A Bijection from Shi Arrangement Regions to Parking Functions via Mixed Graphs

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Abstract

Consider all the different regions in three dimensions bounded by the planes $x_1 = x_2$, $x_1 = x_3$ and $x_2 = x_3$. This is a three-dimensional *braid arrangement*. More generally, in dimension n, B_n is the braid arrangement of hyperplanes of the form $x_i = x_j$ for $1 \leq i < j \leq n$. A *Shi arrangement*, $\text{Shi}(n)$ is an expansion of B_n that includes the hyperplanes $x_i = x_j + 1$. Pak and Stanley have shown that there exists a bijection between the regions of $\text{Shi}(n)$ and a combinatorial object called *parking functions* of length n where both objects have the cardinality of $(n+1)^{n-1}$. Our goal is to expand on the work of Pak and Stanley and provide a different approach to this bijection. We will establish our bijection by linking the regions of $\text{Shi}(n)$ and parking functions to *mixed graphs*, which are well known in the area of Graph Theory. This work provides an exciting link among three areas of mathematics.

1 Introduction

Consider a one-way street that has n available parking spots, ordered $1, 2, \ldots, n$. Imagine that there are n cars entering this street and want to park in one of the n parking spots. Each car has a parking spot preference $i \in \{1, 2, \ldots, n\}$, If a car's preferred parking spot is filled, it must continue past the i^{th} spot until it finds an empty spot or reaches the end, unable to park. A sequence of parking preferences in which every car can park is a *parking function*. We define $\mathcal{P}_n$ to be the set of all parking functions of length n. For example, when the number of cars is 3, the sequence $(1, 3, 3)$ states that car 1 wants to park in spot 1, car 2 wants to park in spot 3, and car 3 wants to park in spot 3. However, car 3 cannot park because car 2 has filled the third parking spot. Thus $(1, 3, 3)$ is not a parking function because not all cars were able to park. On the other hand, the sequence $(1, 3, 2)$ is a parking function because each car can successfully fill a spot. Notice that $(3, 2, 1)$ is also a parking function. In fact, only the set of preferences matter, not the preference of each car. Stanley defines a parking function of length n as a sequence $P = (p_1, p_2, \ldots, p_n)$ such that if $(q_1, q_2, \ldots, q_n)$ is a permutation of P where $q_1 \leq q_2 \leq \cdots \leq q_n$ then for all $1 \leq i \leq n$, $q_i \leq i$ [4]. It is known that the number of parking functions is $(n+1)^{n-1}$ [1].

A *hyperplane arrangement* in $\mathbb{R}^n$ is a finite set of hyperplanes in $\mathbb{R}^n$ [2]. Many hyperplane arrangements are connected to the symmetric group. One such special arrangement, the *braid arrangement* B_n, is the arrangement of hyperplanes of the form $x_i = x_j$ for $1 \leq i < j \leq n$ [5]. An expansion of B_n is the hyperplane arrangement of the form

$$x_s - x_t = 0,\ x_s - x_t = 1, \text{ for } 1 \leq s < t \leq n.$$

This expansion is known as the *Shi arrangement*, denoted by Shi(n) [5]. The set $\mathcal{R}_n$ contains the *regions* of $Shi(n)$, which are the connected components of $\mathbb{R}^n$ with the hyperplanes of Shi(n) removed. It has been shown that Shi(n) partitions $\mathbb{R}^n$ into $(n+1)^{n-1}$ regions [2].

Pak and Stanley have established a bijection between parking functions and the regions of $Shi(n)$, a result prompted by the fact that both objects have the same size $(n+1)^{n-1}$ [5]. Athanasiadis and Linusson have also found a bijection between the two objects through a different method [1]. The purpose of this paper is to establish a new bijective function

$$\Upsilon : \mathcal{R}_n \longrightarrow \mathcal{P}_n$$

that links the regions of the Shi arrangement to the set of parking functions.

Let the set of labeled complete mixed graphs $\mathcal{M}_n$ be defined as the set of graphs whose n vertices are labelled $\{1, 2, \ldots, n\}$ and have directed or undirected edges between each pair of vertices. We will prove Υ is a bijection by introducing an intermediary bijection involving a specific subset $\mathcal{B}_n$ of $\mathcal{M}_n$.

We will begin this paper by considering Υ as being composed of two separate bijections. We first (in section 2) consider the function $\varphi : \mathcal{R}_n \longrightarrow \mathcal{B}_n$ that maps the regions of Shi(n) to a subset of the labeled complete mixed graphs with n vertices. We will later (in section 3) consider another bijection, $\Omega : \mathcal{B}_n \longrightarrow \mathcal{P}_n$ that maps the same subset of labeled complete mixed graphs with n vertices to the set of parking functions. It will follow that $\Upsilon = \Omega \circ \varphi$ is a bijection.

For the purpose of this paper, given two vertices s and t of a labeled mixed graph, we will denote a directed edge from s to t as $\vec{st}$, and an undirected edge between s and t as st. Also, we denote $E(G)$ as the edges and $V(G)$ as the vertices of the graph G .

2 A Bijection From Regions of Shi(n) to Mixed Graphs

Consider $R \in \mathcal{R}_n$. We denote $\mathcal{I}(R)$ as the set of inequalities which define R. $\mathcal{I}(R)$ contains inequalities of the form $x_s - x_t < 0$, $0 < x_s - x_t < 1$, or $x_s - x_t > 1$, for each $1 \leq s < t \leq n$.

We want to construct a unique labeled mixed graph for each region of $Shi(n)$. We construct the labeled mixed graph, $\varphi(R)$, as follows: Let $s < t$. If $x_s - x_t < 0 \in \mathcal{I}(R)$, then $\vec{st} \in E(\varphi(R))$. If $0 < x_s - x_t < 1 \in \mathcal{I}(R)$, then $st \in E(\varphi(R))$. If $x_s - x_t > 1 \in \mathcal{I}(R)$, then $\vec{ts} \in E(\varphi(R))$. Since $\mathcal{I}(R)$ contains an inequality relating x_s and x_t for all $1 \leq s < t \leq n$, $\varphi(R)$ is a complete labeled mixed graph.

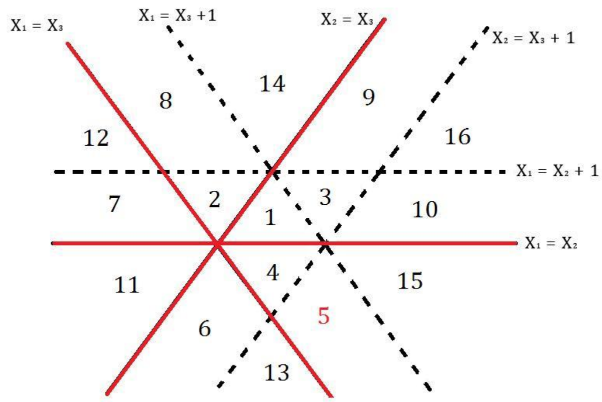
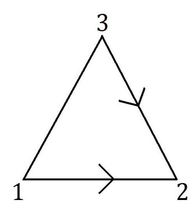


Figure 1: $Shi(3)$ is composed of B_3, the solid lines, together with the dotted lines.

Example 1. *In Figure* 1, $Shi(3)$ *is intersected with the hyperplane* $x_1 + x_2 + x_3 = 0$, *which shows the* $(3+1)^{(3-1)} = 16$ *regions of* $Shi(3)$. *Given the inequalities* $0 < x_1 - x_3 < 1$, $x_1 - x_2 < 0$, $x_2 - x_3 > 1 \in \mathcal{I}(5)$ *in Figure 1, the labeled mixed graph* $\varphi(5)$ *is*



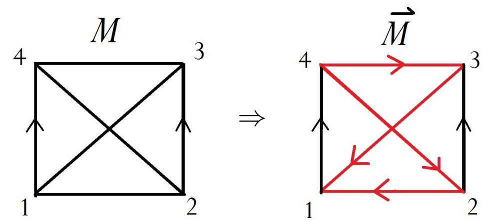
Using this method of construction, clearly φ is well defined. The following lemma shows φ is injective.

Lemma 1. *For every region* R *of* Shi(n) *there is a unique labeled mixed graph representation* $\varphi(R)$.

Proof. Suppose that there exist two distinct regions R_1 and R_2 of Shi(n) that give rise to the same labeled mixed graph $M = \varphi(R_1) = \varphi(R_2)$. By definition, $\mathcal{I}(R_1) \neq \mathcal{I}(R_2)$. Since $R_1 \neq R_2$ there exist s and t such that the inequality relationship between x_s and x_t is different between R_1 and R_2. This produces a different orientation on the edge between s and t. Thus R_1 and R_2 do not give rise to the same labeled mixed graph. By contradiction, no two regions R_1 and R_2 of Shi(n) can give rise to the same M. □

Let $\vec{M}$ be the complete directed graph defined by directing the undirected edges of M from t to s, where $s < t$. A directed graph is acyclic if it contains no cycles [3]. The *source* of a graph G is a vertex with no edges directed towards it. Similarly, a *sink* of a graph G is the vertex with all edges directed towards it. A labeled mixed graph M is acyclic if $\vec{M}$ is acyclic. The *source* of an acyclic graph M, is the source of $\vec{M}$. In addition the *sink* of M is the sink of $\vec{M}$. The *in-degree*, denoted $\#in(v)$, of a vertex $v \in \vec{M}$ is the number of edges directed toward v. The in-degree of $v \in M$ is the in-degree of $v \in \vec{M}$.

Example 2. *How to orient a labeled mixed graph M.*



We will show later that every complete acyclic graph has a unique source and a unique sink, signifying that this is well defined. Let $\mathcal{B}_n$ be the set of acyclic labeled mixed graphs with n vertices containing no subgraphs with an undirected edge between the source and sink and a directed edge from t to s, with $s < t$.

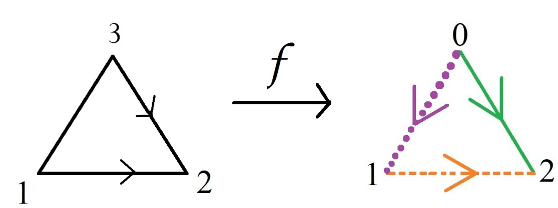
Theorem 1. *The function $\varphi : \mathcal{R}_n \longrightarrow \mathcal{B}_n$ is a bijection.*

We have shown that φ is well defined and injective. However, it remains to show that $\text{Im}(\varphi) = \mathcal{B}_n$. In the next section, we call on the use of *tri-colored graphs* to prove surjectivity.

2.1 Tri-Colored Graphs

We relabel the elements of $\mathcal{B}_n$ via the bijection f as follows: Let $s < t$. A directed edge $\vec{ts}$ will be colored green (solid), and a directed edge $\vec{st}$ will be colored orange (dashed). Lastly, an undirected edge st will be directed to form $\vec{ts}$ and also colored purple (dotted). This assignment relabels $\mathcal{B}_n$ into $\mathcal{A}_n$ where $\mathcal{A}_n$ is the set of acyclic directed complete colored graphs containing no subgraph with both a purple edge between the source and sink and a green edge. We call this condition the *purple-green condition*. The following example shows how to color a mixed graph:

Example 3.



The function f maps $B \in \mathcal{B}_n$ to an acyclic directed graph. We will now prove that a directed graph with n vertices is acyclic if and only if it has a unique vertex v_i with in-degree i for each $i \in \{0, 1, \ldots, n-1\}$. This fact will allow us to construct a bijection $h : \mathcal{A}_n \longrightarrow \mathcal{R}_n$.

Lemma 2. *Every directed acyclic complete graph G has a source.*

Proof. Consider the directed acyclic complete graph with one vertex. Then this vertex is a source. Suppose by induction, that all directed acyclic complete graphs with n vertices contain a source. Consider a directed acyclic complete graph G with $n+1$ vertices. Select v, a vertex of G with the lowest in-degree. Let G' be G without the vertex v. By our inductive hypothesis G' has a source, w. Hence, the number of edges directed toward w in G' is zero. It follows that the number of edges directed toward w in G is less than or equal to 1. In G $\#in(v) \leq \#in(w) \leq 1$. Suppose $\#in(v) = 1$, for G. This will result in one of two cases: If $\vec{wv} \in E(G)$ then w is the source, meaning $\#in(w) = 0$. Therefore w has the smallest in-degree and cannot be in G' which is a contradiction. Furthermore, if $\vec{vw} \in E(G)$ then there exist $x \in G$ such that $\vec{xv} \in E(G)$. However, we also know that $x \in G'$. Since w is the source of G', $\vec{wx} \in E(G') \subset E(G)$. This results in a cycle created by $\vec{vw}$, $\vec{wx}$, and $\vec{xv}$. Thus $\#in(v) \neq 1$, which implies that $\#in(v) = 0$, and so v is a source. □

Lemma 3. *A directed graph with n vertices is acyclic if and only if it has a unique vertex v_i with $\#(in(v_i)) = i$, for each $0 \leq i \leq n-1$.*

Proof. ($\Rightarrow$) Let G be a directed acyclic complete graph with 1 vertex, and so there exists a unique source $v_0 \in V(G)$. Suppose by induction that the lemma holds for all directed acyclic complete graphs with n vertices. Let G be a complete directed acyclic graph with $n+1$ vertices. Let $G' = G \backslash v_0$ By our inductive hypothesis, we know that for each $0 \leq i \leq n-1$ there exists a unique $v_i \in V(G')$ such that $\#(in(v_i)) = i$. Now add v_0 to G' and direct all edges away from v_0 to get back G, adding 1 to each in-degree of the vertex set of G'. Therefore G has a vertex v_i such that $\#(in(v_i)) = i$ for all $0 \leq i \leq n$.
($\Leftarrow$) Let G be a complete directed graph with 1 vertex. Then G is acyclic. Suppose by induction that the lemma holds for all complete directed graphs with n vertices. Let G be a complete directed graph with $n+1$ vertices such that for each $0 \leq i \leq n$ there exists a vertex v_i with $\#in(v_i) = i$. Let G' be G without the vertex v_0. Then by the inductive hypothesis, G' is acyclic. Now add v_0 to G' and direct all edges away from v_0 to get back to G. Since all edges are directed away from v_0 there is no cycle involving v_0, and since G' had no cycles, there are no cycles in G. □

Corollary 1. *Any complete directed acyclic oriented graph G has a unique source and a unique sink.*

We are now ready to define the function $h : \mathcal{A}_n \longrightarrow \mathcal{R}_n$. For any graph $A \in \mathcal{A}_n$, $h(A)$ is the region containing a point $(x_1, x_2, \ldots, x_n)$ that satisfies the following conditions: there exists a $\sigma \in S_n$ such that $x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}$, $\#in(i) = \sigma^{-1}(i) - 1$, and $x_{\sigma(j)} - x_{\sigma(i)}$ is given by the color of $\overrightarrow{\sigma(i)\sigma(j)} \in E(A) as follows$. Let h be defined as the map that correlates $E(\mathcal{A}_n)$ to $\mathcal{I}(R)$. More specifically, an orange directed edge $\vec{st}$ maps to the inequality $x_s - x_t < 0$, the purple directed edge st corresponds to the inequality $0 < x_s - x_t < 1$, and the green directed edge $\vec{ts}$ maps to the inequality $x_s - x_t > 1$.

Lemma 4. *The map h is well defined.*

Proof. Given a labeled mixed graph $A \in \mathcal{A}_n$, by Lemma 3 there exists $\sigma \in S_n$ such that $x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}$, with $\#in(i) = \sigma^{-1}(i) - 1$, i.e. we can order the x_i by their in-degree. Therefore, the only contradictory set of inequalities that can occur are of the form $0 < x_{\sigma(i)} - x_{\sigma(m)} < 1$ and $x_{\sigma(j)} - x_{\sigma(k)} > 1$, where $m \leq k < j \leq i$. But these inequalities correspond to A having the purple-green condition, which is forbidden. Therefore $h(A)$ must map to a region $R \in \mathcal{R}_n$. The uniqueness follows by construction since $h(A)$ is defined by one particular set of inequalities. □

Theorem 2. *The map $h : \mathcal{A}_n \longrightarrow \mathcal{R}_n$ is a bijection.*

Proof. By Lemma 4, h is well defined. Next we will construct the inverse of h. Define $g : \mathcal{R}_n \longrightarrow \mathcal{A}_n$ as follows: Consider the region R of Shi(n) represented by the point $(x_1, x_2, \ldots, x_n)$. Now consider K_n with vertices labeled $\{1, 2, \ldots, n\}$. To each vertex k, assign the value x_k. There exists some permutation $\sigma \in S_n$, such that $x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}$. Direct the edges of K_n such that $\#in(k) = \sigma^{-1}(k) - 1$. Since $\sigma^{-1}(k) - 1$ runs through $\{0, 1, \ldots, n-1\}$ as k runs through $\{1, 2, \ldots, n\}$ we get that K_n is acyclic by Lemma 3. Using our coloring scheme, color the edges of K_n. Our resulting colored graph, A, does not have the purple-green condition since it came from $\mathcal{R}_n$. Therefore, $g(A)$ is well defined.

By construction, $h \circ g(R) = R$ for all $R \in \mathcal{R}_n$ and $g \circ h(A) = A$ for all $A \in \mathcal{A}_n$. Therefore, $g = h^{-1}$, so h is a bijection. □

Proof of Theorem 1. We have shown that φ is injective by Lemma 1, and that the functions f and h are both bijections. What is left to show is that $(h \circ f)^{-1} = \varphi$. We know that for a region $R \in \mathcal{R}_n$, $\varphi(R) = B \in \mathcal{B}_n$ and $h^{-1}(R) = A \in \mathcal{A}_n$. However, we have precisely defined h so that

$$h^{-1}(R) = f(B) \Rightarrow h^{-1}(R) = f(\varphi(R)) \Rightarrow \varphi(R) = f^{-1} \circ h^{-1}(R) = (h \circ f(R))^{-1}.$$

□

3 An Injection from Mixed Graphs to Parking Functions

Recall the definition of a parking function as a sequence $P = (p_1, p_2, \ldots, p_n)$ such that if $(q_1, q_2, \ldots, q_n)$ is a permutation of P where $q_1 \leq q_2 \leq \cdots \leq q_n$ then for all $1 \leq i \leq n$, $q_i \leq i$.

For any graph $B \in \mathcal{B}_n$, let $\{1, 2, \ldots, n\}$ be the set of vertices of B. Define $\tau(i) = \#in(i) + 1$. Then define $\Omega : \mathcal{B}_n \to \mathcal{P}_n$ by $\Omega(B) = (\tau(1), \tau(2), \ldots, \tau(n))$.

We first will prove that Ω is well defined and injective.

Lemma 5. *$\Omega : \mathcal{B}_n \to \mathcal{P}_n$ is well defined.*

Proof. By Lemma 3, we know that for any complete directed acyclic graph $\vec{B} \in \mathcal{B}_n$ the in-degree of each vertex is labeled from $\{0, 1, \ldots, n-1\}$. Let B be any mixed graph in $\mathcal{B}_n$ such that the directed version of B is $\vec{B}$. By construction, $\Omega(\vec{B})$ yields a parking function that is a permutation of $(1, 2, \ldots, n)$. Since the set of in-degrees of the vertices of B is $\{i_1, i_2, \ldots, i_n\}$ where $i_k \leq k-1$, then $\Omega(B)$ is a permutation of $(\tau(1), \tau(2), \ldots, \tau(n))$ where $\tau(i_k) \leq k$. Therefore $\Omega(B)$ is a parking function, which indicates that Ω is well defined. □

Theorem 3. *$\Omega : \mathcal{B}_n \longrightarrow \mathcal{P}_n$ is an injection.*

Proof. Consider the set $\mathcal{B}_1$. By construction, $\Omega : \mathcal{B}_1 \to \mathcal{P}_1$ is an injection. Similarly, it is clear to see that $\Omega : \mathcal{B}_2 \to \mathcal{P}_2$ is also an injection. Depicted below in Figure 2 are the regions of $Shi(3)$ with their corresponding mixed graphs given by φ and their corresponding parking functions given by Ω. This figure gives an explicit bijection between the three objects for $n = 3$, and in particular it shows Ω is an injection for $n = 3$. Now suppose that $\Omega : \mathcal{B}_k \to \mathcal{P}_k$ is an injection. For our induction step, we want to show that $\Omega : \mathcal{B}_{k+1} \to \mathcal{P}_{k+1}$ is an injection.

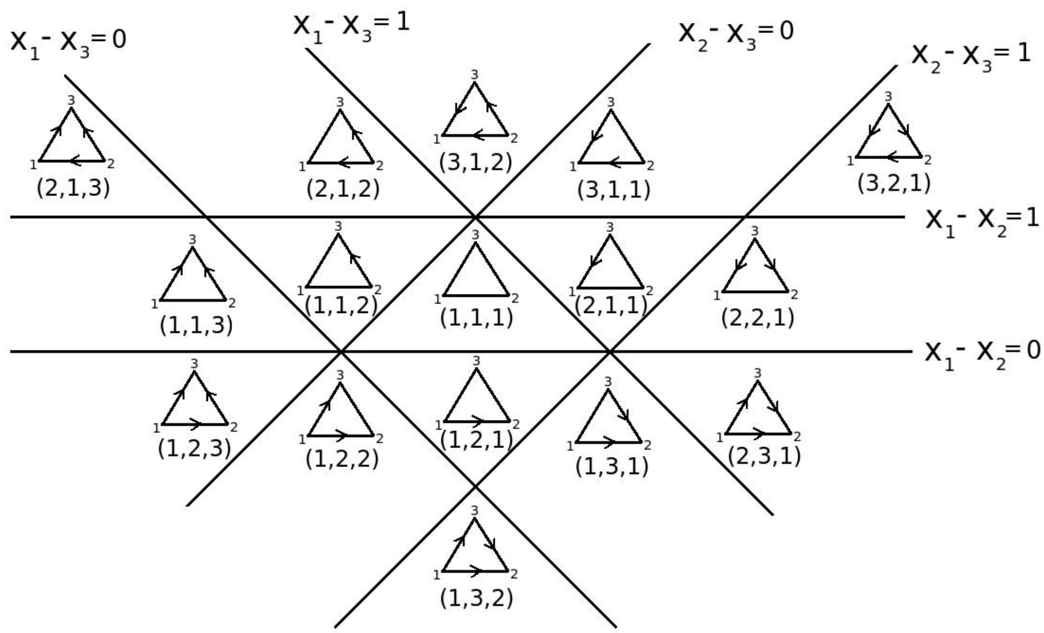


Figure 2: Regions of $Shi(3)$ with $\mathcal{B}_3, \mathcal{P}_3$.

Suppose towards a contradiction that $\Omega : \mathcal{B}_{k+1} \to \mathcal{P}_{k+1}$ is not an injection. This implies that there exist two distinct graphs $B, C \in \mathcal{B}_{k+1}$ such that $\Omega(B) = \Omega(C)$. This means the in-degree of every vertex m of B must be the same as the in-degree of vertex m in C. Some vertex k of B and C is the sink. We now remove k from both B and C to create two new graphs denoted B' and C'. By properties of the sink, the in-degree of the remaining vertices of B' and C' remain unaffected. This implies that $\Omega(B') = \Omega(C')$, which by our inductive hypothesis means that $B' = C'$. Next add vertex k to B' and C' and direct all edges accordingly to obtain B and C again. By our assumption that $B \neq C$, yet $B' = C'$, there must be at least two edges connected to k that differ between B and C. Since k is the sink, there are no connected edges directed away from k.

Furthermore, if all edges were directed towards k, or no edges connected to k were directed, then it would follow that $B = C$. Thus there must be at least one undirected edge connected to k and at least one edge directed toward k that are switched between B and C, i.e., there exists $ak, \vec{bk} \in E(B)$ and $\vec{ak}, bk \in E(C)$.

Consider the subgraphs $abk = S_1 \subset B$ and $abk = S_2 \subset C$ as elements of $\mathcal{B}_3$. By construction, the vertices of S_1 and the vertices of S_2 have the same in-degree. Therefore, $\Omega(S_1) = \Omega(S_2)$ signifies that $S_1 = S_2$ since Ω for $n = 3$ is injective, as seen in Figure 2. Since $ak \in E(S_1)$ and $\vec{ak} \in E(S_2)$, $S_1 \neq S_2$ which is a contradiction. This indicates that $B = C$. Therefore $\Omega : \mathcal{B}_{k+1} \to \mathcal{P}_{k+1}$ is an injection. □

Surjectivity follows from the fact that $|\mathcal{P}_n| = |Shi(n)|$ and that φ is a bijection from Shi(n) to $\mathcal{B}_n$. Therefore Ω is bijective.

4 Bijection from Regions of Shi to Parking Functions

We define the bijection $\Upsilon : \mathcal{R} \longrightarrow \mathcal{P}$ as $\Upsilon = \Omega \circ \varphi(R)$. Since Ω is bijective and φ is bijective, it follows that Υ is bijective. Hence we have established our bijective function from regions of the Shi arrangement to parking functions via mixed graphs.

5 Future Work

We would like to investigate defining Ω^{-1} explicitly instead of claiming surjectivity solely based on cardinality. In addition, there are other Shi-type hyperplane arrangements that are in bijection with an object analogous to parking functions. The method we used to construct this bijection might be useful in proving other such bijections.

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References

[1] Christos A. Athanasiadis and Svante Linusson. A simple bijection for the regions of the Shi arrangement of hyperplanes. *Discrete Math.*, 204(1-3):27–39, 1999.

[2] Jian Yi Shi. Sign types corresponding to an affine Weyl group. *J. London Math. Soc. (2)*, 35(1):56–74, 1987.

[3] Richard P. Stanley. Acyclic orientations of graphs. *Discrete Math.*, 5:171–178, 1973.

[4] Richard P. Stanley. Parking functions and noncrossing partitions. *Electron. J. Combin.*, 4(2):Research Paper 20, approx. 14 pp. (electronic), 1997. The Wilf Festschrift (Philadelphia, PA, 1996).

[5] Richard P. Stanley. Hyperplane arrangements, parking functions and tree inversions. In *Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996)*, volume 161 of *Progr. Math.*, pages 359–375. Birkhäuser Boston, Boston, MA, 1998.