Nowhere-Zero $\vec{k}$-Flows on Graphs

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Abstract

A flow on an oriented graph Γ is a labeling of edges from a group such that the sum of the values flowing into each node is equal to the sum of values flowing out of each node. When the group is $\mathbb{Z}_k$ or $\mathbb{Z}$ (with the labels bounded by k in absolute value), there is an established theory of counting nowhere-zero flows, i.e., the flows where no edge gets labeled 0. This theory includes results about polynomiality of the flow-counting function and combinatorial reciprocity theorems. We introduce nowhere-zero $\vec{k}$-flows, where each edge has a different range of allowable values, and propose an analogous theory. Our method is to translate $\vec{k}$-flows into parametric polytopes and use Ehrhart theory to obtain piecewise-defined polynomials and reciprocity theorems.

1 Introduction

Consider n different water reservoirs that are connected to each other via a pipe system, where water can flow from one reservoir to another. We will specify constraints such that water flow along a pipe is unidirectional and each reservoir needs to maintain its level of capacity, i.e., the amount of water flowing out of each reservoir is equal to the amount of water flowing into each reservoir. Also, water must flow along each pipe or else unused pipes will rust and contaminate the water supply. We will allow multiple pipes flowing from one reservoir to another, but each pipe must flow into exactly one reservoir. Each pipe has a different capacity for the integral volume of water flowing through it. Consider counting the number of distinct integral volumes of flow though our pipes and reservoirs that meet these constraints. We will translate this to a graphical setting where the reservoirs are vertices and the pipe system gives directed edges between them, and propose the concept of $\vec{k}$-flows.

Let $G = (V, E)$ be a loopless, connected multigraph and let Γ be an oriented version of G. $G(\Gamma)$ denotes the underlying graph of Γ. Specifically, each edge has a direction, meaning that if the end points of the edge e are the vertices v_i and v_j, then e can either be directed $v_i \to v_j$ or $v_j \to v_i$. Given an edge labeling of Γ, the **weight** of a vertex is the difference of the sum of edge labelings pointing into and the sum of edge labelings coming out of a vertex. Here, the sum is computed in an Abelian group; in our case, this will be $\mathbb{Z}$ or $\mathbb{Z}_k$. A vertex is **conservative** if its weight is zero. A **flow** on a graph G is a labeling of the edges with values in an Abelian group, such that each vertex $v \in V$ is conservative. A **nowhere-zero flow** is a flow such that none of the edges are labeled zero.

In this paper, we begin by looking at two types of well-examined nowhere-zero flows, $\mathbb{Z}_k$-flows and k-flows, and then generalize nowhere-zero k-flows to $\vec{k}$-flows, which are of new construct. Tutte [11, 12] first introduced the theory of nowhere-zero $\mathbb{Z}_k$-flows as a generalization of the Four-Color-Problem, and he used the the theory of spanning trees to link the theory of graph colorings to that of electrical networks. He showed that the number of nowhere-zero $\mathbb{Z}_k$-flows is counted with a polynomial function in k. More recently, Kochol [7] discovered that a different polynomial in k counts the number of k-flows.

In 2006 Beck and Zaslavsky [3] examined the problem of nowhere-zero k-flows in a geometrical setting. They linked edge labelings, related to nowhere-zero k-flows, as lattice points "inside" a polytope, but "outside" a hyperplane arrangement, using the term inside-out polytope to describe this object. By manipulating Ehrhart theory to accommodate inside-out polytopes, they interpreted the evaluation of k-flow polynomials at negative integers in terms of the combinatorics of a graph. This method was later extended to nowhere-zero $\mathbb{Z}_k$-flows by Breuer and Sanyal [4], who related this back to Stanley's [9] original combinatorial reciprocity theorem for the chromatic polynomial. By combining the previous work of the aforementioned authors we will demonstrate how these types of combinatorial reciprocity theorems can be generalized from k-flows to $\vec{k}$-flows.

For each edge $e \in E$, $x(e)$ will be used to denote its labeling. Let $[k] = \{1, 2, \ldots, k\}$ and $[\pm k] = \{\pm 1, \pm 2, \ldots, \pm k\}$. A **nowhere-zero $\mathbb{Z}_k$-flow** is a flow such that the edges are labeled with elements in $\mathbb{Z}_k \backslash 0$. A **nowhere-zero k-flow** is an integral nowhere-zero flow such that $x(e) \in [\pm(k-1)]$ for each $e \in E$. Finally, we will define a **nowhere-zero $\vec{k}$-flow**, for $\vec{k} \in \mathbb{Z}^E_{>0}$, as an integral nowhere-zero flow such that each edge value $x(e)$ is in the set $[\pm(k_e - 1)]$, for each $e \in E$; hence we have a different capacity for each edge. For an edge labeling using the set $[k-1]$, $\overline{\varphi}_\Gamma(k)$ is used to denote the total number of nowhere-zero $\mathbb{Z}_k$-flows on Γ. For an edge labeling using the set $[\pm(k-1)]$, $\varphi_\Gamma(k)$ is used to denote the total number of nowhere-zero k-flows on Γ. Similarly, $\varphi_\Gamma(\vec{k})$ is used to denote the total number of nowhere-zero $\vec{k}$-flows on Γ.

To further examine and gain more information about the functions φ_Γ and $\overline{\varphi}_\Gamma$, it is natural to consider the cases where a nowhere-zero flow cannot be found for an orientation Γ of a connected multigraph G. Recall that the sets from which edges are labeled in nowhere-zero $\mathbb{Z}_k$, k, and $\vec{k}$-flows are $[k-1], [\pm(k-1)]$, and $[\pm(k_e - 1)]$ respectively. Hence, each counting function has a zero if $k = 1$ or any entry k_e of our k-vector equals 1, since the set with which the edges are labeled is the empty set. Additionally, we define a type of edge that will be destructive in terms of finding a nonzero nowhere-zero flow counting function for a connected graph G: A **bridge** is an edge of G whose removal from G will increase the number of components. We want to show that the presence of a bridge prevents a nowhere-zero $\vec{k}$-flow by an exercise of [2] and to help prove this we direct the reader to Proposition 7.3.16 in [13], which states that for a flow on G, the net flow out of any set $S \subseteq V(G)$ is zero, to help prove the following exercise of [2].

Proposition 1.1 *If a connected graph G has a bridge, then G does not admit a nowhere-zero flow; consequently $\overline{\varphi}_\Gamma$ and φ_Γ are identically zero.*

Proof. By definition, the removal of a bridge e must partition G into two disjoint subgraphs, G_1 and G_2. In order to count nowhere-zero flows, we must assign some orientation to the graph G. Thus, e is the only directed edge incident to a vertex of G_1 and a vertex of G_2, which implies that there is a net flow out of a set of vertices that is non-zero. Proposition 7.3.16 of [13] implies that there does not exist a nowhere-zero flow on G. ■

Thus far we have assumed we have been given a specific orientation for our graph G. The following lemma will show that the counting functions do not depend on the orientation.

Lemma 1.2 *If $G(\Gamma_1) = G(\Gamma_2)$ then $\varphi_{\Gamma_1} = \varphi_{\Gamma_2}$ and $\overline{\varphi}_{\Gamma_1} = \overline{\varphi}_{\Gamma_2}$.*

Proof. Let Γ_1, Γ_2 be connected and oriented multigraphs, such that $G(\Gamma_1) = G(\Gamma_2)$. Let $S(\Gamma_1, \Gamma_2) \subseteq E(G)$ be the set of edges with differing orientations. Starting with Γ_1, pick an edge $e \in S(\Gamma_1, \Gamma_2)$ and reorient it to obtain Γ_3. Notice $S(\Gamma_3, \Gamma_2) = S(\Gamma_1, \Gamma_2) - \{e\}$. Continue choosing an edge that differs between the two graphs and reorienting it until the edges of Γ_1 are all directed the same as Γ_2. Hence, given a connected and oriented multigraph, Γ, with $G(\Gamma) = G$, its edges can be reoriented in a finite number of steps to obtain any other possible orientation of the edges of G. Let Γ_e and Γ_{e^-} denote two oriented multigraphs sharing the same underlying graph, but differing in orientation by one directed edge e. Thus, it is sufficient to show that Γ_e and Γ_{e^-} have the same nowhere-zero flow counting functions. This will be proven by constructing a bijection between the nowhere-zero flows on Γ_e and Γ_{e^-}.

For each nowhere-zero flow on Γ_e, label each edge of Γ_{e^-} the same except that we label e with $-x(e)$. To check that this construction preserves conservative vertices, take a nowhere-zero flow on Γ_e and reorient edge e to obtain Γ_{e^-} without relabeling the edges. Let e be a directed edge from u to v, then the reorientation does not affect the conservative property of any other vertex. Consider the weight of conservative vertices u and v with edge e removed. The weight of u is $-(-x(e)) = x(e)$. Similarly, the weight of v is $-x(e) = -x(e)$. Thus, the label on the reoriented graph must contribute $-x(e)$ to vertex u and $x(e)$ to vertex v. Hence, labeling the reoriented edge, (directed from v to u) with $-x(e)$ ensures that we still have conservative vertices. Note that this construction is also a bijection. That is, for each flow on Γ_e, when we reorient edge e and label it with $-x(e)$, we have a nowhere-zero flow on Γ_{e^-}, and this flow is unique since additive inverses are unique. ■

As a consequence of the above lemma, we can think about the number of flows on a given connected and oriented multigraph as a property of the underlying graph itself. We will use $\overline{\varphi}_G$ and φ_G for counting the number of nowhere-zero flows on arbitrary orientation of a graph G, to denote that they are in fact a function of the underlying graph itself.

Example 1.3 (K_3, $2K_2$, $3K_2$) *In this example we will give our computations of the three flow polynomials, for small graphs.*

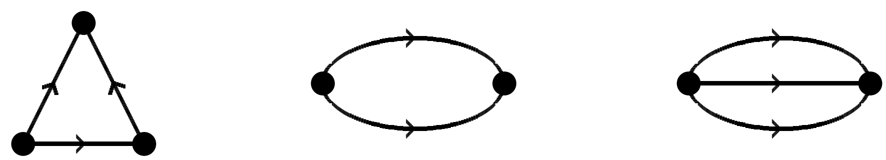

Figure 1.1: small graphs.

For $\Gamma = K_3$:

$\overline{\varphi}_\Gamma(k) =$	$k-1$
$\varphi_\Gamma(k) =$	$2(k-1)$
$\varphi_\Gamma(\vec{k}) =$	$2(k_1-1)$ if $k_1 \leq k_2 \leq k_3$

For $\Gamma = 2K_2$:

$\overline{\varphi}_\Gamma(k) =$	$k-1$
$\varphi_\Gamma(k) =$	$2(k-1)$
$\varphi_\Gamma(\vec{k}) =$	$2(k_1-1)$ if $k_1 \leq k_2$

For $\Gamma = 3K_2$:

$\overline{\varphi}_\Gamma(k) =$	$(k-1)(k-2)$
$\varphi_\Gamma(k) =$	$3(k-1)(k-2)$
$\varphi_\Gamma(\vec{k}) =$	$\begin{cases} (2k_1-2)(2k_2-3) & \text{if } k_2+k_1-1 \leq k_3,\ k_1 \leq k_2 \leq k_3, \\ -k_1^2+2k_1k_2+2k_1k_3-5k_1 \\ \quad -k_2^2+2k_2k_3-3k_2 \\ \quad\quad -k_3^2-k_3+6 & \text{if } k_1 \leq k_2 \leq k_3 \leq k_2+k_1. \end{cases}$

If we look back at each counting function for K_3, we can observe that each counting function has degree 1. Also, notice that $\varphi_\Gamma(k)$ and $\varphi_\Gamma(\vec{k})$ have the same constant term. These same observations can be made for each graph. For $3K_2$, we see our first piecewise-defined polynomial: for $\varphi_\Gamma(\vec{k})$, there will be two cases. The first case is when k_3 has a large enough capacity such that no matter what value is chosen for k_1 and k_2, $x(e_3)$ can be chosen such that the nowhere-zero $\vec{k}$-flow constraint of conservative vertices is preserved. For case 2, k_1 and k_2 were chosen such that $k_3 \leq k_2 + k_1$. Thus, we must have more restriction on what we can chose fro $x(e_3)$ based on our choices for the labels on e_1, e_2.

2 The Flow and Cycle Spaces

Consider an edge e connecting v_1 and v_2. A graph G with e removed is called $G\backslash e$, G with e deleted. Contracting the edge e simply means to delete e and create a new vertex identifying the vertices v_1 and v_2. G/e is the graph G with edge e contracted. For $\overline{\varphi}_\Gamma(k)$, there is a simple contraction-deletion property giving us information about the polynomiality and degree of this counting function.

Theorem 2.1 (Tutte, [11]) *If Γ is a bridgeless connected graph, then $\overline{\varphi}_\Gamma(k)$ is a polynomial of degree $(m - n + 1)$ satisfying $\overline{\varphi}_\Gamma(k) = \overline{\varphi}_{\Gamma/e}(k) - \overline{\varphi}_{\Gamma\backslash e}(k)$.*

It is not possible to use the contraction-deletion method for k- and $\vec{k}$-flows because of our restriction for possible flow values. Upon uncontracting an edge it is possible for the weight in absolute value of v_1 and v_2 to be greater than or equal to k since we are not summing modulo k. Thus, adding an edge e between v_1 and v_2 would require $x(e) \geq k$ which is outside of our range of suitable edge labelings. Hence, a new method is needed to prove the proof the polynomiality of k- and $\vec{k}$-flows.

To gain more information about $\varphi_G(k)$ and $\varphi_G(\vec{k})$, we want to define the terms cycle space and flow space. The **flow space** $Z \subset \mathbb{R}^E$ of a connected, oriented multigraph is described by a system of linear equations that give the space of all possible flows. The **edge space** $\mathcal{E}(G)$ of a graph G is the set of functions $E \to \mathbb{R}$, which can also be written as $\mathbb{R}^E$. A **path** $P = x_0x_1 \ldots x_{k-1}$ is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. If $P = x_0x_1 \ldots x_{k-1}$ is a path and $k \geq 3$, then the graph $C := P \cup x_{k-1}x_0$ is called a **cycle**. The **cycle space** $\mathscr{C} = \mathscr{C}(G)$ is the subspace of $\mathcal{E}(G)$ spanned by all the cycles in G — more precisely, by their edge sets.

The flow space contains all the possible flows on an oriented graph. Hence, it will be the first step in proving the polynomiality and degree of the flow counting function.

A **cut** D is a set of edges whose removal increases the number of components. Note, that if a cut consist of a single edge, it is a bridge. A **spanning tree** is a connected subgraph containing all of the vertices and no cycles. See Figure 2.1 for some examples of spanning trees for the graphs C_4 and K_4.

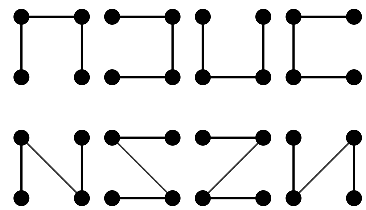

Figure 2.1: Spanning trees

It is well known that a spanning tree T on a graph G with n vertices has exactly $n-1$ edges. Let us consider a cycle formed by adding an edge e, not contained in the spanning tree, between any two vertices. The resulting cycle is unique. We denote this as the **fundamental cycle** C_e formed by e. Similarly, given an edge $e \in T$, where T is the spanning tree, the graph $T - e$ has exactly two components. The set $D_e \subseteq E$ of edges among these two components form the **fundamental cuts** D_e of G with respect to T. Fundamental cuts and edges are functions of a specific spanning tree so they are unique to its spanning tree. A graph Γ is **totally cyclic** if it is directed, connected and if each edge in Γ is contained in a coherently oriented cycle. For these and more graph theoretic definitions see, e.g., [5, 13].

Lemma 2.2 *The flow space equals the cycle space for any graph G.*

Proof. In order to show the cycle space is contained in the flow space, fix a spanning tree T for G. By fixing the T, this fixes the fundamental cycles. Assign an orientation to T such that all the edges are oriented the same, i.e., creating a directed path. These oriented fundamental cycles cover G in a totally cyclic orientation. The direction of T containing e and the direction of the fundamental cycle containing e will be the same. Consider one fundamental cycle, label each edge of the cycle with the same integer. Consider a second fundamental cycle. Once again, label edge of the cycle with the same integer. If there is an edge shared between the first fundamental cycle and the second, sum the integer labels. Label each fundamental cycle in this fashion. This will create a nowhere-zero flow. Thus, $\mathscr{C} \subseteq Z$.

To show the flow space is contained in the cycle space, start with an oriented graph G. Given G, fix a spanning tree T. Reorient T such that it is totally cyclic and the spanning tree is oriented in one direction. Each fundamental cycle contains an edge unique to that cycle. Beginning on the unique edge, subtract the respective edge value from each edge in that fundamental cycle, thus leaving the unique edge a value of zero. Complete this for each fundamental cycle of G. This process provides the cycle space. Thus, $Z \subseteq \mathscr{C}$.

Because $\mathscr{C} \subseteq Z$ and $Z \subseteq \mathscr{C}$, this implies equality. ■

Define the **cyclomatic number** of a graph G as $\xi(G) := |E| - |V| + c$ where c is the number of components of G. For our purposes, we are dealing with connected graphs which have exactly one component. Thus, $\xi(G) = |E| - |V| + 1$.

Proposition 2.3 *dim* $Z =$ *dim* $\mathscr{C} = \xi(G)$.

See [5] for a proof that dim $\mathscr{C} = m - n + 1$.

3 Incidence Matrices and Vector Partition Functions

An incidence matrix shows the relation between edges and vertices in an oriented and connected multigraph. The incidence matrix is an $n \times m$ matrix, where n is the number of vertices and m is the number of edges, of the form

$$\begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \dots & e_m \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \end{array}$$

where each a_{ij} has a value of 0 or ± 1. When the edge e_i is directed into the vertex v_i, then $a_{ij} = 1$, if e_i comes out of v_i, then $a_{ij} = -1$, and if the edge is not incident to a vertex, the entry gets a value of 0.

Each column of the incidence matrix of a graph contains exactly one 1 and one -1, since each column represents an oriented edge, which is incident to exactly two vertices and is directed from one vertex to the other. Hence [8] implies that the incidence matrix is **totally unimodular**, i.e, any square submatrix of A will have a determinant with a value of 0 or ± 1. The incidence matrix can also be used to define the flow space, namely, $Z = \ker(A)$.

Total unimodularity of a matrix is a very strong property that lends well to other theorems about flows. The following well-known Lemma about a property of totally unimodular matrices and the geometry of k-flows gave us motivation to generalize the geometry of $k-$flows and consider the geometry of $\vec{k}$-flows. First, we define a **lattice polytope** to be the convex hull of finitely many integer points in $\mathbb{Z}^d$.

Lemma 3.1 *If a matrix* $A \in \mathbb{Z}^{m \times d}$ *is totally unimodular, then*

$$\{\vec{x} \in \mathbb{R}^d | -1 \leq x_j \leq 1, A\vec{x} = 0\} \tag{3.1}$$

is a lattice polytope.

The above lemma gives us that the flow space intersected with the cube $[-1,1]^E$ is a lattice polytope.

In order to generalize Lemma 3.1 to nowhere-zero $\vec{k}$-flows we consider all of the constraints on nowhere-zero $\vec{k}$-flows. That is, we have that each vertex is conservative, described by the linear system of equations $A\vec{x} = \vec{0}$, where A is the incidence matrix of G, and we have that for each edge e, $-k_e < x_e < 0$ or $0 < x_e < k$. We want to combine the linear system of flow equations with a system of inequalities. To do so, it is necessary to represent the system of inequalities as a system of linear equations.

Define a **slack variable** y_i as the difference $k_i - x_i$ such that $0 < y_i$. Then we can express $0 < x_i < k_i$ as $x_i + y_i = k_i$, where $x_i, y_i > 0$. Let I_m be the $m \times m$ identity matrix, then $\begin{pmatrix} I_m & \vdots & I_m \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \vec{k}$ satisfies the system of inequalities where $0 < x_i < k_i$. With this construction, we can define the matrix

$$M = \begin{pmatrix} A & \vdots & 0 \\ \cdots & & \cdots \\ I_m & \vdots & I_m \end{pmatrix}$$

such that

$$\begin{pmatrix} A & \vdots & 0 \\ \cdots & & \cdots \\ I_m & \vdots & I_m \end{pmatrix} \begin{pmatrix} \vec{x} \\ \cdots \\ \vec{y} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \cdots \\ \vec{k} \end{pmatrix} \tag{3.2}$$

describes the system of equations and inequalities $A\vec{x} = 0$ and $0 < x_i < k_i$ (written as $x_i + y_i = k_i$, with $0 < x_i, y_i$).

Lemma 3.2 *If A is a totally unimodular $n \times m$ matrix, then so is the matrix created by appending all unit vectors as rows and then appending the standard basis vectors e_{n+1}, $e_{n+2}, \ldots, e_{n+m}$ as columns such that we obtain the following matrix:*

$$\begin{pmatrix} A & \vdots & 0 \\ \cdots & & \cdots \\ I_m & \vdots & I_m \end{pmatrix}.$$

Proof. Assume A is totally unimodular. Append the unit vector e_1 of length m where $e_1 = \langle 1, 0, \ldots, 0 \rangle$ to create a matrix A'. Using cofactor expansion along the appended row yields a determinant of ± 1, thus A' is totally unimodular. Append another unit vector, $e_2 = \langle 0, 1, 0, \ldots, 0 \rangle$, to create a matrix A'' and consider cofactor expansion along e_2. Continuing in this fashion, we can ultimately append the identity matrix of dimension $m \times m$ and preserve total unimodularity. Denote this new matrix M.

Now, consider appending the standard basis vector e_{n+1} of length $n + m$ as a column to M. Using cofactor expansion along e_{n+1} for the matrix M yields a totally unimodular matrix M'. This pattern can be continued until we have appended an $m \times n + m$ matrix comprised of the standard basis vectors e_{n+1}, $e_{n+2}, \ldots, e_{n+m}$ and total unimodularity will be preserved. ∎

Definition 3.3 *Let* $M \in \mathbb{Z}^{d\times n}$*, such that* $\ker M \cap \mathbb{R}^n_{\geq 0} = 0$ *The* **vector partition function** *associated to* M *is the function*

$$F_M: \quad \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$$

$$\vec{b} \mapsto \left| \left\{ \vec{x} \in \mathbb{Z}^n_{\geq 0} | M\vec{x} = \vec{b} \right\} \right|$$

The condition that $\ker M \cap \mathbb{R}^n_{\geq 0} = 0$ forces the set $\left\{ x \in \mathbb{Z}^n_{\geq 0} | Mx = b \right\}$ to be compact, in which case it is a polytope P_b, and the vector partition function counts the number of lattice points inside it. Additionally, an $m \times n$ matrix M of rank $M = r$ is **unimodular** if every square submatrix of order r has determinant $0, \pm 1$.

Theorem 3.4 *[10] Let* $F_M(\vec{b}) := \# \left\{ \vec{x} \in \mathbb{Z}^d_{\geq 0} | M\vec{x} = \vec{b} \right\}$*. If* M *is unimodular then* $F_M(\vec{b})$ *is a piecewise-defined polynomial of degree* $d - \operatorname{rank}(A)$*.*

Corollary 3.5 *If* $M = \begin{pmatrix} A & \vdots & 0 \\ \dots & & \dots \\ I_m & \vdots & I_m \end{pmatrix}$*, then the the counting function*

$$F^\circ_M(\vec{k}) := \left| \left\{ \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \mathbb{Z}^{2m}_{>0} | M \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{k} \end{pmatrix} \right\} \right|$$

is a piecewise polynomial in $k_1, \dots, k_m$ *of degree* $\xi(G)$*.*

Proof. By Lemma 3.2, M is a totally unimodular matrix. By definition $F^\circ_M(\vec{k})$ is a vector partition function, hence Theorem 3.4 proves that $F^\circ_M(\vec{k})$ is a piecewise polynomial in $k_1, \dots, k_m$ of degree $d - \operatorname{rank}(A)$. We know that $\operatorname{rank} I_m = m$ and that $\operatorname{rank}(A) = n + 1$. Hence, $\operatorname{rank}(M) = 2m - (m + n + 1) = m - n + 1$. Thus, $F^\circ_M(\vec{k})$ is a piecewise polynomial in $k_1, \dots, k_m$ of degree $\xi(G)$. ■

It follows that $F^\circ_M(\vec{k})$ is the number of integral solutions to (3.2) satisfying $0 < x_i, y_i$ and gives us a subset of the nowhere-zero $\vec{k}$-flows for a given $\vec{k}$. Specifically, it gives us the nowhere-zero $\vec{k}$-flows such that each edge is labeled with a positive value. Notice, this can be done only for a totally cyclic orientation of G. Hence, given a connected multigraph G we can find a (possibly unique) P and corresponding vector partition function for each totally cyclic orientation of G.

Let θ be a totally cyclic orientation of a connected multigraph G and Θ be the set of all totally cyclic orientations of G. Then for each $\theta \in \Theta$ define $A(\theta)$ to be the incidence matrix corresponding to the orientation θ of G. Specifically, given an unoriented, connected multigraph and a totally cyclic orientation $\theta \in \Theta$, we can create a connected and oriented multigraph Γ for which an incidence matrix is defined. It is important to notice that for each $\theta \in \Theta$, there is a unique oriented and connected multigraph Γ, formed given the pair G and θ and thus a unique $A(\theta)$–the incidence matrix of Γ. Additionally, let

$$M(\theta) := \begin{pmatrix} A(\theta) & \vdots & 0 \\ \cdots & & \cdots \\ I_m & \vdots & I_m \end{pmatrix}.$$

Given θ we can also define two vector partition functions associated to $M(\theta)$:

$$F^\circ_{M(\theta)}(\vec{k}) := \left\{ \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \mathbb{Z}^{2m}_{>0} | M \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{k} \end{pmatrix} \right\} \tag{3.3}$$

and

$$F_{M(\theta)}(\vec{k}) := \left\{ \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \mathbb{Z}^{2m}_{\geq 0} | M \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{k} \end{pmatrix} \right\}. \tag{3.4}$$

For nowhere-zero $\vec{k}$-flows $F^\circ_M(\vec{k})$ is very useful since it corresponds to counting positive integer solutions to (3.2), which do not include zeros. However, positive integer solutions to (3.2) do not directly correspond to finding the number of nowhere-zero $\vec{k}$-flows. However, in the next sections we will describe how to use the totally cyclic orientations and the corresponding vector partition functions to obtain $\varphi_G(\vec{k})$.

4 Geometry

A **hyperplane arrangement** $\mathscr{H}$ is a finite collection of hyperplanes in $\mathbb{R}^d$. Let $\mathscr{H}_0 := \{x_i = 0 \mid 1 \leq i \leq d\}$, the **$d$-dimensional Boolean hyperplane arrangement**, consisting of the coordinate hyperplanes in $\mathbb{R}^d$. Also, $\mathscr{H}_0^Z$ is the arrangement in the cycle space Z induced by $\mathscr{H}_0$.

Lemma 4.1 (Greene-Zaslavsky [6]) *Given a bridgeless graph G, the regions of $\mathscr{H} = \mathscr{H}_0^Z$ are in one-to-one correspondence with the totally cyclic orientations of G.*

The geometry of $\vec{k}$-flows starts with $Z \cap [-1, 1]^E$, i.e., the flow space confined within the cube. However, we are going to dilate the cube $[-1, 1]^E$ irregularly according to $\vec{k}$. Specifically, we can think of dilating the first dimension by k_1, then dilating the second dimension by k_2, and so on and so forth until dimension m has been dilated by k_m. We denote this irregular dilation by $\vec{k}[-1, 1]^E$. Alternatively we can think of shifting the hyperplanes $x_i = \pm 1$ that define the cube according to $\vec{k}$ to create the bounding hyperplanes given by the equations $x_i = \pm k_i$. Then we need to remove the Boolean hyperplane arrangement consisting of the coordinate hyperplanes to get nowhere-zero flows. Enumerating the lattice points in $\vec{k}[-1, 1]^E \setminus \bigcup \mathscr{H}_0^Z$ gives the counting function for nowhere-zero $\vec{k}$-flows.

In terms of a reasonable way to count the lattice points we can focus in on individuals regions in the geometric picture. Let

$$P^\circ(\vec{k}) := \left\{ \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \mathbb{R}^{d}_{>0} | M \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{k} \end{pmatrix} \right\},$$

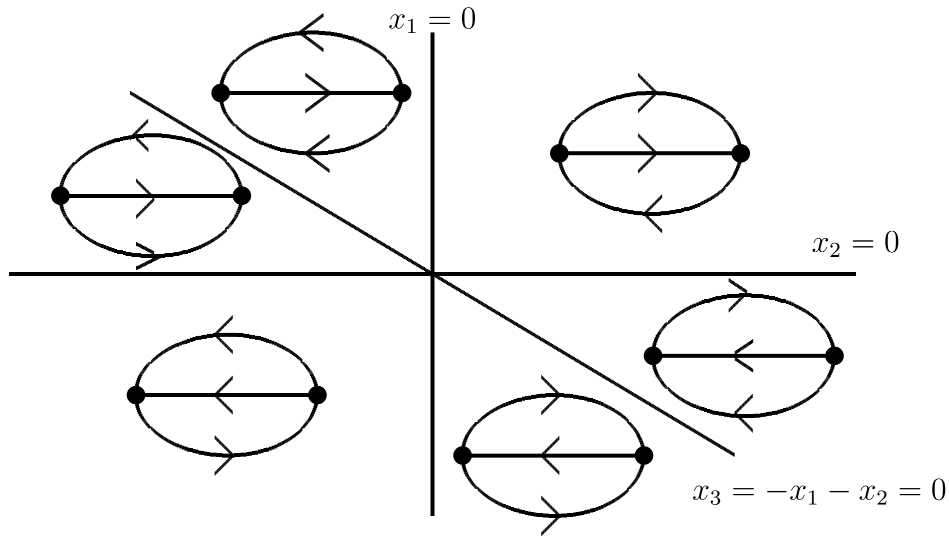

Figure 4.1: [2] The regions of $\mathscr{H}_0^Z$ for $G = 3K_2$ (projected to the x_1, x_2-plane) and their corresponding totally cyclic orientations.

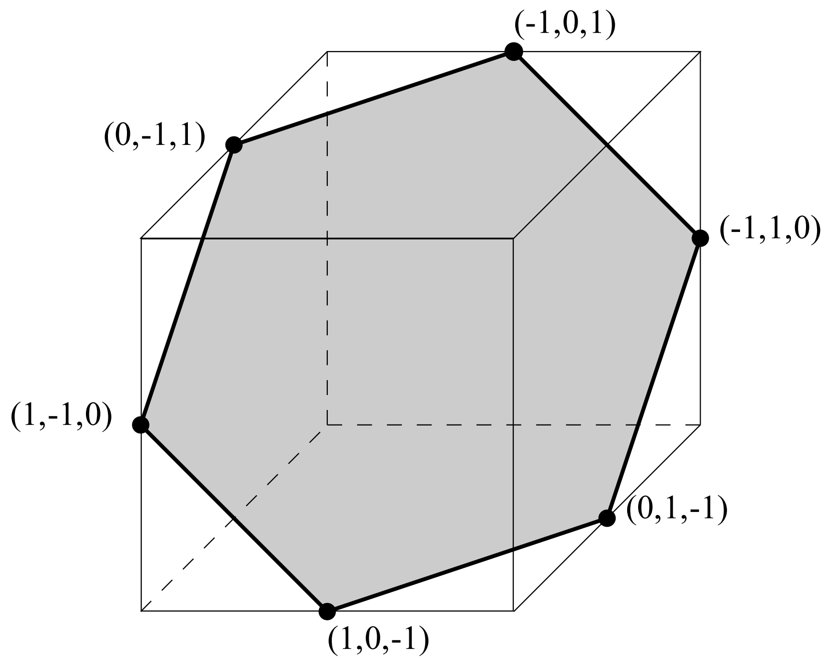

Figure 4.2: $Z \cap [-1, 1]^E$ for $G = 3K_2$

given an initial totally cyclic orientation of G and restricting the labeling of our edges to only positive values. By fixing a totally cyclic orientation θ of G, this determines the flow space Z and also the matrix $M := M(\theta)$, such that $P^\circ(\vec{k})$ corresponds to the bounded open region of $\vec{k}[-1, 1]^E \setminus \bigcup \mathscr{H}_0^Z$ in the all positive orthant of $\mathscr{H}_0$. Hence, the corresponding counting function $F^\circ_M(\vec{k})$ is counting the number of all-positive $\vec{k}$-flows on θ. In order to count the total number of nowhere-zero $\vec{k}$-flows on G, we need to also consider the other regions of $\vec{k}[-1,1]^E \setminus \bigcup \mathscr{H}_0^Z$, that is, we can have $-k_j < x_j < 0$ for some j.

Without the bounding hyperplanes $x_i = \pm k_i$, we can just consider regions of $\mathscr{H}_0^Z$. Each region corresponds to being on one side of each coordinate hyperplane, that is either $x_j < 0$ or $0 < x_j$. Also, by Lemma 4.1 we know that the regions of $\mathscr{H}_0^Z$ are in one-to-one correspondence with the totally cyclic orientations of G. We let the fixed totally cyclic orientation θ of G, correspond to the region of $\mathscr{H}_0^Z$ in the all-positive orthant. Then, lying in a region of $\mathscr{H}_0^Z$ other than the all-positive orthant corresponds to some having a hyperplane description of some $x_j < 0$ and some $x_j > 0$, which we can describe using a **compatible** totally cyclic orientation. To find a compatible totally cyclic orientation to a region, reorient θ according to the following rules:

1. if $x_j > 0$, leave the orientation of e_j as is;
2. if $x_j < 0$, switch the orientation of e_j.

Any orientation of G that can be obtained in this way, by first fixing a totally cyclic orientation θ and then reorienting θ following the above rules, is called compatible with the nowhere-zero $\vec{k}$-flow. Thus, each nowhere-zero $\vec{k}$ flow has exactly one compatible orientation, but a general nowhere-zero $\vec{k}$- flow x gives rise to several orientations that are compatible with x. Notice that following this method for each region we obtain a totally cyclic orientation θ_i specific to a region of $\mathscr{H}_0^Z$. From this we can construct an $M(\theta_i)$ and $P^\circ_{M(\theta_i)}(\vec{k})$ for that region, which corresponds to the all-positive flows on θ_i. Hence, by finding a vector partition function and enumerating the number of lattice points in $P^\circ_{M(\theta_i)}(\vec{k})$, we are actually counting the number of lattice points in the region of $\vec{k}[-1,1]^E \setminus \bigcup \mathscr{H}_0^Z$, whose compatible orientation is θ_i!

In order, to move towards $\varphi(\vec{k})$ we need to sum over all the regions of $\vec{k}[-1,1]^E \setminus \bigcup \mathscr{H}_0^Z$, which by Theorem 4.1 are in one-to-one correspondence with the totally cyclic orientations. Thus

$$\varphi(\vec{k}) = \sum_{\theta \in \Theta} F^\circ_{M(\theta)}(\vec{k}), \tag{4.1}$$

where we are summing over each totally cyclic orientation θ of the set Θ of all totally cyclic orientations.

5 The Counting Function $\varphi(\vec{k})$

Theorem 5.1 *The counting function $\varphi_G(\vec{k})$ for the number of nowhere-zero $\vec{k}$-flows is a piecewise defined polynomial in the components of $\vec{k}$, with degree $\xi(G)$.*

Proof. As described in the above section, the number of lattice points in each region of $\vec{k}[-1,1]^E \setminus \bigcup \mathscr{H}_0^Z$ is equal to the number of lattice points in $P^\circ_{M(\theta_i)}(\vec{k})$, where θ_i

is the compatible totally cyclic orientation of G to that region. As a result of the one-to-one correspondence of the regions of $\mathscr{H}_0^Z$ with the totally cyclic orientations of G, by Lemma 4.1, enumerating the lattice points in each region of $\vec{k}[-1,1]^E \setminus \bigcup \mathscr{H}_0^Z$ is equivalent to $\sum_{\theta\in\Theta} F^\circ_{M(\theta)}(\vec{k})$. And since $\varphi_G(\vec{k}) = \#\left(\mathbb{Z}^m \cap \left(\vec{k}[-1,1]^E \setminus \bigcup \mathscr{H}_0^Z\right)\right)$,

$$\varphi_G(\vec{k}) = \sum_{\theta\in\Theta} F^\circ_{M(\theta)}(\vec{k}). \tag{5.1}$$

Hence, $\varphi_G(\vec{k})$ is the sum of piecewise polynomials in $k_1, \ldots, k_m$ of degree $\xi(G)$. Thus, the sum will also be a piecewise polynomial of degree $\xi(G)$. ■

Since Stanley's first combinatorial reciprocity theorem for the chromatic polynomial [9], it has become very natural to consider the significance of evaluating counting functions for values outside of their given domains, such as negative integers. In fact, combinatorial reciprocity theorems have been discovered for the counting functions of the two other types of nowhere-zero flows mentioned in this paper—nowhere-zero $\mathbb{Z}_k$-flows and nowhere-zero k-flows. Thus, we present the next two theorems as motivation for developing a similar combinatorial reciprocity theorem for the counting function for nowhere-zero $\vec{k}$-flows.

Theorem 5.2 (Beck-Zaslasky [3]) *Let $G = (V,E)$ be a bridgeless graph. Then $(-1)^{\xi(G)}\varphi(-k)$ equals the number of pairs consisting of a $(k+1)$-flow and a compatible totally cyclic orientation of G.*

Theorem 5.3 (Breuer-Sanyal [4]) *Let $G = (V,E)$ be an oriented graph and let k be a positive integer. Then $(-1)^{\xi(G)}\overline{\varphi}_G(-k)$ counts pairs (x,θ), where x is a $\mathbb{Z}_k$-flow and θ is a totally cyclic orientation of G*

Before the next theorem we define $(\vec{k}+\vec{1})$ as $\vec{k}$ with 1 added to each entry, that is

$$(\vec{k}+\vec{1}) = \begin{pmatrix} k_1+1 \\ \vdots \\ k_m+1 \end{pmatrix}.$$

Theorem 5.4 *Let $G = (V,E)$ be a bridgeless graph. Then $(-1)^{\xi(G)}\varphi(-\vec{k})$ equals the number of pairs consisting of a $(\vec{k}+\vec{1})$-flow and a compatible totally cyclic orientation of G.*

Proof. As before let,

$$F_{M(\theta)}(\vec{k}) = \#\left(P_{M(\theta)}(\vec{k}) \cap \mathbb{Z}^n\right)$$

and

$$F^\circ_{M(\theta)}(\vec{k}) = \#\left(P^\circ_{M(\theta)}(\vec{k}) \cap \mathbb{Z}^n\right)$$

Then by [1]

$$F^\circ_{M(\theta)}(-\vec{k}) = (-1)^{\xi(G)} F_{M(\theta)}(\vec{k}).$$

To obtain $\varphi(-\vec{k})$, it is necessary to sum $F^{\circ}_{M(\theta)}(-\vec{k})$ over each totally cyclic orientation of G. That is,

$$\sum_{\theta\in\Theta} F^{\circ}_{M(\theta)}(-\vec{k}) = (-1)^{\xi(G)} \sum_{\theta\in\Theta} F_{M(\theta)}(\vec{k}).$$

However the right-hand side of the above expression $F_{M(\theta)}(\vec{k})$ also counts possibly zero-flows along edges and as a result also counts edges with k units of flow flowing across them. Thus, $F_{M(\theta)}(\vec{k})$, can be thought of as counting $(\vec{k}+\vec{1})$-flows. However, this also creates multiplicities when summing $F_{M(\theta)}(\vec{k})$ over all totally cyclic orientations. Hence, to get the lattice point counting function $\varphi(-\vec{k})$ that sums over all parametric regions created by intersecting the flow space with the boolean lattice, we count each point with multiplicity, which is equal to the number of closed regions the point lies in. The regions are in one-to-one correspondence with the totally cyclic orientations, and so the theorem follows. ■

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