Interval-Vector Polytopes

Jessica De Silva
California State University, Stanislaus

Gabriel Dorfsman-Hopkins
Dartmouth College

Joseph Pruitt
California State University, Long Beach

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Abstract

An interval vector is a $(0,1)$-vector where all the ones appear consecutively. Polytopes whose vertices are among these vectors have some astonishing properties that are highlighted in this paper. We present a number of interval-vector polytopes, including one class whose volumes are the Catalan numbers and another class whose face numbers mirror Pascal's triangle.

1 Introduction

In this paper, we will be analyzing the properties of certain groups of *convex polytopes* which are formed by taking the convex hull of finitely many points in $\mathbb{R}^d$. The *convex hull* of a set $A = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$, denoted $\text{conv}(A)$, is defined as

$$\left\{ \lambda_1 v_1 + \lambda_2 v_2 + \ldots \lambda_n v_n | \lambda_1, \lambda_2, \ldots \lambda_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}. \tag{1}$$

The polytope $\text{conv}(A)$ is contained in the the *affine hull* of A, or $\text{aff}(A)$, whose definition is the same as (1) but without the restriction that $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$. We call a set of points *affinely independent* if each point is not in the affine hull of the rest. The *vertex set* of a polytope is the minimal convexly independent set of points whose convex hull form the polytope. These points are called the *vertices*. The *affine space* of a polytope is the affine hull of its vertices. A polytope is *d-dimensional* if the dimension of the affine hull of its vertices is d. Denote the dimension of the polytope $\mathcal{P}$ as $\dim(\mathcal{P})$. We call a d-dimensional polytope a *d-simplex* if it has $d+1$ vertices.

A *lattice point* is a point with integral coordinates. A *lattice polytope* is a polytope whose vertices are lattice points[1]. A *lattice basis* of a d-dimensional affine space is a set of d integral vectors where, fixing a lattice point on the affine space v, any lattice point inside the affine space can be written as v plus an integral linear combination of these vectors. If we take a *unimodular simplex* formed by the vectors of the lattice basis of the affine hull of $\mathcal{P}$ and assign it volume 1, then the *normalized volume* of a polytope $\mathcal{P}$, denoted $\text{vol}(\mathcal{P})$, is the volume with respect to this simplex. We will refer to the normalized volume of a polytope as its *volume*.

A *t-dilate* of a polytope $\mathcal{P}$ is

$$t\mathcal{P} := \{tv \mid v \in \mathcal{P}\}.$$

The *Ehrhart polynomial* of a lattice polytope $\mathcal{P}$, denoted $L_{\mathcal{P}}(t)$, is the number of lattice points in the t^{th} dilate of the polytope. Here t is considered a positive integer variable. It is known(see, e.g., [1]) that the constant term of any Ehrhart polynomial is 1, and that the degree of this polynomial is the dimension d of $\mathcal{P}$. The leading coefficient of the Ehrhart polynomial is the normalized volume of the polytope times $\frac{1}{d!}$.

A *hyperplane* is of the form

$$H := \{x \in \mathbb{R}^n | a_1x_1 + \cdots + a_nx_n = b\},$$

where not all a_j's are 0. The *half-spaces* defined by this hyperplane are formed by the two weak inequalities corresponding to the equation defining the hyperplane. The *half-space description* of a polytope is the smallest finite set of closed half-spaces whose intersection is the polytope. A *face* of $\mathcal{P}$ is the intersection of a hyperplane and $\mathcal{P}$ such that $\mathcal{P}$ lies completely in one half-space of the hyperplane. This face is a polytope called a *k-face* if its dimension is k. A vertex is a 0-face and an *edge* is a 1-face. Given a d-dimensional polytope $\mathcal{P}$ with f_k k-dimensional faces, the *f-vector* of $\mathcal{P}$ is written as $f(\mathcal{P}) := (f_0, f_1, \ldots, f_{d-1})$[4]. E.g., a triangle $\triangle$ which is 2-dimensional polytope with 3 vertices and 3 edges has f-vector $f(\triangle) = (3, 3)$. As we look at the following polytopes we will see interesting patterns in these properties.

2 Complete Interval-Vector Polytopes

In [2] Dahl introduces a class of polytopes based on interval vectors. An *interval vector* is a $(0,1)$-vector $x \in \mathbb{R}^n$ such that, if $x_i = x_k = 1$ for $i < k$, then $x_j = 1$ for every $i \leq j \leq k$. Let $\alpha_{i,j} := e_i + e_{i+1} + \cdots + e_j$ for $i \leq j$ where e_i is the i^{th} standard unit vector. The *interval length* of α_{ij} is $j - i + 1$. If $\mathcal{I}$ is a set of interval vectors then we define the polytope $P_{\mathcal{I}} := \text{conv}(\mathcal{I})$. We are interested in a number of polytopes that arise when we consider various such sets $\mathcal{I}$. In [2] Dahl provides a method for determining the dimension of these polytopes which we will use throughout this paper.

Denote $\{1, \cdots, n\}$ by $[n]$. Let $\mathcal{I}_n = \{\alpha_{i,j} : i, j \in [n], i \leq j\}$. The *complete interval-vector polytope* is defined as $\mathcal{P}_{\mathcal{I}_n} := \text{conv}(\mathcal{I}_n)$. Computing the Ehrhart polynomials and volumes of small-dimensional polytopes with the aid of a computer, we notice an astounding connection. We computed the volume of the first 9 complete interval vector polytopes, and found that in each case

$$\text{vol}(\mathcal{P}_{\mathcal{I}_n}) = C_n$$

where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number. We will prove that this is the case for any n.

In [5], Postnikov defines the *complete root polytope* $Q_n \subset \mathbb{R}^n$ as the convex hull of 0 and $e_i - e_j$ for all $i < j$. It is shown that the volume of Q_n is C_{n-1}, the same as expected for $\mathcal{P}_{\mathcal{I}_{n-1}}$. In fact, we prove, in a discrete-geometric sense, that the two polytopes are interchangeable, that is, the two polytopes have the same Ehrhart polynomial.

Theorem 1. *$L_{Q_n}(t) = L_{\mathcal{P}_{\mathcal{I}_{n-1}}}(t)$.*

Proof. Each of the vertices of Q_n are vectors with entries that sum to zero, so any linear combination (and specifically any convex combination) of these vertices also has entries who sum to zero:

$$\sum_i x_i = \sum_j y_j = 0 \Longrightarrow \sum_i ax_i + \sum_j by_j = a\sum_i x_i + b\sum_j y_j = 0.$$

Define $B := \{x \in \mathbb{R}^n | \sum_{i=1}^n x_i = 0\}$; then $Q_n \subset B$. B is an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$.

Consider the linear transformation T given by the $n \times n$ lower triangular $(0,1)$-matrix where $t_{ij} = 1$ if $i \geq j$ and $t_{ij} = 0$ otherwise. Then the image

$$T(B) \subset A = \{x \in \mathbb{R}^n | x_n = 0\}.$$

Since T has determinant 1, it is injective when restricting the domain to B. For the same reason, we know that for any $y \in A$, there exists $x \in \mathbb{R}^n$ such that $y = T(x)$. But since $y_n = \sum_{i=1}^n x_i = 0$, then $x \in B$, so that $T|_B : B \to A$ is surjective, and therefore a linear bijection.

Now consider the projection $\Pi : A \to \mathbb{R}^{n-1}$ given by

$$\Pi\left((x_1, \ldots, x_{n-1}, 0)\right) = (x_1, \ldots, x_{n-1}).$$

The transformation is clearly linear, and has the inverse

$$\Pi^{-1}\left((x_1, \ldots, x_{n-1})\right) = (x_1, \ldots, x_{n-1}, 0),$$

so that Π is a bijection.

Now we show that the linear bijection $\Pi \circ T|_B : B \to \mathbb{R}^{n-1}$ is a lattice-preserving map. First we find a lattice basis for B. Consider

$$C = \{e_{i,n} = e_i - e_n | i < n\}.$$

We notice that any integer point of B

$$\left(a_1, \ldots, a_{n-1}, -\sum_{i=1}^{n-1} a_i\right) = \sum_{i=1}^{n-1} a_i e_{i,n}.$$

Any integer point is an integer combination elements of C, so C is a lattice basis.

Note that $\Pi \circ T(e_{i,n}) = e_i + \cdots + e_{n-1} =: u_i$. Therefore

$$\Pi \circ T(C) = \{u_i | i \leq n-1\} =: U.$$

We notice that $e_{n-1} = u_{n-1}$ and $e_i = u_i - u_{i+1}$, so that each of the standard unit vectors e_i of $\mathbb{R}^{n-1}$ is an integral combination of the vectors in U. Since the standard basis is a lattice basis, so is F, thus $\Pi \circ T|_B$ is a lattice-preserving map. Since our bijection is linear and lattice-preserving, all we have left to show is that the vertices of Q_n map to those of $P_{\mathcal{I}_{n-1}}$. By linearity, $\Pi \circ T(0) = 0$, and given any vertex $\alpha_{i,j}$ for $P_{\mathcal{I}_{n-1}}$, we know that $\Pi \circ T(e_{i,j+1}) = \alpha_{i,j}$ where $i < j+1 \leq n$ so that $\Pi \circ T|_B$ is surjective. ☐

The volume of $\mathcal{P}_{\mathcal{I}_n}$ now follows directly from this theorem, since the leading coefficient of the Ehrhart polynomial of $\mathcal{P}_{\mathcal{I}_n}$ is the volume of $\mathcal{P}_{\mathcal{I}_n}$ times $\frac{1}{n!}$.

Corollary 1. $\text{vol}(\mathcal{P}_{\mathcal{I}_n}) = C_n$.

3 Fixed Interval Vector Polytopes

The following construction is due to [2]. Let $e_{i,j} := e_i - e_j$ for $i < j$. We define the set of *elementary vectors* as containing all such $e_{i,j}$, each unit vector e_i, and the zero vector. Let T be the lower triangular $(0,1)$-matrix, as in the proof of Theorem 1. We notice that $T(e_i) = \alpha_{i,n}$ and $T(e_{i,j}) = \alpha_{i,j-1}$. So the image of an elementary vector is an interval vector. Since T is invertible, for any set of interval vectors $\mathcal{I}$, there is a unique set $\mathcal{E}$ of elementary vectors such that $T(\mathcal{E}) = \mathcal{I}$, namely $T^{-1}(\mathcal{I}) = \mathcal{E}$.

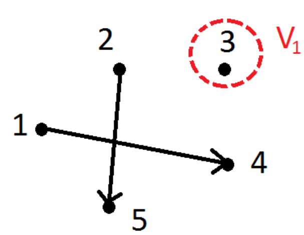
Thus for any interval polytope $\mathcal{P}_{\mathcal{I}} \subset \mathbb{R}^n$, we can construct the corresponding *flow-dimension graph* $G_{\mathcal{I}} = (V, E)$ as follows. Let $\mathcal{E} = T^{-1}(\mathcal{I})$. We let the vertex set $V = [n]$, specify a subset $V_1 = \{j \in V \mid e_j \in \mathcal{E}\}$, and define the edge set $E = \{(i,j) \mid e_{i,j} \in \mathcal{E}\}$. Also we let k_0 denote the number of connected components $\mathcal{C}$ of the graph G (ignoring direction) so that $\mathcal{C} \cap V_1 = \emptyset$.

Given an interval length i and a dimension n we define the fixed interval vector polytope $\mathcal{Q}_{n,i}$ as the convex hull of all vectors in $\mathbb{R}^n$ with interval length i.

Example 3.1. *The fixed interval-vector polytope with $n = 5$, $i = 3$ is*

$$\mathcal{Q}_{5,3} = \text{conv}\left((1,1,1,0,0), (0,1,1,1,0), (0,0,1,1,1)\right).$$

Flow-dimension graph of $\mathcal{Q}_{5,3}$:



Theorem 2 (Dahl, [2]). *If $0 \in \text{aff}(\mathcal{I})$, then the dimension of $P_{\mathcal{I}}$ is $n - k_0$. Else, if $0 \notin \text{aff}(\mathcal{I})$ then the dimension of $P_{\mathcal{I}}$ is $n - k_0 - 1$.*

For $\mathcal{Q}_{n,i}$, we have $\mathcal{I} = \{\alpha_{j,j+i-1} \mid j \leq n-i+1\}$ which translates to the elementary vector set $\mathcal{E} = \{e_{k,k+i} \mid k \leq n-i\} \cup \{e_{n-i+1}\}$. We can define the corresponding flow-dimension graph $G_{\mathcal{Q}_{n,i}} = (V, E)$ where $V = \{1, \ldots, n\}$ and $E = \{(k, k+i) \mid k \in [n-i]\}$ corresponding to each $e_{i,j} \in \mathcal{E}$. Then $V_1 := \{n - i + 1\}$ corresponds to $e_{n-i+1} \in \mathcal{E}$.

Two nodes a, b in a graph $G = (V, E)$ are said to be *connected* if there exists a *path* from a to b, that is there exist $q_0, \ldots, q_s \in V$ such that $(a, q_0), (q_0, q_1), \ldots, (q_s, b) \in E$.

Lemma 1. *Let a, b be nodes in the flow-dimension graph $G_{\mathcal{Q}_{n,i}}(V, E)$. Then a and b are connected iff $a \equiv b \mod i$.*

Proof. Assume without loss of generality $a \leq b$. Suppose a and b are connected by the path $q_0, \ldots, q_s \in V$. Therefore by definition of E, we have

$$\begin{aligned} q_0 &= a + i \\ q_1 &= q_0 + i = a + 2i \\ &\vdots \\ q_s &= q_{s-1} + i = a + (s+1)i \\ b &= q_s + i = a + (s+2)i \end{aligned}$$

Thus $a \equiv b \mod i$ by definition.

Now suppose that $a \equiv b \mod i$ where $a \leq b$, then there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} b &= a + mi \\ &= a + (m-1)i + i. \end{aligned}$$

Since b and $a + (m-1)i$ differ by i, then by definition of E, there is an edge between these nodes. Call this edge $(q_t, b) \in E$. Similarly, we have

$$\begin{aligned} a + (m-1)i &= a + (m-2)i + i & &\Rightarrow (q_t, q_{t-1}) \in E \\ a + (m-2)i &= a + (m-3)i + i & &\Rightarrow (q_{t-1}, q_{t-2}) \in E \\ &\vdots \\ a + 2i &= (a + i) + i & &\Rightarrow (q_1, q_0) \in E \\ a + i &= a + i & &\Rightarrow (q_0, a) \in E. \end{aligned}$$

Hence $q_0, q_1, \ldots, q_t \in V$, define a path from a to b, so a and b are connected. $\square$

Theorem 3. *$\mathcal{Q}_{n,i}$ is an $(n-i)$-dimensional simplex.*

Proof. For a given dimension and interval length, an interval vector is uniquely determined by the location of the first 1, hence we can determine the number of vertices of $\mathcal{Q}_{n,i}$ by counting all possible placements of the first 1 in an interval of i 1's. Since the string must have length i, the number of spaces before the first 1 must not exceed $n-i$ and so there are $n-i+1$ possible locations for the first 1 in the interval to be placed. Thus, $\mathcal{Q}_{n,i}$ has $n-i+1$ vertices.

By Lemma 1 we know there are i connected components in the flow-dimension graph $G_{\mathcal{Q}_{n,i}}$ and since V_1 has only one element, $k_0 = i-1$. Thus by Theorem 2 the dimension of $\mathcal{Q}_{n,i}$ is $n-i$. Therefore $\mathcal{Q}_{n,i}$ is an $(n-i)$-dimensional simplex by definition. □

Theorem 4. *$\mathcal{Q}_{n,i}$ is an $(n-i)$-dimensional unimodular simplex.*

Proof. Consider the affine space where the sum over every i^{th} coordinate is 1,

$$A = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{j \equiv k \bmod i} x_j = 1, \ \forall\, k \in [i] \right\}.$$

Since the vertices of $\mathcal{Q}_{n,i}$ have interval length i, they are in A. Thus $\mathcal{Q}_{n,i} \subset A$.

We want to show that the $w_1, w_2, \ldots, w_{n-i}$ of $\mathcal{Q}_{n,i}$ form a lattice basis for A where

$$\begin{array}{lcl} w_1 & = & \alpha_{1,i} - \alpha_{n-i+1,n} \\ w_2 & = & \alpha_{2,i+1} - \alpha_{n-i+1,n} \\ & \vdots & \\ w_{n-i} & = & \alpha_{n-i,n-1} - \alpha_{n-i+1,n} \end{array} .$$

We will do this by showing that any integer point $p \in A$ can be expressed as a integral linear combination of the proposed lattice basis, that is, there exist integer coefficients $C_1, \ldots, C_{n-i}$ so that $C_1 w_1 + \ldots + C_{n-i} w_{n-i} + \alpha_{n-i+1,n} = p$.

We first notice that p can be expressed as

$$\left(p_1, p_2, \ldots, p_{n-i}, \sum_{\substack{j \le n-i \\ j \equiv t-i+1 \bmod i}} (-p_j) + 1, \sum_{\substack{j \le n-i \\ j \equiv t-i+2 \bmod i}} (-p_j) + 1, \ldots, \sum_{\substack{j \le n-i \\ j \equiv n = \bmod i}} (-p_j) + 1 \right)$$

by solving for the last term in each of the equations defining A. Let

$$C_t = \begin{cases} p_1 & t = 1 \\ p_t - p_{t-1} & 1 < t \le i \\ p_t - C_{t-i} & i < t \le n-i \end{cases}$$

Then each C_t is an integer since it is a sum of integers. We claim that

$$C_1 w_1 + \cdots + C_{n-i} w_{n-i} + \alpha_{n-i+1,n} = p.$$

Clearly the first coordinate is p_1 since w_1 is the only vector with an element in the first coordinate. Next consider the t^{th} coordinate of this linear combination for $1 < t \leq i$, by summing the coefficients of all the vectors who have a 1 in the t^{th} position:

$$C_t + C_{t-1} + C_{t-2} + \cdots + C_1 = p_t - p_{t-1} + p_{t-1} - p_{t-2} + \cdots + p_2 - p_1 + p_1 = p_t$$

We next consider the t^{th} coordinate of the combination for $i < t \leq n - i$ by summing the coefficients of the vectors who have a 1 in the t^{th} position.

$$C_t + C_{t-1} + \cdots + C_{t-i+1} = (p_t - C_{t-1} - \cdots - C_{t-i+1}) + C_{t-1} + \cdots + C_{t-i+1} = p_t$$

Finally, we consider the t^{th} coordinate of the combination for $n - i < t \leq n$, noticing that each coordinate from w_1 to w_t has a -1 in the $(t-i)^{\text{th}}$ position and $\alpha_{n-i+1,n}$ has a 1 in this position. Thus we get:

$$-(C_1 + C_2 + \cdots + C_{t-i}) + 1.$$

Applying the two relations we have defined between coordinates, and calling $\langle t \rangle$ the least residue of $t \bmod i$, we see:

$$\begin{aligned}
-(C_1 + C_2 + \cdots + C_{t-i}) + 1 &= -(C_1 + C_2 + \cdots + C_{t-2i} + p_{t-i}) + 1 \\
&= -(C_1 + C_2 + \cdots + C_{t-3i} + p_{t-2i} + p_{t-i}) + 1 \\
&= -\left(C_1 + C_2 + \cdots + C_{\langle t \rangle} + \sum_{\substack{i < j \leq n-i \\ j \equiv t \bmod i}} p_j \right) + 1 \\
&= -\left(\sum_{\substack{j \leq n-i \\ j \equiv t \bmod i}} p_j \right) + 1.
\end{aligned}$$

Thus $p = C_1 w_1 + C_2 w_2 + \cdots + C_{n-i} w_{n-i} + \alpha_{n-i+1,n}$ and so $w_1, \ldots, w_{n-i}$ form a lattice basis of A.

By the definition of normalized volume, a simplex defined by a lattice basis has volume 1, so $\mathcal{Q}_{n,i}$ has volume 1 and is a unimodular simplex. □

4 Another Interesting Polytope

Given a dimension n and an interval length i define $\mathcal{P}_{n,i}$ to be the convex hull of all vectors in $\mathbb{R}^n$ with interval length 1 or $n - i$.

Example 4.1. *For $n = 4$, $i = 1$,*

$$\mathcal{P}_{4,1} = \operatorname{conv}\big((1,0,0,0)\,,\,(0,1,0,0)\,,\,(0,1,0,0)\,,\,(0,0,1,0)\,,\,(0,0,0,1)\,,\,(1,1,1,0)\,,\,(0,1,1,1)\big).$$

Proposition 1. *The dimension of $\mathcal{P}_{n,1}$ is n.*

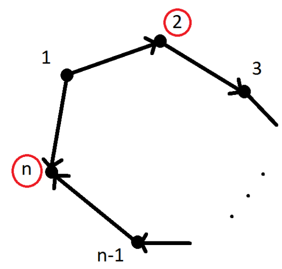


Figure 1: $G_{\mathcal{P}_{n,1}}$

Proof. For $n \geq 3$, the vertices of $\mathcal{P}_{n,1}$ form the set

$$\mathcal{I} = \left\{ \begin{array}{cl} e_1 & = (1, 0, \dots, 0, 0) \\ e_2 & = (0, 1, \dots, 0, 0) \\ \vdots & \\ e_n & = (0, 0, \dots, 0, 1) \\ \alpha_{1,n-1} & = (1, 1, \dots, 1, 0) \\ \alpha_{2,n} & = (0, 1, \dots, 1, 1) \end{array} \right\}.$$

We convert the interval vectors to the corresponding elementary vector set

$$\mathcal{E} = \{e_{1,2}, e_{2,3}, \dots, e_{n-1,n}, e_{1,n}, e_2, e_n\}.$$

From this we construct the flow-dimension graph $G_{\mathcal{P}_{n,1}} = (V, E)$ as seen in Figure 1, where $V = [n]$ and

$$E = \{(k, k+1) | k \in [n-1]\} \cup \{(1, n)\}$$

corresponding to each $e_{i,j}$ in $\mathcal{E}$. The subset of vertices $V_1 = \{2, n\}$ (circled in Figure 1) corresponds to each e_i in $\mathcal{E}$. Since the underlying graph is connected, we know

$$k_0 = \ \#\{\text{connected components } C \text{ in } G_{\mathcal{P}_{n,1}} \text{ such that } C \cap V_1 = \emptyset\} = 0.$$

Next we notice that

$$\frac{1}{n-2}e_1 + \frac{1}{n-2}e_2 + \dots + \frac{1}{n-2}e_{n-1} - \frac{1}{n-2}\alpha_{1,n-1} = \mathbf{0}$$

where the sum of the coefficients is

$$\frac{n-1}{n-2} - \frac{1}{n-1} = \frac{n-2}{n-2} = 1$$

So $\mathbf{0} \in \text{aff}(\mathcal{I})$ and by Theorem 2, $\dim(P_{n,1}) = n - k_0 = n$. □

4.1 f-Vectors of $\mathcal{P}_{n,i}$

Recall that the f-vector of a polytope tells us the number of faces a polytope has of each dimension. We will see that the f-vector of $\mathcal{P}_{n,1}$ with $n \geq 3$, is precisely the n^{th} row of the Pascal 3-triangle without 1's. The *Pascal 3-triangle* is an analogue of Pascal's Triangle, where the third row, instead of being 1 2 1, is replaced with 1 3 1, and then the same addition pattern is followed as in Pascal's triangle.

$$\begin{array}{lccccccccccc} n=1: & & & & & & 3 & & & & & \\ n=2: & & & & & 4 & & 4 & & & & \\ n=3: & & & & 5 & & 8 & & 5 & & & \\ n=4: & & & 6 & & 13 & & 13 & & 6 & & \\ n=5: & & 7 & & 19 & & 26 & & 19 & & 7 & \\ n=6: & 8 & & 26 & & 45 & & 45 & & 26 & & 8 \end{array} \tag{2}$$

The proof of this correspondence requires a few preliminary results.

Lemma 2. *Let* $\mathcal{P}_{n,1} = \text{conv}(\mathcal{I})$ *where* $\mathcal{I} := \{e_1, e_2, \ldots, e_n, \alpha_{1,n-1}, \alpha_{2,n}\}$ *with* $n \geq 3$. *Then* $\mathcal{B} = \text{conv}(e_1, e_n, \alpha_{1,n-1}, \alpha_{2,n})$ *is a 2-dimensional face of* $\mathcal{P}_{n,1}$.

Proof. We first consider $\mathcal{A} = \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$. The corresponding elementary vectors of the vertex set are $\{e_{1,n}, e_2, e_n\}$. So we build the flow-dimension graph as seen in Figure 2, $G_{\mathcal{A}} = (V, E)$ where $V = [n]$, $E = \{(1, n)\}$ corresponding to $e_{1,n}$. The subset $V_1 = \{2, n\}$ (circled in Figure 2) corresponds to e_2 and e_n. This graph has $n-1$ connected components, two of which contain elements of V_1 so that $k_0 = n-3$.

If we let $\lambda_1 e_n + \lambda_2 \alpha_{1,n-1} + \lambda_3 \alpha_{2,n} = \mathbf{0}$, we first notice that $\lambda_2 = 0$ since $\alpha_{1,n-1}$ is the only vector with a nonzero first coordinate. But this implies that $\lambda_1 = \lambda_3 = 0$. Since the coefficients cannot sum to 1, we conclude that $\mathbf{0} \notin \text{aff}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$.

So now by Theorem 2,

$$\dim(\text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})) = n - k_0 - 1 = n - (n-3) - 1 = 2.$$

Finally $e_1 = (1)\alpha_{1,n-1} + (-1)\alpha_{2,n} + (1)e_n$ is in the affine hull of $\mathcal{A}$ and does not add a dimension. Thus we conclude that $\dim(\mathcal{B}) = 2$. □

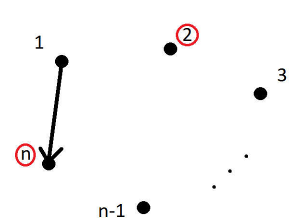


Figure 2: $G_{\mathcal{A}}$

Corollary 2. *Let $\mathcal{I}$ be as in Lemma 2. Then each e_i for $2 \leq i \leq n-1$ adds a dimension to $\mathcal{P}_{n,1}$, that is $e_i \notin \text{aff}(\mathcal{I} \setminus \{e_i\})$.*

Proof. This follows from Theorem 1 and Lemma 2. Since $\mathcal{B}$ has dimension 2 and $\mathcal{P}_{n,1}$ has dimension n, then the $n-2$ remaining vertices must add the remaining $n-2$ dimensions. Clearly none can add more than one, so each must add precisely one dimension. □

Lemma 3. *Let $\mathcal{B}$ as in Lemma 2. Then $\mathcal{B}$ has f-vector $(4, 4)$.*

Proof. Since $\mathcal{B}$ has dimension 2, $f_1 = f_0$. We know that $\{e_n, \alpha_{1,n-1}, \alpha_{2,n}\}$ are three vertices of $\mathcal{B}$ since they form a 2-dimensional object. If $e_1 \in \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$ then

$$e_1 = \lambda_1 e_n + \lambda_2 \alpha_{1,n-1} + \lambda_3 \alpha_{2,n} \tag{3}$$

where the coefficients sum to 1. Since $\alpha_{1,n-1}$ is the only vector with a nonzero coordinate in the first position, that implies $\lambda_2 = 1$. This in turn implies that $\lambda_1 = \lambda_3 = 0$, contradicting (3). So $e_1 \notin \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$ and therefore forms a fourth vertex. Thus $f_0 = 4 = f_1$ completing the proof. □

We can tie all this together with the following theorem. First we define a *d-pyramid* P^d as the convex hull of the union of a $(d-1)$-dimensional polytope K^{d-1} (the *basis* of P^d) and a point $A \notin \text{aff}(K^{d-1})$ (the *apex* of P^d).

Theorem 5 (see, e.g., [4])**.** *If P^d is a d-pyramid with $(d-1)$-dimensional basis K^{d-1} then*

$$\begin{aligned} f_0(P^d) &= f_0(K^{d-1}) + 1 \\ f_k(P^d) &= f_k(K^{d-1}) + f_{k-1}(K^{d-1}) \quad \textit{for } 1 \leq k \leq d-2 \\ f_{d-1}(P^d) &= 1 + f_{d-2}(K^{d-1}). \end{aligned}$$

We notice that the rows of Pascal's 3-triangle act in the same manner and we claim the face numbers for $\mathcal{P}_{n,1}$ can be derived from Pascal's 3-triangle.

Theorem 6. *The f-vector for $\mathcal{P}_{n,1}$ for $n \geq 3$ is the n^{th} row of the Pascal 3-triangle.*

Proof. Let $\mathcal{I} = \{e_1, e_2, \ldots, e_n, \alpha_{1,n-1}, \alpha_{2,n}\}$ be the vertex set for $\mathcal{P}_{n,1}$ with $n \geq 3$, and call $\mathcal{R}_k = \text{conv}(\mathcal{I} \setminus \{e_k, e_{k+1}, \ldots, e_{n-1}\})$ for $1 \leq k < n$. Then it is clear that $\mathcal{P}_{n,1}$ is the convex hull of the union of the $(n-1)$-dimensional polytope $\mathcal{R}_{n-1}$ and $e_{n-1} \notin \text{aff}(\mathcal{R}_{n-1})$ (by Corollary 2), and thus is a pyramid and its face numbers can be computed as in Theorem 5 from the face numbers of $\mathcal{R}_{n-1}$.

Notice next that $\mathcal{R}_{n-1}$ is the convex hull of the union of the $(n-2)$-dimensional polytope $\mathcal{R}_{n-2}$ and $e_{n-2} \notin \text{aff}(\mathcal{R}_{n-2})$ (again by Corollary 2), so we can compute the face numbers of $\mathcal{R}_{n-1}$ from those of $\mathcal{R}_{n-2}$ as in Theorem 5.

We can continue this process until we get that $\mathcal{R}_3$ is the convex hull of $\mathcal{R}_2$ and $e_2 \notin \text{aff}(\mathcal{R}_2)$. But we notice that $\mathcal{R}_2 = \mathcal{B}$, so by Lemma 3, $f_0(\mathcal{R}_2) = f_1(\mathcal{R}_2) = 4$. From here we can build using f-vectors of $\mathcal{P}_{n,1}$ from Theorem 5 which are exactly those of the Pascal 3-triangle. We do this $n-1$ times to reach $\mathcal{P}_{n,1}$, and since (4,4) is the second row of the triangle, then the f-vector of $\mathcal{P}_{n,1}$ is the n^{th} row of the Pascal 3-triangle, as desired. □

Corollary 3. *For $n \geq 3$, $\mathcal{P}_{n,1}$ is self dual.*

We can rewrite (2) as

$$
\begin{array}{ccccccccccc}
 & & & & & 2+1 & & & & & \\
 & & & & 3+1 & & 3+1 & & & & \\
 & & & 4+1 & & 6+2 & & 4+1 & & & \\
 & & 5+1 & & 10+3 & & 10+3 & & 5+1 & & \\
 & 6+1 & & 15+4 & & 20+6 & & 15+4 & & 6+1 & \\
7+1 & & 21+5 & & 35+10 & & 35+10 & & 21+5 & & 7+1
\end{array} \tag{4}
$$

which is Pascal's triangle added to a shifted Pascal's triangle. Thus we can derive a closed formula for the number of k-faces for $\mathcal{P}_{n,1}$:

Corollary 4. *For $n \geq 3$, $f_k(\mathcal{P}_{n,1}) = \binom{n-1}{k} + \binom{n+1}{k+1}$.*

5 Volume of $\mathcal{P}_{n,1}$

Lemma 4. *The determinant of the $n \times n$-matrix*

$$
\begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
 & & \ddots & & \\
1 & \cdots & 1 & 0 & 1 \\
1 & 1 & \cdots & 1 & 0
\end{bmatrix}
$$

is $(-1)^{n-1}(n-1)$.

Proof. Let A_n be the $n \times n$ matrix whose diagonal entries are 0, and all entries off the diagonal are 1. E.g.,

$$
A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
$$

and so $\det(A_2) = -1$. Assume $\det(A_k) = (-1)^{k-1}(k-1)$. A_{k+1} is the $(k+1) \times (k+1)$-matrix of the form

$$
A_{k+1} = \begin{bmatrix}
0 & 1 & \cdots & 1 & 1 \\
1 & 0 & 1 & \cdots & 1 \\
 & & \ddots & & \\
1 & 1 & \cdots & 0 & 1 \\
1 & 1 & \cdots & 1 & 0
\end{bmatrix}.
$$

Subtracting the second row from the first, which does not change the value of the determinant, will give us the matrix

$$\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 \\ & & \ddots & & \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

Now the determinant of A_{k+1} is the sum of two determinants by cofactor expansion. Specifically it is $(-1)\det(A_k)$ minus the determinant of the matrix obtained by taking out the first row and second column. We know that $(-1)\det(A_k) = (-1)^k(k-1)$ by the inductive hypothesis. So what we have left to compute is the determinant of the $(k \times k)$-matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ & & \ddots & & \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

We will subtract the first row from each of the rows below it, also not changing the determinant, to give us the upper triangular matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ & & & \ddots & & \\ 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}$$

whose determinant is $(-1)^{k-1}$. Furthermore,

$$\begin{aligned} \det(A_{k+1}) &= (-1)\det(A_k) - (-1)^{k-1} \\ &= (-1)^k(k-1) + (-1)^k \\ &= (-1)^k k. \end{aligned}$$

Therefore, by induction, $\det(A_n) = (-1)^{n-1}(n-1)$, for all $n \in \mathbb{Z}_{\geq 2}$. □

Theorem 7. $\operatorname{vol}(\mathcal{P}_{n,1}) = 2(n-2)$ *for* $n \geq 3$

Proof. In order to calculate the volume of $\mathcal{P}_{n,1}$ we will first triangulate the 2-dimensional base of the pyramid from Lemma 2

$$\triangle_1 = \operatorname{conv}(e_1 e_n \alpha_{1,n-1}) \text{ and } \triangle_2 = \operatorname{conv}(e_n \alpha_{1,n-1} \alpha_{2,n}).$$

Let x be a point in the base, then for some $\lambda_i \geq 0$, where $\sum_{i=1}^{4} \lambda_i = 1$,

$$\begin{aligned} x &= \lambda_1 e_1 + \lambda_2 e_n + \lambda_3 \alpha_{1,n-1} + \lambda_4 \alpha_{2,n} \\ &= (\lambda_1 + \lambda_3, \lambda_3 + \lambda_4, \cdots, \lambda_3 + \lambda_4, \lambda_2 + \lambda_4) \\ &= (\lambda_1 - \lambda_4) e_1 + (\lambda_2 + \lambda_4) e_n + (\lambda_3 + \lambda_4) \alpha_{1,n-1} \\ &= (\lambda_1 + \lambda_2) e_n + (\lambda_1 + \lambda_3) \alpha_{1,n-1} + (\lambda_4 - \lambda_1) \alpha_{2,n}. \end{aligned}$$

So x is a point in $\triangle_1$ if $\lambda_1 \geq \lambda_4$ and x is a point in $\triangle_2$ if $\lambda_4 \geq \lambda_1$. Thus $\triangle_1$ and $\triangle_2$ is a triangulation of the 2-dimensional base of the pyramid.

By Corollary 2, each $e_2, \cdots, e_{n-1}$ adds a dimension so that the convex hull of these points and $\triangle_1$ is an n-dimensional simplex. The same can be said of $\triangle_2$. Call these simplices S_1 and S_2 respectively. Thus S_1 and S_2 triangulate $\mathcal{P}_{n,1}$. Therefore the sum of their volumes is equal to the volume of $\mathcal{P}_{n,1}$. In order to calculate the volume of S_1 and S_2, we will use the Cayley Menger determinant [3] once again. Consider S_1, whose volume is the determinant of the matrix

$$\begin{bmatrix} e_1 - \alpha_{1,n-1} & e_2 - \alpha_{1,n-1} & \dots & e_n - \alpha_{1,n-1} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 0 & -1 & \cdots & -1 & -1 \\ -1 & -1 & 0 & -1 & \cdots & -1 \\ & & & \ddots & & \\ -1 & -1 & \cdots & -1 & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Cofactor expansion on the last row will leave us with the determinant, up to a sign, of the $(n-1) \times (n-1)$ matrix

$$\begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ & & \ddots & & \\ -1 & \cdots & -1 & 0 & -1 \\ -1 & -1 & \cdots & -1 & 0 \end{bmatrix}, \tag{5}$$

which when ignoring sign by Lemma 4 is $n-2$. Therefore the volume of S_1 is $n-2$.

Now consider the Cayley Menger determinant of S_2, the determinant of

$$\begin{bmatrix} \alpha_{1,n-1} - \alpha_{2,n} & e_2 - \alpha_{2,n} & e_3 - \alpha_{2,n} & \cdots & e_n - \alpha_{2,n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & -1 & 0 & -1 & \dots & -1 \\ & & & \ddots & & \\ 0 & -1 & -1 & \cdots & 0 & -1 \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{bmatrix}.$$

By cofactor expansion on the first row we are left with the positive determinant of the matrix (5) which is $n-2$. Therefore the volume of S_2 is $n-2$ and so the volume of $\mathcal{P}_{n,1}$ is $2(n-2)$, as desired. □

6 Conclusion

We have looked at several interval-vector polytopes, including complete interval-vector polytopes, fixed interval-vector polytopes, and $\mathcal{P}_{n,1}$. The volume of the n-dimensional complete interval-vector polytope is the n^{th} Catalan number. We also formed a bijection between the complete interval-vector polytope and Postnikov's complete root polytope. We proved that the fixed interval-vector polytope with interval length i is an $(n-i)$-dimensional unimodular simplex. Finally, $\mathcal{P}_{n,1}$ is a pyramid and its f-vector is the n^{th} row of the Pascal 3-triangle. Also, the volume of $\mathcal{P}_{n,1}$ is $2(n-2)$.

Because of these properties of $\mathcal{P}_{n,1}$, we studied the related polytopes

$$\mathcal{P}_{n,i} := \text{conv}(e_1, \ldots, e_n, \alpha_{1,n-i}, \alpha_{2,n-i+1}, \ldots, \alpha_{i+1,n}).$$

We observed that the f-vectors of $\mathcal{P}_{n,i}$ correspond to the sum of multiple shifted Pascal triangles. Also, we conjecture that the volume of $\mathcal{P}_{n,i}$ is equal to $2^i(n-(i+1))$. Our future work entails proving this volume conjecture and establishing a more concrete conjecture regarding the f-vectors of $\mathcal{P}_{n,i}$.

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