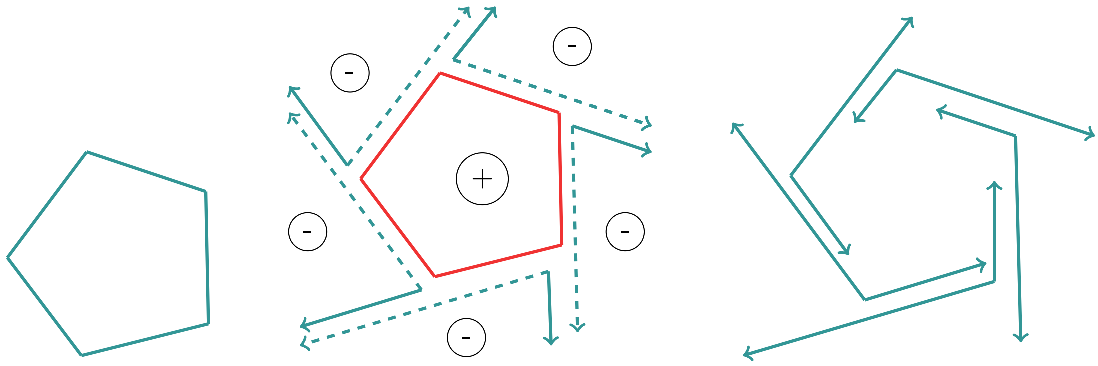
$$K_v^\xi = v + \sum_{w \in E_v^+(\xi)} \mathbb{R}_{\geq 0} w + \sum_{w \in E_v^-(\xi)} \mathbb{R}_{<0} w$$

Theorem 1.4 (Lawrence-Varchenko, [16])
At the rational generating functions level we have

$$\rho_P(\mathbf{z}) = \sum_{v \text{ a vertex in } P} (-1)^{|E_v^-(\xi)|} \rho_{K_v^\xi}(\mathbf{z})$$

□

In other words, for each flip when constructing the vertex cone we multiply the rational generating function of the cone by -1. In the following figures, we see in action the application of a vertex cone decomposition for a pentagon. [2]



1.3.3 Barvinok's Algorithm

Barvinok's algorithm [12, 28], if the dimension is fixed, is a polynomial-time algorithm that computes the rational generating function of a rational polyhedron.

The main idea behind the algorithm is to compute a unimodular decomposition of the given polyhedron. To this end, we first triangulate in order to obtain simplicial cones. If we try to decompose a simplicial cone C into cones that sum up to C (possibly using inclusion-exclusion), in the worst case, we will need as many cones as the number of lattice points in the fundamental parallelepiped of the cone. This means that instead of having a potentially exponentially big numerator polynomial, we have a sum of very simple rational functions (with numerator equal to 1), but with potentially exponentially many summands.

[2] These figures were shown to me by Felix Breuer.

Barvinok proposed the use of signed decompositions. Instead of decomposing the cone C into unimodular cones that are contained in C, one can find a unimodular decomposition where cones with generators in the exterior of C are involved. In that case, the generating functions of the unimodular cones involved in the sum will have a sign. The advantage of such a signed unimodular decomposition is that it is computable in polynomial time (in fixed dimension), which implies that the number of unimodular cones is not exponential as well.

Algorithm 1 Summary of Barvinok's algorithm

Require: $P \subset \mathbb{R}^d$ is a rational polyhedron in h-representation

1: Compute the vertices of P, denoted by v_i.
2: Compute the vertex cones P_{v_i}.
3: If necessary, triangulate P_{v_i}.
4: Compute a signed unimodular decomposition (find a shortest vector).
5: Compute the lattice points in the fundamental parallelepiped of each cone.
6: **return** $\sum_{i=1,2\ldots,(\text{number of cones})} \epsilon_i \frac{1}{\prod_{j=1}^{d}\left(1-\mathbf{z}_{]}^{\mathbf{b}}\mathbf{ij}\right)}$, where ϵ_i is the sign of the i-th cone in the signed decomposition and $b_{i1}, b_{i2} \ldots, b_{id}$ are the generators of the i-th cone.

Chapter 2

Linear Diophantine Systems

> 'Here lies Diophantus,' the wonder behold. Through art algebraic, the stone tells how old: 'God gave him his boyhood one-sixth of his life, One twelfth more as youth while whiskers grew rife; And then yet one-seventh ere marriage begun; In five years there came a bouncing new son. Alas, the dear child of master and sage After attaining half the measure of his father's life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life.'
>
> Diophantus tombstone

The main focus of this thesis is on the solution of linear Diophantine systems. In this chapter, we introduce some of their properties and provide a classification that is useful when considering algorithms for solving linear Diophantine systems.

2.1 Introduction

Definition

As the name suggests we have a linear system of equations/inequalities. From linear algebra, the standard representation of linear systems is in matrix notation. Since we respect Diophantus' viewpoint, our matrices will always be in $\mathbb{Z}^{m\times d}$. In addition, there is the restriction that the solutions are non-negative integers.

Let $\mathbb{M}$ be the set $\mathbb{Z} \times [t_1, t_2, \ldots, t_k]$, where $[t_1, t_2, \ldots, t_k]$ denotes the multiplicative monoid generated by $t_1, t_2, \ldots, t_k$. In other words, $\mathbb{M}$ is the set of monomials in the variables t_i with integer coefficients. We first give the definition of a linear Diophantine system.

Definition 2.1
Given $A \in \mathbb{Z}^{m \times d}$, $b \in \mathbb{M}^m$ and $\Diamond \in \{\geq, =\}$, the triple $(A, b, \Diamond)$ is called a linear Diophantine system. Note that the relation is considered componentwise (one inequality/equation per row). Let $S = \{x \in \mathbb{N}^d | Ax \Diamond b\}$. □

We denote the family of solution sets S as $S(\mathbf{t})$ in order to indicate the dependence on the parameters. Moreover, we will denote by $|S(\mathbf{t})|$ the function on $\mathbf{t}$ mapping each set in the family to its cardinality. We will refer to the family of sets $S(\mathbf{t})$ as the solution set of the linear Diophantine system.
Assuming that b lives in $(\mathbb{Z} \times [t_1, t_2, \ldots, t_k])^m$ is essential for the understanding of a certain category of problems, but for many interesting problems one can restrict to having constant right-hand side, i.e., $b \in \mathbb{Z}^m$. In the latter case, we use S and $|S|$, since the family contains only one member.

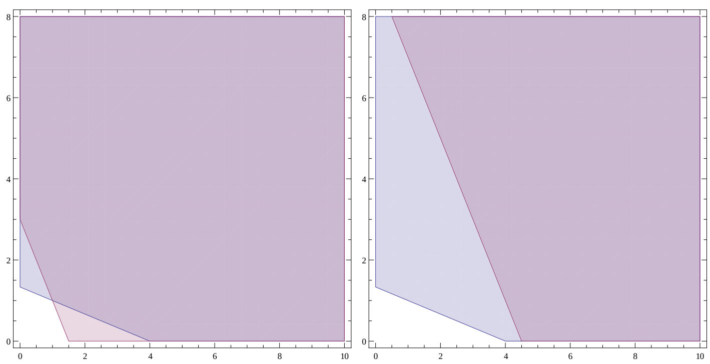
Characteristics

A linear Diophantine system has two important characteristics. These are related to its geometry and influence greatly the algorithmic treatment. A linear Diophantine system defines a polyhedron, whose lattice points are the solutions to the system. If the polyhedron is bounded, i.e, it is a polytope, then we say that the linear Diophantine system is bounded. If the polyhedron is parametric, we say that the linear Diophantine system is parametric. We will clarify the notion of parametric with two examples.

The (vector partition function) problem

$$\left(\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \geq \right)$$

is parametric, since the right hand side contains (honest) elements of $\mathbb{Z} \times [b_1, b_2]$. The choice of b_1 and b_2 can change considerably the geometry of the problem. The change in geometry is apparent for $b_1 = 4, b_2 = 9$ and $b_1 = 4, b_2 = 3$ in the following figures.



On the other hand, the problem (of symmetric 3×3 magic squares with magic sum t)

$$\left(\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} t \\ t \\ t \end{pmatrix}, =\right)$$

is not a parametric problem, since the (positive integer) parameter t does not change the geometry of the problem (every element of the right hand side is a non-constant univariate monomial in t). In other words, we can consider the polytope P, defined by the linear Diophantine system

$$\left(\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, =\right)$$

and the problem is restated as counting (or listing) lattice points in tP, i.e., in dilations of P. Thus t is a dilation parameter and is not changing the geometry of the problem.

It is important for any further analysis to define properly the size of a linear Diophantine problem. There are three input size quantities that are all relevant concerning the size of a problem:

- B is the number of bits needed to represent the matrix A.
- d is the number of variables.
- m is the number of relations (equations/inequalities).

In the number of relations we do not count the non-negativity constraints for the variables. In the number of variables we do not count parameters in parametric problems. These two rules are essential in order to differentiate between problems and to define problem equivalence.

Counting vs Listing

Given a linear Diophantine system, one can ask two questions:

- How many solutions are there?
- What are the solutions?

The first is called the **counting** problem, while the second is the **listing** problem. We are interested in solving these two problems in an efficient way. The listing of the solutions may be exponentially big in comparison to the input size B (or even infinite), thus we need a representation of the answer that encodes the solutions in an efficient way. We resort to the use of generating functions for that reason. More precisely, the solution to the listing problem is a full (or multivariate) generating function, while for the counting problem is a counting generating function.

In the literature, depending on the motivation of each author, the problem specification is (sometimes silently) altered. In order to tackle the problem in an algorithmic way, we have to first resolve the specification issue. The formal definitions we use for the two problems we are interested in are

Definition 2.2 (Listing Linear Diophantine Problem)
Given a linear Diophantine system $(A, b, \Diamond) \in \mathbb{Z}^{m\times d} \times \mathbb{M}^m \times \{=, \geq\}$, denote by $S(\mathbf{t})$ the solution set of the linear Diophantine system. Compute the generating function

$$\mathcal{L}_{A,b,\Diamond}(\mathbf{z}, \mathbf{q}) = \sum_{\mathbf{t}\in\mathbb{Z}^m} \left(\sum_{\mathbf{x}\in S(\mathbf{t})} \mathbf{z}^{\mathbf{x}} \right) \mathbf{q}^{\mathbf{t}}.$$

□

Definition 2.3 (Counting Linear Diophantine Problem)
Given a linear Diophantine system $(A, b, \Diamond) \in \mathbb{Z}^{m\times d} \times \mathbb{M}^m \times \{=, \geq\}$, denote by $S(\mathbf{t})$ the solution set of the system. Compute the generating function

$$\mathcal{C}_{A,b,\Diamond}(\mathbf{t}) = \sum_{\mathbf{t}\in\mathbb{Z}^m} |S(\mathbf{t})| \mathbf{q}^{\mathbf{t}}.$$

□

Complexity

The reason why we restrict to linear systems is that algorithmic treatment of the general (polynomial) problem is not feasible. This is the celebrated Hilbert's 10th problem, which Matiyasevich [32] solved in the negative direction. In other words, there is no algorithm to decide if a polynomial has integer solutions.

Although the restriction to linear systems makes the problem algorithmically solvable, it does not make it efficiently solvable. The problem remains NP-hard.

Reduction

The decision problem of linear Diophantine systems (LDS) is

decide if there exists	$x \in \mathbb{N}^d$
such that	$Ax \geq b$

In order to show that LDS is NP-hard we reduce from 3-SAT [26]:

given a boolean expression $B(x) = \bigwedge_{i=1}^m C_i$ in d-variables,
where C_i is a disjunction of three literals
decide if there exists $x \in \{0, 1\}^d$ such that $B(x) = 1$

For each clause $Y_i \vee Y_j \vee Y_k$, where Y_u is either y_u or $\neg y_u$, we construct an inequality $X_i + X_j + X_k \geq 1$ where X_u is x_u or $(1 - x_u)$ accordingly.

Let $a_i x \geq b_i$ be the inequality corresponding to the i-th clause of $B(x)$. Moreover, let $Q = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{Z}^{m \times d}$, $b^+ = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{Z}^m$, $b^- = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix} \in \mathbb{Z}^d$ and I_d be the rank d unit matrix.

Then if $A = \begin{bmatrix} Q \\ -I_d \end{bmatrix}$ and $b = \begin{bmatrix} b^+ \\ b^- \end{bmatrix}$ we have

$$x \in \mathbb{N}^d \text{ such that } Ax \geq b \Leftrightarrow B(x) = 1.$$

We note that this reduction is polynomial.

2.2 Classification

We examine the hierarchy of problems related to linear Diophantine systems from the perspective of the nature of the problem. Although many problems admit the same algorithmic solution, their nature may be very diverse. Similarly, problems that look similar may differ a lot when it comes to their algorithmic solution. There is a twofold motivation for this classification. On one hand, it clarifies (by refining) the specifications of the linear Diophantine problems. On the other hand, it provides useful insight concerning the algorithmics, by providing base, degenerate or special cases for the "general" problems.

Moreover, since many problems are equivalent, it is useful to have an overview of the alternatives one has for solving a particular instance of a problem by reformulating it as an instance of another.

Another important aspect is that a problem (and its solution) may change a lot depending on whether one is interested in the full generating function $\mathcal{L}$ (listing problem) or the counting generating function $\mathcal{C}$ (counting problem).

Each of these two types of problems contains several subproblems, depending on the domain of the coefficients and the relation involved (equation or inequality).

2.2.1 Classification

For a problem $(A, b, \Diamond)$, we will use a four letter notation MHRT, where M is the domain of the matrix A, H is $\mathbb{M}$ if the problem is parametric with respect to b, O if the problem is homogeneous or the domain of b if the problem is inhomogeneous, R is $\geq$ or $=$ depending on whether we have inequalities or equations and T is B for bounded or U for unbounded problems.

In the diagram we see the inclusion relations (if a class A is included in class B, then an instance from A is also an instance of B).

The following list gives some rules eliminating some of the cases:

1. $*O*B$ has $\{0\}$ as its solution set.

2. $\mathbb{N}* \geq *$ is unbounded.

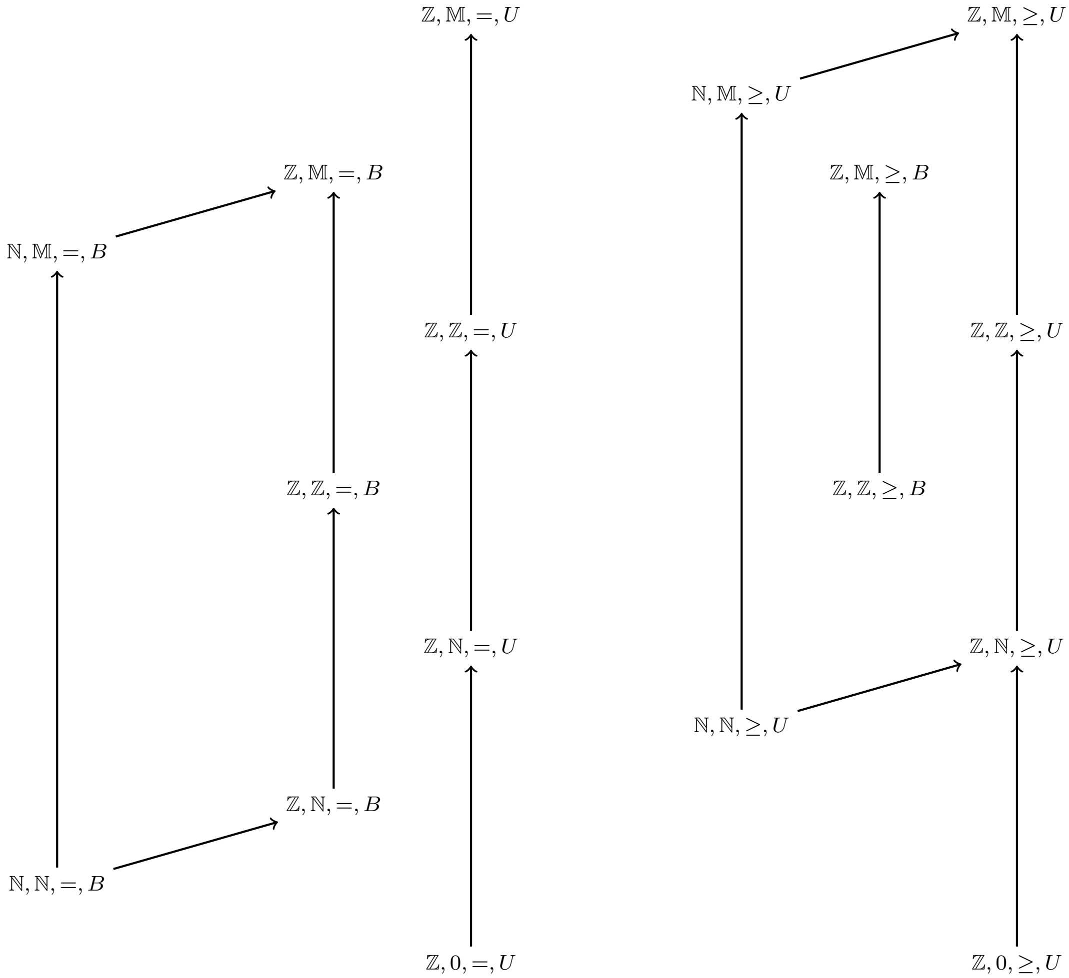
3. $\mathbb{N}O**$ is trivially satisfied ($\mathbb{N}^d$ is its solution set).

4. $\mathbb{N}\mathbb{Z} \geq *$ is trivially satisfied ($\mathbb{N}^d$ is its solution set).

5. $\mathbb{N}\mathbb{Z}{=}*$ is infeasible.

We note that a problem is assumed to belong to one of the above classes and not any of its subclasses in order to apply the rules. For example, the last rule means that the inhomogeneous part has a negative entry, otherwise we would consider the problem in the class of problems with inhomogeneous part in $\mathbb{N}$.

Omitting the cases covered above, we obtain the following diagram:



We now present examples for some of the cases and provide names and references used in literature when available.

$\mathbb{N}, \mathbb{N}, =, B$

Name:	Magic Squares
References:	[4], Ch.4. Prop 4.6.21 in [35], §6.2 in [17]
Description:	3×3 symmetric matrices of row and column sum equal to n

Matrix A	vector b	Parameters
$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	n

Listing	$\frac{(z_0z_1z_2z_3z_4z_5q^3-1)}{(z_1z_5q-1)(z_0z_4q-1)(z_2z_3q-1)(z_0z_3z_5q-1)(z_1z_2z_4q^2-1)}$
Counting	$\frac{(1+q+q^2}{(1-q)^4(1+q)}$

Name:	Magic Pentagrams
References:	§3, page 14 in [5]
Description:	Partitions of n into parts satisfying relations given by a pentagram with the parts as vertices and inequalities given by the edges.

Matrix A	vector b
$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$

Listing	$\frac{qz_{10}^2z_3^2z_7^3z_4^3}{(-z_3z_6+z_7z_{10})(-z_3z_4+z_8z_{10})(z_4z_7^2z_9z_{10}q-1)(z_2z_3z_4z_7z_{10}q-1)} \cdot \frac{1}{(z_3z_4^2z_5z_7q-1)(z_3z_4^2z_7^2z_{10}q-z_1)}$
Counting	$\frac{(q^2+3q+1)(q^2+13q+1)}{(q-1)^6}$

$\mathbb{Z}, \mathbb{Z}, \geq, B$

Name:	Solid Partitions on a Cube
References:	§4.1 in [4]
Description:	Partitions of n subject to relations given by a cube where the vertices are the parts and the directed edges indicate the inequalities.

Matrix A	vector b
$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}$

Listing	$\frac{z_4 z_2 z_1 q^3}{(1-z_6 q)(z_5 q-1)(z_1 q-1)(z_1 z_2 q^2-1)(z_1 z_2 z_4 q^3-1)(z_1 z_2 z_3 z_4 q^4-1)} \cdot \frac{1}{(z_1 z_2 z_3 z_4 z_5 z_7 q^6-1)(z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8 q^8-1)}$
Counting	$\frac{(q^{16}+2q^{14}+2q^{13}+3q^{12}+3q^{11}+5q^{10}+4q^9+8q^8+4q^7+5q^6+3q^5+3q^4+2q^3+2q^2+1)}{(q-1)^8(q+1)^4(q^2+1)^2(q^2+q+1)^2(q^2-q+1)(q^4+1)(q^4+q^3+q^2+q+1)} \cdot \frac{1}{(q^6+q^5+q^4+q^3+q^2+q+1)}$

$\mathbb{N}, \mathbb{M}, =, B$

Name:	Vector Partition Function
References:	Section 4 in [15]
Description:	Non-negative integer counts the natural solutions of $\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & x_3 & & & = & b_1 \\ x_1 & + & x_2 & & & + & x_4 & = & b_2 \end{array}$

Matrix A	vector b	Parameters
$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$	b_1, b_2

Counting	$\begin{cases} \frac{b_1^2}{4} + b_1 + \frac{7+(-1)^{b_1}}{8} & , b1 \leq b_2 \\ b_1 b_2 - \frac{b_1^2}{4} - \frac{b_2^2}{2} + \frac{b_1+b_2}{2} + \frac{7+(-1)^{b_1}}{8} & , b1 > b_2 > \frac{b_1-3}{2} \\ \frac{b_2^2}{2} - \frac{3 \cdot b_2}{2} + 1 & , b_2 \leq \frac{b_1-3}{2} \end{cases}$

$\mathbb{Z}, 0, \geq, B$

Name:	Lecture Hall Partitions
References:	[18]
Description:	Partitions of n where $2a \leq b$, $3b \leq 2c$ and $4c \leq 3d$.

Matrix A	vector b
$\begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 4 & -3 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Listing	$\frac{-z_1 z_2^3 z_3^4 z_4^5 q^{13}}{(z_2 z_3 z_4 q^3-1)(z_1 z_2^2 z_3^2 z_4^2 q^7-1)(z_1 z_2^2 z_3^3 z_4^3 q^9-1)(z_1 z_2^2 z_3^3 z_4^4 q^{10}-1)}$
Counting	$\frac{(q^2-q+1)(q^4+1)}{(q-1)^4(q^2+q+1)(q^4+q^3+q^2+q+1)}$

$\mathbb{Z}, \mathbb{M}, \geq, B$

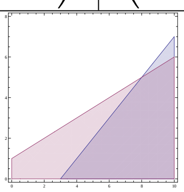
Name:	Vector Partition Function
References:	Section 4 in [15]
Description:	Non-negative integer solutions of $\begin{aligned} -x_1 - 2x_2 &\geq b_1 \\ -x_1 - x_2 &\geq b_2 \end{aligned}$

Matrix A	vector b	Parameters
$\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$	b_1, b_2

Counting	$\begin{cases} \frac{b_1^2}{4} + b_1 + \frac{7+(-1)^{b_1}}{8} & , b1 \leq b_2 \\ b_1 b_2 - \frac{b_1^2}{4} - \frac{b_2^2}{2} + \frac{b_1+b_2}{2} + \frac{7+(-1)^{b_1}}{8} & , b1 > b_2 > \frac{b_1-3}{2} \\ \frac{b_2^2}{2} - \frac{3 \cdot b_2}{2} + 1 & , b_2 \leq \frac{b_1-3}{2} \end{cases}$

$\mathbb{Z}, \mathbb{Z}, \geq, U$

Matrix A	vector b
$\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$	$\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

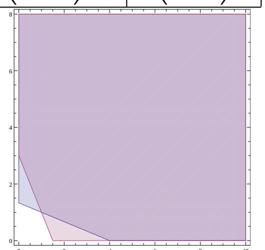


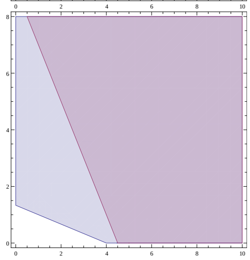
Listing	$\frac{-z_1^2}{(z_1-1)(-z_1^2+q)(z_1^2 z_2-1)}$
Counting	$\frac{5q(2q+3)}{(q-1)^3}$

$\mathbb{N}, \mathbb{M}, \geq, U$

Name:	Vector Partition Function

Matrix A	vector b
$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$





Listing	
	$\frac{N}{(1-z_1)(1-z_2)(1-q_1z_1)\left(1-q_2^2z_1\right)\left(1-q_1q_2^2z_1\right)\left(1-q_1^3z_2\right)(1-q_2z_2)\left(1-q_1^3q_2z_2\right)}$ where $N = q_2^5z_1^3z_2^3q_1^7 + q_2^4z_1^3z_2^3q_1^7 - q_2^4z_1^3z_2^2q_1^7 - q_2^3z_1^2z_2^2q_1^7 + q_2^5z_1^3z_2^3q_1^6 + q_2^4z_1^3z_2^3q_1^6 - q_2^5z_1^2z_2^3q_1^6 - q_2^4z_1^2z_2^3q_1^6 - q_2^3z_1^2z_2^3q_1^6 - q_2^2z_1^2z_2^3q_1^6 - q_2^4z_1^3z_2^2q_1^6 + q_2^4z_1^2z_2^2q_1^6 - q_2^3z_1^2z_2^2q_1^6 + q_2^2z_1^2z_2^2q_1^6 + q_2^3z_1z_2^2q_1^6 + q_2z_1z_2^2q_1^6 + q_2^5z_1^3z_2^3q_1^5 + q_2^4z_1^3z_2^3q_1^5 - q_2^5z_1^2z_2^3q_1^5 - q_2^4z_1^2z_2^3q_1^5 - q_2^3z_1^2z_2^3q_1^5 - q_2^2z_1^2z_2^3q_1^5 + q_2^3z_1z_2^3q_1^5 + q_2^2z_1z_2^3q_1^5 - q_2^4z_1^3z_2^2q_1^5 + q_2^4z_1^2z_2^2q_1^5 - q_2^3z_1^2z_2^2q_1^5 + q_2^2z_1^2z_2^2q_1^5 - q_2z_2^2q_1^5 + q_2^3z_1z_2^2q_1^5 - q_2^2z_1z_2^2q_1^5 + q_2z_1z_2^2q_1^5 - q_2^4z_1^3z_2^2q_1^4 - q_2^5z_1^2z_2^2q_1^4 - q_2^4z_1^2z_2^2q_1^4 - 3q_2^3z_1^2z_2^2q_1^4 - q_2^2z_1^2z_2^2q_1^4 - q_2z_1^2z_2^2q_1^4 - q_2z_2^2q_1^4 + 2q_2^3z_1z_2^2q_1^4 + q_2^2z_1z_2^2q_1^4 + 2q_2z_1z_2^2q_1^4 + q_2^4z_1^2z_2q_1^4 + 2q_2^3z_1^2z_2q_1^4 + 2q_2^2z_1^2z_2q_1^4 + q_2z_1^2z_2q_1^4 - q_2^2z_1z_2q_1^4 - q_2z_1z_2q_1^4 - q_2^4z_1^3z_2^2q_1^3 + q_2^4z_1^2z_2^2q_1^3 - q_2^3z_1^2z_2^2q_1^3 + q_2^2z_1^2z_2^2q_1^3 - q_2z_2^2q_1^3 + 2q_2^3z_1z_2^2q_1^3 + q_2^2z_1z_2^2q_1^3 + 2q_2z_1z_2^2q_1^3 + q_2^3z_1^2z_2q_1^3 + q_2^2z_1^2z_2q_1^3 - q_2^3z_1z_2q_1^3 - 2q_2^2z_1z_2q_1^3 - 2q_2z_1z_2q_1^3 - z_1z_2q_1^3 - q_2^4z_1^3z_2^2q_1^2 + q_2^4z_1^2z_2^2q_1^2 - q_2^3z_1^2z_2^2q_1^2 + q_2^2z_1^2z_2^2q_1^2 - q_2z_2^2q_1^2 + q_2^3z_1z_2^2q_1^2 - q_2^2z_1z_2^2q_1^2 + q_2z_1z_2^2q_1^2 + q_2^3z_1^2z_2q_1^2 + q_2^2z_1^2z_2q_1^2 + q_2z_2q_1^2 - q_2^3z_1z_2q_1^2 - q_2^2z_1z_2q_1^2 - q_2z_1z_2q_1^2 - z_1z_2q_1^2 + z_2q_1^2 - q_2^3z_1^2q_1 - q_2^2z_1^2q_1 - q_2z_1^2q_1 - q_2^3z_1^2z_2^2q_1 - q_2z_2^2q_1 + q_2^3z_1z_2^2q_1 + q_2z_1z_2^2q_1 + q_2z_1q_1 + q_2^4z_1^2z_2q_1 + 2q_2^3z_1^2z_2q_1 + 2q_2^2z_1^2z_2q_1 + q_2z_1^2z_2q_1 + q_2z_2q_1 - q_2^3z_1z_2q_1 - 2q_2^2z_1z_2q_1 - 2q_2z_1z_2q_1 - z_1z_2q_1 + z_2q_1 + q_2z_1 - q_2^2z_1z_2 - q_2z_1z_2 + 1$

Chapter 3

Partition Analysis

3.1 Partition Analysis Revisited

Partition analysis is a general methodology for the treatment of linear Diophantine systems. The methodology is commonly attributed to MacMahon [31], since he was the first to apply it in a way similar to the one used today, that is, for the solution of combinatorial problems subject to linear Diophantine systems.

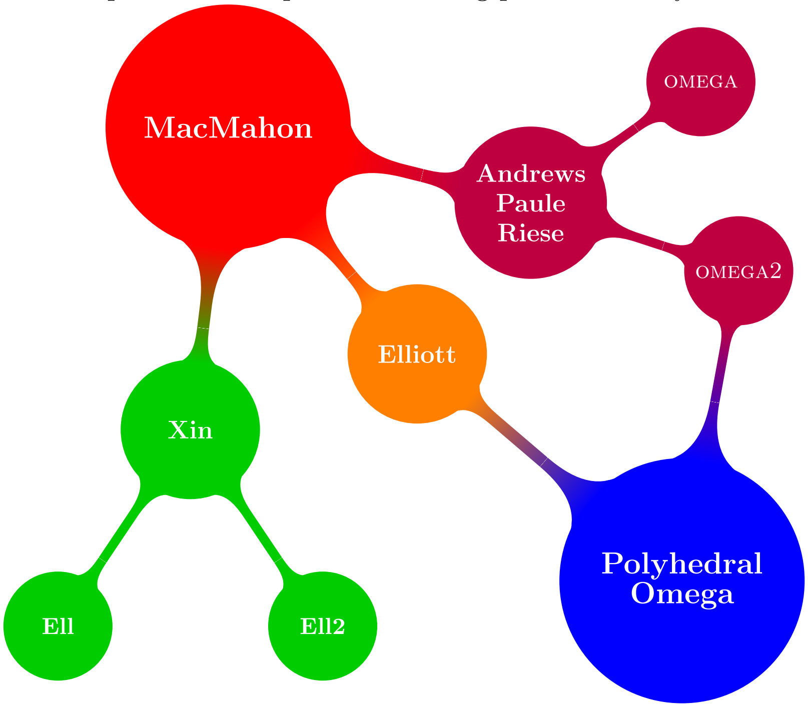
MacMahon's motivation was the proof of his conjectures about plane partitions. This was his declared goal in Combinatory Analysis. The fact that he failed to actually give an answer to the problem was like signing the death certificate of his own child. In combination with other reasons, including the lack of interest for computational procedures in mathematics for most of the 20th century, MacMahon's method did not become mainstream among mathematicians.

Except for the theoretical side of the method, there is an algorithmic aspect. The computational and algorithmic nature of the method was evident since the beginning. MacMahon employed an algorithm of Elliott in order to turn his method into an algorithm. In [24], Elliott introduced the basics of the methodology and used it to solve linear homogeneous Diophantine equations. In particular, Elliott's method is of interest because it is algorithmic (in the strict modern sense of the term). Elliott himself, although lacking modern terminology, is arguing on the termination of the procedure. Naturally, given the lack of computers at that time, MacMahon resorted in finding shortcuts (rules) for the most usual cases. This list of rules, included in Combinatory Analysis, he expanded as needed for the problem at hand.

It was Andrews who observed the computational potential of MacMahon's partition analysis and waited for the right problem to apply it. In the late 1990's, the seminal paper of Bousquet–Mélou and Eriksson [18] on lecture-hall partitions appeared. Andrews suggested to Paule to explore the capacity of partition analysis combined with symbolic computation. This collaboration gave a series of papers, among which [4] and [5] deal with implementations of two fully algorithmic versions of partition analysis, called Omega, empowered by symbolic computation.

There are different algorithmic (and non-algorithmic) realizations of the general idea.

Their relation is illustrated in the following figure. In what follows we will see in chronological order the milestones of partition analysis. Although the goal is to present the history, we will allow for a modern view. In particular, we will see some geometric and algebraic aspects that help understanding partition analysis.



3.1.1 Elliott

One of the first references relevant to partition analysis is Elliott's article "On linear homogeneous Diophantine equation" [24]. The work of Elliott is exciting, if not for anything else, because it addresses mathematicians that lived a century apart. It was of interest to MacMahon, who based his method on Elliott's decomposition, but also to 21st century mathematicians for explicitly giving an important algorithm. Although, as we shall see, the algorithm has very bad complexity, it can be considered as an early algorithm for the enumeration of lattice points in cones (among other things).

The problem Elliott considers is to find all non-negative integer solutions to the equation

$$\sum_{i=1}^{m} a_i x_i - \sum_{i=m+1}^{m+n} b_i x_i = 0 \qquad \text{for } a_i, b_i \in \mathbb{N}. \tag{3.1}$$

In other words, we consider one homogeneous linear Diophantine equation. It should be noted that Elliott himself (as well as subsequent authors) expressed the equation in the form Diophantus would prefer, without using negative coefficients.

The starting point for Elliott's work is the fact that even if one computes the set of "simple solutions", i.e., solutions that are not combinations of others, there are syzygies preventing us from writing down formulas giving each and every solution of the equation exactly once. He proceeds by explaining that his method computes a generating function whose terms are in one-to-one correspondence with the solutions of the equation. Elliott states that this generating function is obtained "by a finite succession of simple stages". This sentence exhibits an impressive insight on what an algorithm is, as well as Elliott's realization that he has an algorithm at hand. We quote Elliott outlining his idea in [24]:

> *The principle is that in the infinite expansion which is the formal product of the infinite expansions*
> $1+\xi_1 u^{a_1}+\xi_1^2 u^{2a_1}+\cdots$
> $1+\xi_2 u^{a_2}+\xi_2^2 u^{2a_2}+\cdots$
> $\vdots$
> $1+\xi_m u^{a_m}+\xi_m^2 u^{2a_m}+\cdots$
> $1+\xi_{m+1} u^{-b_{m+1}}+\xi_{m+1}^2 u^{-2b_{m+1}}+\cdots$
> $1+\xi_{m+2} u^{-b_{m+2}}+\xi_{m+2}^2 u^{-2b_{m+2}}+\cdots$
> $\vdots$
> $1+\xi_{m+n} u^{-b_{m+n}}+\xi_{m+n}^2 u^{-2b_{m+n}}+\cdots$
>
> *the terms free from u are of the form* $\xi_1^{x_1}\xi_2^{x_2}\cdots\xi_m^{x_m}\xi_{m+1}^{x_{m+1}}\xi_{m+2}^{x_{m+2}}\cdots\xi_{m+n}^{x_{m+n}}$ *with 1 for numerical coefficient, where* $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}$ *are a set of positive integers (zero included), which satisfy our Diophantine equation 3.1, and that there is just one such term for each set of solutions.*
>
> *In other words, the generating function for determining the sets of solutions, each once, of 3.1, as sets of exponents of the* ξ_i*'s in its several terms, is the expression for the part which is free from u of the expansion of*
>
> $$\frac{1}{(1-\xi_1 u^{a_1})\cdots(1-\xi_m u^{a_m})(1-\xi_{m+1}u^{-b_{m+1}})\cdots(1-\xi_{m+n}u^{-b_{m+n}})} \tag{3.2}$$
>
> *in the positive powers of the* ξ_i*'s.*
>
> *The problem is, in any case, to extract from 3.2, and to examine, this generating function. The extraction may be effected by a finite number of easy steps as follows.*

The principle presented here by Elliott is the basis of partition analysis, i.e., introducing an extra variable, denoted by u in Elliott and by λ in modern partition analysis. We switch from ξ and u to z and λ in order to conform with more modern notational conventions. The method of Elliott computes a partial fraction decomposition of an expression

$$\prod_i^k \frac{1}{1-m_i}, \tag{3.3}$$

where $m_i \in \left[z_1, z_2, \ldots, z_k, \lambda, \lambda^{-1}\right]$ and $k \in \mathbb{N}$, into a sum of the form

$$\sum \frac{\pm 1}{\prod(1-p_i)} + \sum \frac{\pm 1}{\prod(1-q_j)} \tag{3.4}$$

with $p_i \in [z_1, z_2, \ldots, z_k, \lambda]$ and $q_j \in [z_1, z_2, \ldots, z_k, \lambda^{-1}]$.
We note that some authors refer to rational functions of the form 3.3 as Elliott rational functions. The algorithm is based on the fact

$$\begin{aligned}\frac{1}{(1-x\lambda^{\alpha})(1-\frac{y}{\lambda^{\beta}})} &= \frac{1}{1-xy\lambda^{\alpha-\beta}}\left(\frac{1}{1-x\lambda^{\alpha}}+\frac{1}{1-\frac{y}{\lambda^{\beta}}}-1\right) \\ &= \frac{1}{(1-xy\lambda^{\alpha-\beta})(1-x\lambda^{\alpha})}+\frac{1}{(1-xy\lambda^{\alpha-\beta})(1-\frac{y}{\lambda^{\beta}})}-\frac{1}{(1-xy\lambda^{\alpha-\beta})}.\end{aligned} \tag{3.5}$$

Observe that given α and β positive integers, after applying 3.5, we obtain a sum of terms where in each of them either the number of factors containing λ reduced or the exponent of λ reduced (in absolute value) in one of the factors while it did not change in the other. This observation is the proof of termination Elliott gives for his algorithm.

Let's see a very simple example, where by applying (3.5) once, we obtain the desired partial fraction decomposition.

Example 7.

$$\begin{aligned}\frac{1}{(1-x\lambda)(1-\frac{y}{\lambda})} &= \frac{1}{(1-xy)(1-x\lambda)}+\frac{1}{(1-xy)(1-\frac{y}{\lambda})}-\frac{1}{(1-xy)} \\ &= \left(\frac{1}{(1-xy)(1-x\lambda)}-\frac{1}{(1-xy)}\right)+\frac{1}{(1-xy)(1-\frac{y}{\lambda})}.\end{aligned}$$

Notice that the terms in the parenthesis contain only non-negative λ exponents, while the last term contains only non-positive ones as requested by (3.4). □

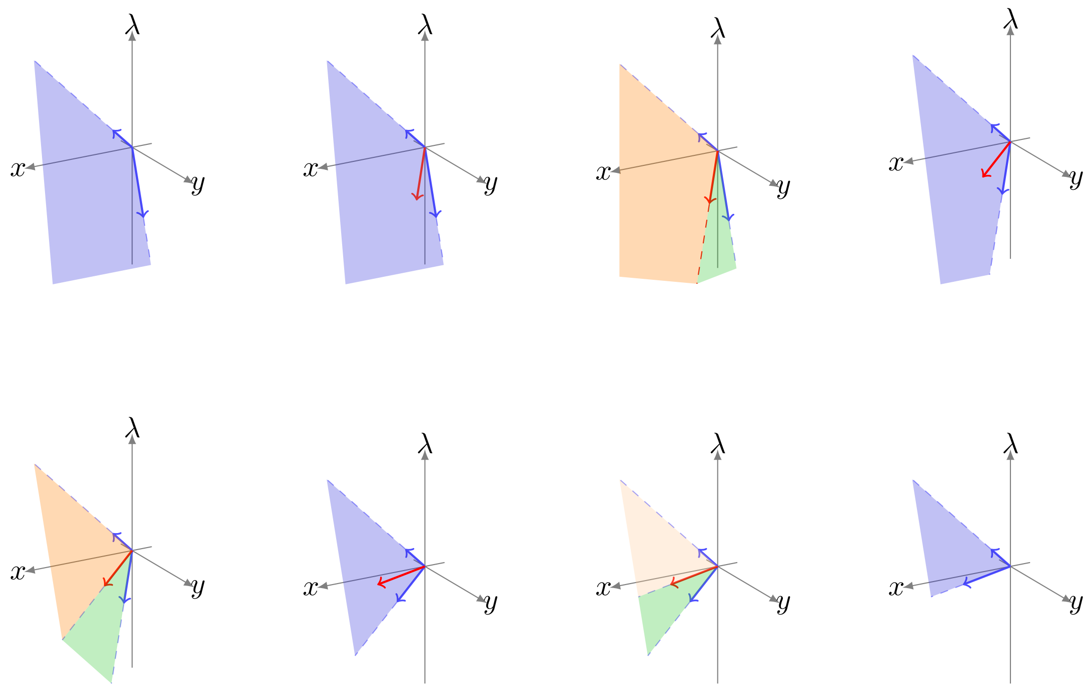
Each of the summands in $\sum_i \frac{\pm 1}{\prod_j (1-p_i)}$ can be expanded using the geometric-series expansion formula. It is easy to see that the summands contributing to the generating function for the solution of Equation 3.1 are exactly the ones where in their expansion λ does not appear. This happens only for the terms $\frac{\pm 1}{\prod_k (1-p_k)}$ where all p_k are λ-free. Summing up only these terms we get the desired generating function.

In order to translate this rational function identity to an identity about cones we first observe that

$$\frac{1}{(1-x\lambda^{\alpha})(1-\frac{y}{\lambda^{\beta}})}$$

is the generating function of the 2-dimensional cone $C = \mathcal{C}_{\mathbb{R}}\left((1,0,\alpha),(1,0,-\beta)\right)$, since Lemma 2 guarantees that the numerator in the rational generating function of C is indeed 1.

Observe that the point $(1,1,\alpha-\beta)$ is in the interior of the cone C. This means that the cones $A = \mathcal{C}_{\mathbb{R}}\left((1,0,\alpha),(1,1,\alpha-\beta)\right)$ and $B = \mathcal{C}_{\mathbb{R}}\left((1,0,-\beta),(1,1,\alpha-\beta)\right)$ subdivide the cone C. Their intersection is exactly the ray starting from the origin and passing through $(1, 1, \alpha - \beta)$. By a simple inclusion-exclusion argument we have the signed decomposition $C = A + B - (A \cap B)$. This decomposition translated to the generating function level is exactly the partial fraction decomposition employed by Elliott. In the example shown below, after three applications of Elliott's decomposition we obtain the desired cone.



Elliott's decomposition

The Algorithm

It is easy to see that, after a finite number of steps, we end up with a sum of cones of three types:

1. the generators contain only zero last coordinate (λ-coordinate);
2. the generators contain only non-negative (but not all zero) last coordinate;
3. the generators contain only non-positive (but not all zero) last coordinate.

Note that all the cones involved are unimodular.

For equations we sum up the generating functions corresponding to cones of the 1st type. For inequalities, we intersect each cone with the non-negative λ halfspace. This means that we discard the cones of the 3^{rd} type and sum up the generating functions of the rest of the cones. Then we (orthogonally) project with respect to the λ-coordinate. The sum of rational generating functions for the lattice points in each cone we obtained is the partial fraction decomposition obtained by Elliott's algorithm.

Elliott for Systems

It is important to note that Elliott's method relies on the fact that the numerators in all the expressions involved are equal to 1. Of course after bringing all the terms the algorithm returns over a common denominator, there is no guarantee that the numerator will be 1. But for each term returned the condition is preserved. Thus it is possible to iteratively apply the algorithm to eliminate λ's in order to solve systems of equations or inequalities. We note that Elliott proves that the numerators will always be ± 1. No repetitions occur in the computed expression.

Example 8. *The system of linear homogeneous inequalities* $\{7a - b \geq 0, a - 3b \geq 0\}$ *can be solved by Elliott, resulting in* $\frac{1}{(1-x)(1-x^3y)}$, *if the inequality* $a - 3b \geq 0$ *is treated first. If the order is reversed then the intermediate expression*

$$\frac{-1 - \frac{xy^6}{\lambda^{17}} - \frac{xy^5}{\lambda^{14}} - \frac{xy^4}{\lambda^{11}} - \frac{xy^3}{\lambda^8} - \frac{xy^2}{\lambda^5} - \frac{xy}{\lambda^2}}{(1 - x\lambda)(1 - \frac{xy^7}{\lambda^{20}})}$$

is violating the condition that numerators are equal to 1.

The first inequality of the system is redundant, i.e., if we ignore it the solution set does not change. Thus, choosing the right order, the intermediate expression $\frac{1}{(1-x\lambda^7)(1-x^3y\lambda^{20})}$ *behaves well. We could still apply Elliott's method if we did not bring the intermediate expression over a common denominator.* □

3.1.2 MacMahon

MacMahon presented his investigations concerning integer partition theory in a series of seven memoirs, published between 1895 and 1916. In the second memoir [30], MacMahon observes that the theory of partitions of numbers develops in parallel to that of linear Diophantine equations and in his masterpiece "Combinatory Analysis" [31] he notes:

> The most important part of the volume. Sections VIII et seq., arises from basing the theory of partitions upon the theory of Diophantine inequalities. This method is more fundamental than that of Euler, and leads directly to a high degree of generalization of the theory of partitions, and to several investigations which are grouped together under the title of "Partition Analysis".

In order to attack the problem of solving linear Diophantine systems, MacMahon (like Elliott) introduced extra variables. Let $A \in \mathbb{Z}^{m \times d}$ and $\mathbf{b} \in \mathbb{Z}^m$. We want to find all $\mathbf{x} \in \mathbb{N}^d$ satisfying $A\mathbf{x} \geq \mathbf{b}$. The generating function of the solution set is

$$\Phi_{A,b}(\mathbf{z}) = \sum_{A\mathbf{x} \geq \mathbf{b}} \mathbf{z}^{\mathbf{x}}.$$

Let's introduce extra variables $\lambda_1, \lambda_2, \ldots, \lambda_m$ to encode the inequalities $A\mathbf{x} \geq \mathbf{b}$, transforming the generating function to

$$\sum_{\mathbf{x} \in \mathbb{N}^d, A\mathbf{x} \geq \mathbf{b}} \lambda^{A\mathbf{x}} \mathbf{z}^{\mathbf{x}}$$

The λ variables are introduced to encode the inequalities, but the solutions we are searching for live in dimension d. The λ dimensions are used to control that after a certain decomposition we can filter out the solutions involving negative exponents. Following this principle, MacMahon introduced the concept of the crude generating function. But before that, we need to see the $\Omega_{\geq}$ operator. MacMahon defined it in Article 66 of [30]:

> Suppose we have a function
>
> $$F(x, a) \tag{3.6}$$
>
> which can be expanded in ascending powers of x. Such expansion being either finite or infinite, the coefficients of the various powers of x are functions of a which in general involve both positive and negative powers of a. We may reject all terms containing negative powers of a and subsequently put a equal to unity. We thus arrive at a function of x only, which may be represented after Cayley (modified by the association with the symbol $\geq$) by
>
> $$\Omega_{\geq} F(x, a) \tag{3.7}$$
>
> the symbol $\geq$ denoting that the terms retained are those in which the power of a is ≥ 0.

The notation $\Omega_{\geq}$ is now a standard and usually referred to as "MacMahon's Omega". It is worth noting, at least in order to give credit where credit is due, as MacMahoon did, that the first use of this notation is by Cayley in [22]. MacMahon extends the notation to multivariate functions and defines the $\Omega_{=}$ operator.

A more modern definition of the $\Omega_{\geq}$ operator is given in [4], by Andrews,Paule and Riese:

> The $\Omega_{\geq}$ operator is defined on functions with absolutely convergent multisum expansions
>
> $$\sum_{s_1=-\infty}^{\infty} \sum_{s_2=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,s_2,\ldots,s_r} \lambda_1^{s_1} \lambda_2^{s_2} \cdots \lambda_r^{s_r}$$
>
> in an open neighborhood of the complex circles $|\lambda_i| = 1$. The action of $\Omega_{\geq}$ is given by
>
> $$\Omega_{\geq} \sum_{s_1=-\infty}^{\infty} \sum_{s_2=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,s_2,\ldots,s_r} \lambda_1^{s_1} \lambda_2^{s_2} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,s_2,\ldots,s_r}.$$

This definition is concerned with convergence issues, while MacMahon considered $\Omega_{\geq}$ acting on purely formal objects. In what follows we take the purely formal side (being on safe ground due to geometry). But we also address the justified concerns about convergence, relating them to the geometric issues.

In order to use the $\Omega_{\geq}$ operator for the solution of linear Diophantine systems, MacMahon resorts to the construction of a generating function he calls crude. This is in principal a multivariate version of Elliott's construction.

Definition 3.1 (Crude Generating Function)
Given $A \in \mathbb{Z}^{m\times d}$ and $\mathbf{b} \in \mathbb{Z}^m$ we define the crude generating function as

$$\Phi^{\Omega}_{A,b}\left(\mathbf{z};\lambda\right) := \Omega_{\geq} \sum_{\mathbf{x}\in\mathbb{N}^n} \mathbf{z}^{\mathbf{x}} \prod_{i=1}^{m} \lambda_i^{A_i\mathbf{x}-b_i}.$$

□

We stress the fact that the assignment is meant formally, since it is easy to see that

$$\Omega_{\geq}\, \Phi^{\lambda}_{A,b}(z;\lambda) = \Phi_{A,b}(\mathbf{z})$$

as well as

$$\Phi^{\Omega}_{A,b}(\mathbf{z};\lambda) = \sum_{\mathbf{x}\in\mathbb{N}^n, A\mathbf{x}\geq b} \mathbf{z}^{\mathbf{x}} = \Phi_{A,b}(\mathbf{z}).$$

This means that $\Omega_{\geq} \sum_{\mathbf{x}\in\mathbb{N}^n} \mathbf{z}^{\mathbf{x}} \prod_{i=1}^{m} \lambda_i^{A_i\mathbf{x}-b_i}$ is the answer to the problem of solving a linear Diophantine system. Thus, it is expected that the computation of the action of the $\Omega_{\geq}$ operator is not easy.

We define the λ-generating function as an intermediate step, both in order to increase clarity in this section and because it is essential when discussing geometry later. In principal, the λ-generating function is the crude generating function without prepending $\Omega_{\geq}$. Given $A \in \mathbb{Z}^{m\times d}$ and $\mathbf{b} \in \mathbb{Z}^m$ we define the λ generating function as

$$\Phi^{\lambda}_{A,b}(\mathbf{z},\lambda) = \sum_{\mathbf{x}\in\mathbb{N}^d} \mathbf{z}^{\mathbf{x}} \prod_{i=1}^{m} \lambda_i^{A_i\mathbf{x}-b_i}.$$

Based on the geometric series expansion formula

$$(1-z)^{-1} = \sum_{x\geq 0} z^x$$

we can transform the series into a rational function. The rational form of $\Phi^{\lambda}_{A,b}(\mathbf{z})$ is denoted by $\rho^{\lambda}_{A,b}(\mathbf{z})$ and it has the form

$$\rho^{\lambda}_{A,b}(\mathbf{z},\lambda) = \lambda^{-\mathbf{b}} \prod_{i=1}^{m} \frac{1}{\left(1 - z_i\lambda^{(A^T)_i}\right)}$$

We present an example that we will also use for the geometric interpretation of $\Omega_\geq$, Let $A = [\ 2 \quad 3\ ]$ and $b = 5$.

Then

$$\Phi^\lambda_{A,b}(z_1, z_2, \lambda) = \sum_{x_1,x_2 \in \mathbb{N}} \lambda^{2x_1+3x_2-5} z_1^{x_1} z_2^{x_2}$$

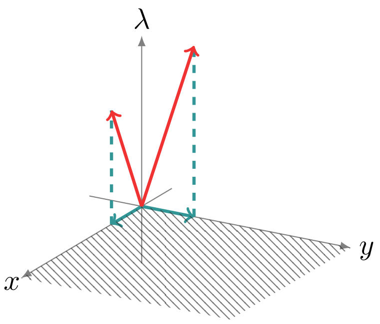
and

$$\rho^\lambda_{A,b}(z_1, z_2, \lambda) = \frac{\lambda^{-5}}{(1 - z_1\lambda^2)(1 - z_2\lambda^3)}$$

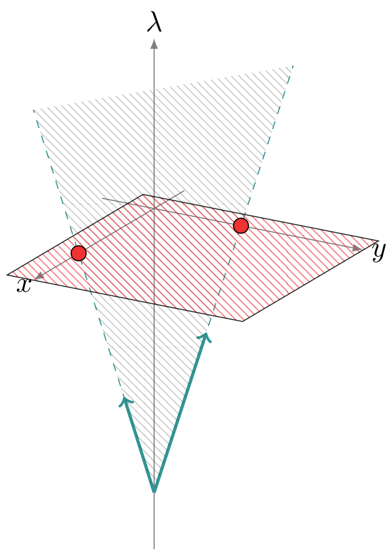
These generating functions can be directly translated to cones by examining their rational form and recalling the form for the generating function of lattice points in cones.

In the following figure, we see the geometric steps for creating the λ-cone (equivalent of the λ-generating function) and then applying the $\Omega_\geq$ operator.

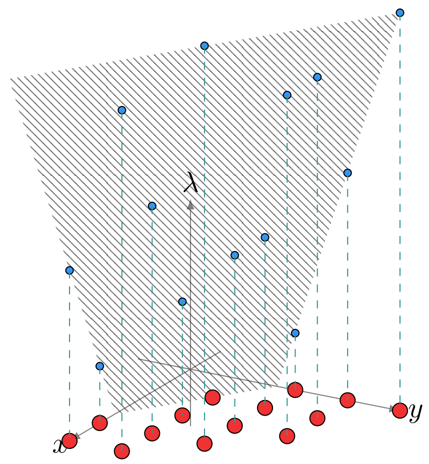
At first we lift the standard generators of the positive quadrant of $\mathbb{R}^d$ by appending the exponents of the λ variables (thus lifting the generators in $\mathbb{R}^{d+m}$). We translate the lifted cone by the exponent of λ in the numerator. Then we intersect with the positive quadrant of $\mathbb{R}^{d+m}$ and project the intersection to $\mathbb{R}^d$. The generating function of the obtained polyhedron is the result of the action of $\Omega_\geq$ on the λ-generating function.



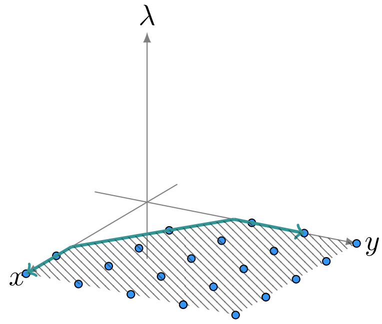
Consider the first quadrant and lift the standard generators according to the input.



Shift according to the inhomogeneous part in the negative λ direction the cone generated by the lifted generators. Intersect with the xy-plane.



Take the part of the cone (a polyhedron) living above the xy-plane and project its lattice points.



The projected polyhedron contains the solutions to the inequality.

Given the geometric interpretation of the crude generating function, one could argue that it is actually a refined one. Or if we consider the generating series as a cloth-hanging rope, then the crude generating function is a cloth-hanging grid allowing us to parametrize the hanging.

3.1.3 MacMahon's Rules

Although the machinery of MacMahon was very powerful, it was not very practical. Lacking computing machines, he had to resort to lookup tables in order to solve problems he was interested in. In his partition theory memoirs and in "Combinatory Analysis", he presents a set of rules. These are adhoc rules he was inventing when needed in order to solve some combinatorial problem he was presenting.

Here we present nine of the rules MacMahon used for the evaluation of the $\Omega_{\geq}$ operator, taken from [4]. Most of the proofs are easy, so we restrict to observations using geometric insight.

MacMahon Rules 1 & 2

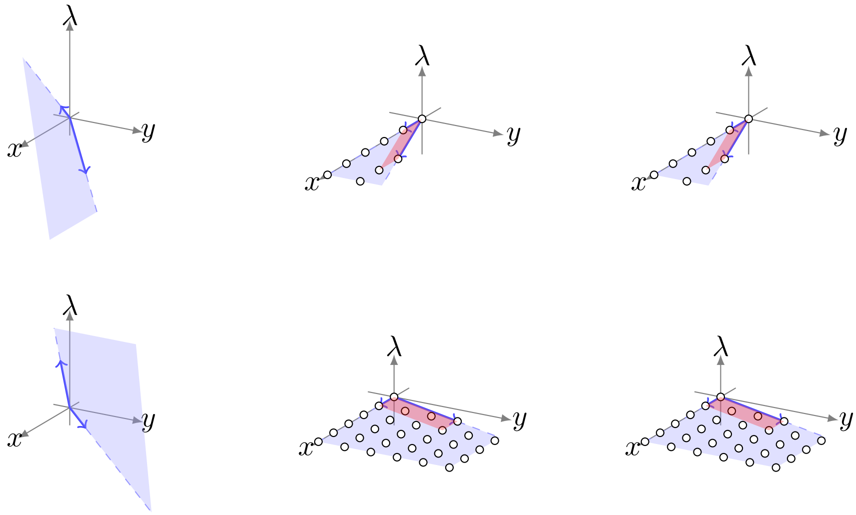
The first two rules in MacMahon's list are:
For $s \in \mathbb{N}^*$:

$$\Omega_{\geq} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda^s}\right)} = \frac{1}{(1-x)\left(1-x^s y\right)}$$

and

$$\Omega_{\geq} \frac{1}{(1-\lambda^s x)\left(1-\frac{y}{\lambda}\right)} = \frac{1+xy\frac{1-y^{s-1}}{1-y}}{(1-x)\left(1-xy^s\right)}.$$

The rules are about evaluating $\Omega_{\geq} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda^s}\right)}$ and $\Omega_{\geq} \frac{1}{(1-\lambda^s x)\left(1-\frac{y}{\lambda}\right)}$. There is an apparent symmetry in the input, but elimination results in structurally different numerators, i.e., 1 versus $1 + xy\frac{1-y^{s-1}}{1-y}$. This phenomenon is better understood if we look at the saturated semigroup that is the solution set of each linear Diophantine system. These two cases correspond to the inequalities $x - sy \geq 0$ and $sx - y \geq 0$ respectively. The two cones generated by following the Cayley Lifting construction are



The geometry of the first two MacMahon rules.

The numerator in the rational generating function expresses the lattice points in the fundamental parallelepiped. Since the first cone is unimodular, the numerator is 1. For the second cone, we observe that the fundamental parallelepiped, except for the origin, contains a vertical segment (truncated geometric series) at $x = 1$. The length of the segment is s and since the fundamental parallelepiped is half-open, the lattice points of this segment are $(1, q)$ for $q = 1, 2, \ldots, s$. The rational function

$$xy\frac{1-y^{s-1}}{1-y} = \frac{xy - xy^s}{1-y}$$

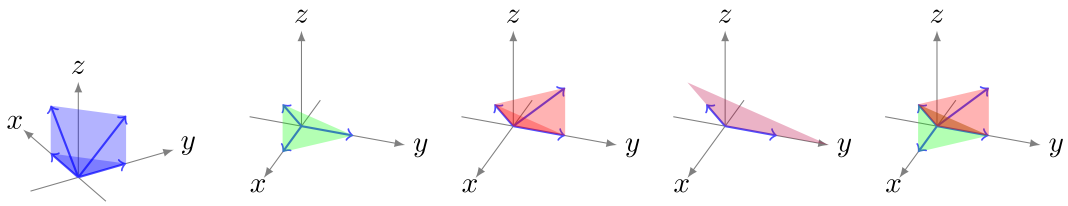
is exactly the generating function of that segment.

MacMahon Rule 4

$$\Omega_{\geq} \frac{1}{(1-\lambda x)(1-\lambda y)\left(1-\frac{z}{\lambda}\right)} = \frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)}.$$

Although the rational function on which $\Omega_{\geq}$ acts has three factors in the denominator, the resulting rational generating function has four factors in the denominator. Moreover, we observe that the numerator has terms with both positive and negative signs. We will give a geometric view on these observations. The λ-cone defined in the left hand side of the MacMahon rule is $\mathcal{C}_{\mathbb{R}}\left((1,0,0,1),(0,1,0,1),(0,0,-1,1)\right)$. If we compute the intersection of this cone with the positive quadrant of $\mathbb{R}^4$, we obtain a cone generated by

$(1,0,0,1),(0,1,0,1),(1,0,1,0)$ and $(0,1,1,0)$. Let's project this into $\mathbb{R}^3$ (by eliminating the last coordinate). We obtain a non-simplicial cone. We provide two decompositions of the cone into simplicial cones. We used Normaliz [21] for the computations.
The first decomposition is by triangulation as shown here:



The generating function of the green cone $\mathcal{C}_{\mathbb{R}}\left((1,0,0),(0,1,0),(1,0,1)\right)$ is

$$\frac{1}{(1-x)(1-y)(1-xz)}$$

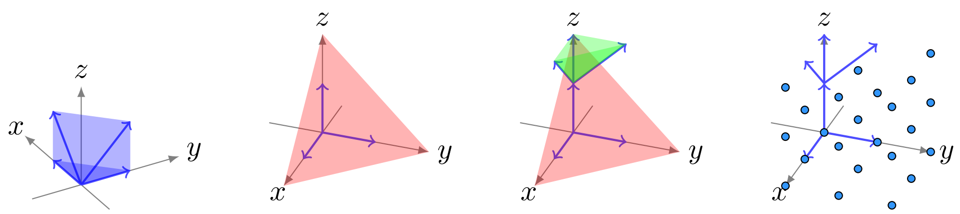
while that of the red cone $\mathcal{C}_{\mathbb{R}}\left((0,1,0),(0,1,1),(1,0,1)\right)$ is

$$\frac{1}{(1-y)(1-xz)(1-yz)}.$$

If we perform inclusion-exclusion (subtracting the purple cone), we obtain:

$$\begin{aligned}\frac{1}{(1-x)(1-y)(1-xz)}+\frac{1}{(1-y)(1-xz)(1-yz)}-\frac{1}{(1-y)(1-xz)} &= \\ \frac{(1-xy)(1-y)}{(1-x)(1-y)(1-xz)(1-yz)}-\frac{1}{(1-y)(1-xz)} &= \\ \frac{(1-xy)(1-y)-(1-xz)}{(1-y)(1-y)(1-xz)(1-yz)} &= \\ \frac{1-xyz}{(1-y)(1-y)(1-xz)(1-yz)}\end{aligned}$$

The second decomposition is not a triangulation, but a signed cone decomposition (à la Barvinok).



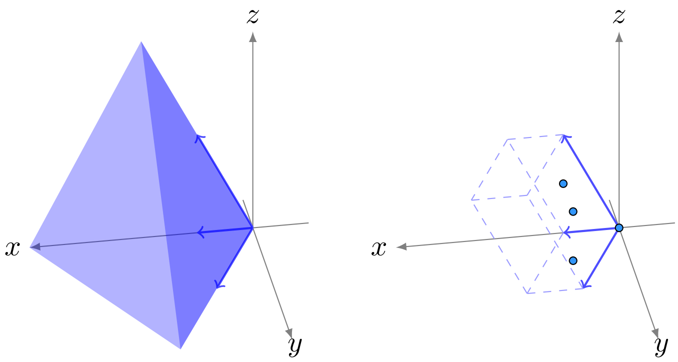
We first consider the whole positive quadrant and then subtract the half-open cone $\mathcal{C}_{\mathbb{R}}^{(0,0,1)}\left((1,0,1),(0,1,1),(0,0,1)\right)$. Half-opening the cone is equivalent (as far as lattice points are concerned) to shifting the cone by the generator $(0,0,1)$. Thus we have at the generating function level:

$$\begin{aligned}
\frac{1}{(1-x)(1-y)(1-z)} - \frac{z}{(1-xz)(1-yz)(1-z)} &= \\
\frac{(1-xz)(1-yz) - z(1-x)(1-y)}{(1-x)(1-y)(1-z)(1-xz)(1-yz)} &= \\
\frac{(1-z) - xyz(1-z)}{(1-y)(1-y)(1-z)(1-xz)(1-yz)} &= \\
\frac{1-xyz}{(1-y)(1-y)(1-xz)(1-yz)}
\end{aligned}$$

MacMahon Rule 6

$$\Omega_{\geq} \frac{1}{(1-\lambda^2 x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)} = \frac{1+xy+xz+xyz}{(1-x)(1-xy^2)(1-xz^2)}.$$

In this rule we observe that we have a positive sum of four terms in the numerator. The Ω-polyhedron is a cone that is simplicial but not unimodular. Observe in the figure the four points in the fundamental parallelepiped, corresponding to the four terms in the sum.



Other MacMahon Rules

Although the list of rules is not comprehensive, in the sense that they cannot cover all cases needed to treat linear Diophantine systems, they can solve certain combinatorial problems. For the rest of MacMahon's rules similar geometric observations apply, but the geometry becomes more complicated and the reasoning behind the numerators or the denominators appearing is not as simple anymore. A list of other MacMahon rules is given here:

MacMahon Rule 3

$$\Omega_{\geq} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)(1-xz)}.$$

MacMahon Rule 5

$$\Omega_{\geq} \frac{1}{(1-\lambda x)(1-\lambda y)\left(1-\frac{z}{\lambda^2}\right)} = \frac{1+xyz-x^2yz-xy^2z}{(1-x)(1-y)(1-x^2z)(1-y^2z)}.$$

MacMahon Rule 7

$$\Omega_{\geq} \frac{1}{(1-\lambda^2 x)(1-\lambda y)\left(1-\frac{z}{\lambda}\right)} = \frac{1+xz-xyz-xyz^2}{(1-x)(1-y)(1-yz)(1-xz^2)}.$$

MacMahon Rule 8

$$\frac{1}{(1-\lambda x)(1-\lambda y)(1-\lambda z)\left(1-\frac{w}{\lambda}\right)} = \frac{1-xyw-xzw-yzw+xyzw+xyzw^2}{(1-x)(1-y)(1-z)(1-xw)(1-yw)(1-zw)}.$$

MacMahon Rule 9

$$\Omega_{\geq} \frac{1}{(1-\lambda x)(1-\lambda y)\left(1-\frac{z}{\lambda}\right)\left(1-\frac{w}{\lambda}\right)} = \frac{1-xyz-xyw-xyzw+xy^2zw+x^2yzw}{(1-x)(1-y)(1-xz)(1-xw)(1-yz)(1-yw)}.$$

3.1.4 Andrews-Paule-Riese

In 1997, Bousquet–Mélou and Eriksson presented a theorem on lecture-hall partitions in [18]. This theorem gathered a lot of attention from the community (and it still does, with many lecture-hall type theorems appearing still today). Andrews, who had already studied MacMahon's method and was aware of its computational potential, figured that lecture-hall partitions offered the right problems to attack algorithmically via partition analysis. At the same time he planned to spend a semester during his sabbatical at the Research Institute for Symbolic Computation (RISC) in Austria to work with Paule. It is only natural that the result was a fully algorithmic version of MacMahon's method powered by symbolic computation. This collaboration gave a series of 10 papers [4, 1, 10, 5, 6, 7, 3, 8, 9, 2]. Many interesting theorems and different kinds of partitions are defined and explored in this series, but for us the two most important references are [4] and [5], which contain the algorithmic improvements on partition analysis. Namely, in [4] the authors introduce Omega, a Mathematica package based on a fully algorithmic partition analysis version, while in [5] they present a more advanced partial fraction decomposition (and the related Mathematica package Omega2) solving some of the problems appearing in Omega. While presenting the two methods, we will see some geometric aspects of theirs.

Omega

The main tool in Omega is the Fundamental Recurrence for the $\Omega_{\geq}$ operator, given by Lemma 6. Following MacMahon, or Elliott for that matter, iterative application of this recurrence is enough for computing the action of $\Omega_{\geq}$.

Lemma 6 (Fundamental Recurrence, Theorem 2.1 in [4]). *For $m, n \in \mathbb{N}^*$ and $a \in \mathbb{Z}$:*

$$\Omega_{\geq} \frac{\lambda^a}{(1-x_1\lambda)\cdots(1-x_n\lambda)(1-\frac{y_1}{\lambda})\cdots(1-\frac{y_m}{\lambda})} = \frac{P_{n,m,a}(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_m)}{\prod_{i=1}^n(1-x_i)\prod_{i=1}^n\prod_{j=1}^m(1-x_iy_j)}$$

where for $n > 1$

$$P_{n,m,a}(x_1,\ldots,x_n;y_1,\ldots,y_m) =$$

$$\frac{1}{x_n - x_{n-1}}\left(x_n(1-x_{n-1})\prod_{j=1}^m(1-x_{n-1}y_j)P_{n-1,m,a}(x_1,\ldots,x_{n-2},x_n;y_1,\ldots,y_m)\right.$$

$$\left.-x_{n-1}(1-x_n)\prod_{j=1}^m(1-x_ny_j)P_{n-1,m,a}(x_1,\ldots,x_{n-2},x_{n-1};y_1,\ldots,y_m)\right)$$

and for $n = 1$

$$P_{1,m,a}(x_1;y_1,\ldots,y_m) = \begin{cases} x_1^{-a} & \text{if } a \leq 0 \\ x_1^{-a} + \prod_{j=1}^m(1-x_1y_j)\sum_{j=0}^a h_j(y_1,\ldots,y_m)(1-x_1^{j-a}) & \text{if } a > 0 \end{cases}$$

□

The recurrence looks intimidating, but bear in mind that it is supposed to be a computational tool used in combination with symbolic computation. It is this power that earlier authors did not have. Even the ones with extreme computational (by hand) abilities, like MacMahon, had to resort to lookup tables and sets of rules for the more usual cases.

The base cases for the recurrence are when either all the terms have positive λ-exponents or all the terms have negative-λ exponents. For the two base cases we recall the definition of complete homogeneous symmetric polynomials and some related notation.

Definition 3.2 (Complete Homogeneous Symmetric Polynomials, see [5])
We define $h_i(z_1, z_2, \ldots, z_n)$ through the generating function

$$\sum_{i=0}^{\infty} h_i(z_1,z_2,\ldots,z_n)t^i = \frac{1}{(1-z_1t)(1-z_2t)\cdots(1-z_nt)}.$$

What is needed for the base cases of the fundamental recurrence, is the partial sum of this generating function (series). The following definition is useful to set notation.

For $a \in \mathbb{Z}$, let

$$H_a(z_1, z_2, \ldots, z_n) = \begin{cases} \sum_{i=0}^{a} h_i(z_1, z_2, \ldots, z_n) & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases}.$$

□

Now, returning to [4], we have the two base cases:

Lemma 7 (Lemma 2.1 in [4]). *For any integer a,*

$$\begin{aligned} \Omega_{\geq} \frac{\lambda^{\alpha}}{(1 - x_1\lambda)(1 - x_2\lambda)\ldots(1 - x_n\lambda)} &= \Omega_{\geq} \sum_{j=0}^{\infty} h_j(x_1, x_2, \ldots, x_n)\lambda^{a+j} \\ &= \frac{1}{(1-x_1)(1-x_2)\ldots(1-x_n)} - H_{-a-1}(x_1, x_2, \ldots, x_n). \end{aligned}$$

□

Lemma 8 (Lemma 2.2 in [4]). *For any integer a,*

$$\begin{aligned} \Omega_{\geq} \frac{\lambda^{\alpha}}{(1 - \frac{y_1}{\lambda})(1 - \frac{y_2}{\lambda})\ldots(1 - \frac{y_m}{\lambda})} &= \Omega_{\geq} \sum_{j=0}^{\infty} h_j(y_1, y_2, \ldots, y_m)\lambda^{a-j} \\ &= H_a(x_1, x_2, \ldots, x_n). \end{aligned}$$

□

The fundamental recurrence in Lemma 6 assumes that the exponents of λ in the denominator are ± 1. This is not a strong assumption, as noted in [4], since we can always employ the following decomposition.

$$(1 - x\lambda^r) = \prod_{j=0}^{r-1}(1 - \rho^j x^{\frac{1}{r}}\lambda)$$

$$(1 - \frac{y}{\lambda^s}) = \prod_{j=0}^{s-1}(1 - \frac{\sigma^j y^{\frac{1}{s}}}{\lambda})$$

where $\rho = e^{\frac{2\pi i}{r}}$ and $\sigma = e^{\frac{2\pi i}{s}}$. The obvious drawback of this approach is that we introduce complex coefficients instead of just ± 1, which were the only possible coefficients before the decomposition. This motivates the desire for a better recurrence.

Omega2

In [5] the authors introduce an improved partial fraction decomposition method, given by the recurrence of Theorem 3.1.

Theorem 3.1 (Generalized Partial Fraction Decomposition, see [5])
Let $\alpha \geq \beta \geq 1$, $\gcd(\alpha, \beta) = 1$ and let $\operatorname{rmd}(a, b)$ denote the division remainder of a by b. Then

$$\frac{1}{(1 - z_1 z_3^{\alpha})(1 - z_2 z_3^{\beta})} = \frac{1}{(z_2^{\alpha} - z_1^{\beta})} \left(\frac{\bar{P}(z_3)}{(1 - z_1 z_3^{\alpha})} + \frac{\bar{Q}(z_3)}{(1 - z_2 z_3^{\beta})} \right) \tag{3.8}$$

where

$$\bar{P}(z_3) := \sum_{i=0}^{\alpha-1} \bar{a}_i z_3^i \text{ for } \bar{a}_i = \begin{cases} -z_1^{\beta} z_2^{\frac{i}{\beta}} & \text{if } \beta | i \text{ or } i = 0, \\ -z_1^{\mathrm{rmd}((\alpha^{-1} \bmod \beta)i,\beta)} z_2^{\mathrm{rmd}((\beta^{-1} \bmod \alpha)i,\alpha)} & \text{otherwise,} \end{cases}$$

while

$$\bar{Q}(z_3) := \sum_{i=0}^{\beta-1} \bar{b}_i z_3^i \text{ for } \bar{b}_i = \begin{cases} z_2^{\alpha} & \text{if } i = 0, \\ z_1^{\mathrm{rmd}((\alpha^{-1} \bmod \beta)i,\beta)} z_2^{\mathrm{rmd}((\beta^{-1} \bmod \alpha)i,\alpha)} & \text{otherwise.} \end{cases}$$

□

Moreover, two new base cases are introduced. As before, we define some notation for homogeneous polynomials.

Definition 3.3 (Oblique Complete Homogeneous Polynomials, see [5])
For $\zeta_1, \zeta_2, \ldots, \zeta_n \in \mathbb{N}^*$, we define $h_i(z_1, z_2, \ldots, z_n; \zeta_1, \zeta_2, \ldots, \zeta_n)$ through the generating function

$$\sum_{i=0}^{\infty} h_i(z_1, z_2, \ldots, z_n; \zeta_1, \zeta_2, \ldots, \zeta_n) t^i = \frac{1}{(1 - z_1 t^{\zeta_1})(1 - z_2 t^{\zeta_2}) \cdots (1 - z_n t^{\zeta_n})}.$$

□

The related partial sums are defined for $a \in \mathbb{Z}$, as

$$H_a(z_1, z_2, \ldots, z_n; \zeta_1, \zeta_2, \ldots, \zeta_n) = \begin{cases} \sum_{i=0}^{a} h_i(z_1, z_2, \ldots, z_n; \zeta_1, \zeta_2, \ldots, \zeta_n) & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases}.$$

Lemma 9 (Case $m = 0$, Section 2 in [5]). *For any integer a,*

$$\begin{aligned} \underset{\geq}{\Omega} \frac{\lambda^a}{(1 - x_1 \lambda^{j_1})(1 - x_2 \lambda^{j_2}) \ldots (1 - x_n \lambda^{j_n})} &= \underset{\geq}{\Omega} \sum_{j=0}^{\infty} h_j(x_1, x_2, \ldots, x_n; j_1, j_2, \ldots, j_n) \lambda^{a+j} \\ &= \frac{1}{(1 - x_1)(1 - x_2) \ldots (1 - x_n)} \\ &\quad - H_{-a-1}(x_1, x_2, \ldots, x_n; j_1, j_2, \ldots, j_n). \end{aligned}$$

□

Lemma 10 (Case $n = 0$, Section 2 in [5]). *For any integer a,*

$$\begin{aligned} \underset{\geq}{\Omega} \frac{\lambda^a}{(1 - \frac{y_1}{\lambda^{j_1}})(1 - \frac{y_2}{\lambda^{j_2}}) \ldots (1 - \frac{y_m}{\lambda^{j_m}})} &= \underset{\geq}{\Omega} \sum_{j=0}^{\infty} h_j(x_1, x_2, \ldots, x_n; j_1, j_2, \ldots, j_n) \lambda^{a-j} \\ &= H_a(x_1, x_2, \ldots, x_n; j_1, j_2, \ldots, j_n). \end{aligned}$$

□

Lemma 11 (Case $m = 1$, Section 2 in [5]). *For any integer $a < j$,*

$$\Omega_{\geq} \frac{\lambda^a}{(1 - x_1\lambda^{j_1}) \ldots (1 - x_n\lambda^{j_n})(1 - y\lambda^{-k})} = \frac{1}{(1 - x_1)(1 - x_2) \ldots (1 - x_n)(1 - y)} - \frac{\sum_{\tau_1,\tau_2,\ldots,\tau_n=0}^{k-1} \prod x_i^{\tau_i} y^{\left\lfloor \frac{\sum j_i\tau_i + a}{k} \right\rfloor + 1}}{(1 - x_1^k y^{j_1}) \ldots (1 - x_n^k y^{j_n})(1 - y)}.$$

□

Lemma 12 (Case $n = 1$, Section 2 in [5]). *For any integer $a > -k$,*

$$\Omega_{\geq} \frac{\lambda^a}{(1 - x\lambda^j)(1 - y_1\lambda^{-k_1}) \ldots (1 - y_m\lambda^{-k_m})} = \frac{\sum_{\tau_1,\tau_2,\ldots,\tau_n=0}^{j-1} \prod y_i^{\tau_i} x^{\left\lceil \frac{\sum k_i\tau_i - a}{j} \right\rceil}}{(1 - x)(1 - x^{k_1}y_1^j) \ldots (1 - x^{k_m}y_m^j)}.$$

□

Omega2 was used to obtain results in a series of papers by Andrews-Paule and their coauthors, dealing with various problems from partition theory. For the project in general see `http://www.risc.jku.at/research/combinat/software/Omega/`.

Now we proceed with the geometric interpretation of the Generalized Partial Fraction Decomposition. We first rewrite (3.8) by pulling out $-z_1^\beta$ from the denominator of $\frac{1}{(z_2^\alpha - z_1^\beta)}$,i.e.,

$$\frac{1}{(1 - z_1 z_3^\alpha)(1 - z_2 z_3^\beta)} = \frac{-z_1^{-\beta}\bar{P}(z_3)}{(1 - z_1^{-\beta}z_2^\alpha)(1 - z_1 z_3^\alpha)} - \frac{z_1^{-\beta}\bar{Q}(z_3)}{(1 - z_1^{-\beta}z_2^\alpha)(1 - z_2 z_3^\beta)}, \tag{3.9}$$

and define $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ to be

- $P_{\alpha,\beta} := -z_1^{-\beta}\bar{P}(z_3) = \sum_{i=0}^{\alpha-1} a_i z_3^i$
- $Q_{\alpha,\beta} := z_1^{-\beta}\bar{Q}(z_3) = \sum_{i=0}^{\beta-1} b_i z_3^i$

where

- $a_i = \begin{cases} z_2^{\frac{i}{\beta}} & \text{if } \beta | i \text{ or } i = 0, \\ z_1^{\mathrm{rmd}((\alpha^{-1} \bmod \beta)i, \beta) - \beta} z_2^{\mathrm{rmd}((\beta^{-1} \bmod \alpha)i, \alpha)} & \text{otherwise,} \end{cases}$
- $b_i = \begin{cases} z_1^{-\beta} z_2^\alpha & \text{if } i = 0, \\ z_1^{\mathrm{rmd}((\alpha^{-1} \bmod \beta)i, \beta) - \beta} z_2^{\mathrm{rmd}((\beta^{-1} \bmod \alpha)i, \alpha)} & \text{otherwise.} \end{cases}$

The following theorem provides the basis for the geometric interpretation of the Generalized Partial Fraction Decomposition.

Theorem 3.2 (Polyhedral geometry interpretation of OMEGA2)
Application of the generalized partial fraction decomposition

$$\frac{1}{(1-z_1z_3^{\alpha})(1-z_2z_3^{\beta})} = \frac{P_{\alpha,\beta}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_1z_3^{\alpha})} - \frac{Q_{\alpha,\beta}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_2z_3^{\beta})} \tag{3.10}$$

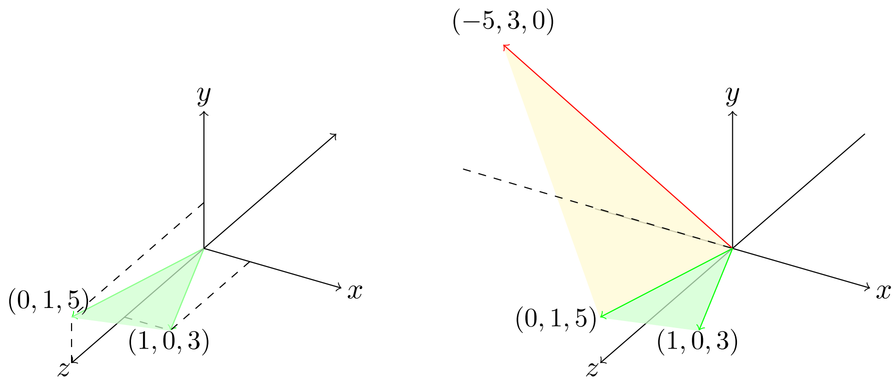
on $\rho_C(\mathbf{z})$ induces a signed cone decomposition [1] of the cone $C = \mathcal{C}_{\mathbb{R}}((1,0,\alpha),(0,1,\beta))$. □

Proof Strategy:

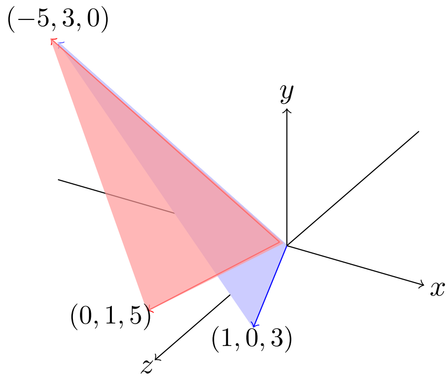
- Determine the structure of the fundamental parallelepiped of the cones $A = \mathcal{C}_{\mathbb{R}}((-\beta,\alpha,0),(1,0,\alpha))$ and $B = \mathcal{C}_{\mathbb{R}}((-\beta,\alpha,0),(0,1,\beta))$.

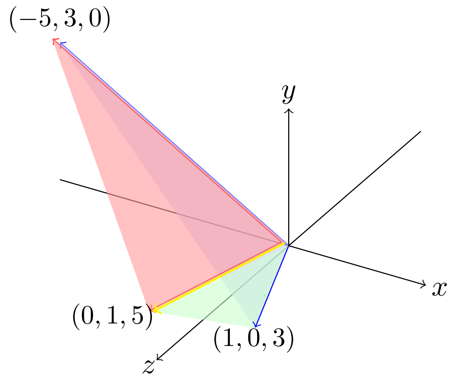
- Prove that $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ are the generating functions of these fundamental parallelepipeds.

For the detailed proof of Theorem 3.2 see Appendix A. Here we provide the picture of the decomposition for the cone $C = \mathcal{C}_{\mathbb{R}}((1,0,3),(0,1,5))$, into the cones $A = \mathcal{C}_{\mathbb{R}}((-5,3,0),(1,0,3))$ and $B = \mathcal{C}_{\mathbb{R}}^{0,1}((-5,3,0),(0,1,5))$.



[1] A decomposition into a sum where the summands may have positive or negative sign.





The statement (from the proof in Appendix A)

$$\rho_{\Pi(B)} = Q_{\alpha,\beta} + 1 - z_1^{-\beta} z_2^{\alpha}$$

means that $\frac{Q_{\alpha,\beta}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_2z_3^{\beta})}$ is the generating function of the half-open cone B that is open on the ray generated by $(-\beta, \alpha, 0)$.

One can see the full signed decomposition on the generating-function level as follows:

$$\rho_C = \rho_A - \rho_B + \rho_{\mathcal{C}_{\mathbb{R}}((0,1,\beta))}.$$

According to the previous analysis $\rho_A = \frac{P_{\alpha,\beta}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_1z_3^{\alpha})}$.
Since $\rho_{\mathcal{C}_{\mathbb{R}}((0,1,\beta))} = \frac{1}{(1-z_2z_3^{\beta})}$ we have

$$\begin{aligned}-\rho_B + \rho_{\mathcal{C}_{\mathbb{R}}((0,1,\beta))} &= -\frac{Q_{\alpha,\beta} + 1 - z_1^{-\beta}z_2^{\alpha}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_2z_3^{\beta})} + \frac{1}{\left(1-z_2z_3^{\beta}\right)} \\ &= -\frac{Q_{\alpha,\beta}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_2z_3^{\beta})}\end{aligned}$$

which means

$$\frac{1}{(1-z_1z_3^{\alpha})(1-z_2z_3^{\beta})} = \frac{P_{\alpha,\beta}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_1z_3^{\alpha})} - \frac{Q_{\alpha,\beta}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_2z_3^{\beta})}.$$

In other words, $Q_{\alpha,\beta}$ encodes the inclusion-exclusion step in the decomposition.

3.2 Algebraic Partition Analysis

Having reviewed the history of partition analysis both from the perspective of the original authors and from a more geometric perspective, we give now an algebraic presentation the $\Omega_{\geq}$ operator .

In particular, we will use graded vector spaces and rings in order to compute the generating functions of the solutions of a linear Diophantine system.

3.2.1 Graded Rings

Let $R = \mathbb{K}[z_1, z_2, \ldots, z_d]$ be the ring of polynomials in d variables $z_1, z_2, \ldots, z_d$. In what follows we will grade this ring in a way that is useful for solving linear Diophantine systems. Let us first define a graded vector space, as we will look at the polynomial ring as a vector space over the coefficient field.

Definition 3.4 (Graded vector space)
Let $\mathbb{K}$ be a field and V a vector space of $\mathbb{K}$. If there exists a direct sum decomposition

$$V = \bigoplus_{i \in I} V_i$$

of V into linear subspaces V_i for some index set I, then we say that V is I-graded and the V_i's are called homogeneous components, while the elements of V_k are called homogeneous elements of degree k. □

The most usual grading of the polynomial ring $\mathbb{K}[z_1, z_2, \ldots, z_d]$ is the one induced by the usual polynomial degree. The homogeneous component V_k is the linear span of all monomials of degree k, i.e., the set of all homogeneous polynomials of degree k and the polynomial 0.

A notion that connects graded algebraic structures with generating functions is that of the Hilbert-Poincaré series. Given a graded vector space $V = \bigoplus_{i \in I} V_i$ for some (appropriate) index set I, then the Hilbert-Poincaré series of V over $\mathbb{K}$ is the formal powerseries

$$\mathcal{HP}(V) = \sum_{i \in I} \dim_{\mathbb{K}}(V_i)\, t^i.$$

Note that:

- The most usual definition of Hilbert-Poincaré series in the literature, restricts the index set I to be $\mathbb{N}$ and all linear subspaces V_i to be finite-dimensional.
- If the index set I is not a subset of $\mathbb{N}^k$ for some k, but all elements of I are bounded from below, then the Hilbert-Poincaré series is a formal Laurent series. If the index set contains elements that are not bounded from below, then the Hilbert-Poincaré series is not defined.
- According to the definition of a graded vector space, it is possible that not all V_i are finite-dimensional. In that case the Hilbert-Poincaré series is not defined.

The algebraic approach to partition analysis has its roots in the very early work and motivation of Cayley, MacMahon and Elliott. Their motivation to deal with the problem stem from invariant theory, which was very fashionable these days. We will present here a setup of algebraic structures and associated series, mostly Hilbert series, which allows to view partition analysis algebraically. In the definition of graded vector spaces, and as is customary in literature, we first defined the homogeneous components and then assign to the elements of each component the corresponding degree. For our purposes though, it is more natural to follow the reverse path. We will first define a degree on the elements of the polynomial ring and then construct the homogeneous components as the linear span of all elements of the same degree. In particular, we want to construct a grading of $\mathbb{K}[z_1, z_2, \ldots, z_d]$ starting from a matrix A in $\mathbb{Z}^{m \times d}$. We assume that A is full-rank.

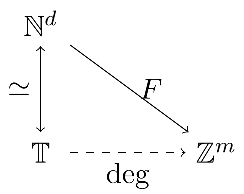
Let f_{a_i} be the linear functionals defined by the rows of A for $i \in [m]$. We define the function $F : \mathbb{N}^d \mapsto \mathbb{Z}^m$ by

$$F(v) = (f_{a_1}(v), f_{a_2}(v), \ldots, f_{a_m}(v)) \text{ for } v \in \mathbb{N}^d.$$

Since we want to use this function to grade the polynomial ring, it is necessary to translate from vectors to monomials and back. Given a set of formal variables $Z = \{z_1, z_2, \ldots, z_d\}$, we define the term monoid $\mathbb{T}$ to be the subset of $\mathbb{K}[z_1, z_2, \ldots, z_d]$ consisting of powerproducts, i.e., $\{z_1^{a_1} z_2^{a_2} \cdots z_d^{a_d} | a_i \in \mathbb{N}\}$ and we set $1 = z^0$. This is a multiplicative monoid with the standard monomial multiplication. We define ϕ as

$$\begin{aligned} \phi : &\mathbb{N}^d \to \mathbb{T} \\ &v \mapsto z^v \end{aligned}$$

establishing the monoid isomorphism $\mathbb{N}^d \simeq \mathbb{T}$. Now, we can define a degree in $\mathbb{T}$ via F.



In words, let $\deg : \mathbb{T} \to \mathbb{Z}^m$ be defined as $\deg(z^v) = F\left(\phi^{-1}(z^v)\right)$.
Define V_i for $i \in \mathbb{Z}^m$ to be the linear span over $\mathbb{K}$ of all elements in $\mathbb{T}$ with degree i, i.e.,

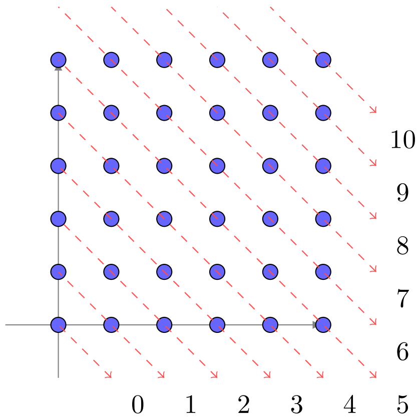
$$\begin{aligned} V_i &= \left\{ \sum_{k=0}^{n} c_k t_k \,\middle|\, n \in \mathbb{N}, c_k \in \mathbb{K}, t_k \in \mathbb{T} \text{ with } \deg t_k = i \right\} \text{ for } i \in \mathbb{Z}^m \\ &= \langle t \in \mathbb{T} : \deg(t) = i \rangle_{\mathbb{K}}, \text{ for } i \in \mathbb{Z}^m \end{aligned}$$

and $\langle \emptyset \rangle_{\mathbb{K}} = \{0\}$. Then we have that

$$\mathcal{R} = \oplus_{i \in \mathbb{Z}^m} V_i.$$

This means that the collection of V_i's defines a grading on $\mathbb{K}[z_1, z_2, \ldots, z_d]$, which now can be seen as a graded vector space. Let us see an example of such a grading.

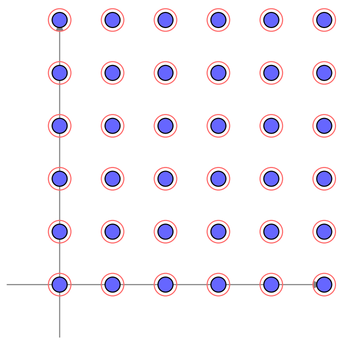
Example 9 (Total degree or ℓ_1-grading). *The degree function is given by* $F(s) = |s|_1$*, i.e., the matrix* A *is the unit matrix of rank* d*. This is the usual total degree for polynomials in* d *variables. In the figure we see the grading for* $d = 2$*.*



V_i *is the vector space of homogeneous polynomials of degree* i*.*

□

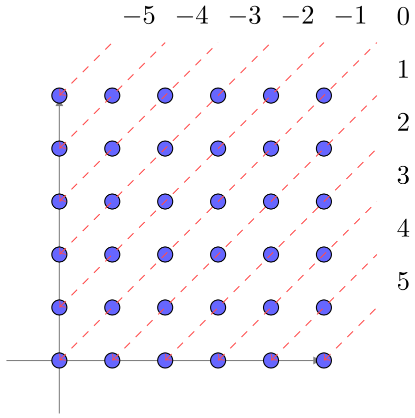
Example 10. *Let the degree map be defined by* $F(s) = s$*. The grading this map induces is called trivial because each* $\mathbb{K}$*-vector space* V_a *is generated by* z^a *alone.*



□

We note that some (or all) V_i may be infinite dimensional depending on the matrix A we started with, as in the following example.

Example 11. *Let the degree map be defined by* $F(x, y) = x - y$*. The grading this map induces is infinite, because each* $\mathbb{K}$*-vector space* V_i *is generated by infinitely many monomials.*



□

The Hilbert-Poincaré series provide a useful, refined counting tool for the number of solutions of linear Diophantine systems. Nevertheless, we would like to allow for non finite-dimensional graded components as well as for a tool enumerating rather than counting solutions.
Each V_i is a vector space containing homogeneous polynomials of degree i. It is always possible to pick a basis of V_i consisting of terms of degree i, i.e., elements of $\mathbb{T}$. We will denote such a basis by $\mathcal{B}_i$. We define the basis of $\{0\}$ to be the empty set.

In the direction of obtaining an enumerating generating function we define the truncated multivariate Hilbert-Poincaré series,

Definition 3.5 (truncated multivariate Hilbert-Poincaré series)
Given a graded vector space $V = \bigoplus_{i\in\mathbb{Z}^k} V_i$ over the field $\mathbb{K}$ and a vector $b \in \mathbb{Z}^k$ for some $k \in \mathbb{N}$, we define the truncated multivariate Hilbert-Poincaré series of V as

$$t\mathcal{HP}_b^V(z,t) = \sum_{b\leq i\in\mathbb{Z}^k} \left(\sum_{e\in\mathcal{B}_i} e \right) t^i.$$

The truncated multivariate Hilbert-Poincaré series is a formal Laurent series in the t variables and a formal powerseries in the z variables. □

The truncation in the indices is necessary in order to ensure that the formal Laurent series is well defined, while the formal sum of the basis elements avoid the problem of infinite-dimensionality. At the same time, this formal series enumerates all solutions of a linear Diophantine system $Ax \geq b$, if the vector space V is graded via the procedure described above for the matrix $A \in \mathbb{Z}^{m\times d}$. We note that in that case $k = m$.

The following theorem provides the connection between the $\Omega_\geq$ operator and Hilbert-Poincaré series.

Theorem 1. *Given $A \in \mathbb{Z}^{m\times d}$, $b \in \mathbb{Z}^m$ and $t\mathcal{HP}_b^V(z,t)$ as above, we have*

$$\Omega_\geq \sum_{x\in\mathbb{N}^n} z^x \prod_{i=1}^{m} \lambda_i^{A_i x - b_i} = t\mathcal{HP}_b^V(z,t)|_{t=1}.$$

□

Proof. Let S be the set of solutions to the linear Diophantine system $Ax \geq b$. By the definition of the $\Omega_\geq$ operator, we have that $\Omega_\geq \sum_{\mathbf{x}\in\mathbb{N}^n} \mathbf{z^x} \prod_{i=1}^m \lambda_i^{A_i\mathbf{x}-b_i}$ is $\Phi_{A,b}(z)$, the generating function of S. Now we have:

$$\begin{aligned}
\alpha \in S \quad &\Leftrightarrow \quad A\alpha \geq b\\
&\Leftrightarrow \quad A\alpha = i \text{ with } i \geq b\\
&\Leftrightarrow \quad F(\alpha) = i \text{ with } i \geq b\\
&\Leftrightarrow \quad \deg(z^\alpha) = i \text{ with } i \geq b\\
&\Leftrightarrow \quad z^\alpha \in R_i \text{ with } i \geq b\\
&\Leftrightarrow \quad z^\alpha \in B_i \text{ with } i \geq b.
\end{aligned}$$

The last equivalence implies that

$$[z^x]\Phi_{A,b}(z) = 1 \Leftrightarrow [z^x]t\mathcal{HP}_b^V(z,t) = t^{F(x)}$$

and the theorem follows. □

Note that this relation is all but new. It is actually as old as Cayley who first used the $\Omega_\geq$ operator in the context of invariant theory.

The construction of the grading provides a way to describe the solutions of a linear Diophantine system. Given a matrix A in $\mathbb{Z}^{m\times d}$, the solution set of the homogeneous linear Diophantine system $Ax \geq 0$ is

$$\bigcup_{i\in\mathbb{N}^m} \phi^{-1}(\mathcal{B}_i) \subseteq \mathbb{N}^d.$$

For the inhomogeneous linear Diophantine system $Ax \geq b$ for some $b \in \mathbb{Z}^m$ we have that the solution set is

$$\bigcup_{b\leq i\in\mathbb{N}^m} \phi^{-1}(\mathcal{B}_i) \subseteq \mathbb{N}^d.$$

In other words, the solutions of a linear Diophantine system or equivalently the lattice points of a polyhedron can be described via the construction of the appropriate grading given by the inequality description.

We will now proceed with a different grading construction, relevant to polytopes and Ehrhart theory.

Ehrhart grading

As we saw, the description of the lattice points in a polytope can be given by the above procedure. Continuing in the same direction, we will construct a grading that instead of just describing the lattice points in a polytope, it gives the lattice points in a cone over a polytope. The cone over a polytope is a fundamental construction in Ehrhart theory, used to enumerate the lattice points in the dilations of a polytope. Given a polytope R in $\mathbb{R}^d$, we embed it in $\mathbb{R}^{d+1}$ at height 1, i.e., $P' = \left\{(x,1) \in \mathbb{R}^{d+1} \;\middle|\; x \in P\right\}$.

In our grading construction, we respect the decomposition of lattice points according to their height. Given a matrix A in $\mathbb{Z}^{m\times d}$ and a vector b in $\mathbb{Z}^m$ we construct the matrix

$$E = \left[\begin{array}{ccc|c} & & & -b_1 \\ & A & & \vdots \\ & & & -b_m \\ \hline 0 & \dots & 0 & 1 \end{array}\right]$$

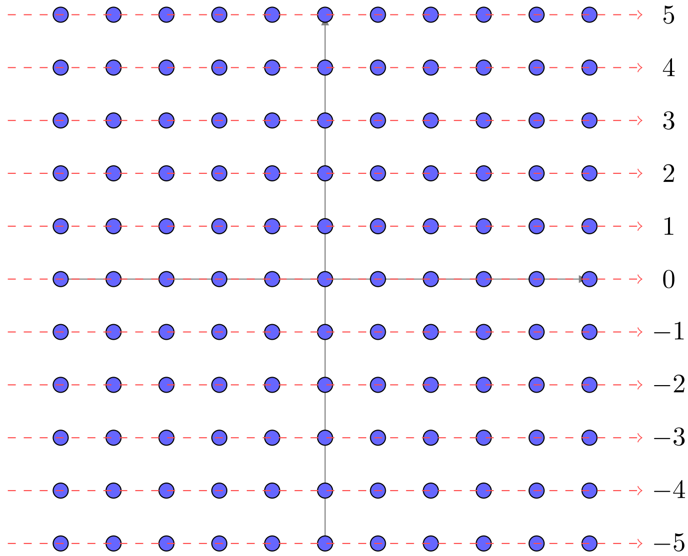
This matrix has the property that if the matrix A defines a polytope, then the first m rows of matrix E define the t-dilate of that polytope. The last row of E is used to keep track of the dilate t.

Since we allow the polytope P to contain negative coordinates, we consider the Laurent polynomial ring in $d+1$ variables $\mathbb{K}\left[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_d^{\pm 1}, t^{\pm 1}\right]$ and the corresponding Laurent term monoid $\mathbb{T} = [z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_d^{\pm 1}, t^{\pm 1}]$. We construct the V_i as before using the matrix E.

Now, on top of the set of V_i we need some extra structure. For j in $\mathbb{Z}$, define A_j to be the direct sum of all V_i for which the last coordinate of i is equal to j, i.e.,

$$A_j = \bigoplus \langle t \in \mathbb{T} \mid \deg(t) = (\alpha_1, \alpha_2, \dots, \alpha_m, j)\rangle_{\mathbb{K}}$$

The vector space V is now graded by the collection of A_i's. One should not confuse the $\deg(v)$ of an element v of V, which is a vector in $\mathbb{Z}^{m+1}$ and the degree of v induced by the grading, which is a non-negative integer indicating the height (last coordinate) of $\phi^{-1}(v)$ in the cone over the polytope.



Although this grading respects the height given by the cone over the polytope construction, it gives no information about which lattice points lie in the polytope. Note that we are only interested in non-negative dilations. For this reason, we define the subspaces L_j as

$$L_j = \langle t \in \mathbb{T} \mid \deg(t) = (\alpha_1, \alpha_2, \dots, \alpha_m, j), \alpha_k \geq 0 \text{ for all } k \in [m] \rangle_{\mathbb{K}}$$

for $j \in \mathbb{N}$, i.e., each L_j is generated by the monomials of A_j that correspond to lattice points in the j-th dilation of the polytope. This is expressed by the non-negativity condition on the first m coordinates of the degree vector.

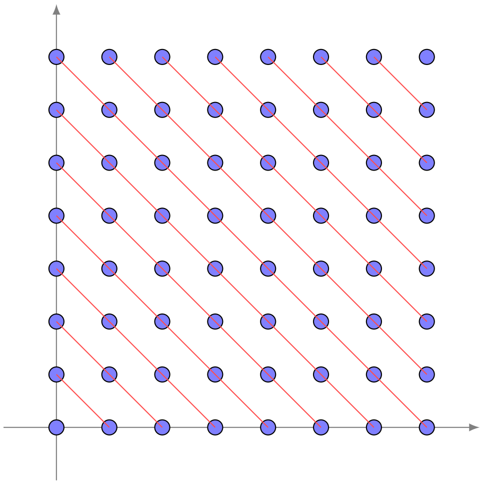
Contrary to the general construction from an arbitrary matrix $A \in \mathbb{Z}^{m\times d}$, the construction of a cone over a polytope guarantees that the homogeneous components L_j are finite-dimensional subspaces. This is because at each height k, the L_k is generated by the finitely many lattice points in th k-th dilation of the (by definition bounded) polytope P.

Now, the Ehrhart series of P is given by $\sum_{i\in\mathbb{N}} dim_{\mathbb{K}}(L_i)\, t^i$.

Oblique Graded Simplices

A useful example for truncated multivariate Hilbert-Poincaré series is the case of Oblique Graded Simplices. Let's see two cases:

- If we use the grading induced by the total degree (ℓ_1 grading) on the positive quadrant we have the following picture

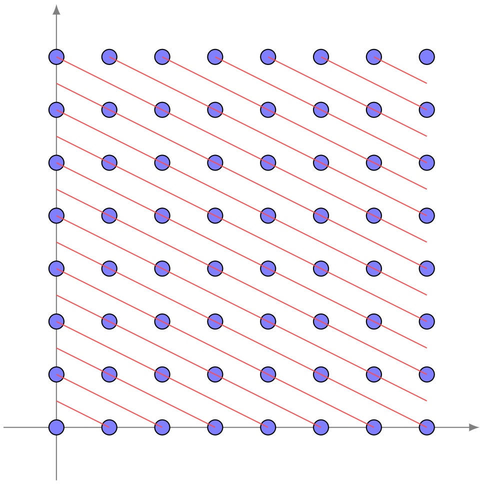


and the corresponding truncated multivariate Hilbert-Poincaré series is the generating function for the lattice points in the positive quadrant graded by the given grading. Note that if V is the grading constructed as above, then

$$h'_i(z) = [t^i]t\mathcal{HP}^V(z,t)$$

are the usual complete homogeneous symmetric polynomial of degree i.

- If we use the degree function given by $\mathrm{degree}(x,y) = x + 2y$ the picture is



and then

$$h'_i(z) = [t^i]t\mathcal{HP}^V(z,t)$$

are the oblique complete homogeneous polynomials $h_i(z;1,2)$.

In [5], the authors introduce $h_i(z_1, z_2, \ldots, z_n; \zeta_1, \zeta_2, \ldots, \zeta_n)$ for $\zeta_i \in \mathbb{N}$, "a variant of the symmetric functions", given by their generating function

$$\sum_{i=0}^{\infty} h_i(z_1, z_2, \ldots, z_n; \zeta_1, \zeta_2, \ldots, \zeta_n)t^i = \frac{1}{(1 - z_1 t^{\zeta_1})(1 - z_2 t^{\zeta_2}) \cdots (1 - z_n t^{\zeta_n})}.$$

Chapter 4

Partition Analysis via Polyhedral Geometry

The main idea behind all implementations of the $\Omega_\geq$ operator is to obtain a partial-fraction decomposition of the crude generating function and exploit the linearity of the $\Omega_\geq$ operator. In particular, we want a partial-fraction decomposition such that in the denominator of each fraction appear either only non-negative or only non-positive powers of λ's. Given such a partial-fraction decomposition, we can be sure that in the series expansions of the crude generating function with respect to the λ's, the fractions with only non-positive powers of λ cannot contribute terms with positive λ exponents. In other words, we should only keep the fractions that have non-negative powers of λ and the fractions that are λ-free. There are different ways to compute such a partial-fraction decomposition, like Elliott's algorithm [24] and Omega2 [5].

In this chapter, we present an algorithmic geometric approach. The main goal of the method presented here is to compute a cone decomposition of the Ω-polyhedron. Instead of going directly for the rational generating function of the lattice points in the Ω-polyhedron, we first compute a set of simplicial cones that sum up to the desired polyhedron. This step is done by using only rational linear algebra. In order to obtain the rational generating function, one can either use Barvinok's algorithm or explicit formulas. Some more custom-tailored tools will appear in [19]. Here we restrict to computing a set of "symbolic cones", i.e., instead of using the actual generating function we say "the cone generated by $a_1, a_2, \ldots, a_k$ with apex q". This work is part of [19].

Recall the definitions of the rational form of the crude generating function

$$\rho^{\Omega}_{A,b}(z;\lambda) = \Omega_\geq \frac{\lambda^{-b}}{\prod_{i=1}^{m}(1 - z_i \lambda^{A_i})}$$

and of the formal Laurent series form

$$\Phi^{\Omega}_{A,b}(z;\lambda) = \Omega_\geq \sum_{x_1,\ldots,x_d \in \mathbb{N}} \lambda^{Ax-b} z_1^{x_1} \ldots z_d^{x_d},$$

as well as that of the formal Laurent series form of the λ-generating function

$$\Phi^{\lambda}_{A,b}(z) = \sum_{x_1,\ldots,x_d \in \mathbb{N}} \lambda^{Ax-b} z_1^{x_1} \ldots z_d^{x_d}$$

and its rational form

$$\rho^{\lambda}_{A,b}(z) = \frac{\lambda^{-b}}{\prod_{i=1}^{n}(1 - z_i\lambda^{A_i})}.$$

We will denote by A the matrix with $A_1, A_2, \ldots, A_n$ as columns. We note that $\rho_{\Pi_C}(z)$ and $\Phi_{\Pi_C}(z)$, where Π_C denotes the fundamental parallelepiped of the cone C, are identical since the fundamental parallelepiped is finite and both objects collapse to a Laurent polynomial.

4.1 Eliminating a single λ

Elimination of a single λ amounts to the solution of a single linear Diophantine inequality. All known implementations of the $\Omega_{\geq}$ operator are based on the recursive elimination of λ's for the solution of linear Diophantine systems. In order to employ polyhedral geometry, we need to translate the problem to a problem about cones. Recall the definitions of the λ-cone and the Ω-polyhedron:

$$\mathcal{C}^{\lambda,\mathbb{Z}}(A) = \mathcal{C}_{\mathbb{R}}((e_1 : A_1), (e_2 : A_2), \ldots, (e_d : A_m))$$
$$\mathcal{P}^{\Omega_{\geq},\mathbb{R}}_{\mathbf{b}}(A) = \{\mathbf{x} \in \mathbb{R}^d_+ : A\mathbf{x} \geq \mathbf{b}\}$$

Let $H_b = \{(x_1, \ldots, x_d, x_{d+1}) \ : \ b \leq x_{d+1}\}$ be the set of vectors with last component greater or equal to b. Since we have one λ, the A_i is an integer denoted by a_i.

Computing $\Omega_{\geq} \frac{\lambda^{-b}}{(1-z_1\lambda^{a_1})\cdot\ldots\cdot(1-z_d\lambda^{a_d})}$ amounts to computing the generating function $\rho_{\pi(C\cap H_b)}(z_1, \ldots, z_d)$, where π denotes projection with respect to the last coordinate, as follows:

1. Compute a vertex description of the vertex cones $\mathcal{K}_v$ at the vertices of $C \cap H_b$.
2. Compute the generating functions $\rho_{\mathcal{K}_v}(z_1, z_2, \ldots, z_{d+1})$, either using Barvinok's algorithm [12] or by explicit formulas using modular arithmetic (similar to the expressions in [5]).
3. Substitute $z_{d+1} \mapsto 1$ to obtain λ-free generating functions (projection of the polyhedron).
4. Sum all the projected generating functions for the tangent cones. By Brion's theorem [12], this yields the desired generating function $\Omega_{\geq} \frac{\lambda^{-b}}{(1-z_1\lambda^{a_1})\cdot\ldots\cdot(1-z_d\lambda^{a_d})}$.

Let P be the polyhedron $C \cap H_b$. We need to compute the vertices and vertex cones of P. We note that any polyhedron can be written as the Minkowski sum of a cone and a polytope. In our case

$$P = \text{conv}(0, u_i : i \in \{1, \ldots, \ell\}) + \mathcal{C}_{\mathbb{R}}(w_{i,j} : i \in \{1, \ldots, \ell\}, j \in \{\ell+1, \ldots, d\})$$

where

$$\begin{aligned} u_i &= \frac{b}{a_i} v_i = (\ldots \frac{b}{a_i} \ldots b) \\ w_{i,j} &= -a_j v_i + a_i v_j = (0 \ldots 0 \; -a_j \; 0 \ldots 0 \; a_i \; 0 \ldots 0). \end{aligned}$$

The vertices of P, denoted by u_i, are the intersection points of the hyperplane at height $x_{d+1} = b$ with the generators of P that are pointing "up" ($a_i \geq 0$). The $w_{i,j}$ are positive linear combinations of the generators v_i that are pointing "up" and the generators v_j that are pointing "down" ($a_j < 0$) such that $w_{i,j}$ has last coordinate equal to zero. Combinatorially, this is a cone over a product of simplices.

Next, we want to compute the tangent cones of our polyhedron P. The tangent cones are then given by the following lemma.

Lemma 13. *The generators of the cone $\mathcal{K}_{u_i}$ are*

$$-u_i \text{ and } u_{i'} - u_i \text{ for } i' \in \{1, \ldots, \ell\} \setminus \{i\} \text{ and } w_{i,j} \text{ for } j \in \{\ell+1, \ldots, d\}.$$

These are d generators in total that are linearly independent. In particular, $\mathcal{K}_{u_i}$ is a simplicial cone.

□

Proof. First, we argue that $w_{i',j}$ is not a generator of $\mathcal{K}_{u_i}$ for all $i' \in \{1, \ldots, \ell\} \setminus \{i\}$ and all $j \in \{\ell+1, \ldots, n\}$. This follows from the simple calculation

$$u_i - \frac{b}{a_{i'} a_j} w_{i',j} = \begin{bmatrix} \frac{b}{a_i} \\ 0 \\ 0 \\ b \end{bmatrix} - \frac{a}{a_{i'} a_j} \begin{bmatrix} 0 \\ -a_j \\ a_{i'} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{b}{a_{i'}} \\ 0 \\ b \end{bmatrix} - \frac{a}{a_i a_j} \begin{bmatrix} -a_j \\ 0 \\ a_i \\ 0 \end{bmatrix} = u_{i'} - \frac{b}{a_i a_j} w_{i,j}$$

where the four components of the vectors that are shown have indices $i, i', j, d+1$, in that order, all other components are zero. Note that the coefficients $-\frac{b}{a_{i'} a_j}$ and $-\frac{b}{a_i a_j}$ are *positive* rational numbers as a_j is negative.

Moreover, none of the vectors $u_{i'} - u_k$ for $i' \neq i \neq k$ are in the tangent cone. This can be seen by observing that the k-th component of $u_{i'} - u_k$ is negative, while the k-th component of u_i is zero. However, $P \subset \mathbb{R}^d_{\geq 0} \times \mathbb{R}$.

So now we know that the only directions that can appear as generators are those given in the lemma. That all of these are, in fact, generators follows from a dimension argument: There are precisely d directions given in the lemma and we know that the polyhedron P and all of its tangent cones have dimension d as well. So the tangent cones have to have at least d generators. □

We denote by U_i the matrix that has as columns the vectors $a_i u_i$, $\operatorname{lcm}(a_i, a_{i'})(u_{i'} - u_i)$ for $i' \in \{1, \ldots, \ell\} \setminus \{i\}$ and $w_{i,j}$ for $j \in \{\ell+1, \ldots, d\}$. U_i is a $(d+1) \times d$ integer matrix that has a full-rank diagonal submatrix.

By Brion's theorem, the generating function for $\pi(P \cap H_b)$ is the sum for all i of the generating function of the cone generated by the columns of U_i and equal to $\rho_C(z_1, z_2, \ldots, z_d)$. We can now present the algorithm:

Algorithm 2 Elimination of a single λ

Require: $b \in \mathbb{Z}$ and $A \in \mathbb{Z}^d$ such that $a_1, a_2, \ldots, a_\ell > b$ and $a_{\ell+1}, a_{\ell+2}, \ldots, a_d < b$

$I^+ \leftarrow \{1, \ldots, \ell\}$
$I^+ \leftarrow \{\ell + 1, \ldots, d\}$
for $i \in I^+$ **do**
  $v_i \leftarrow (e_i : a_i)$
$U \leftarrow \{(\ldots \frac{b}{a_i} \ldots b)\}$ for $i \in I^+$
for $i \in I^+$ **do**
  for $j \in I^-$ **do**
    $w_{i,j} \leftarrow -a_j e_i + a_i e_j$
for $i \in I^+$ **do**
  $U_i \leftarrow \{-u_i, u_{i'} - u_i, w_{i,j} : i' \in I^+ \setminus \{i\}, j \in I^-\}$
  $T_i \leftarrow$ a triangulation of the vertex cone $\mathcal{C}_{\mathbb{R}}(U_i)$
  $\rho_{\mathcal{C}_{\mathbb{R}}(U_i)}(z_1, z_2, \ldots, z_{d+1}) \leftarrow \sum_{s \in T_i} \rho_s(z_1, z_2, \ldots, z_{d+1})$
$\rho(z_1, z_2, \ldots, z_{d+1}) \leftarrow \sum_{i \in I^+} \rho_{U_v}(z_1, z_2, \ldots, z_{d+1})$
return $\rho(z_1, z_2, \ldots, z_d, 1)$

To illustrate the algorithm we present an example.

Example 12. *Given an inequality* $2x_1 + 3x_2 - 5x_3 \geq 4$, *we want to compute*

$$\Omega_{\geq} \frac{\lambda^{-4}}{(1 - z_1\lambda^2)(1 - z_2\lambda^3)(1 - z_3\lambda^{-5})}.$$

We first compute the generators of $\mathcal{C}^{\lambda,\mathbb{Z}}(A)$*:*

$$v_1 = (1, 0, 0, 2), v_2 = (0, 1, 0, 3), v_3 = (0, 0, 1, -5)$$

and construct the matrix V*:*

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & -5 \end{bmatrix}.$$

We want to compute the generating function of the intersection of $\mathcal{C}^{\lambda,\mathbb{Z}}(A)$ *and* $H_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 \geq 4\}$.

The vectors needed to construct the tangent cones are:

$$u_1 = \frac{4}{2} v_1 = (2, 0, 0, 4)$$

$$u_2 = \frac{4}{3} v_2 = (0, \frac{4}{3}, 0, 4)$$

$$w_{1,3} = (-5, 0, 2, 0)$$

$$w_{2,3} = (0, -5, 3, 0)$$

and the two tangent cones are:

$$K_{u_1} = \mathcal{C}_{\mathbb{R}} \left((-2, 0, 0, -4), (-2, \frac{4}{3}, 0, 0), (-5, 0, 2, 0); (2, 0, 0, 4) \right)$$

$$K_{u_2} = \mathcal{C}_{\mathbb{R}} \left((0, -\frac{4}{3}, 0, -4), (2, -\frac{4}{3}, 0, 0), (0, -5, 3, 0); (0, \frac{4}{3}, 0, 4) \right).$$

The generating functions of the lattice points in the projected cones generated by the columns of U_1 and U_2 are

$$\frac{z_1^4 \left(z_1 + z_2 + z_1^4 z_3 + z_1^2 z_2 z_3\right)}{(1 - z_1) \left(z_1^3 - z_2^2\right) \left(1 - z_1^5 z_3^2\right)}$$

and

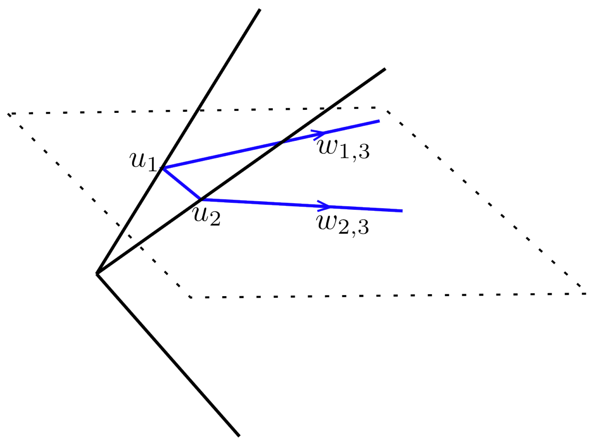
$$-\frac{z_2^2 \left(z_1^2 + z_1 z_2 + z_2^2 + z_1^2 z_2^2 z_3 + z_2^3 z_3 + z_1 z_2^3 z_3 + z_1 z_2^4 z_3^2 + z_1^2 z_2^4 z_3^2 + z_2^5 z_3^2\right)}{(1 - z_2) \left(z_1^3 - z_2^2\right) \left(1 - z_2^5 z_3^3\right)}.$$

By Brion their sum is the generating function for $\pi(C \cap H_4)$, which is equal to

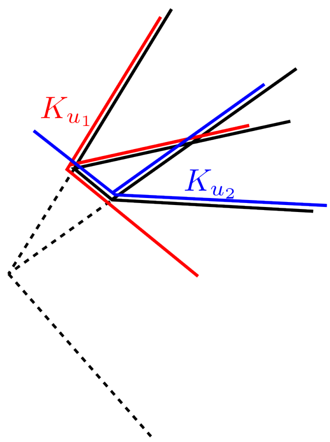
$$\Omega_{\geq} \frac{\lambda^{-4}}{(1 - z_1 \lambda^2)(1 - z_2 \lambda^3)(1 - z_3 \lambda^{-5})}.$$

□

In the following figure we can see the Ω-polyhedron and the two vertex cones K_{u_1} and K_{u_2}. Note that in the drawing our vectors are supposed to live in $\mathbb{R}^4$.



The two vertex cones.



Elimination of a single λ.

The discussion above and Lemma 13 give us the following theorem that summarizes the algorithm for the elimination of one λ. Note that instead of taking the halfspace H_b, we translate the cone by $-b$ and use the positive halfspace with respect to the λ coordinate denoted by H_λ. The subscript does not mean the offset but denotes the coordinate.

Theorem 2. *Let $a_1, \ldots, a_k \in \mathbb{Q}^n$ be linearly independent vectors, let $A \in \mathbb{Q}^{n\times k}$ denote the matrix with the a_i as columns, let $b \in \mathbb{Q}^n$, and let $I \subset [k]$. Then $\mathcal{C}^I_{\mathbb{R}}(A; b)$ is an inhomogeneous simplicial cone. Assume that the affine hull of $\mathcal{C}^I_{\mathbb{R}}(A; b)$ contains integer points. Let $\lambda \in [n]$ and define $H^{\geq}_\lambda = \{x \in \mathbb{R}^n \mid x_\lambda \geq 0\}$ to be the non-negative half-space with respect to the coordinate λ. Let $\mathcal{I}^+$, $\mathcal{I}^-$, $\mathcal{I}^0$ denote the sets of indices $i \in [k]$ with $a_{i\lambda} > 0$, $a_{i\lambda} < 0$ and $a_{i\lambda} = 0$, respectively. Let P denote the polyhedron $P = H^{\geq}_\lambda \cap (\mathcal{C}^I_{\mathbb{R}}(A; b))$. Under these assumptions, the following hold.*

If $b_\lambda < 0$, then:

1. *The vertices of P are*

$$v_i = b - \frac{b_\lambda}{a_{i\lambda}} a_i \quad \text{for } i \in \mathcal{I}^+,$$

2. *For every $i \in \mathcal{I}^+$, the extreme rays of the tangent cone of P at v_i are generated by the vectors $g_1, \ldots, g_k$ defined by*

$$\begin{aligned} g_i &= a_i \\ g_{i'} &= a_{i'} - a_i \quad \text{for all } i' \in \mathcal{I}^+ \setminus \{i\} \\ g_j &= \frac{1}{a_{j\lambda}} a_j - \frac{1}{a_{i\lambda}} a_i \quad \text{for all } j \in \mathcal{I}^- \\ g_l &= a_l \quad \text{for all } l \in \mathcal{I}^0. \end{aligned}$$

   The vectors g_i are linearly independent whence the tangent cone at v_i is simplicial. Let G_i denote the matrix with these vectors as columns.

3. *The half-open tangent cone of P at v_i is*

$$\mathcal{K}_{v_i} = v_i + \mathcal{C}_{\mathbb{R}}^{I\setminus i}\left(G_i;0\right).$$

In particular, the tangent cone is "open in direction g_s if and only if the cone we started out with is open in direction a_s" for all $s \neq i$. It is closed in direction i, because we intersect with a closed half-space.

4. *For any vertex v_i, let J_i denote the set of indices $s \in [k]$ where $s \in J_i$ if and only if the first non-zero component of g_s is negative (i.e. if g_s is backward). Let G_i' denote the matrix obtained from G_i by multiplying all columns with an index in J_i by -1. (Now all columns of G_i are forward.) Then*

$$\Phi_{(\mathcal{C}_{\mathbb{R}}^{I}(A;b))\cap H_{\lambda}^{\geq}} = \sum_{i\in\mathcal{I}^{+}} (-1)^{|J_i|}\Phi_{\mathcal{C}_{\mathbb{R}}()^{I\setminus i\,\Delta\, J_i}(G_i')+v_i}$$

where Δ denotes the symmetric difference. The same identity also holds on the level of rational functions.

5. *Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection that forgets the λ-th coordinate. If* $\operatorname{aff}(C) \cap L = 0$ *for some linear subspace L that contains* $\ker(\pi)$*, then*

$$\Omega_{\lambda}\Phi_{\mathcal{C}_{\mathbb{R}}^{I}(A;b)} = \sum_{i\in\mathcal{I}^{+}} (-1)^{|J_i|}\Phi_{\mathcal{C}_{\mathbb{R}}^{I\setminus i\,\Delta\, J_i}\left(\hat{G}_i;v_i\right)}.$$

where $\hat{G}_i$ denotes the matrix obtained from G_i' by deleting the λth row. Moreover, all the cones $\hat{C}_i = \mathcal{C}_{\mathbb{R}}^{I\setminus i\,\Delta\, J_i}\left(\hat{G}_i;v_i\right)$ have the property that $\operatorname{aff}(\hat{C}_i) \cap \pi(L) = 0$.

□

We note that for the case when $b_\lambda > 0$, the theorem is completely analogous and the only difference is that the apex of the cone is a vertex of the polyhedron. Each other vertex is visible from the apex, thus an extra ray is added in all vertex cones (from each vertex to the apex of the cone). If $b_\lambda = 0$, then not all vertex cones are simplicial. Thus we employ a deformation argument, presented by Fu Liu in [29] , and reduce the case to the case $b_\lambda < 0$.

The following lemma says that after we project we can repeat the same procedure, thus allowing for recursive elimination of λs in order to solve systems of inequalities.

Lemma 14. *Assume* $\ker(\pi) \subset L$*. If $J \cap L = 0$, then $\pi(J) \cap \pi(L) = 0$.* □

Proof. Let $v \in \pi(J) \cap \pi(l)$. Then there exist $j \in J$ and $l \in L$ such that $v = \pi(j) = \pi(l)$. Because π is linear, we have that $j - l \in \ker(\pi) \subset L$. Therefore $j = j - l + l \in L$ and so $j \in J \cap L$ which means by assumption $j = 0$. But then $v = \pi(j) = 0$. □

4.2 Eliminating multiple λ

Using the algorithm for single λ elimination recursively (due to Lemma 14, we essentially compute triangulations of the dual vertex cones and their generating functions. This is an interesting fact which comes directly from the particular choice of geometry that partition analysis makes. Nevertheless, recursive elimination may not always be the most desirable strategy, as it is well known that the choice of the elimination order may considerably alter the running time in the traditional partition analysis algorithms (intermediate expressions swell).

In order to eliminate multiple λ we need to follow the same procedure, but now the dimension of the λ-cone is more than one lower than the ambient space, since we have more than one λ. This makes the computation of explicit expressions for the vertices and the vertex cones considerably harder. In what follows we give a description of how this could be done algorithmically.

Given $A \in \mathbb{Z}^{m\times d}$ and $b \in \mathbb{Z}^m$ for $m \geq 1$, we define the vectors $v_i = (e_i : A_i) \in \mathbb{R}^{d+m}$ where A_i denotes the i-th column of A and e_i the i-th standard unit vector in $\mathbb{R}^d$. Let V denote the matrix with the v_i as columns. Then V is a vertex description of the λ-cone

$$C = \mathcal{C}_{\mathbb{R}}^{\lambda,\mathbb{Z}}(A) = \mathcal{C}_{\mathbb{R}}(V).$$

The Ω-polyhedron we want to compute is now

$$P = \mathcal{C}_{\mathbb{R}}(V) \cap \left\{ (z,\lambda) \in \mathbb{R}^{d+m} \;\middle|\; \lambda_i \geq b_i \text{ for all } i = 1,\ldots,m \right\}.$$

Note that this definition of P is neither a vertex-description nor a hyperplane-description. We need to compute the vertices and the tangent cones at the vertices of our polyhedron and we will follow a standard method for that [38].
It is straightforward to give a hyperplane-description of C. Note that C is a simplicial d-dimensional cone. Its facets are spanned by all combinations of $d-1$ generators of the cone.

Lemma 15.

$$\operatorname{span}(C) = \left\{ \begin{pmatrix} x \\ \ell \end{pmatrix} \in \mathbb{R}^{d+m} \;\middle|\; \begin{pmatrix} A & -I \end{pmatrix} \begin{pmatrix} x \\ \ell \end{pmatrix} = 0 \right\}.$$

□

Proof. There exist α_i such that $x = \sum_i \alpha_i v_i$ if and only if there exist α_i such that $x_i = \alpha_i$ for $i = 1,\ldots,d$ and $\ell_j = \sum_i \alpha_i \lambda_{j,i}$ for $j = 1,\ldots,m$. Such α_i exist if and only if $\ell_j = \sum_{i=1}^d x_i \lambda_{j,i}$ for all $j = 1,\ldots,m$. □

Given the linear span of C, we can write down a hyperplane description of P.

Lemma 16.

$$P = \left\{ \begin{pmatrix} x \\ \ell \end{pmatrix} \in \mathbb{R}^{d+m} \;\middle|\; \begin{pmatrix} x \\ \ell \end{pmatrix} \geq \begin{pmatrix} 0 \\ b \end{pmatrix} \textit{ and } \begin{pmatrix} A & -I \end{pmatrix} \begin{pmatrix} x \\ \ell \end{pmatrix} = 0 \right\}.$$

□

Proof. From the proof of the previous lemma, we see that $x = \sum_i \alpha_i v_i$ for $\alpha_i \geq 0$ if and only if $x_i \geq 0$ for $i = 1, \ldots, d$ and

$$\begin{pmatrix} A & -I \end{pmatrix} \begin{pmatrix} x \\ \ell \end{pmatrix} = 0.$$

So

$$C = \left\{ \begin{pmatrix} x \\ \ell \end{pmatrix} \in \mathbb{R}^{d+m} \;\middle|\; x \geq 0 \text{ and } \begin{pmatrix} A & -I \end{pmatrix} \begin{pmatrix} x \\ \ell \end{pmatrix} = 0 \right\}$$

whence the theorem follows from the definition of P. □

Having the H-description of the Ω-polyhedron, what we need to do first is compute its vertices. This is done by considering all the full-rank square subsystems of the system

$$\begin{aligned} \begin{pmatrix} x \\ \ell \end{pmatrix} &= \begin{pmatrix} 0 \\ b \end{pmatrix} \\ \begin{pmatrix} A & -I \end{pmatrix} \begin{pmatrix} x \\ \ell \end{pmatrix} &= 0. \end{aligned}$$

Each system has (at most) one rational solution. If the solution satisfies the full system (the point lies in the polyhedron), then that solution is a vertex of the Ω-polyhedron.

Unfortunately, unlike the single λ case, explicit formulas are no more easy to obtain for the vertex cones. This is due to the fact that there are many combinations of positive/negative entries in the inhomogeneous part. Thus, one can not simply label generators as pointing up or down, since the same generator may be pointing up with respect to one λ coordinate and down with respect to another.

We resort to the use of standard tools from polyhedral geometry.

- Compute the vertex cones, which is possible since we have both an H-description and a V-description of the polyhedron.
- Triangulate if needed. The nice simplicial structure of the vertex cones is no longer guaranteed.
- Use Brion's theorem and Barvinok's algorithm in order to compute the desired rational generating function.

This method is algorithmic, but no closed formulas for the vertices and the vertex cones are provided. Nevertheless, while its performance will be poorer than that of traditional recursive elimination algorithms in general, it is expected to perform better than traditional algorithms on problems that show intermediate expression swell. This latter case is very often met in combinatorial problems, which have a nice [1] generating function due to the structure of the problem, but the intermediate expressions do not see the overall structure. One could say that this is a difference between local or global knowledge about the problem structure.

[1]this usually means that a lot of cancellations occur in the rational function.

4.3 Improvements based on geometry

4.3.1 Lattice Points in Fundamental Parallelepipeds

In this section we develop a closed formula for the generating function of the set of lattice points in the fundamental parallelepiped of a simplicial cone. We will first illustrate the basic idea for finding this lattice point set using an example.

Motivating Example

Let $J \subset \mathbb{R}^n$ be a lattice and L a sublattice with basis $a_1, \ldots, a_k \in J$. Our goal is to come up with a closed formula (or simply a way of enumerating) the J-lattice points in the fundamental parallelepiped generated by the a_i, i.e., we want to come up with a generating function for the set $\Pi(A)$ where A is the $n \times k$-matrix with the vectors a_i as columns.

As an example we will take $J = \mathbb{Z}^2$ and

$$A = \begin{bmatrix} 2 & 6 \\ -2 & 2 \end{bmatrix}.$$

The fundamental parallelepiped $\Pi(A)$ and the lattice points contained therein are illustrated in Figure 4.1.

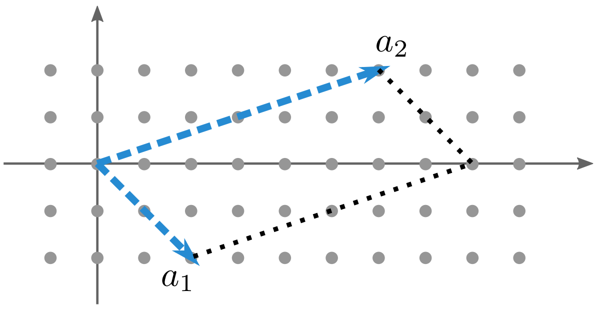


Figure 4.1: The lattice points in the fundamental parallelepiped $\Pi(A)$.

If $\Pi(A)$ were a rectangle, or, more precisely, if A were a diagonal matrix, then $J \cap \Pi(A)$ would be particularly easy to describe. For example if A is an $n \times n$ diagonal matrix with diagonal entries $l_1, \ldots, l_n$, then we have

$$\begin{aligned} \Pi(A) &= [0, l_1) \times \cdots \times [0, l_n) \\ \mathbb{Z}^n \cap \Pi(A) &= \{0, \ldots, l_1 - 1\} \times \cdots \times \{0, \ldots, l_n - 1\} \\ \Phi_{\Pi(A)}(z) &= \frac{1 - z_1^{l_1}}{1 - z_1} \cdot \ldots \cdot \frac{1 - z_n^{l_n}}{1 - z_n}. \end{aligned}$$

How can we make use of this simple observation about rectangles to handle general matrices A such as our example above? One idea is to transform A into a diagonal matrix and control the corresponding changes to $\Pi(A)$. To implement this approach we can use the Smith normal form of a matrix.

Theorem 4.1
(Smith normal form) Let A be a non-zero $n \times k$ integer matrix. Then there exist matrices $U \in \mathbb{Z}^{n\times n}, S \in \mathbb{Z}n \times k, V \in \mathbb{Z}^{k\times k}$ such that $A = USV$, the matrices U and V are invertible over the integers (i.e., they are lattice transformations), and

$$S = \begin{bmatrix} s_1 & 0 & 0 & \cdots & & & 0 \\ 0 & s_2 & 0 & \cdots & & & 0 \\ 0 & 0 & \ddots & & & & \\ \vdots & \vdots & & s_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & 0 & & & & & 0 \end{bmatrix}.$$

where r is the rank of A and $s_i | s_{i+1}$ and $A = USV$. □

In our case this can be interpreted as follows:

- Intuitively, multiplying A from the left with a lattice transform U^{-1} means performing elementary row operations on A, which corresponds to changing the basis of J. This simply applies a change of coordinates to the lattice points in the fundamental parallelepiped.
- Multiplying A from the right with a lattice transform V^{-1} means performing elementary column operation on A, which corresponds to changing the basis of L and thus changing the fundamental parallelepiped. We will return to the question how to relate $J \cap \Pi(A)$ and $J \cap \Pi(AV^{-1})$ in a moment.

The result of applying both of these transformations to A is a diagonal matrix S. So the Smith normal form presents a method of changing bases for both J and L such that the fundamental parallelepiped is a rectangle with respect to the new bases.

Let's see how this plays out in our example, see Figure 4.2. We start out with e_1, e_2 as a basis for $\mathbb{Z}^2$ and a_1, a_2 as our basis for L. Our first step is to change bases on L by replacing e_1 with $e'_1 = e_1 - e_2$ and keeping $e'_2 = e_2$. We then have $a_1 = 2e'_1$. This corresponds to the multiplication

$$U^{-1}A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 0 & 8 \end{bmatrix}.$$

As we can see, the colums of the matrix on the right hand side give the coordinates of a_1, a_2 in terms of e'_1, e'_2.

The next step is to changes bases of L and replace a_2 with $a_2' = a_2 - 3a_1$ and keeping $a_1' = a_1$. This corresponds to the multiplication

$$(U^{-1}A)V^{-1} = \begin{bmatrix} 2 & 6 \\ 0 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} = S,$$

which gives the Smith normal form $S = U^{-1}AV^{-1}$ of A. We have now found a base a_1', a_2' of L in which each basis vector is a multiple of the corresponding basis vector e_1', e_2'. Equivalently, the matrix S is in diagonal form, which means that with respect to the basis e_1', e_2', the fundamental parallelepiped $\Pi(a_1', a_2')$ is a rectangle:

$$J \cap \Pi(a_1', a_2') = \left\{ \mu_1 e_1' + \mu_2 e_2' \mid \mu_1 \in [0,2), \mu_2 \in [0,8), \mu_1, \mu_2 \in \mathbb{Z} \right\}.$$

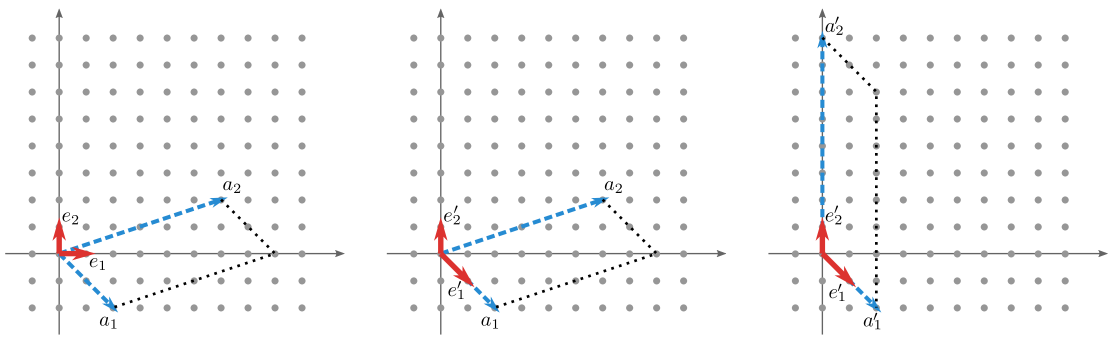


Figure 4.2: By changing bases on J and L we can transform the parallelepiped $\Pi(A)$ into a parallelepiped $\Pi(a_1', a_2')$ whose generators a_1', a_2' are multiples of the e_1', e_2' respectively. The figure on the left show the bases e_1, e_2 of J and a_1, a_2 of L that we start out with. In the center figure, we pass from basis e_1, e_2 to e_1', e_2', thereby lining up e_1' and a_1. In the right-hand figure, we pass from a_1, a_2 to a_1', a_2' thereby lining up e_2' and a_2'. This process corresponds to the computation of the Smith normal form of A.

We have now found a simple description of $J \cap \Pi(a_1', a_2')$. How can we transform this lattice-point set into the set $J \cap \Pi(a_1, a_2)$ that we are interested in? Since both a_1, a_2 and a_1', a_2' are bases of L, the fundamental parallelepipeds contain the same number of lattice points. However there is no linear transformation that gives a bijection between these lattice-point sets.

For any basis A of L, the fundamental parallelepiped $\Pi(A)$ tiles the plane. This means that for any basis A and any point $z \in \mathbb{Z}^2$ there is a unique $y \in L$ such that $z - y \in \Pi(A)$. If we can find a formula for this map $f : z \mapsto z - y$ for any given A, we can use this map to transform $\Pi(a_1', a_2')$ into $\Pi(a_1, a_2)$. Fortunately this is straightforward. We simply find the corrdinates λ of z with respect to the basis A, i.e., we write $z = \lambda_1 a_1 + \lambda_2 a_2$. If we now take fractional parts, we obtain

$$f(z) = \{\lambda_1\}a_1 + \{\lambda_2\}a_2 \in \Pi(a_1, a_2).$$

Note that $y = z - f(z) = \lfloor \lambda_1 \rfloor a_1 + \lfloor \lambda_2 \rfloor a_2 \in L$. Using matrices and the fractional part function, f can be described simply via $f = A \circ \mathrm{frac} \circ A^{-1}$. This function f is illustrated in Figure 4.3.

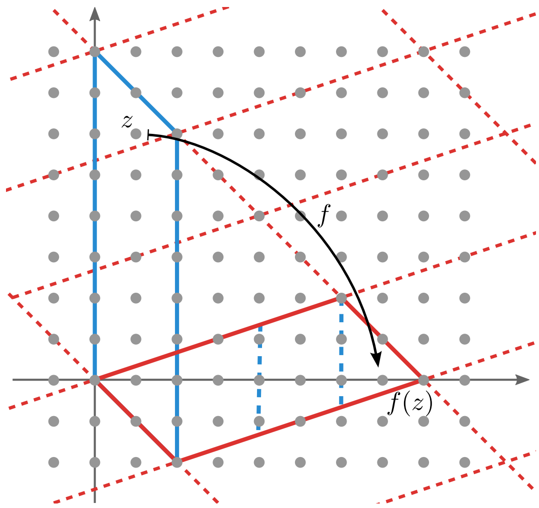


Figure 4.3: The lattice points in the fundamental parallelepiped $\Pi(a_1', a_2')$ can be mapped into $\Pi(A)$ by means of the map $f = A \circ \mathrm{frac} \circ A^{-1}$.

On the whole, this means that we can enumerate the lattice points in the fundamental parallelepiped we are interested in via

$$\begin{aligned} J \cap \Pi(a_1, a_2) &= A \circ \mathrm{frac} \circ A^{-1}(J \cap \Pi(a_1', a_2')) \\ &= \left\{ (A \circ \mathrm{frac} \circ A^{-1} \circ U)\mu \,\middle|\, \mu \in \mathbb{Z}^2 \cap ([0,2) \times [0,8)) \right\} \end{aligned}$$

where

$$A^{-1} \circ U = \begin{bmatrix} \frac{1}{8} & -\frac{3}{8} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{8} \\ 0 & \frac{1}{8} \end{bmatrix}.$$

General Method

We now describe the method illustrated in the above example in full detail.

Let $J \subset \mathbb{R}^n$ be a lattice and L a sublattice. Most of the time, we will have $J = \mathbb{Z}^n$. Let $a_1, \ldots, a_k$ be a basis for L and let A denote the $n \times k$-matrix with the a_i as columns. We are then interested in describing the set of J-lattice points contained in the fundamental parallelepiped $\Pi(a_1, \ldots, a_n)$ of L with respect to the generators $a_1, \ldots, a_n$.

The starting point for this method is the observation that $\Pi(A)$ is easy to describe if the basis of L is a multiple of a basis of J.

Lemma 17. *Let J be a lattice with basis $e_1, \ldots, e_n$ and let L be a sublattice with basis $a_1, \ldots, a_k$ for $k \leq n$. Let $I \subset [k]$. If $a_i = \alpha_i e_i$ for $i = 1, \ldots, k$ and $\alpha_i \in \mathbb{Z}_{\geq 1}$, then the* *set of J-lattice points in the fundamental parallelepiped $\Pi_I(A)$ of L with respect to the basis A is*

$$J \cap \Pi_I(A) = \left\{ \sum_{i=1}^{k} \lambda_i e_i \;\middle|\; \lambda_i \in \mathbb{Z}\ \forall i \in [k], 0 \le \lambda_i < \alpha_i\ \forall i \notin I, 0 < \lambda_i \le \alpha_i\ \forall i \in I \right\}$$

and its generating function has the rational function representation

$$\Phi_{\Pi_I(A)}(z) = \prod_{i \in I} z_i \cdot \prod_{i=1}^{k} \frac{1 - z^{\alpha_i e_i}}{1 - z^{e_i}}.$$

□

Proof. This follows immediately from the definition of $\Pi_I(A)$ and the fact that

$$\prod_{i=1}^{k} \frac{1 - z^{\alpha_i e_i}}{1 - z^{e_i}} = \sum_{x \in S} z^x \text{ for } S = \left\{ \sum_{i=1}^{k} \lambda_i e_i \;\middle|\; \lambda_i \in \{0, \ldots, \alpha_i - 1\}\ \forall i \right\}.$$

□

Changing the basis of L changes the fundamental parallelepiped. However, the number of lattice points in the fundamental parallelepiped remains the same. Moreover there is a natural bijection between the lattice points in the fundamental parallelepipeds. This bijection is essentially given by the "fractional part" or the "division with remainder".

To make this precise in the next lemma, we need some additional notation.

For any real number $x \in \mathbb{R}$ and $e \in \{0, 1\}$ we define $\mathrm{integ}^e(x)$ and $\mathrm{fract}^e(x)$ to be the unique real numbers such that

$$x = \mathrm{integ}^e(x) + \mathrm{fract}^e(x)$$

and

$$\mathrm{integ}^e(x) \in \mathbb{Z}, \quad 0 \le \mathrm{fract}^0(x) < 1, \quad 0 < \mathrm{fract}^1(x) \le 1.$$

Note that $\mathrm{integ}^0 x = \lfloor x \rfloor$ and $\mathrm{fract}^0 x = \{x\}$. For any vector of real numbers $v \in \mathbb{R}^k$ and any set $I \subset [k]$, we also write $\mathrm{integ}^I v$ and $\mathrm{fract}^I v$ to denote the vectors

$$\mathrm{integ}^I(v) = (\mathrm{integ}^{i \in I}(v_i))_{i \in [k]} \text{ and } \mathrm{fract}^I(v) = (\mathrm{fract}^{i \in I}(v_i))_{i \in [k]}$$

where we interpret the exponent $i \in I$ to denote 1 if $i \in I$ and to denote 0 if $i \notin I$.

In analogy to the above notation we define an extension of the mod function. Let $a, b \in \mathbb{R}$. We define $a \mod^0 b$ to be the unique real number $0 \le a \mod^0 b < b$ such that $a = kb + a \mod^0 b$ for some $k \in \mathbb{Z}$. And we define $a \mod^1 b$ to be the unique real number $0 < a \mod^1 b \le b$ such that $a = kb + a \mod^1 b$ for some $k \in \mathbb{Z}$. Again, we will write $\mod^{i \in I}$ to denote $\mod^0$ if $i \notin I$ and $\mod^1$ if $i \in I$, etc. mod without exponent is understood to denote $\mod^0$. Note that $\mathrm{fract}^e(\frac{a}{b}) = \frac{a \mod^e b}{b}$ with this notation.

Lemma 18. *Let J be a lattice in $\mathbb{R}^n$ with basis $e_1, \ldots, e_n$ and let L be a sublattice. Let $a_1, \ldots, a_k$ be a basis of L and let A denote the corresponding $n \times k$-matrix. Let A^{-1} denote any left inverse of A, i.e., let A^{-1} be a matrix with the property $A^{-1}z = \lambda$ for any $z \in \operatorname{span}(L)$ with $z = \sum_{i=1}^{k} \lambda_i a_i$. Let $I \subset [k]$ and let $q \in \mathbb{Q}^n$.*

1. *For all $z \in J \cap (q + \operatorname{span}(L))$ there exists a unique lattice point $y \in \Pi_I(A, q)$ such that $z - y \in L$. Let $z = q + \sum_{i=1}^{k} \lambda_i a_i$ and let $\lambda = (\lambda_i)_{i=1,\ldots,k}$. Then $y = q + A(\operatorname{fract}^I(\lambda))$.*
2. *$J \cap \Pi_I(A, q)$ is empty if and only if $J \cap (q + \operatorname{span}(L))$ is empty.*
3. *Suppose $p \in J \cap (q + \operatorname{span}(L))$. Let $b_1, \ldots, b_k$ be a second basis of L and let B denote the corresponding $n \times k$-matrix and let $I' \subset [k]$. Then*

$$J \cap \Pi_I(A, q) = A(\operatorname{fract}^I(A^{-1}((J \cap \Pi_{I'}(B)) - (q - p)))) + q$$

and the above map induces a bijection between the sets $J \cap \Pi_I(A, q)$ and $J \cap \Pi_{I'}(B)$.

□

Note that the map $A \circ \operatorname{fract}^I \circ A^{-1}$ is not linear, as we have seen in the example.

Proof. 1. Let $z \in J \cap (q + \operatorname{span}(L))$. Then there exist uniquely determined λ_i such that

$$z = q + \sum_{i=1}^{k} \lambda_i a_i = q + \sum_{i=1}^{k} \operatorname{integ}^{i \in I}(\lambda_i) a_i + \sum_{i=1}^{k} \operatorname{fract}^{i \in I}(\lambda_i) a_i.$$

Notice that

$$\sum_{i=1}^{k} \operatorname{integ}^{i \in I}(\lambda_i) a_i \in L \subset J$$

and

$$y = q + \sum_{i=1}^{k} \operatorname{fract}^I(\lambda_i) a_i = q + A(\operatorname{fract}^I(\lambda)) \in \Pi_I(A, q).$$

Moreover, $y \in J$ because $z \in J$. Notice also that y is uniquely determined as changing any coefficient by an integer amount takes us out of $\Pi_I(A, b)$.

2. By the first part of the theorem, we already know that if there exists a $z \in J \cap (q + \operatorname{span}(L))$, then there exists a $y \in J \cap \Pi_I(A, q)$. Conversely, if $y \in J \cap \Pi_I(A, q)$ then $y \in J \cap (q + \operatorname{span}(L))$ because $\Pi_I(A, q) \subset q + \operatorname{span}(L)$.

3. Let T_1 denote the translation $T_1(x) = x - (q - p)$ and let T_2 denote the translation $T_2(x) = x + q$. Using this notation we have to show that the map

$$f = T_2 \circ A \circ \operatorname{fract}^I \circ A^{-1} \circ T_1$$

gives a bijection from $J \cap \Pi_{I'}(B)$ to $J \cap \Pi_I(A, q)$. We will proceed in five steps.

Step 1: The image of $J \cap \operatorname{span}(L)$ under f is contained in $J \cap \Pi_I(A, q)$. Let $z \in f(J \cap (q + \operatorname{span}(L)))$. Then z has the form $z = q + \sum_{i=1}^{k} \operatorname{fract}^I(\lambda_i) a_i$ where the λ_i are such that $\sum_{i=1}^k \lambda_i a_i \in (J \cap \operatorname{span} L) - (q-p)$. Therefore z is contained in $\Pi_I(A,q)$ and it remains to show $z \in J$. By construction, the coefficients λ_i have the property

$$\sum_{i=1}^k \lambda_i a_i = -\sum_{i=1}^k \nu_i a_i + \sum_{i=1}^k \mu_i a_i$$

where $q - p = \sum_{i=1}^k \nu_i a_i$ and $\sum_{i=1}^k \mu_i a_i \in J$. Therefore

$$\begin{aligned} z &= q + \sum_{i=1}^k \operatorname{fract}^I(\lambda_i) a_i \\ &= p + (q-p) + \sum_{i=1}^k \operatorname{fract}^I(\lambda_i) a_i \\ &= p + \sum_{i=1}^k \nu_i a_i + \sum_{i=1}^k \operatorname{fract}^I(-\nu_i + \mu_i) a_i \\ &= p + \sum_{i=1}^k \nu_i a_i + \sum_{i=1}^k (-\nu_i + \mu_i) a_i - \sum_{i=1}^k \operatorname{integ}^I(-\nu_i + \mu_i) a_i \\ &= p + \sum_{i=1}^k \mu_i a_i - \sum_{i=1}^k \operatorname{integ}^I(-\nu_i + \mu_i) a_i. \end{aligned}$$

Now p and $\sum_{i=1}^k \mu_i a_i$ are elements of J by assumption and $\sum_{i=1}^k \operatorname{integ}^I(-\nu_i + \mu_i) a_i$ is an element of J because the coefficients are integer and L is a sublattice of J. Thus $z \in J$.

Step 2: If $u, v \in J \cap \operatorname{span}(L)$ and $f(u) = f(v)$, then $u - v \in L$. Let $u = \sum_{i=1}^k \mu_i a_i$ and $v = \sum_{i=1}^k \nu_i a_i$. Let $q - p = \sum_{i=1}^k \kappa_i a_i$. Then $f(u) = f(v)$ is equivalent to

$$\sum_{i=1}^k \operatorname{fract}^I(\mu_i - \kappa_i) a_i = \sum_{i=1}^k \operatorname{fract}^I(\nu_i - \kappa_i) a_i.$$

Then

$$\begin{aligned} u - v &= \sum_{i=1}^k (\mu_i - \kappa_i) a_i - \sum_{i=1}^k (\nu_i - \kappa_i) a_i \\ &= \sum_{i=1}^k \operatorname{fract}^I(\mu_i - \kappa_i) a_i - \sum_{i=1}^k \operatorname{fract}^I(\nu_i - \kappa_i) a_i + \sum_{i=1}^k \operatorname{integ}^I(\mu_i - \kappa_i) a_i - \sum_{i=1}^k \operatorname{integ}^I(\nu_i - \kappa_i) a_i \\ &= \sum_{i=1}^k \operatorname{integ}^I(\mu_i - \kappa_i) a_i - \sum_{i=1}^k \operatorname{integ}^I(\nu_i - \kappa_i) a_i \end{aligned}$$

which is an integral combination of basis vectors of L and therefore contained in L as claimed.

Step 3: $f|_{J\cap\Pi_{I'}(B)}$ is injective. As B is a basis of L, no two distinct elements of $J\cap\Pi_{I'}(B)$ differ by an element of L. Therefore Step 2 implies that $f|_{J\cap\Pi_{I'}(B)}$ is injective.

Step 4: Both $J\cap\Pi_{I'}(B)$ and $J\cap\Pi_I(A,q)$ contain the same finite number of lattice points. This follows directly from the fact that B and A are bases of the same lattice.

Step 5: $f|_{J\cap\Pi_{I'}(B)}$ is bijective. Combining Steps 3 and 4 yields bijectivity. □

The previous lemma gives us a way to keep track of $\Pi_I(A,q)$ when we change bases of L. Handling changes of bases of J is far easier, as the next lemma tells us.

Lemma 19. *If T is a lattice transformation of the lattice J, then*

$$J\cap\Pi(A;q) = T(J\cap\Pi(T^{-1}A;T^{-1}q)).$$

□

Proof. This follows from $J = TJ = T^{-1}J$ and $A\lambda+q = T(T^{-1}A\lambda+T^{-1}q)$ for all $\lambda\in[0,1)^k$. □

So far, Lemma 17 tells us that lattice-point sets in "rectangular" parallelepipeds are easy to describe and Lemmas 18 and 19 allow us to control $J\cap\Pi_I(A,q)$ when we change bases on L or J, respectively. So what is left to do is to find a way to transform an arbitrary fundamental parallelepiped $\Pi_I(A,q)$ into a "rectangular" parallelepiped using the transformations given in Lemmas 18 and 19. To this end we are going to use the Smith normal form as illustrated with the above example.

Lemma 20. *Let $J=\mathbb{Z}^n$ be a lattice and L a sublattice of J with basis $a_1,\dots,a_k$ and corresponding $n\times k$-matrix A. Let $q\in\mathbb{R}^n$. If $q+\operatorname{span}(L)$ does not contain any integer point, then $\mathbb{Z}^n\cap\Pi_{A,q}$ is empty. Otherwise, let $p\in\mathbb{Z}^n\cap\operatorname{span}(L)$. Then*

$$\mathbb{Z}^n\cap\Pi_I(A,q) = A\circ\operatorname{fract}^I\circ V^{-1}\circ S^{-1}(\mathbb{Z}^k\cap\Pi(S)-U^{-1}(q-p))+q$$

where $A=USV$ is the Smith normal form of A, the numbers $s_1,\dots,s_k$ are the diagonal entries of $n\times k$-matrix S and S^{-1} is the diagonal $k\times n$-matrix with diagonal entries $\frac{1}{s_1},\dots,\frac{1}{s_1}$. □

Note that in the above lemma

$$\mathbb{Z}^n\cap\Pi(S) \;=\; \mathbb{Z}^n\cap[0,s_1)\times\ldots\times[0,s_k)\times\{0\}\times\ldots\times\{0\},$$

is a rectangle of lattice points.

Proof. In order to be able to apply Lemma 18 we need to pick a left inverse A^{-1} of A. To this end we define $A^{-1}:=V^{-1}S^{-1}U^{-1}$. As $S^{-1}S=I$ is the identity, A^{-1} is indeed a left inverse of A since $A^{-1}A=V^{-1}S^{-1}U^{-1}USV=I$. We then have

$$\begin{aligned}
\mathbb{Z}^n\cap\Pi_I(A,q) &= A\circ\operatorname{fract}^I\circ A^{-1}((\mathbb{Z}^n\cap\Pi(AV^{-1}))-(q-p))+q\\
&= A\circ\operatorname{fract}^I\circ A^{-1}\circ U((\mathbb{Z}^n\cap\Pi(U^{-1}AV^{-1}))-U^{-1}(q-p))+q\\
&= A\circ\operatorname{fract}^I\circ A^{-1}\circ U((\mathbb{Z}^n\cap\Pi(S))-U^{-1}(q-p))+q\\
&= A\circ\operatorname{fract}^I\circ V^{-1}\circ S^{-1}((\mathbb{Z}^n\cap\Pi(S))-U^{-1}(q-p))+q
\end{aligned}$$

where the first identity follows from Lemma 18 using the fact that AV^{-1} is a lattice basis for L, the second identity follows from Lemma 19 using the fact that U is a lattice transformation, the third identitiy follows from $A = USV$ and the last identity follows from our choice of A^{-1} and S^{-1}. □

Theorem 4.2
Let $a_1, \ldots, a_k \in \mathbb{Z}^n$ be linearly independent and let $A = (a_{ji})_{j\in[n],i\in[k]}$ be the corresponding matrix with Smith normal form $A = USV$. Let $q \in \mathbb{R}^n$ and let $I \subset [k]$. Let $s_1, \ldots, s_k$ denote the diagonal entries of S and define $t_i = \frac{s_k}{s_i}$. Note that $t_i \in \mathbb{Z}$ as $s_i | s_{i+1}$. Let $V^{-1} = (v_{ji})_{ji}$ where j is the row-index and i the column index. If $p \in \mathbb{Z}^n \cap \operatorname{span}(A)$, then the generating function of the set of lattice points in the fundamental parallelepiped $\Pi_I(A, q)$ is given by

$$\Phi_{\mathcal{C}_{\mathbb{Z}}(A;q)}(x) = \sum_{\lambda_1=0}^{s_1-1} \cdots \sum_{\lambda_k=0}^{s_k-1} x^{\left(\frac{1}{s_k}\sum_{j=1}^{k} a_{lj}\left(\sum_{i=1}^{k}(\lambda_i - w_i)v_{ji}t_i \mod^{i\in I} s_k\right)+q_l\right)_{l=1,\ldots,n}},$$

where $w = U^{-1}(q-p)$. □

Proof. From Lemmas 17 and 20 we immediately get

$$\sum_{\lambda_1=0}^{s_1-1} \cdots \sum_{\lambda_k=0}^{s_k-1} x^{A \operatorname{fract}^I V^{-1}S^{-1}((\lambda_1,\ldots,\lambda_k,0,\ldots,0)^t - w)+q}.$$

We then calculate

$$\begin{aligned} V^{-1}S^{-1}((\lambda_1,\ldots,\lambda_k,0,\ldots,0)^t - w) &= \left(\sum_{i=1}^{k} \frac{\lambda_i - w_i}{s_i} v_{ji}\right)_{j=1,\ldots,k} \\ &= \frac{1}{s_k}\left(\sum_{i=1}^{k} (\lambda_i - w_i)t_i v_{ji}\right)_{j=1,\ldots,k} \end{aligned}$$

where we denote the entries of V^{-1} by v_{ji}. Recall that for $e \in \{0,1\}$ we have $\operatorname{fract}^e(\frac{a}{b}) = \frac{a \mod^e b}{b}$. With this notation we get

$$\operatorname{fract}^I \left(\frac{1}{s_k}\sum_{i=1}^{k}(\lambda_i - w_i)t_i v_{ji}\right)_{j=1,\ldots,k} = \left(\frac{1}{s_k}\sum_{i=1}^{k}(\lambda_i - w_i)t_i v_{ji} \mod^{i\in I} s_k\right)_{j=1,\ldots,k}.$$

Finally, we calculate

$$A\left(\frac{1}{s_k}\sum_{i}(\lambda_i - w_i)v_{ji}t_i \mod^{i\in I} s_k\right)_{j=1,\ldots,k} + q =$$

$$\left(\frac{1}{s_k}\sum_{j=1}^{k} a_{lj}\left(\sum_{i=1}^{k}(\lambda_i - w_i)v_{ji}t_i \mod^{i\in I} s_k\right) + q_l\right)_{l=1,\ldots,n}.$$

□

4.4 Conclusions

One should note that the iterative algorithm is in the intersection of MacMahon's approach and polyhedral geometry. During the execution of the algorithm, geometric objects such as the vertices and the vertex cones of the Ω-polyhedron, as well as a triangulation of the duals of the vertex cones, are computed, without use of any tool from polyhedral geometry. Moreover, by using the intermediate step of computing cones symbolically, some of the cones are dropped without computing their generating functions, i.e., if all generators of a cone are pointing down and the apex lies below the intersecting halfspace, the cone can be safely ignored.

On the other hand, the algorithm for simultaneous multiple λ elimination leans clearly towards polyhedral geometry, using exclusively tools from that area and ignoring the recursive λ elimination, prominent in the traditional partition analysis implementations. In the polyhedral geometry world, already for a decade, there exist tools, like the algorithm of Barvinok-Woods [13] for computing Hadamard products, that can be applied to perform simultaneous λ elimination. Nevertheless, in the partition analysis world these tools are not yet widespread and they are not tested concerning possible advantages, either practical or theoretical.

A practical implementation of the algorithms and further theoretical developments are under preparation and will be presented in [19].

Appendix A

Proof of "Geometry of Omega2"

In what follows we assume $\gcd(\alpha,\beta)=1$ as indicated in Theorem 3.1.

Fundamental Parallelepipeds

Let $\Pi_\zeta = \Pi_A \cap (\mathbb{Z}^2 \times \{\zeta\})$ and $\Xi_\zeta = \Pi_B \cap (\mathbb{Z}^2 \times \{\zeta\})$ for $\zeta \in \mathbb{Z}$. Π_ζ (resp. Ξ_ζ) is the intersection of Π_A (resp. Π_B) with the affine subspace $\{(x,y,\zeta) \in \mathbb{R}^3\}$.

Lemma 21. *There is at most one lattice point in each fundamental parallelepiped of the cones A and B at any given height (x_3-value) and if there is one then its form is determined as follows:*

- $\Pi_\zeta = \{(x_1,x_2,\zeta) \in \mathbb{Z}^3 | x_1 = \frac{\zeta - x_2\beta}{\alpha}, x_2 \in \{0,1,\dots,\alpha-1\}\}$.
- $\Xi_\zeta = \{(x_1,x_2,\zeta) \in \mathbb{Z}^3 | x_2 = \frac{\zeta - x_1\alpha}{\beta}, x_1 \in \{-\beta+1,-\beta+2,\dots,0\}\}$.

Moreover $|\Pi_\zeta| \le 1$ *and* $|\Xi_\zeta| \le 1$. □

Proof. Let $(x_1,x_2,x_3) \in \Pi_A \cap \mathbb{Z}^3$. Then there exist $k,l \in [0,1)$ such that

$$k(-\beta,\alpha,0) + l(1,0,\alpha) = (x_1,x_2,x_3) \in \mathbb{Z}^3.$$

This translates to the system

$$\begin{cases} x_1 = l - k\beta \\ x_2 = k\alpha \\ x_3 = l\alpha \end{cases} \to \begin{cases} x_1 = \frac{\zeta}{\alpha} - k\beta \\ x_2 \in [0,\alpha) \cap \mathbb{Z} \\ l = \frac{x_3}{\alpha} \end{cases} \to \begin{cases} x_1 = \frac{x_3 - x_2\beta}{\alpha} \\ x_2 \in \{0,1,\dots,\alpha-1\} \\ x_3 \in \{0,1,\dots,\alpha-1\} \end{cases}$$

Fix $x_3 = \zeta \in \{0,1,\dots,\alpha-1\}$. Assume $|\Pi_\zeta| > 1$ and let $(x_1',x_2',\zeta),(x_1'',x_2'',\zeta) \in \Pi_\zeta$. Then

$$\begin{cases} \alpha | \zeta - x_2'\beta \\ \alpha | \zeta - x_2''\beta \end{cases} \to \alpha | x_2'\beta - x_2''\beta \to \begin{cases} \alpha | \beta(x_2' - x_2'') \\ \gcd(\alpha,\beta) = 1 \end{cases} \to \alpha | x_2' - x_2''$$

Since $|x_2' - x_2''| < \alpha$ we have a contradiction. Thus $|\Pi_\zeta| \le 1$.

The proof for Ξ_ζ is completely analogous. □

The first summand

Our goal is to show that $\rho_A = \frac{P_{\alpha,\beta}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_1z_3^{\alpha})}$.

Proposition 4.

$$\rho_{\Pi(A)} = P_{\alpha,\beta} = \sum_{i=0}^{\alpha-1} a_i z_3^i$$

for

$$a_i = \begin{cases} z_2^{\frac{i}{\beta}} & \textit{if } \beta | i \textit{ or } i = 0, \\ z_1^{rmd((\alpha^{-1} \bmod \beta)i,\beta)-\beta} z_2^{rmd((\beta^{-1} \bmod \alpha)i,\alpha)} & \textit{otherwise,} \end{cases}$$

□

Proof. Since $\Pi_\zeta = \emptyset$ for $\zeta \geq \alpha$, we need to prove the following three statements

1. $\rho_{\Pi_\zeta} = 1 = a_0$.
2. Let $\zeta \in \{1, 2, \dots, \alpha - 1\}$ such that $\beta | \zeta$. Then $\rho_{\Pi_\zeta} = z_2^{\frac{\lambda}{\beta}} z_3^{\zeta} = a_\zeta z_3^\zeta$.
3. Let $\zeta \in \{1, 2, \dots, \alpha - 1\}$ such that $\beta \nmid \zeta$. Then $\rho_{\Pi_\zeta} = a_\zeta z_3^\zeta$.

Using the equations from the proof of Lemma 21 we have

$$\begin{cases} x_1 = l - k\beta \\ x_2 = k\alpha \\ 0 = l\alpha \end{cases} \xrightarrow{l=0} \begin{cases} k\beta \in \mathbb{Z} \\ k\alpha \in \mathbb{Z} \end{cases} \xrightarrow{\exists n \in \mathbb{N}} \begin{cases} k\beta \in \mathbb{Z} \\ k = \frac{n}{\alpha} \end{cases}$$

$$\rightarrow \begin{cases} \frac{n\beta}{\alpha} \in \mathbb{Z} \\ k = \frac{n}{\alpha} \end{cases} \rightarrow \begin{cases} \alpha | n \text{ or } \alpha | \beta \\ k = \frac{n}{\alpha} \end{cases} \xrightarrow{\gcd(\alpha,\beta)=1} \begin{cases} \alpha | n \\ \frac{n}{\alpha} = k \in [0, 1) \end{cases}$$

Thus $k = 0$, which means $x_1 = x_2 = x_3 = 0$ and $\rho_{\Pi_0} = 1$.

By the definition of a_0 we have that $a_0 = 1$.

■ of statement 1.

From Lemma 21 we know that $|\Pi_\zeta| \leq 1$ and if equality holds the lattice point is of the form $(\frac{\zeta - x_2\beta}{\alpha}, x_2, \zeta)$ for some $x_2 \in \{0, 1, \dots, \alpha - 1\}$. We will check if the desired exponent is actually a lattice point in Π_ζ. Let $x_2 = \frac{\zeta}{\beta}$. Since $\zeta < \alpha$ we have that $x_2 \in \{0, 1, \dots, \alpha - 1\}$. Moreover

$$x_1 = \frac{\zeta - x_2\beta}{\alpha} = \frac{\zeta - \frac{\zeta}{\beta}\beta}{\alpha} = 0 \in \mathbb{Z}$$

which means that $(0, \frac{\zeta}{\beta}, \zeta) \in \Pi_\zeta$. Thus $\rho_{\Pi_\zeta} = z_2^{\frac{\zeta}{\beta}} z_3^\zeta$, which by definition is $a_\zeta z_3^\zeta$.

■ of statement 2.

We will proceed in two steps. First show that $\text{rmd}((\beta^{-1} \bmod \alpha)\zeta, \alpha)$ is the x_2-coordinate of a lattice point in Π_ζ and then that $\text{rmd}((\alpha^{-1} \bmod \beta)\zeta, \beta) - \beta$ is the x_1-coordinate of a lattice point in Π_ζ. Since $|\Pi_\zeta| \leq 1$, we have that $\rho_{\Pi_\zeta} = a_\zeta z_3^\zeta$.

- In order to prove that $\text{rmd}((\beta^{-1} \bmod \alpha)\zeta, \alpha)$ is the x_2-coordinate of a lattice point in Π_ζ we have to show that

  1. $\text{rmd}((\beta^{-1} \bmod \alpha)\zeta, \alpha) \in \{0, 1, \ldots, \alpha - 1\}$;
  2. $\frac{\zeta - \left(\text{rmd}((\beta^{-1} \bmod \alpha)\zeta, \alpha)\right)\beta}{\alpha} \in \mathbb{Z}$.

  Let $x_2 = \text{rmd}((\beta^{-1} \bmod \alpha)\zeta, \alpha)$. By definition the remainder of division by α is in $\{0, 1, \ldots, \alpha - 1\}$.

  By the definitions of remainder and modular inverse we have

  $$x_2 = r\zeta - \alpha m$$

  for some $m \in \mathbb{Z}$ and r such that $r\beta = \alpha n + 1$ for some $n \in \mathbb{Z}$.

  Then

  $$\begin{aligned} x_2 = r\zeta - \alpha m &\Leftrightarrow \beta x_2 = \beta r\zeta - \alpha\beta m \\ &\Leftrightarrow \beta x_2 = (\alpha n + 1)\zeta - \alpha\beta m \\ &\Leftrightarrow \beta x_2 = \alpha n\zeta + \zeta - \alpha\beta m \\ &\Leftrightarrow \zeta - \beta x_2 = \alpha(\beta m - n\zeta) \\ &\Leftrightarrow \frac{\zeta - \beta x_2}{\alpha} = \beta m - n\zeta \in \mathbb{Z}. \end{aligned}$$

- In order to prove that $\text{rmd}((\alpha^{-1} \bmod \beta)\zeta, \beta) - \beta$ is the x_1-coordinate of a lattice point in Π_ζ we have to show that there exist $x_2 \in \{0, 1, \ldots, \alpha - 1\}$ such that $\text{rmd}((\alpha^{-1} \bmod \beta)\zeta, \beta) - \beta = \frac{\zeta - x_2\beta}{\alpha}$.

  Let $b = \text{rmd}((\alpha^{-1} \bmod \beta)\zeta, \beta)$. By definition

  $$b = r\zeta - \beta m$$

  for some $m \in \mathbb{Z}$ and r such that $\alpha r = \beta n + 1$ for some $n \in \mathbb{Z}$. Then we want to prove that

  $$b - \beta = \frac{\zeta - x_2\beta}{\alpha}.$$

  We have

  $$\begin{aligned} b - \beta = \frac{\zeta - x_2\beta}{\alpha} &\Leftrightarrow r\zeta - \beta m - \beta = \frac{\zeta - x_2\beta}{\alpha} \\ &\Leftrightarrow \alpha r\zeta - \alpha\beta m - \alpha\beta = \zeta - x_2\beta \\ &\Leftrightarrow \beta n\zeta + \zeta - \alpha\beta m - \alpha\beta = \zeta - x_2\beta \\ &\Leftrightarrow x_2 = \alpha(m + 1) - n\zeta. \end{aligned}$$

Thus we need to show that $\alpha(m+1) - n\zeta \in \{0, 1, \ldots, \alpha - 1\}$. First observe that $\alpha(m+1) - n\zeta \in \mathbb{Z}$.

If m is the quotient in the division of $r\zeta$ by β we have $m + 1 > \frac{r\zeta}{\beta}$. In order to show that $\alpha(m+1) - n\zeta > 0$ we have

$$\begin{aligned}
m + 1 > \frac{r\zeta}{\beta} &\Rightarrow m > \frac{r\zeta}{\beta} - 1 \\
&\Rightarrow m > \frac{\left(\frac{\beta n+1}{\alpha}\right)\zeta}{\beta} - 1 \\
&\Rightarrow m > \frac{\beta n\zeta + \zeta}{\alpha\beta} - 1 \\
&\Rightarrow m > \frac{\beta n\zeta}{\alpha\beta} - 1 \\
&\Rightarrow m > \frac{n\zeta}{\alpha} - 1 \\
&\Rightarrow \alpha(m+1) > n\zeta \\
&\Rightarrow \alpha(m+1) - n\zeta > 0.
\end{aligned}$$

For the other direction, i.e. $\alpha(m+1) - n\zeta < \alpha$ we observe that since $\beta \nmid \zeta$ we have $0 < b = r\zeta - m\beta$. We know that $\zeta < \alpha$ and

$$\begin{aligned}
\zeta < \alpha b &\Rightarrow \frac{\zeta}{\alpha} < b \\
&\Rightarrow 0 < b - \frac{\zeta}{\alpha} \\
&\Rightarrow m\beta < m\beta + b - \frac{\zeta}{\alpha} \\
&\Rightarrow m\beta < r\zeta - \frac{\zeta}{\alpha} \\
&\Rightarrow m\beta < \frac{(n\beta + 1)\zeta}{\alpha} - \frac{\zeta}{\alpha} \\
&\Rightarrow m\beta < \frac{n\beta\zeta}{\alpha} \\
&\Rightarrow m < \frac{n\zeta}{\alpha} \\
&\Rightarrow \alpha m < n\zeta \\
&\Rightarrow \alpha m + \alpha < n\zeta + \alpha \\
&\Rightarrow \alpha(m+1) - n\zeta < \alpha.
\end{aligned}$$

■ of statement 3.

□

Corollary 1. *From Proposition 4 and Lemma 4 we have that* $\rho_A = \frac{P_{\alpha,\beta}}{(1 - z_1^{-\beta} z_2^{\alpha})(1 - z_1 z_3^{\alpha})}$.

□

The second summand

We want to show that $\rho_B = \frac{Q_{\alpha,\beta}+1-z_1^{-\beta}z_2^{\alpha}}{(1-z_1^{-\beta}z_2^{\alpha})(1-z_1z_3^{\alpha})}$.

Proposition 5.

$$\rho_{\Pi(B)} = Q_{\alpha,\beta} + 1 - z_1^{-\beta}z_2^{\alpha} = \sum_{i=0}^{\beta-1} b_i z_3^i + 1 - z_1^{-\beta}z_2^{\alpha}$$

for

$$b_i = \begin{cases} z_1^{-\beta}z_2^{\alpha} & \text{if } i = 0, \\ z_1^{rmd((\alpha^{-1} \bmod \beta)i,\beta)-\beta} z_2^{rmd((\beta^{-1} \bmod \alpha)i,\alpha)} & \text{otherwise.} \end{cases}$$

□

Proof. Since $\Xi_\zeta = \emptyset$ for $\zeta \geq \beta$, we need to prove the following three statements

1. $\rho_{\Xi_0}(\mathbf{z}) = 1$ and $b_0 = z_1^{-\beta}z_2^{\alpha}$.
2. If $\zeta \in \{1, 2, \dots, \beta-1\}$, then $\rho_{\Xi_\zeta}(\mathbf{z}) = a_\zeta z_3^\zeta$.

The proof is analogous to that of Proposition 4

$$\begin{cases} x_1 = -k\beta \\ x_2 = l + k\alpha \\ 0 = l\beta \end{cases} \xrightarrow{l=0} \begin{cases} k\beta \in \mathbb{Z} \\ k\alpha \in \mathbb{Z} \end{cases} \xrightarrow{\exists n \in \mathbb{N}} \begin{cases} k\beta \in \mathbb{Z} \\ k = \frac{n}{\alpha} \end{cases}$$

$$\to \begin{cases} \frac{n\beta}{\alpha} \in \mathbb{Z} \\ k = \frac{n}{\alpha} \end{cases} \to \begin{cases} \alpha|n \text{ or } \alpha|\beta \\ k = \frac{n}{\alpha} \end{cases} \xrightarrow{\gcd(\alpha,\beta)=1} \begin{cases} \alpha|n \\ \frac{n}{\alpha} = k \in [0,1) \end{cases}$$

Thus $k = 0$, $x_1 = x_2 = x_3 = 0$ and $\rho_{\Xi_0} = 1$.
By the definition of b_0 we have that $b_0 = z_1^{-\beta}z_2^{\alpha}$.

■ of statement 1.

We need to show that $\Pi_\zeta = \Xi_\zeta$ for $\zeta \in \{1, 2, \dots, \beta-1\}$.
From the analysis in the proof of Proposition 4, we know that

$$(\mathrm{rmd}((\alpha^{-1} \bmod \beta)\zeta, \beta) - \beta, \mathrm{rmd}((\beta^{-1} \bmod \alpha)\zeta, \alpha), \zeta) \in \Pi_\zeta$$

for $\zeta \in \{0, 1, \dots, \beta-1\} \subset \{0, 1, \dots, \alpha-1\}$.
The proof follows from the fact that $x_1 = \frac{\zeta - x_2\beta}{\alpha} \Leftrightarrow x_2 = \frac{\zeta - x_1\alpha}{\beta}$.

■ of statement 2.

□

Now the proof of Theorem 3.2 follows from Proposition 4, Proposition 5 and considering the signs in the two summands of (3.10).

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