On Weak Chromatic Polynomials of Mixed Graphs

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Abstract

Modeling of metabolic pathways in biology and process management in operating systems are applications of mixed graphs. A *mixed graph* is a graph with directed edges, called arcs, and undirected edges. The *weak (resp. strong) chromatic polynomial* of a mixed graph is a counting function that counts proper k-colorings, that is, assigning colors to vertices such that colors are different on vertices connected by an edge, while colors have to obey the $\leq$ (resp. $<$) relation along an arc. We find a contraction-deletion analogue for mixed graphs in which the weak chromatic polynomial of any mixed graph can be reduced to a linear combination of weak chromatic polynomials of simpler mixed graphs, such as trees. Following closely previous work on reciprocity theorems for other types of chromatic polynomials, we also find a reciprocity theorem for weak chromatic polynomials using partially ordered sets and order polynomials.

1 Introduction

Modeling of metabolic pathways in biology and process management in operating systems are applications of mixed graphs. A *mixed graph* is a graph with both directed and undirected edges. We represent this as $G = (V,E,A)$, where V represents the vertices, E represents the undirected edges, and A represents the directed edges. We will refer to the elements of E as *edges* and the elements of A as *arcs*. Given two vertices a and b, an edge between a and b is denoted by $\bar{ab}$, and an arc from a to b is denoted by $\vec{ab}$. We denote the set of edges of G as E_G and the set of arcs as A_G.

Given a mixed graph G and a set of k colors, $[k] = \{1,2,...,k\}$, we define a k-coloring of G to be a map $x : V \to [k]$ where each vertex of G is assigned a color from $[k]$. A *weak (resp. strong) proper k-coloring* of G is a k-coloring such that $x(v) \neq x(w)$ for all $\bar{vw} \in E$ and $x(v) \leq x(w)$ (resp. $x(v) < x(w)$) for all $\vec{vw} \in A$.

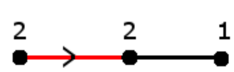


Figure 1: A weak proper 2-coloring for a path with one arc and one edge

The function that counts the number of proper k-colorings is a polynomial of k; this *weak (resp. strong) chromatic polynomial* $\chi_G(k)$ (resp. $\hat{\chi}_G(k)$) counts the number of weak (resp. strong) proper k-colorings of G [1].

An *orientation* of G is obtained by maintaining the direction of arcs and orienting edges. For any edge $\bar{ab}$ there are two possible orientations: $a \to b$ and $b \to a$. A *mixed cycle* is a mixed graph such that there exists an orientation of it that results in a directed cycle. A mixed graph G is *cyclic* if it has a mixed cycle as a subgraph. A mixed graph is *acyclic* if it is not cyclic. Finally, a coloring c and an orientation of G are *compatible* if for every $u \to v$ in the orientation, $c(u) \leq c(v)$.

Stanley proved a reciprocity theorem stating that, for any graph $G = (V, E, \emptyset)$ and positive integer k, $(-1)^{|V|}\chi_G(-k)$ enumerates the pairs of k-colorings and compatible acyclic orientations of G [2]. Beck, Bogart, and Pham proved the following analogue of Stanley's reciprocity theorem for mixed graphs that connects the strong chromatic polynomials of mixed graphs to their acyclic orientations: for any $G = (V, E, A)$, $(-1)^{|V|}\hat{\chi}_G(-k)$ equals the number of k-colorings of G, each counted with multiplicity equal to the number of compatible acyclic orientations of G [3].

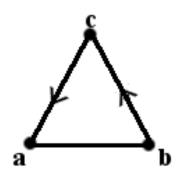


Figure 2: Using the reciprocity for strong chromatic polynomials for the above mixed graph, $\hat{\chi}_G(-2)$ will equal 4. There are only four 3-colorings with one compatible acyclic orientation: colorings (1, 1, 1), (2, 1, 1), (2, 1, 2), and (2, 2, 2) with orientation $b \to a$.

The goal of this paper is to develop an analogous reciprocity theorem for weak chromatic polynomials $\chi_G(k)$ of mixed graphs. We first define a coloring c and an orientation of G to be *intercompatible* if for every $u \to v$ in the orientation, $c(u) \leq c(v)$ when $\bar{uv} \in E_G$, and $c(u) < c(v)$ when $\vec{uv} \in A_G$. Theorem 1.1 is a generalization of Stanley's chromatic reciprocity theorem for weak chromatic polynomials of mixed graphs.

Theorem 1.1. *Let $G = (V, E, A)$ be an acyclic mixed graph and let $\chi_G(k)$ be the weak chromatic polynomial of G. Then for any positive integer k, $(-1)^{|V|}\chi_G(-k)$ is equal to the number of k-colorings of G, each counted with multiplicity equal to the number of intercompatible acyclic orientations of G.*

2 A Poset Approach

Throughout the rest of this paper, a proper k-coloring will refer only to a weak proper k-coloring, and a chromatic polynomial will refer only to a weak chromatic polynomial. For a mixed graph $G = (V, E, A)$, the chromatic polynomial $\chi_G(k)$ can be written as

$$\chi_G(k) = \#\{(x_1, \ldots, x_n) \in [k]^n : x_i \leq x_j \text{ if } \vec{ij} \in A_G, x_i \neq x_j \text{ if } \bar{ij} \in E_G\}.$$

Each proper k-coloring corresponds to an acyclic orientation of G since either $x_i < x_j$ or $x_i > x_j$ for every $\bar{ij} \in E_G$. A cyclic orientation of G cannot admit a proper k-coloring since there would be a situation in which $x_i < x_j < \cdots < x_i$. On the other hand, given an acyclic orientation of G, a proper k-coloring consistent with the acyclic orientation must follow certain inequalities, that is, for all $i \to j$ in the acyclic orientation, $x_i \le x_j$ if $\vec{ij} \in A_G$, and $x_i < x_j$ if $\bar{ij} \in E_G$.

A *poset* (partially ordered set) is a set P with a relation $\preceq$ such that for each $a,b \in P$, $a \preceq a$, $a = b$ if $a \preceq b$ and $b \preceq a$, and $a \preceq c$ if $a \preceq b$ and $b \preceq c$. Each acyclic orientation of G can be translated into a poset by the following construction: for each $i \to j$ in the orientation, $i \prec j$. Such a relationship cannot occur in a cylic oriented graph since the occurence of $i \prec j \prec \cdots \prec i$ is not allowed in a poset. An *edge* of the poset P is an ordered pair (a,b) such that $a,b \in P, a \prec b$, and there is no $c \in P$ such that $a \prec c \prec b$. Denote the set of all edges of P by E_P. We define a *bi-colored labeling* of a poset P as a function $\beta : E_P \to \{U,D\}$. A poset with such a bi-colored labeling is a *bi-colored poset (BP)*. A *BP of a mixed graph G* is any BP formed by taking an orientation of G, such that for all $\vec{ab} \in A_G$, $\beta(a,b) = U$, and for all $\bar{ab} \in E_G$, $\beta(a,b) = D$. Note that this means there can be multiple BPs for a given G; each such BP we call *consistent* with G.

The *order polynomial* for a BP with n elements is defined as

$$\Omega_{BP}(k) = \#\{(x_1,\ldots,x_n) \in [k]^n : x_i \le x_j \text{ if } \beta(i,j) = U, x_i < x_j \text{ if } \beta(i,j) = D\}.$$

Lemma 2.1. *If the mixed graph G is acyclic, then $\chi_G(k) = \sum_i \Omega_{BP_i}(k)$, where we sum over all BPs that are consistent with G.*

Proof. If every orientation of G is acyclic, each orientation of G has a corresponding BP. $\chi_G(k)$ counts the number of all possible proper k-colorings of G, while each $\Omega_{BP}(k)$ counts the number of proper k-colorings for a possible acyclic orientation of G. Thus, since the colorings giving rise to different BPs are mutually exclusive, $\chi_G(k) = \sum_i \Omega_{BP_i}(k)$. □

For a poset P with cardinality n, we call a bijection $\omega : P \to [n]$ an *ω-labeling*, while $\bar{\omega}$ is the *complementary labeling* to ω defined by $\bar{\omega}(v) = n + 1 - \omega(v)$. The *order polynomial* [2] $\Omega_{P,\omega}$ is defined as

$$\Omega_{P,\omega}(k) = \#\{(x_1,x_2,\ldots,x_n) \in [k]^n : x_i \le x_j \text{ if } i \preceq j \text{ and } \omega(i) < \omega(j), x_i < x_j \text{ if } i \preceq j \text{ and } \omega(i) > \omega(j)\}.$$

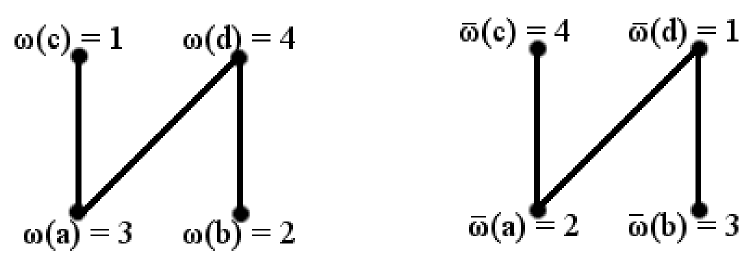


Figure 3: A four element poset with an ω-labeling and $\bar{\omega}$-labeling, respectively.

A theorem by Stanley [4] gives the following reciprocity relation:

Theorem 2.2 (Stanley)**.** *Let the poset P have an ω-labeling. Then* $\Omega_{P,\omega}(-k) = (-1)^{|P|}\Omega_{P,\bar{\omega}}(k)$.

This reciprocity theorem is useful for us because it accomodates the strong and weak inequalities present in $\Omega_{BP}(k)$. However, this reciprocity theorem does not apply to order polynomials of all bi-colored posets. So, if P is the poset that BP came from, we want to know when we can find a labeling ω such that $\Omega_{BP}(k) = \Omega_{P,\omega}(k)$. A bi-colored poset has a *natural ω-labeling* if $\omega(a) < \omega(b)$ when $\beta(a,b) = U$, and $\omega(a) > \omega(b)$ when $\beta(a,b) = D$. Thus, if a BP has a natural ω-labeling, ω, then $\Omega_{BP}(k) = \Omega_{P,\omega}(k)$, and so we wish to know which BPs have a natural ω labeling.

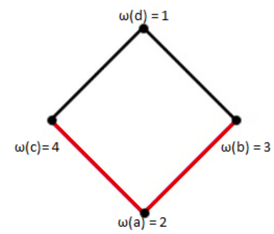


Figure 4: A bi-colored poset with a natural ω-labeling. Edges $\bar{bd}$ and $\bar{cd}$ are labeled D, while edges $\bar{ab}$ and $\bar{ac}$ are labeled U.

3 Proof of Theorem 1.1

The first step towards proving Theorem 1.1 requires knowing the conditions on a BP such that there exists a natural ω-labeling. Define a *bi-colored poset orientation (BPO)* of a BP as a directed graph, $BPO = (V, \emptyset, A)$, such that for every $\bar{ab} \in E_P$, $\vec{ab} \in A_{BPO}$ if $\beta(a,b) = U$ and $\vec{ba} \in A_{BPO}$ if $\beta(a,b) = D$.

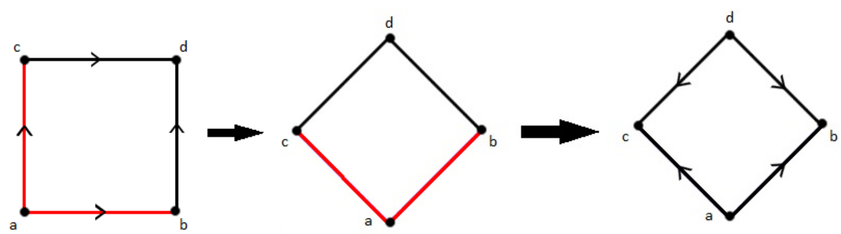


Figure 5: A corresponding acyclicly oriented mixed graph, bi-colored poset, and bi-colored poset orientation.

Lemma 3.1. *A BP has a natural ω-labeling iff its corresponding BPO is acyclic.*

Proof. If a BPO is cyclic, then there is a situation in which a natural ω-labeling must obey $\omega(a) < \omega(b) < \cdots < \omega(n) < \omega(a)$, which cannot happen, so there does not exist a natural ω-labeling.

If a BPO is acyclic, then there exists at least one vertex such that all arcs connected to that vertex are oriented away from it. Denote this vertex v. Assign v the lowest possible unused value (so, in the first iteration, $\omega(v) = 1$). This is consistent with a natural ω-labeling since the value of ω for all other vertices must be strictly greater than $\omega(v)$. Now remove v and its adjacent arcs. Since the BPO is acyclic, then the remaining directed graph must also be acyclic. This process can be repeated for each vertex in the directed graph until all vertices are labeled; thus, a natural ω-labeling exists. $\square$

Next, we deduce the conditions on a mixed graph such that all of its corresponding BPs have a natural ω-labeling.

Proposition 3.2. *If $\vec{ab} \in A_G$, then for any BPO of G, $\vec{ab} \in A_{BPO}$.*

Proof. Let $\vec{ab} \in A_G$. This means for all BPs that can be obtained from G, the relationship between a and b is labeled with a U. So, for a given BP, its corresponding BPO has $\vec{ab} \in A_{BPO}$. $\square$

Given a mixed graph G, *deorienting* an arc $\vec{ab} \in A_G$ means converting $\vec{ab} \in A_G$ into $\bar{ab} \in E_G$.

Corollary 3.3. *Given that a BPO comes from G, G can be obtained by deorienting certain arcs of the BPO.*

Proof. Given a BPO that comes from G, for all $\bar{cd} \in E_G$, deorient arcs $\vec{cd}$ or $\vec{dc} \in A_{BPO}$. From Proposition 3.2, if $\vec{ab} \in A_{BPO}$, then $\vec{ab} \in A_G$. Thus, the resulting graph G' will be a mixed graph such that $\vec{ab} \in A_{G'}$ for all $\vec{ab} \in A_G$ and $\bar{cd} \in E_{G'}$ for all $\bar{cd} \in E_G$. So $G' = G$, which ends the proof. $\square$

Lemma 3.4. *A mixed graph G is acyclic iff every BPO that can be obtained from G is acyclic.*

Proof. Given a cyclic BPO of G, any mixed graph obtained by deorienting arcs of the BPO must be cyclic since there exists a way to orient its edges to obtain a cyclic subgraph. By Corollary 3.3, we know that G can be obtained by deorienting certain arcs of the BPO. Thus, G must be cyclic.

If G is cyclic, G has a mixed cycle. Denote this mixed cycle g. Consider the orientation of G in which for all $\bar{ab} \in E_g$, if orienting $a \to b$ would create a directed cycle, instead orient a and b as $b \to a$. In other words, take g and orient its edges opposite of what would give a directed cycle. Denote this oriented subgraph of g as g'. Since g' is a subgraph of an orientation of G, all that must be shown is that the BPO of g' is cyclic.

Recall that the arcs of g' that correspond to the arcs of G will have $\beta(a,b) = U$ in the BP of g', while the arcs of g' that correspond to the edges of G will have $\beta(a,b) = D$ in the BP of g'. The arcs of g' that correspond to the arcs of G will preserve direction in the BPO of g' by Proposition 3.2. If $\bar{ab} \in E_G$ and $b \to a$ in g' then $a \to b$ in the BPO of g'. By the construction of g', the BPO stemming from g' is cyclic. $\square$

Proof of Theorem 1.1. Let $G = (V,E,A)$ be an acyclic mixed graph. Since G is acyclic, every orientation of G must also be acyclic. By Lemma 2.1, $\chi_G(k) = \sum_i \Omega_{BP_i}(k)$. By Lemma 3.4, every BPO that can be obtained from G must be acyclic. By Lemma 3.1, every BP of G has a natural ω-labeling. Thus, for each BP of G that stemmed from a poset P,

$\Omega_{BP_i}(k) = \Omega_{P_i,\omega}(k)$ for all i. This means that $\chi_G(k) = \sum_i \Omega_{BP_i,\omega}(k) = \sum_i \Omega_{P_i,\omega}(k)$. Since $\chi_G(k)$ and $\Omega_{P_i,\omega}(k)$ are polynomials in k, the relation between the two must be true for all k, and so we have $\chi_G(-k) = \sum_i \Omega_{P_i,\omega}(-k)$. Applying Stanley's Theorem 2.2 for order polynomials and noting that $|P_i| = |V|$ for all i gives

$$\chi_G(-k) = \textstyle\sum_i \Omega_{P_i,\omega}(-k) = \sum_i (-1)^{|P_i|}\Omega_{P_i,\bar{\omega}}(k) = (-1)^{|V|}\sum_i \Omega_{P_i,\bar{\omega}}(k).$$

Recalling that ω is a natural ω-labeling, for each i,

$$\begin{aligned}\Omega_{P_i,\bar{\omega}} &= \#\{(x_1,x_2,\ldots,x_n) \in [k]^n : x_u < x_v \text{ if } u \preceq v \text{ and } \omega(u) < \omega(v),\, x_u \leq x_v \text{ if } u \preceq v \text{ and } \omega(u) > \omega(v)\}\\ &= \#\{(x_1,x_2,\ldots,x_n) \in [k]^n : x_u < x_v \text{ if } \beta(u,v) = U,\, x_u \leq x_v \text{ if } \beta(u,v) = D\}\\ &= \#\{(x_1,x_2,\ldots,x_n) \in [k]^n : x_u < x_v \text{ if } \vec{uv} \in A_G,\, x_u \leq x_v \text{ if } \bar{uv} \in E_G\}.\end{aligned}$$

Thus, $\Omega_{P_i,\bar{\omega}}(k)$ counts the number of k-colorings that are intercompatible with the orientation of G that gives rise to the BP that corresponds to P_i. And, for each possible k-coloring of G, $\sum_i \Omega_{P_i,\bar{\omega}}(k)$ counts the number of acyclic orientations of G that are intercompatible with that k-coloring. So, the multiplicity of each k-coloring in $\sum_i \Omega_{P_i,\bar{\omega}}(k)$ will be the number of intercompatible acyclic orientations of G. □

This proof is reminiscent of the *inside-out polytope* approach that Beck, Bogart, and Pham use to prove a reciprocity theorem for strong chromatic polynomials [3]. Both theorems also result in relating k-colorings of a mixed graph to the acyclic orientations of that mixed graph; although the strong case involves compatible orientations, while the weak case involves intercompatible orientations. The key difference is that the reciprocity theorem for strong chromatic polynomials applies to all mixed graphs, whereas the reciprocity theorem for weak chromatic polynomials has the necessary condition that G be an acyclic mixed graph. This necessary condition for reciprocity shall be demonstrated in the examples in Section 4.2.

4 Computing Weak Chromatic Polynomials

4.1 Deletion-Contraction for Mixed Graphs

The examples in the next section that demonstrate Theorem 1.1 will require a deletion-contraction formula for mixed graphs. Let $G = (V,E,A)$ be a mixed graph and consider an edge $e \in E_G$ (resp. $a \in A_G$). Denote $G - e$ (resp. $G - a$) as the mixed graph G with e (resp. a) deleted. Denote G/e (resp. G/a) as the mixed graph G with e (resp. a) removed and the two endpoints of e (resp. a) are identified.

Proposition 4.1. *Let $\chi_G(k)$ be the chromatic polynomial of G. Then $\chi_{G-e}(k) = \chi_G(k) + \chi_{G/e}(k)$.*

Proof. Proof by counting. Let G be a graph such that e has vertices v and w, and let $c(v)$ be the color of vertex v. Since the relationship between v and w is not specified, $\chi_{G-e}(k)$ on the left-hand side counts the number of proper k-colorings when either $c(v) \neq c(w)$ or $c(v) = c(w)$. On the-right hand side, $\chi_G(k)$ counts the number of proper k-colorings when $c(v) \neq c(w)$, while $\chi_{G/e}(k)$ counts the number of proper k-colorings when $c(v) = c(w)$. Thus, each quantity on both sides count the same number of proper k-colorings. □

Denote G_{aR} as the mixed graph G with arc a directed in the reverse direction, i.e., if $a = \vec{vw} \in A_G$, then $a = \vec{wv} \in A_{G_{aR}}$.

Proposition 4.2. *Let $\chi_G(k)$ be the chromatic polynomial of G. Then $\chi_{G-a}(k) = \chi_G(k) + \chi_{G_{aR}}(k) - \chi_{G/a}(k)$.*

Proof. Proof by counting. Let G be a graph such that a has vertices v and w, and let $c(v)$ be the color of vertex v. Without loss of generality, let $a = \vec{vw}$. Since the relationship between v and w is not specified, $\chi_{G-a}(k)$ on the left-hand side counts the number of proper k-colorings when either $c(v) < c(w), c(v) > c(w)$, or $c(v) = c(w)$. On the right-hand side, $\chi_G(k)$ counts the number of proper k-colorings when either $c(v) < c(w)$ or $c(v) = c(w)$, while $\chi_{G_{aR}}(k)$ counts the number of proper k-colorings when either $c(v) > c(w)$ or $c(v) = c(w)$. Since $\chi_G(k) + \chi_{G_{aR}}(k)$ counts the number of proper k-colorings in which $c(v) = c(w)$ by a multiplicative factor of 2, subtracting $\chi_{G/a}(k)$ — which counts the number of proper k-colorings in which $c(v) = c(w)$ — from the sum will remove any overcounting and result in having the same number of proper k-colorings on both sides of the equation. □

So, for deletion-contraction of an arc, we get

$$\begin{aligned} &\chi_{G-a}(k) = \chi_G(k) + \chi_{G_{aR}}(k) - \chi_{G/a}(k) \\ \Rightarrow\ &\chi_G(k) = \chi_{G-a}(k) + \chi_{G/a}(k) - \chi_{G_{aR}}(k). \end{aligned}$$

The issue with this deletion-contraction formula, however, is that applying it does not result in solving for a chromatic polynomial of a graph less complicated than the original. This is because the right-hand side still has a chromatic polynomial for a graph that is equally complicated, i.e., one that has the same number of edges, arcs, and vertices as the original. To circumvent this issue, we need to introduce the following terms: A directed graph S is *connected* if there exists a path between any two vertices and *strongly connected* if there exists a directed path from any vertex to any other vertex, where a *directed path* is simply a path consisting only of coherently oriented arcs.

Lemma 4.3. *Let S be a strongly connected directed graph, and p the graph consisting of only a single vertex. Then $\chi_S(k) = \chi_p(k) = k$.*

Proof. By definition, there exists a directed path between v and any $w \in S$. So if v has color i, w must have color i (recall that our definition of a proper k-coloring is of the weak case). This is true for all vertices $v, w \in G$. Since there are k available colors to choose from, we have $C_S = C_p = k$. □

Given a subgraph S of G, denote G/S as the mixed graph G with all edges and arcs of S removed and all vertices of S identified.

Theorem 4.4. *Let $G = (V, E, A)$, S be a strongly connected directed subgraph of G. Then $\chi_G(k) = \chi_{G/S}(k)$.*

Proof. Denote the vertex that S contracts to as v. It is sufficient to show that a bijection exists between the set of proper k colorings of G and the set of proper k-colorings of G/S. Take a proper k-coloring that is found in G. Since S is a subgraph of G, by Lemma 4.3, we can map the proper k-coloring of G in which all vertices of S are colored $i \in [k]$ to the proper k-coloring of G/S in which v is also colored i. So, any proper k-coloring for G can be mapped to a proper k-coloring for G/S.

Without loss of generality, take a proper k-coloring that is found in G/S, and v has color i for some $i \in [k]$. By Lemma 4.3, we can map this proper k-coloring of G/S to the proper k-coloring of G in which every vertex of S is also colored i. So, any proper k-coloring for G/S can be mapped to a proper k-coloring for G. This proves the existence of a bijection. □

The above theorem allows the deletion-contraction formulas for mixed graphs to have wider applications, since they can now be used recursively to rewrite chromatic polynomials of mixed graphs as chromatic polynomials of comparatively less complicated mixed graphs. In other words, we can remove edges and flip orientations until we encounter a strongly connected directed subgraph (such as a cycle, for instance), contract that subgraph to a point, and compute chromatic polynomials for the resulting mixed graph instead.

4.2 Examples

We denote the total number of k-colorings of G each counted with multiplicity equal to the number of intercompatible acyclic orientations of G as α_k.

4.2.1 Path with 2 arcs separated by an edge

Let P be a path with 4 vertices a, b, c and d such that $A = \{\vec{ab}, \vec{cd}\}$ and $E = \{\bar{bc}\}$. To find the chromatic polynomial of P, we apply deletion-contraction. So delete $\bar{bc} = e$ to create $G - e$, and contract $\bar{bc}$ to create a new vertex b' for G/e. $G - e$ results in two separate paths with one arc each, i.e., $A = \{\vec{ab}, \vec{cd}\}$ and $E = \emptyset$. G/e results in a path with two arcs, i.e., $A = \{\vec{ab'}, \vec{b'd}\}$ and $E = \emptyset$.

To calculate $\chi_{G/e}$, let R be an arbitary path with n nodes and k colors with two arcs, $\vec{ae}$ and $\vec{ed}$. Let's add two arbitary colors $*_1, *_2$ to our colors such that if we choose the colors $(*_1, y, z)$ (where $x, y, z \in [k]$ and the ordered triple represents the color of each respective vertex), then $a = e$. If we choose the colors $(x, *_2, z)$, then $e = d$, and if we choose the colors $(*_1, *_2, z)$, then $a = e = d$. Therefore, we have $k+2$ coloring choices. This gives us a total of $\binom{k+2}{3}$ coloring choices for R. Therefore,

$$\chi_P(k) = \binom{k+2}{3} = \frac{k(k+1)(k+2)}{6}$$

is the chromatic polynomial for a path with one arc.

Since the deletion of an edge results in two one arc'd paths, we just need to calculate the chromatic polynomial of a path with one arc and then square it. To calculate $\chi_{G-e}(k)$, we use a method similar to the above. Let Q be an arbitrary path with n nodes, k colors, and one arc. Let a and b be arbitary vertices in Q such that $\vec{ab}$. Introduce a new color $*$ such that if we choose the colors $(*, y)$, then $a = b$. This gives us $k+1$ coloring choices. And as with the above reasoning

$$\chi_G(k) = \binom{k+1}{2} = \frac{k(k+1)}{2}.$$

Using the equation for deletion contraction, we get that

$$\chi_G = \left(\frac{k(k+1)}{2}\right)^2 - \frac{k(k+1)(k+2)}{6} = \frac{k(k+1)(k-1)(3k+4)}{12}$$

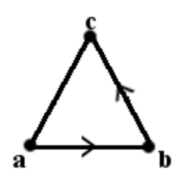
is the chromatic polynomial for P.

Consider $\chi_G(-2) = 1$. The only possible labeling is $(1,2,1,2)$, so $\alpha_2 = \chi_G(-2) = 1$. At $k = 3$, $\chi_G(-3) = 10$. For this case, our possible colorings are:

$$(1,2,1,2),(1,2,1,3),(1,2,2,3),(1,3,1,2),(1,3,1,3),$$
$$(1,3,2,3),(2,3,1,2),(2,3,2,3),(1,3,2,3).$$

The coloring (1,2,2,3) has a multiplicity of 2 as it is found in 2 orientations, so $\alpha_3 = \chi_G(-3) = 10$. Thus, Theorem 1.1 is consistent with this acyclic mixed graph.

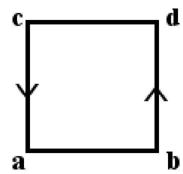
4.2.2 Triangle with 2 arcs in the same direction



Let G be a triangle with vertices a, b, and c such that $A = \{\vec{ab}, \vec{bc}\}$ and $E = \{\bar{ac}\}$. If we give vertex b the color $i \in [k]$, then we have i colors that we can choose for vertex a and $(k+1-i)$ colors that we can choose for vertex c. Since $\bar{ac}$ is an edge, they cannot have the same color. This results in the loss of 1 color. The resulting chromatic polynomial is

$$\chi_G(k) = \sum_{i=1}^{k}[i(k+i-1)-1] = \frac{k(k+1)(k+2)}{6} - k.$$

Consider $\chi_G(-2) = 2$. Should Theorem 1.1 apply to this graph, there should be two pairs of 2-colorings with intercompatible acyclic orientations of this graph. However, an intercompatible coloring x would require that $x(b) > x(a)$ and $x(c) > x(b)$, so $\alpha_2 = 0 \neq 2$. Thus, the reciprocity theorem does not apply to this cyclic mixed graph, which illustrates the necessity of the condition in Theorem 1.1 that G be acyclic.



4.2.3 Square with two non-adjacent arcs

Consider a square G with two arcs and vertices a, b, c, and d such that $A = \{\vec{ca}, \vec{bd}\}$ and $E = \{\bar{ab}, \bar{cd}\}$. We apply deletion-contraction to calculate the chromatic polynomial. Delete and contract $\bar{cd}$. When we contract $\bar{cd}$, we get a triangle wtih two arcs going in the same direction. When we delete an edge, we get a path with two arcs separated by an edge. Both being chromatic polynomials we have already computed, we use the deletion-contraction formula:

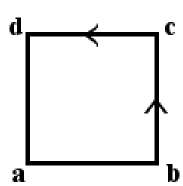
$$\begin{aligned}\chi_G(k) &= \chi_{G-e}(k) - \chi_{G/e}(k) \\ &= \left(\frac{k(k+1)(k-1)(3k+4)}{12}\right) - \left(\frac{k(k+1)(k+2)}{6} - k\right) \\ &= \frac{k(k+1)(k-1)(3k+4)}{12} - \frac{k(k+1)(k+2)}{6} + k.\end{aligned}$$

For $k = -3, \chi_G(-3) = 8$. In this case, our possible colorings with an intercompatible acyclic orientation of the square are

$$(2,1,1,2),(2,1,1,3),(2,2,1,3),(3,1,1,2),(3,1,1,3),$$
$$(3,2,1,3),(3,1,2,2),(3,1,2,3),(3,2,2,3).$$

The colorings $(2,2,1,3)$ and $(3,1,2,2)$ have a multiplicity of 2, since $(2,2,1,3)$ is intercompatible with the orientations that have $A = \{\vec{ca}, \vec{bd}, \vec{cd}, \vec{ab}\}$ or $A = \{\vec{ca}, \vec{bd}, \vec{cd}, \vec{ba}\}$, while $(3,1,2,2)$ is intercompatible with the orientations that have $A = \{\vec{ca}, \vec{bd}, \vec{ba}, \vec{cd}\}$ or $A = \{\vec{ca}, \vec{bd}, \vec{ba}, \vec{dc}\}$. This gives $\alpha_3 = 11 \neq 8$, so Theorem 1.1 does not apply for this cyclic mixed graph, which again illustrates the necessity of the condition that G be acyclic.

4.2.4 Square with two adjacent arcs - same direction



Consider a square G with two arcs and vertices a, b, c, and d such that $A = \{\vec{bc}, \vec{cd}\}$ and $E = \{\bar{ab}, \bar{ad}\}$. As with the above example, we use deletion-contraction to calculate the chromatic polynomial. Delete-contract $\bar{ad}$. When we contract $\bar{ad}$, we get a triangle with two arcs going in the same direction. When we delete $\bar{ad}$, we get a path with two arcs adjacent to each other and an edge. Since the chromatic polynomial for the triangle has already been computed, only the chromatic polynomial of the path need be computed.

For the path, we have an edge $\bar{ab}$ extending off of two connected arcs. For a coloring x, the only rule we have to follow is that $x(b) \neq x(a)$. Since there are k colors for the mixed graph, we have $k-1$ choices for b so that its color differs from a. So we just need to multiply our previous χ_G for the path by $k-1$. Therefore, the chromatic polynomial for the square using deletion and contraction is

$$\begin{aligned}
\chi_G(k) &= \chi_{G-e}(k) - \chi_{G\setminus e}(k) \\
&= \left(\frac{k(k+1)(k-1)^2(3k+4)}{12}\right) - \left(\frac{k(k+1)(k+2)}{6} - k\right) \\
&= \frac{k(k+1)(k-1)(3k+4)}{12} - \frac{k(k+1)(k+2)}{6} + k.
\end{aligned}$$

Consider $\chi_G(-2) = 1$. However, since there are two arcs with the same orientation adjacent to each other, an intercompatible coloring for this square requires at least 3 colors. So $\alpha_2 = 0 \neq 1$, and again Theorem 1.1 does not apply, once more illustrating the necessity of the condition in Theorem 1.1 that G be acyclic.

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